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Equivalence of second order optimality conditions for bang–bang control problems. Part 2 : Proofs, variational derivatives and representations

by

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Abstract: In Part 1 of this paper (Osmolovskii and Maurer, 2005), we have summarized the main results on the equivalence of two quadratic forms from which second order necessary and sufficient conditions can be derived for optimal bang-bang control problems. Here, in Part 2, we give detailed proofs and elaborate explicit relations between Lagrange multipliers and elements of the critical cones in both approaches. The main analysis concerns the derivation of formulas for the first and second order derivatives of trajectories with respect to variations of switching times, initial and final time and initial point. This leads to explicit representations of the second order derivatives of the Lagrangian for the induced optimization problem. Based on a suitable transformation, we obtain the elements of the Hessian of the Lagrangian in a form which involves only first order variations of the nominal trajectory. Finally, a careful regrouping of all terms allows us to find the desired equivalence of the two quadratic forms.

Keywords: bang–bang control, second order necessary and sufficient conditions, critical cones, equivalence of quadratic forms, representation of the Hessian of the Lagrangian.

1. Introduction

Second order necessary and sufficient conditions for optimal bang-bang controls are obtained from the property that a certain quadratic form be positive (semi)-definite on a finite-dimensional critical cone. Two different quadratic forms have been developed by Agrachev, Stefani and Zezza (2002) and Osmolovskii (1988, 2004); see also Milyutin and Osmolovskii (1998), Part 2, Chapter, Section 12. In Part 1 of this paper (Osmolovskii and Maurer, 2005) we have summarized the main results on the equivalence of both forms and have derived explicit relations between the corresponding Lagrange multipliers and elements of the critical cones. The main purpose of the present Part 2 is to give detailed proofs, which make extensive use of explicit formulas for first and second order derivatives of the trajectory with respect to variations of the optimization variable comprising the switching times, the free initial and final times and the free initial state.

In Section 2, some basic facts from Part 1 (Osmolovskii and Maurer, 2005) are summarized for the convenience of the reader. In particular, we formulate the induced optimization problem with optimization variable ζ, which is associated with the bang-bang control problem. In Section 3, we give formulas for the first order derivatives of trajectories with respect to ζ, which follow from elementary properties of ODEs. The formulas are used in Section 4 to establish the explicit relations between the multipliers of Pontryagin's minimum principle and the Lagrange multipliers of the induced optimization problem. Elements of the corresponding critical cones are related in Section 5. Second order order derivatives of trajectories with respect to ζ are elaborated in Section 6. In our opinion, the resulting formulas seem to be mostly unknown in the literature. These formulas provide the main technical tools to obtain explicit representations of the second order derivatives of the Lagrangian (Section 7). The remarkable fact to be noted here is that using a suitable transformation these derivatives are seen to involve only *first order* variations of the trajectory w.r.t ζ. This property facilitates considerably the numerical computation of the Hessian of the Lagrangian. Thus, we arrive at a representation of the quadratic form associated with the Hessian of the Lagrangian. In Section 8, we carefully regroup the terms in the quadratic form associated with the Hessian of the Lagrangian and finally obtain the desired equivalence of the two quadratic forms.

Due to space limitations, no illustrative examples are discussed here. The time-optimal control of a van der Pol oscillator has been discussed in Maurer and Osmolovskii (2004) using ideas along the lines of this paper. The explicit computations of the variations of the trajectory w.r.t. ζ on the basis of the formulas given here may become quite involved and tedious. Maurer, Büskens, Kim and Kaya (2005) propose methods to compute the Hessian of the Lagrangian by finite differences. The efficiency and accuracy of this approach has been demonstrated there by several examples.

