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On asymptotic behaviour of the dynamical systems generated by von Foerster-Lasota equations

by

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Abstract: We consider two semidynamical systems, $(\widetilde{T}_t)_{t\geqslant 0}$ and $(T_t)_{t\geqslant 0}$, generated by different partial differential equations of von Foerster-Lasota type. We investigate some of their common properties in the integrable space with the p-exponent. We show that their chaotic or stable behaviour depends on the common value of the parameter $\gamma = \lambda(0)$.

Keywords: von Foerster-Lasota equation, chaos, stability.

1. Introduction

We investigate von Foerster-Lasota equations which are a part of mathematical description of the reproduction of a population of red blood cells. Such description appeared in Ważewska-Czyżewska, Lasota (1976), and found application in research on anemia (Lasota, Mackey, Ważewska-Czyżewska, 1981). There was a high degree of matching between this mathematical theory and medical experience. The chaotic behaviour of this equation is still the matter of interest of many mathematicians: Lasota, Pianigiani (1977), Rudnicki (1985), Łoskot (1985), Lasota, Szarek (2004), as well as Dawidowicz (1982, 1983), Dawidowicz, Poskrobko (2006), or Brzeźniak, Dawidowicz (2006). Research of the periodic or chaotic solutions of the Lasota equation is interesting from the medical point of view. So, we fix our attention on the existence of such solutions, the problem of chaos and stability of the equation in $\widehat{L}^p$ space, i.e. the space of functions, which are close to zero in a neighbourhood of zero point.

2. Formulation of the problem

We consider two semidynamical systems, $(\widetilde{T}_t)_{t\geqslant 0}$ and $(T_t)_{t\geqslant 0}$. They are connected with two partial differential equations. We wish to investigate some common properties of these systems.

DEFINITION 1 A function $v_0 \in V$ is a *periodic point* of the semigroup $(T_t)_{t\geqslant 0}$ with a period $t_0 \geqslant 0$ if and only if $T_{t_0}v_0 = v_0$. A number $t_0 > 0$ is called a *principal period* of a periodic point v_0 if and only if the set of all periods of v_0 is equal $\mathbb{N}t_0$.

DEFINITION 2 The semigroup $(T_t)_{t\geqslant 0}$ is *strongly stable* in V iff for every $v \in V$,

$$\lim_{t\to\infty} T_t v = 0 \quad \text{in } V.$$

DEFINITION 3 The semigroup $(T_t)_{t\geqslant 0}$ is *exponentially stable* on V iff there exists $D < \infty$ and $\omega > 0$ such that

$$\|T_t\| \leqslant D\mathrm{e}^{-\omega t}, \quad \text{for } t \geqslant 0$$

where $\|.\|$ is the norm of V.

First, we consider the partial differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \gamma u, \quad t \geqslant 0,\ 0 \leqslant x \leqslant 1 \tag{1}$$

with the initial condition

$$u(0,x) = v(x), \quad 0 \leqslant x \leqslant 1 \tag{2}$$

where v belongs to some normed vector space V of functions defined on $[0,1]$. The function $\widetilde{T}_t$ is given by the formula

$$(\widetilde{T}_t v)(x) = \widetilde{u}(t,x) = \mathrm{e}^{\gamma t}v(x\mathrm{e}^{-t}), \quad x \in [0,1] \tag{3}$$

where $\widetilde{u}$ is the unique solution of (1) and (2).
The second considered partial differential equation is

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda(x)u, \quad t \geqslant 0,\ 0 \leqslant x \leqslant 1 \tag{4}$$

with the initial condition

$$u(0,x) = v(x), \quad 0 \leqslant x \leqslant 1 \tag{5}$$

where v belongs to some normed vector space V of functions defined on $[0,1]$ and $\lambda : [0,1] \to \mathbb{R}$ is given continuous function. Let a semidynamical system

$$T_t : V \to V$$

be given by the formula

$$(T_t v)(x) = u(t, x).$$

It is clear that the unique solution of (4), (5) is given by the formula

$$(T_t v)(x) = u(t, x) = \mathrm{e}^{g(x)}\mathrm{e}^{-g(x\mathrm{e}^{-t})}v(x\mathrm{e}^{-t}), \quad x \in [0, 1] \tag{6}$$

where

$$g(x) = -\int_x^1 \frac{\lambda(s)}{s} ds$$

with the condition

$$\int_0^1 \frac{\lambda(s)}{s} ds = \infty. \tag{7}$$

This can be found in Dawidowicz, Poskrobko (2006). There exists a connection between these two equations. It is easy to check that if u and $\widetilde{u}$ are the solutions of the equations (4) and (1), respectively, we have the equality

$$\widetilde{u}(t, x) = \kappa(x)u(t, x), \tag{8}$$

where

$$\kappa(x) = \mathrm{e}^{\int_0^x \frac{\lambda(0)-\lambda(s)}{s} ds} \quad \text{and} \quad \gamma = \lambda(0). \tag{9}$$

All properties of the systems $(\widetilde{T}_t)_{t\geqslant 0}$ and $(T_t)_{t\geqslant 0}$ depend on the value of the constant $\gamma = \lambda(0)$. In Brzeźniak, Dawidowicz (2006), Dawidowicz, Poskrobko (2006) the chaotic and stable behaviour of these systems in V_α (the subspace of Hölder continuous functions) and L^p space was described. The main results of this paper are theorems relating to similar behaviour (chaos and stability), but in spaces $\widetilde{L}^p$ and $\widehat{L}^p$.

Let v be a continuous function on $[0, 1]$ such that $v(0) = 0$. For any interval $A \subset (0, 1]$ define

$$\mathcal{S}_A(v) = \sup_{x\in A} \left(\frac{1}{x}\int_0^x |v(s)|^p ds\right)^{\frac{1}{p}}.$$

Definition 4 Denote by $\widetilde{L}^p$ the space of all functions $v \in L^p(0, 1)$ $(1 \leqslant p < \infty)$ satisfying the following condition: $\mathcal{S}_{(0,1]}(v) < \infty$.

Proposition 1 *The space* $\widetilde{L}^p$ *with the norm* $\mathcal{S}_{(0,1]}$ *is a Banach space.*

Proof. Let $\{v_n\}$ be a Cauchy sequence in $\widetilde{L}^p$ space, so for any $\epsilon > 0$ there is n_0 such that for all numbers $n, m \geqslant n_0$ $\left(\frac{1}{x}\int_0^x |v_n(s) - v_m(s)|^p ds\right)^{\frac{1}{p}} < \epsilon$ for all $x \in (0, 1]$. $\{v_n\}$ is also a Cauchy sequence in L^p, according to the standard norm $\|.\|$. Because L^p is a Banach space, so there exists $v_0 \in L^p$ such that $v_n \to v_0$ in L^p. For any $x \in (0, 1]$ we have from Fatou's lemma

$$\left(\frac{1}{x}\int_0^x |v_n(s) - v_0(s)|^p ds\right)^{\frac{1}{p}} \leqslant \liminf_{m\to\infty} \left(\frac{1}{x}\int_0^x |v_n(s) - v_m(s)|^p ds\right)^{\frac{1}{p}} < \epsilon.$$

Therefore $v_n \to v_0$, as $n \to \infty$. Similarly, using Fatou's lemma we can prove that $v_0 \in \widetilde{L}^p$. ■

Proposition 2 *If* $0 < a < b < c \leqslant 1$, *then*

$$\mathcal{S}_{(a,c]} \leqslant \mathcal{S}_{(a,b]} + \mathcal{S}_{(b,c]}.$$

Proof. It is sufficient to prove that for a continuous function v defined on the interval $[0, 1]$ and $x \in (a, c]$

$$\left(\frac{1}{x}\int_0^x |v(s)|^p ds\right)^{\frac{1}{p}} \leqslant \mathcal{S}_{(a,b]}(v) + \mathcal{S}_{(b,c]}(v).$$

For $x \in (a, b]$ we have

$$\left(\frac{1}{x}\int_0^x |v(s)|^p ds\right)^{\frac{1}{p}} \leqslant \mathcal{S}_{(a,b]}(v) \leqslant \mathcal{S}_{(a,b]}(v) + \mathcal{S}_{(b,c]}(v).$$

An analogous inequality is true for $x \in (b, c]$. ■

Remark 1 If $1 \leqslant p_1 < p_2 < \infty$, then

$$\widetilde{L}^{p_2} \subset \widetilde{L}^{p_1}$$

and

$$\sup_{x\in(0,1]} \left(\frac{1}{x}\int_0^x |v(s)|^{p_1} ds\right)^{\frac{1}{p_1}} \leqslant \sup_{x\in(0,1]} \left(\frac{1}{x}\int_0^x |v(s)|^{p_2} ds\right)^{\frac{1}{p_2}}.$$