2. Review of basic notations for bang–bang control problems

We review basic definitions and notations for bang-bang controls from Part 1 of this paper (Osmolovskii and Maurer, 2005). There, the following *main problem* was considered, where $x(t) \in \mathbb{R}^{d(x)}$ denotes the state variable and $u(t) \in \mathbb{R}^{d(u)}$ the control variable in the time interval $t \in \Delta = [t_0, t_1]$ with the non–fixed initial time t_0 and final time t_1:

$$\text{Minimize} \qquad \mathcal{J}(t_0, t_1, x, u) = J(t_0, x(t_0), t_1, x(t_1)) \tag{1}$$

subject to the constraints

$$\dot{x}(t) = f(t, x(t), u(t)), \quad u(t) \in U, \quad (t, x(t)) \in \mathcal{Q}, \quad t_0 \le t \le t_1, \tag{2}$$

$$\begin{gathered} F(t_0, x(t_0), t_1, x(t_1)) \le 0, \quad K(t_0, x(t_0), t_1, x(t_1)) = 0, \\ (t_0, x(t_0), t_1, x(t_1)) \in \mathcal{P}, \end{gathered} \tag{3}$$

where the control variable appears linearly in the system dynamics,

$$f(t, x, u) = a(t, x) + B(t, x)u\,. \tag{4}$$

Here, F, K, a are vector functions, B is a $d(x) \times d(u)$ matrix function, $\mathcal{P} \subset \mathbb{R}^{2+2d(x)}$, $\mathcal{Q} \subset \mathbb{R}^{1+d(x)}$ are open sets and $U \subset \mathbb{R}^{d(u)}$ is a convex polyhedron. The functions J, F, K are assumed to be twice continuously differentiable on $\mathcal{P}$ and the functions a, B are twice continuously differentiable on $\mathcal{Q}$. The dimensions of F, K are denoted by $d(F), d(K)$. We shall use the abbreviations

$$x_0 = x(t_0),\ x_1 = x(t_1),\ p = (t_0, x_0, t_1, x_1).$$

Let

$$\hat{\mathcal{T}} = \{\, (\hat{x}(t), \hat{u}(t)) \mid t \in [\hat{t}_0, \hat{t}_1] \,\}$$

be a fixed admissible pair of functions such that the control $\hat{u}(\cdot)$ is a piecewise constant function on the interval $\hat{\Delta} = [\hat{t}_0, \hat{t}_1]$. Denote by

$$\hat{\theta} = \{\hat{\tau}_1, \ldots, \hat{\tau}_s\}, \quad \hat{t}_0 < \hat{\tau}_1 < \ldots < \hat{\tau}_s < \hat{t}_1 \tag{5}$$

the finite set of all discontinuity points (jump points) of the control $\hat{u}(t)$. Then, $\dot{\hat{x}}(t)$ is a piecewise continuous function whose discontinuity points belong to $\hat{\theta}$, and hence $\hat{x}(t)$ is a piecewise smooth function on $\hat{\Delta}$. Assume that $\hat{\theta} = \{\hat{\tau}_1, \ldots, \hat{\tau}_s\}$ is the set of switching points of the bang-bang control $\hat{u}(\cdot)$ that takes values in the vertex set $\mathrm{ex}(U)$ of the polyhedron U,

$$\hat{u}(t) = u^i \in \mathrm{ex}(U) \ \text{ for } \ t \in (\hat{\tau}_{i-1}, \hat{\tau}_i), \quad i = 1, \ldots, s+1,$$

where $\hat{\tau}_0 = \hat{t}_0$ and $\hat{\tau}_{s+1} = \hat{t}_1$. Put $n = d(x)$ and

$$\hat{x}(\hat{t}_0) = \hat{x}_0, \quad \hat{\tau} = (\hat{\tau}_1, \ldots, \hat{\tau}_s) \in \mathbb{R}^s, \quad \hat{\zeta} = (\hat{t}_0, \hat{t}_1, \hat{x}_0, \hat{\tau}) \in \mathbb{R}^{2+n+s}. \tag{6}$$

Take a small neighbourhood $\mathcal{V}$ of the point $\hat{\zeta}$ and let $\zeta = (t_0, t_1, x_0, \tau) \in \mathcal{V}$, where $\tau = (\tau_1, \ldots, \tau_s)$ satisfies $t_0 < \tau_1 < \tau_2 < \ldots < \tau_s < t_1$. Put $\tau_0 = t_0$ and $\tau_{s+1} = t_1$ and define the function $u(t;\tau)$ by the condition

$$u(t;\tau) = u^i \text{ for } t \in (\tau_{i-1}, \tau_i), \quad i = 1, \ldots, s+1. \tag{7}$$

For uniqueness, define the values $u(\tau_i;\tau)$, $i = 1,\dots,s$, by the condition of continuity of the control from the left: $u(\tau_i;\tau) = u(\tau_i - 0;\tau)$, $i = 1,\dots,s$.