Definition 5 Denote by $\widehat{L}^p$ the space of all functions $v \in \widetilde{L}^p$ $(1 \leqslant p < \infty)$, satisfying the following condition

$$\lim_{a\to 0} \mathcal{S}_{(0,a]}(v) = 0.$$

Proposition 3 *The space* $\widehat{L}^p$ *with the norm* $\mathcal{S}_{(0,1]}$ *is a Banach space.*

Proof. The space $\widetilde{L}^p$ with the norm $\mathcal{S}_{(0,1]}$ is a Banach space. Since $\widehat{L}^p$ is its linear subspace, it is sufficient to show that $\widehat{L}^p$ is its closed subspace. Let $\{v_n\}$ be the sequence of functions belonging to $\widehat{L}^p$ such that $v_n \to v_0$, as $n \to \infty$. We need to show that $v_0 \in \widehat{L}^p$. Let $\epsilon > 0$. There exists n_0 such that for every $n \geqslant n_0$,

$$\sup_{x\in(0,1]} \left(\frac{1}{x}\int_0^x |v_n(s) - v_0(s)|^p ds\right)^{\frac{1}{p}} < \frac{\epsilon}{2}.$$

Since $v_{n_0} \in \widehat{L}^p$, there exists x_0 such that, for every $x < x_0$,

$$\left(\frac{1}{x}\int_0^x |v_{n_0}(s)|^p ds\right)^{\frac{1}{p}} < \frac{\epsilon}{2}.$$

For every $x < x_0$, we have

$$\left(\frac{1}{x}\int_0^x |v_0(s)|^p ds\right)^{\frac{1}{p}} \leqslant \sup_{x<x_0} \left(\frac{1}{x}\int_0^x |v_0(s)|^p ds\right)^{\frac{1}{p}}$$
$$\leqslant \sup_{x<x_0} \left(\frac{1}{x}\int_0^x |v_0(s) - v_{n_0}(s)|^p ds\right)^{\frac{1}{p}} + \sup_{x<x_0} \left(\frac{1}{x}\int_0^x |v_{n_0}(s)|^p ds\right)^{\frac{1}{p}} < \epsilon.$$

Therefore, for sufficiently small x_0

$$\mathcal{S}_{(0,x_0]}(v_0) < \epsilon.$$

This is our claim. ■

3. Chaos and stability of dynamical system $(\widetilde{T}_t)_{t\geqslant 0}$

THEOREM 1 *If $\gamma > 0$, then for any t_0 there exists $v_0 \in \widehat{L}^p$ such that*

$$\widetilde{T}_{t_0} v_0 = v_0. \tag{10}$$

Moreover,

$$\widetilde{T}_t v_0 = v_0 \quad \textit{if and only if} \quad t = nt_0 \textit{ for some positive integer } n. \tag{11}$$

Proof. Let w be an arbitrary continuous function such that $|w(x)| \leqslant Cx^\gamma$, where $C > 0$ and $x \in [0,1]$. Let w satisfy the following conditions:

$$\mathrm{e}^{\gamma t_0} w(\mathrm{e}^{-t_0}) = w(1), \tag{12}$$

$$\mathrm{e}^{\gamma t} w(\mathrm{e}^{-t}) \neq w(1) \quad \forall t \in (0, t_0). \tag{13}$$

The function w, defined in such way, belongs to $\widehat{L}^p$ space because

$$\begin{aligned}\mathcal{S}_{(0,a]}(w) &= \sup_{x\in(0,a]}\left(\frac{1}{x}\int_0^x |w(s)|^p ds\right)^{\frac{1}{p}} \leqslant C \sup_{x\in(0,a]}\left(\frac{1}{x}\int_0^x s^{\gamma p} ds\right)^{\frac{1}{p}} \\ &= C\sup_{x\in(0,a]}\left(\frac{x^{\gamma p}}{\gamma p+1}\right)^{\frac{1}{p}} \leqslant C\frac{a^\gamma}{(\gamma p+1)^{\frac{1}{p}}}.\end{aligned}$$

This leads to $\lim_{a\to 0}\mathcal{S}_{(0,a]}(w) = 0$, as $\gamma > 0$. Since $(0,1] = \bigcup_{n=0}^{\infty}(\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$ we can define a function v on the interval $(0,1]$ by squeezing the graph of the function w into intervals $(\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$. We put

$$v(x) = \mathrm{e}^{-n\gamma t_0} w(x\mathrm{e}^{nt_0}) \quad \text{for} \quad x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]. \tag{14}$$

For such function we have

$$|v(x)| = |\mathrm{e}^{-n\gamma t_0} w(x\mathrm{e}^{nt_0})| = \mathrm{e}^{-n\gamma t_0}|w(x\mathrm{e}^{nt_0})| \leqslant \mathrm{e}^{-n\gamma t_0} C(x\mathrm{e}^{nt_0})^\gamma = Cx^\gamma.$$

So, $v(0) = 0$ and we obtain the continuous function v defined on the whole interval $[0,1]$. The properties (10) and (11) follow from (12) and (13), respectively. As we showed above the function which fulfils the condition $|v(x)| \leqslant Cx^\gamma$ for $x \in [0,1]$ belongs to $\widehat{L}^p$ space. This finishes the proof. ∎

Theorem 2 *If $\gamma > 0$ the set of periodic points of (1) is dense in $\widehat{L}^p$.*

Proof. Let $w \in \widehat{L}^p$ be an arbitrary function. Define a new function

$$w_C(x) = \begin{cases} w(x) & \text{for} \quad |w(x)| \leqslant Cx^\gamma \\ \operatorname{sgn}(w(x)) \cdot Cx^\gamma & \text{for} \quad |w(x)| > Cx^\gamma \end{cases}$$

where $C > 0$. It is obvious that $w_C \in \widehat{L}^p$. At first we claim that for such functions $\lim_{C\to\infty}\mathcal{S}_{(\mathrm{e}^{-t_0},1]}(w_C - w) = 0$.

$$\begin{aligned}\mathcal{S}_{(\mathrm{e}^{-t_0},1]}(w_C - w) &= \sup_{x\in(\mathrm{e}^{-t_0},1]}\left(\frac{1}{x}\int_0^x |w_C(s) - w(s)|^p ds\right)^{\frac{1}{p}} \\ &\leqslant \sup_{x\in(\mathrm{e}^{-t_0},1]}\left(\frac{1}{x}\int_{\{s\in[0,1]:w(s)>Cs^\gamma\}} |w_C(s) - w(s)|^p ds\right)^{\frac{1}{p}} \\ &\leqslant \sup_{x\in(\mathrm{e}^{-t_0},1]}\left(\frac{1}{x}\int_{\{s\in[0,1]:w(s)>Cs^\gamma\}} |w(s)|^p ds\right)^{\frac{1}{p}} \\ &\leqslant \mathrm{e}^{\frac{1}{p}t_0}\left(\int_{\{s\in[0,1]:w(s)>Cs^\gamma\}} |w(s)|^p ds\right)^{\frac{1}{p}}.\end{aligned}$$

We know that for arbitrary $\eta > 0$

$$\begin{aligned}\left(\int_0^1 |w(s)|^p ds\right)^{\frac{1}{p}} &\geqslant \left(\int_{\{s\in[0,1]:w(s)>C\eta^\gamma\}} |w(s)|^p ds\right)^{\frac{1}{p}} \\ &\geqslant \left(\mu\left(\{s \in [0,1] : w(s) > C\eta^\gamma\}\right)\right)^{\frac{1}{p}} (C\eta^\gamma)\end{aligned}$$

where μ is the measure in $\mathbb{R}$. We will show that the measure of the interval $\{s \in [0,1] : w(s) > Cs^\gamma\}$ tends to zero when $C \to \infty$. Using the above estimation we have for any $\eta > 0$

$$\begin{aligned}\mu\left(\{s \in [0,1] : w(s) > Cs^\gamma\}\right) &\leqslant \eta + \mu\left(\{s \in [0,1] : s > \eta \wedge w(s) > C\eta^\gamma\}\right) \\ &\leqslant \eta + \frac{1}{C\eta^\gamma}\left(\int_0^1 |w(s)|^p ds\right)^{\frac{1}{p}}.\end{aligned}$$

Since η is arbitrary, it follows that $\mu(\{s \in [0,1] : w(s) > Cs^\gamma\}) \to 0$ when $C \to \infty$. This completes our claim. Fix t_0 and the constant C such that

$$\mathcal{S}_{(\mathrm{e}^{-t_0},1]}(w_C - w) \leqslant \mathrm{e}^{\frac{1}{p}t_0}\left(\int_{\{s\in[0,1]:w(s)>Cs^\gamma\}} |w(s)|^p ds\right)^{\frac{1}{p}} < \frac{\varepsilon}{4}.$$