Let $x(t;t_0,x_0,\tau)$ be the solution of the initial value problem (IVP)

$$\dot{x} = f(t,x,u(t;\tau)), \quad t \in [t_0,t_1], \qquad x(t_0) = x_0. \tag{8}$$

For each $\zeta \in \mathcal{V}$ this solution exists, if the neighborhood $\mathcal{V}$ of the point $\hat{\zeta}$ is sufficiently small. Obviously, we have

$$x(t;\hat{t}_0,\hat{x}_0,\hat{\tau}) = \hat{x}(t), \quad t \in \hat{\Delta}, \qquad u(t;\hat{\tau}) = \hat{u}(t), \quad t \in \hat{\Delta} \setminus \hat{\theta}.$$

We shall make extensive use of the variational system

$$\dot{V} = f_x(t, x(t;t_0,x_0,\tau), u(t;\tau))\, V\,, \quad V(t_0) = E, \tag{9}$$

where E is the identity matrix. The solution $V(t)$ is $n\times n$ matrix-valued function ($n = d(x)$) which is absolutely continuous in $\Delta = [t_0,t_1]$. The solution of (9) is denoted by $V(t;t_0,x_0,\tau)$. Along the reference trajectory $\hat{x}(t),\hat{u}(t)$, i.e. for $\zeta = \hat{\zeta}$, we shall use the notation $V(t)$ for simplicity.

Consider now the following finite dimensional optimization problem in the space $\mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^s$ of variables $\zeta = (t_0,t_1,x_0,\tau)$:

$$\begin{aligned} \mathcal{F}_0(\zeta): &= J(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) \to \min, \\ \mathcal{F}(\zeta): &= F(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) \le 0, \\ \mathcal{G}(\zeta): &= K(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) = 0, \end{aligned} \tag{10}$$

or simply *the induced problem*. The Lagrange function in the induced problem is

$$\begin{aligned} L(\mu,\zeta) &= \alpha_0 J(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) \\ &\quad +\alpha F(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) + \beta K(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) \\ &= l(\mu,t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)), \end{aligned} \tag{11}$$

where $l = \alpha_0 J + \alpha F + \beta K$ and $\mu = (\alpha_0,\alpha,\beta)$. By definition, Λ_0 is the set of multipliers μ such that

$$\alpha_0 \ge 0,\ \alpha \ge 0,\ \alpha_0 + |\alpha| + |\beta| = 1,\ \alpha F(\hat{p}) = 0,\ L_\zeta(\mu,\hat{\zeta}) = 0, \tag{12}$$

where $\hat{p} = (\hat{t}_0,\hat{x}_0,\hat{t}_1,\hat{x}_1)$, $\hat{x}_0 = \hat{x}(\hat{t}_0)$, $\hat{x}_1 = \hat{x}(\hat{t}_1) = x(\hat{t}_1;\hat{t}_0,\hat{x}_0,\hat{\tau})$. Now, let us define the corresponding set of normalized Lagrange multipliers for the trajectory $\hat{\mathcal{T}}$ in the main problem. Denote by Λ the set of multipliers $\lambda = (\alpha_0,\alpha,\beta,\psi,\psi_0)$ such that