We are going to define periodic point by formula $v(x) = \mathrm{e}^{-n\gamma t_0} w_C(x\mathrm{e}^{nt_0})$, $x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$ for a good choice of the parameter t_0. Both functions w_C and v belong to $\widehat{L}^p$ so $\mathrm{e}^{\frac{t_0}{p}}\|v1_{(0,\mathrm{e}^{-t_0}]}\|_{L^p(0,1)} < \frac{\epsilon}{8}$ and $\mathrm{e}^{\frac{t_0}{p}}\|w_C 1_{(0,\mathrm{e}^{-t_0}]}\|_{L^p(0,1)} < \frac{\epsilon}{8}$, where 1_A denotes the indicator of the set A. Moreover fix t_0 such that $\mathcal{S}_{(0,\mathrm{e}^{-t_0}]}(w) < \frac{\epsilon}{4}$ and $\mathcal{S}_{(0,\mathrm{e}^{-t_0}]}(v) < \frac{\epsilon}{4}$. Since for $x \in (\mathrm{e}^{-t_0}, 1]$ $v(x) = w_C(x)$ and so

$$\begin{aligned}\mathcal{S}_{(\mathrm{e}^{-t_0},1]}(v - w_C) &= \sup_{x\in(\mathrm{e}^{-t_0},1]}\left(\frac{1}{x}\int_0^x |v(s) - w_C(s)|^p ds\right)^{\frac{1}{p}} \\ &= \sup_{x\in(\mathrm{e}^{-t_0},1]}\left(\frac{1}{x}\int_0^{\mathrm{e}^{-t_0}} |v(s) - w_C(s)|^p ds\right)^{\frac{1}{p}} \\ &\leqslant \mathrm{e}^{\frac{1}{p}t_0}\left(\int_0^{\mathrm{e}^{-t_0}} |v(s) - w_C(s)|^p ds\right)^{\frac{1}{p}} < \frac{\epsilon}{4}.\end{aligned}$$

Finally,

$$\begin{aligned}\mathcal{S}_{(0,1]}(v - w) &\leqslant \mathcal{S}_{(0,\mathrm{e}^{-t_0}]}(v - w) + \mathcal{S}_{(\mathrm{e}^{-t_0},1]}(v - w) \\ &\leqslant \mathcal{S}_{(0,\mathrm{e}^{-t_0}]}(v) + \mathcal{S}_{(0,\mathrm{e}^{-t_0}]}(w) + \mathcal{S}_{(\mathrm{e}^{-t_0},1]}(v - w_C) \\ &+ \mathcal{S}_{(\mathrm{e}^{-t_0},1]}(w_C - w) < \epsilon.\end{aligned}$$

■

THEOREM 3 *If $\gamma \leqslant 0$, then the semigroup $(\widetilde{T}_t)_{t\geqslant 0}$ is strongly stable. If $\gamma < 0$, $(\widetilde{T}_t)_{t\geqslant 0}$ is exponentially stable.*

Proof. Let $v \in \widehat{L}^p$ be an arbitrary function.

$$\begin{aligned}\mathcal{S}^p_{(0,1]}(\widetilde{T}_t v) &= \sup_{x\in(0,1]}\left(\frac{1}{x}\int_0^x |\widetilde{T}_t v(s)|^p ds\right) = \sup_{x\in(0,1]}\left(\frac{1}{x}\int_0^x |\mathrm{e}^{\gamma t} v(s\mathrm{e}^{-t})|^p ds\right)\\ &= \mathrm{e}^{(\gamma p+1)t}\sup_{x\in(0,1]}\left(\frac{1}{x}\int_0^{x\mathrm{e}^{-t}} |v(s)|^p ds\right)\\ &= \mathrm{e}^{\gamma p t}\sup_{x\in(0,\mathrm{e}^{-t}]}\left(\frac{1}{x}\int_0^x |v(s)|^p ds\right) = \mathrm{e}^{\gamma p t}\,\mathcal{S}^p_{(0,\mathrm{e}^{-t}]}(v),\end{aligned}$$

since $\mathrm{e}^{\gamma p} \leqslant 1$ and by definition $\mathcal{S}_{(0,\mathrm{e}^{-t}]}(v) \to 0$ as $t \to \infty$, we obtain strong stability of the system $(\widetilde{T}_t)_{t\geqslant 0}$ in $\widehat{L}^p$. The second part of the proof follows from the above inequality, too. We gain exponential stability with $D = 1$ and $\omega = -\gamma$. ■

The stability of the system in L^p space implies the stability in $\widehat{L}^p$, but not conversely. It is enough to choose γ from the interval $\left(-\frac{1}{p}, 0\right]$, then we have the stability of the semigroup $(\widetilde{T}_t)_{t\geqslant 0}$ in $\widehat{L}^p$ subspace, but not in L^p space.

4. Properties of dynamical system $(T_t)_{t\geqslant 0}$

THEOREM 4 *Assume that*

$$\exists C, q > 0 \quad \forall x \in [0,1] \quad |\lambda(0) - \lambda(x)| \leqslant C x^q \tag{15}$$

holds, then we have the equivalence: the function u belongs to the space $\widehat{L}^p$ if and only if $\widetilde{u} \in \widehat{L}^p$.

Proof. By (15), $u \in L^p$ iff $\widetilde{u} \in L^p$. This can be found in Dawidowicz, Poskrobko (2006). Assuming that $u \in \widehat{L}^p$ we have

$$\begin{aligned}\mathcal{S}_{(0,a]}(\widetilde{u}) &= \sup_{x\in(0,a]}\left(\frac{1}{x}\int_0^x |\widetilde{u}(t,s)|^p ds\right)^{\frac{1}{p}} = \sup_{x\in(0,a]}\left(\frac{1}{x}\int_0^x |\kappa(s)u(t,s)|^p ds\right)^{\frac{1}{p}}\\ &\leqslant \sup_{x\in(0,a]}\left(\frac{1}{x}\int_0^x \mathrm{e}^{p\int_0^s \frac{|\lambda(0)-\lambda(\sigma)|}{\sigma}d\sigma}|u(t,s)|^p ds\right)^{\frac{1}{p}}\\ &\leqslant \sup_{x\in(0,a]}\left(\frac{1}{x}\int_0^x \mathrm{e}^{\frac{Cp}{q}s^q}|u(t,s)|^p ds\right)^{\frac{1}{p}} \leqslant \mathrm{e}^{\frac{Ca^q}{q}}\mathcal{S}_{(0,a]}(u),\end{aligned}$$

so

$$\lim_{a\to 0}\mathcal{S}_{(0,a]}(\widetilde{u}) = 0.$$

In the same manner we can establish the inverse implication. ■

For convenience, we assume, once and for all, that (15) is satisfied.

THEOREM 5 *If* $\lambda(0) > 0$*, then for any* t_0 *there exists such* $v_0 \in \widehat{L}^p$ *that*

$$T_{t_0} v_0 = v_0. \tag{16}$$

Moreover,

$$T_t v_0 = v_0 \quad \textit{if and only if} \quad t = nt_0 \textit{ for some positive integer } n. \tag{17}$$

Proof. Let w be an arbitrary function belonging to L^p, defined on the interval $[\mathrm{e}^{-t_0}, 1]$ and satisfying the following conditions:

$$\mathrm{e}^{-g(\mathrm{e}^{-t_0})} w(\mathrm{e}^{-t_0}) = w(1), \tag{18}$$

$$\mathrm{e}^{-g(\mathrm{e}^{-t})} w(\mathrm{e}^{-t}) \neq w(1) \quad \forall t \in (0, t_0). \tag{19}$$

Consider the function v on the interval $(0, 1]$

$$v(x) = \mathrm{e}^{g(x)} \mathrm{e}^{-g(x\mathrm{e}^{nt_0})} w(x\mathrm{e}^{nt_0}) \quad \text{for} \quad x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}].$$