$$\begin{aligned} &\alpha_0 \ge 0, \quad \alpha \ge 0, \quad \alpha_0 + |\alpha| + |\beta| = 1, \quad \alpha F(\hat{p}) = 0, \\ &-\dot{\psi}(t) = \psi(t) f_x(t,\hat{x}(t),\hat{u}(t)), \quad -\dot{\psi}_0(t) = \psi(t) f_t(t,\hat{x}(t),\hat{u}(t)), \\ &\psi(\hat{t}_0) = -l_{x_0}(\mu,\hat{p}), \quad \psi(\hat{t}_1) = l_{x_1}(\mu,\hat{p}), \\ &\psi_0(\hat{t}_0) = -l_{t_0}(\mu,\hat{p}), \quad \psi_0(\hat{t}_1) = l_{t_1}(\mu,\hat{p}), \\ &\psi(t) f(t,\hat{x}(t),\hat{u}(t)) + \psi_0(t) = 0 \quad \forall\, t \in \hat{\Delta} \setminus \hat{\theta}, \end{aligned} \tag{13}$$

where $\hat{\Delta} = [\hat{t}_0, \hat{t}_1]$, $\hat{\theta} = \{\hat{\tau}_0, \ldots, \hat{\tau}_s\}$. We shall prove that there is a one-to-one correspondence between elements of the sets Λ and Λ_0 (see Propositions 4.1 and 4.2 in Part 1) and a one-to-one correspondence between elements of the critical cone in the main bang-bang problem for the trajectory $\hat{\mathcal{T}}$ and that in the induced optimization problem at the point $\hat{\zeta}$ (see Propositions 4.3 and 4.4 in Part 1). To this end we shall need formulas for all first order partial derivatives of the function $x(t_1; t_0, x_0, \tau)$.

3. First order derivatives of $x(t_1; t_0, x_0, \tau)$ with respect to t_0, t_1, x_0, and τ

Let $x(t; t_0, x_0, \tau)$ be the solution of the IVP (8) and put

$$g(\zeta) = g(t_0, t_1, x_0, \tau) := x(t_1; t_0, x_0, \tau). \tag{14}$$

Under our assumptions, the operator $g : \mathcal{V} \to \mathbb{R}^n$ is well defined and C^2-smooth if the neighborhood $\mathcal{V}$ of the point $\hat{\zeta}$ is sufficiently small. In this subsection, we shall derive the first order partial derivatives of $g(t_0, t_1, x_0, \tau)$ with respect to t_0, t_1, x_0, and τ at the point $\hat{\zeta}$. We shall use well-known results in theory of ODEd about differentiation of solutions to ODEs with respect to parameters and initial values.

In the sequel, it will be convenient to drop those arguments in $x(t; t_0, x_0, \tau)$, $u(t, \tau)$, $V(t; t_0, x_0, \tau)$ etc. that are kept fixed.

3.1. Derivative $\partial x/\partial x_0$

Let us fix τ and t_0. The following result is well-known in the theory of ODEs.

Proposition 3.1 *We have*

$$\frac{\partial x(t; x_0)}{\partial x_0} = V(t; x_0), \tag{15}$$

where the matrix-valued function $V(t; x_0)$ is the solution to the IVP (9), i.e.,

$$\dot{V} = f_x(t, x(t), u(t))V, \qquad V\,|_{t=t_0} = E, \tag{16}$$

where $x(t) = x(t; x_0)$, $\dot{V} = \frac{\partial V}{\partial t}$.

Consequently, we have

$$g_{x_0}(\hat{\zeta}) := \frac{\partial x(\hat{t}_1; \hat{t}_0, \hat{x}_0, \hat{\tau})}{\partial x_0} = V(\hat{t}_1). \tag{17}$$

3.2. Derivatives $\partial x/\partial t_0$ and $\partial x/\partial t_1$

Fix x_0 and τ and put

$$w(t;t_0)=\frac{\partial x(t;t_0)}{\partial t_0}\,.$$

PROPOSITION 3.2 *The vector function $w(t;t_0)$ is the solution to the IVP*

$$\dot w=f_x(t,x(t),u(t))w,\qquad w\,|_{t=t_0}=-\dot x(t_0),\tag{18}$$

where $x(t)=x(t;t_0)$, $\dot w=\frac{\partial w}{\partial t}$. Therefore, we have $w(t;t_0)=-V(t;t_0)\dot x(t_0)$, where the matrix-valued function $V(t;t_0)$ is the solution to the IVP (9).