The function v is defining on the whole interval $(0, 1] = \bigcup_{n=0}^{\infty} (\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$ and comes into being by squeezing the graph of the function w into each of the intervals $(\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$.
By assumption of the continuity of w on $[\mathrm{e}^{-t_0}, 1]$ follows its boundedness, i.e. $\exists M > 0$ such that $|w(x)| \leqslant M$ for each $x \in [\mathrm{e}^{-t_0}, 1]$. By the above for $x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$ we have the estimation:

$$|v(x)| = \mathrm{e}^{g(x)} \mathrm{e}^{-g(x\mathrm{e}^{nt_0})} |w(x\mathrm{e}^{nt_0})| \leqslant M \mathrm{e}^{g(x)} \cdot \sup_{x \in [\mathrm{e}^{-t_0}, 1]} \mathrm{e}^{-g(x)} \leqslant M_1 \mathrm{e}^{g(x)}$$

where $M_1 = M \cdot \sup_{x \in [\mathrm{e}^{-t_0}, 1]} \mathrm{e}^{-g(x)}$. From the assumption (7) $\lim_{x \to 0} \mathrm{e}^{g(x)} = 0$ so we deduce that $v(0) = 0$. We obtain the continuous function v defined on the whole interval $[0, 1]$. The property (16) follows from (18), while the property (17) from (19). Our next goal is to show that $v \in \widehat{L}^p$. Under Theorem 1, we know that $\widetilde{v} \in \widehat{L}^p$ for $\gamma > 0$, where $\widetilde{v}$ is the solution of the equation (1). It clearly forces the same conclusion for the function v by Theorem 4. ∎

THEOREM 6 *If* $\lambda(0) > 0$ *then the set of periodic points of (4) is dense in* $\widehat{L}^p$*.*

Proof. Let $\epsilon > 0$ and let $w \in \widehat{L}^p$. Let v be a periodic solution of (4) defined by the formula (6). Since v and w belong to $\widehat{L}^p$ there exists such t_0 that $\mathcal{S}_{(0,\mathrm{e}^{-t_0}]}(v) < \frac{\epsilon}{4}$ and $\mathcal{S}_{(0,\mathrm{e}^{-t_0}]}(w) < \frac{\epsilon}{4}$. We know that $v(x) = \frac{\widetilde{v}(x)}{\kappa(x)}$, where $\widetilde{v}$ is the periodic solution of (1). The assumption $\lambda(0) > 0$ guarantees the density of the set of periodic points of (1), so $\mathcal{S}_{(0,1]}(v - \widetilde{v}) < \frac{\epsilon}{4}$ and $\mathcal{S}_{(0,1]}(w - \widetilde{v}) < \frac{\epsilon}{4}$.
Thus

$$\begin{aligned} \mathcal{S}_{(0,1]}(v - w) &\leqslant \mathcal{S}_{(0,\mathrm{e}^{-t_0}]}(v - w) + \mathcal{S}_{(\mathrm{e}^{-t_0},1]}(v - w) \\ &\leqslant \mathcal{S}_{(0,\mathrm{e}^{-t_0}]}(v) + \mathcal{S}_{(0,\mathrm{e}^{-t_0}]}(w) + \mathcal{S}_{(0,1]}(v - \widetilde{v}) + \mathcal{S}_{(0,1]}(\widetilde{v} - w) < \epsilon. \end{aligned}$$

This is the desired conclusion. ∎

Theorem 7 *If* $\lambda(0) \leqslant 0$ *then for every* $v \in \widehat{L}^p$

$$\lim_{t\to\infty} \mathcal{S}_{(0,1]}(T_t v) = 0.$$

Moreover, if $\lambda(0) < 0$*, the semigroup* $(T_t)_{t\geqslant 0}$ *is exponentially stable.*

Proof. Take any $v \in \widehat{L}^p$. Then we have

$$\begin{aligned}\mathcal{S}_{(0,1]}(T_t v) &= \sup_{x\in(0,1]} \left(\frac{1}{x}\int_0^x |u(t,s)|^p ds\right)^{\frac{1}{p}} = \sup_{x\in(0,1]} \left(\frac{1}{x}\int_0^x \left|\frac{\widetilde{u}(t,s)}{\kappa(s)}\right|^p ds\right)^{\frac{1}{p}} \\ &= \sup_{x\in(0,1]} \left(\frac{1}{x}\int_0^x \left|\frac{1}{\kappa(s)}(T_t\widetilde{v})(s)\right|^p ds\right)^{\frac{1}{p}} \leqslant \mathrm{e}^{\frac{C}{q}} \mathcal{S}_{(0,1]}(T_t\widetilde{v}).\end{aligned}$$

Applying Theorem 3 we can assert that $\mathcal{S}_{(0,1]}(T_t v) \to 0$, as $t \to \infty$. This proves the first part of the Theorem. The second one follows immediately from the same above inequality and Theorem 3 with $D = \mathrm{e}^{\frac{C}{q}}$ and $\omega = -\lambda(0)$. ■

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