Hence, we obtain

$$g_{t_0}(\hat\zeta):=\frac{\partial x(\hat t_1;\hat t_0,\hat x_0,\hat\tau)}{\partial t_0}=-V(\hat t_1)\dot{\hat x}(\hat t_0).\tag{19}$$

Obviously, we have

$$g_{t_1}(\hat\zeta):=\frac{\partial x(\hat t_1;\hat t_0,\hat x_0,\hat\tau)}{\partial t_1}=\dot{\hat x}(\hat t_1)\,.\tag{20}$$

3.3. Derivatives $\partial x/\partial\tau_i$

Fix t_0 and x_0. Take some i and fix τ_j for all $j\neq i$. Put

$$y^i(t;\tau_i)=\frac{\partial x(t;\tau_i)}{\partial\tau_i}.$$

and denote by $\dot y^i$ the derivative of y^i with respect to t.

PROPOSITION 3.3 *For $t\geq\tau_i$ the function $y^i(t;\tau_i)$ is the solution to the IVP*

$$\dot y^i=f_x(t,x(t;\tau_i),u(t;\tau_i))\,y^i,\quad y^i\,|_{t=\tau_i}=-[f]^i,\tag{21}$$

where $[f]^i=f(\tau_i,x(\tau_i;\tau_i),u^{i+1})-f(\tau_i,x(\tau_i;\tau_i),u^i)$ is the jump of the function $f(t,x(t;\tau_i),u(t;\tau_i))$ at the point τ_i. For $t<\tau_i$ we have $y^i(t;\tau_i)=0$. Thus, $[y]^i=-[f]^i$, where $[y]^i=y(\tau_i+0;\tau_i)-y(\tau_i-0;\tau_i)$ is the jump of the function $y^i(t;\tau_i)$ at the point τ_i.

Proof. Let us sketch how to obtain the representation (21). For $t\geq\tau_i$ the trajectory $x(t;\tau_i)$ satisfies the integral equation

$$x(t;\tau_i)=x(\tau_i-0;\tau_i)+\int_{\tau_i+0}^{t}f(h,x(h;\tau_i),u(h,\tau_i))\,dh\,.$$

By differentiating this equation with respect to τ_i we obtain

$$\begin{aligned}y^i(t;\tau_i)\;&=\;\dot x(\tau_i-0;\tau_i)-\dot x(\tau_i+0;\tau_i)\\&\quad+\int_{\tau_i+0}^{t}f_x(h,x(h;\tau_i),u(h,\tau_i))y^i(h;\tau_i)\,dh\,,\end{aligned}$$

from which we get $y^i\mid_{t=\tau_i}=-[f]^i$ and the variational equation in (21). ■

In particular, we obtain

$$g_{\tau_i}(\hat{\zeta}):=\frac{\partial x(\hat{t}_1;\hat{t}_0,\hat{x}_0,\hat{\tau})}{\partial\tau_i}=y^i(\hat{t}_1). \tag{22}$$

4. Lagrange multipliers

Here we shall prove Propositions 4.1 and 4.2 of Part 1. Consider the Lagrangian (11) with a multiplier $\mu=(\alpha_0,\alpha,\beta)\in\Lambda_0$, where Λ_0 is the set (12) of normalized Lagrange multipliers at the point $\hat{\zeta}$ in the induced problem (10). Define the absolutely continuous function $\psi(t)$ and the function $\psi_0(t)$ by equations (42) and (43) in Part 1 (see also (13)):

$$-\dot{\psi}=\psi f_x(t,\hat{x}(t),\hat{u}(t)),\quad \psi(t_1)=l_{x_1}(\mu,\hat{p}) \tag{23}$$

$$\psi(t)f(t,\hat{x}(t),\hat{u}(t))+\psi_0(t)=0. \tag{24}$$

We shall show that the function $\psi_0(t)$ is absolutely continuous and the collection $\lambda=(\alpha_0,\alpha,\beta,\psi,\psi_0)$ satisfies all conditions in (13) and hence belongs to the set Λ. The conditions

$$\alpha_0\geq 0,\ \alpha\geq 0,\ \alpha_0+|\alpha|+|\beta|=1,\ \alpha F(\hat{p})=0$$

in the definitions of Λ_0 and Λ are identical. Hence, we must analyze the equations

$$L_\zeta(\mu,\hat{\zeta})=0$$

in the definition of Λ_0, which are equivalent to the system

$$L_{t_0}(\mu,\hat{\zeta})=l_{t_0}(\hat{p})+l_{x_1}(\hat{p})g_{t_0}(\hat{\zeta})=0,$$
$$L_{t_1}(\mu,\hat{\zeta})=l_{t_1}(\hat{p})+l_{x_1}(\hat{p})g_{t_1}(\hat{\zeta})=0,$$
$$L_{x_0}(\mu,\hat{\zeta})=l_{x_0}(\hat{p})+l_{x_1}(\hat{p})g'_{x_0}(\hat{\zeta})=0,$$
$$L_{\tau_i}(\mu,\hat{\zeta})=l_{x_1}(\hat{p})g_{\tau_i}(\hat{\zeta})=0,\quad i=1,\ldots,s.$$

Using the equality $l_{x_1}(\hat{p})=\psi(\hat{t}_1)$ and formulas (19), (20), (17), (22) for the derivatives of g with respect to t_0, t_1, x_0, τ_i, respectively, at the point $\hat{\zeta}$, we get

$$L_{t_0}(\mu,\hat{\zeta})=l_{t_0}(\hat{p})-\psi(\hat{t}_1)V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)=0, \tag{25}$$

$$L_{t_1}(\mu,\hat{\zeta})=l_{t_1}(\hat{p})+\psi(\hat{t}_1)\dot{\hat{x}}(\hat{t}_1)=0, \tag{26}$$

$$L_{x_0}(\mu,\hat{\zeta})=l_{x_0}(\hat{p})+\psi(\hat{t}_1)V(\hat{t}_1)=0, \tag{27}$$

$$L_{\tau_i}(\mu,\hat{\zeta})=\psi(\hat{t}_1)y^i(\hat{t}_1)=0,\quad i=1,\ldots,s. \tag{28}$$

Analysis of (25). The $n \times n$ matrix value function $V(t)$ satisfies the equation

$$\dot V = f_x V, \qquad V(t_0) = E$$

with $f_x = f_x(t, \hat x(t), \hat u(t))$. Then, $\Psi(t) := V^{-1}(t)$ is the solution to the adjoint equation

$$-\dot\Psi = \Psi f_x, \qquad \Psi(t_0) = E.$$

Consequently, $\psi(\hat t_1) = \psi(\hat t_0)\Psi(\hat t_1) = \psi(\hat t_0)V^{-1}(\hat t_1)$. Using these relations in (25), we get

$$l_{t_0}(\hat p) - \psi(\hat t_0)\dot{\hat x}(\hat t_0) = 0.$$

By virtue of (24) we have $\psi(\hat t_0)\dot{\hat x}(\hat t_0) = -\psi_0(\hat t_0)$. Hence, (25) is equivalent to the transversality condition for ψ_0 at the point $\hat t_0$:

$$l_{t_0}(\hat p) + \psi_0(\hat t_0) = 0.$$

Analysis of (26). Since $\psi(\hat t_1)\dot{\hat x}(\hat t_1) = -\psi_0(\hat t_1)$ holds, (26) is equivalent to the transversality condition for ψ_0 at the point $\hat t_1$:

$$l_{t_1}(\hat p) - \psi_0(\hat t_1) = 0.$$

Analysis of (27). Since $\psi(\hat t_1) = \psi(\hat t_0)V^{-1}(\hat t_1)$, equality (27) is equivalent to the transversality condition for ψ at the point $\hat t_0$:

$$l_{x_0}(\hat p) + \psi(\hat t_0) = 0.$$

Analysis of (28). We need the following result.

PROPOSITION 4.1 *Let the absolutely continuous function y be a solution to the system $\dot y = f_x y$ on an interval Δ and let the absolutely continuous function ψ be a solution to the adjoint system $-\dot\psi = \psi f_x$ on the same interval, where $f_x = f_x(t, \hat x(t), \hat u(t))$. Then $\psi(t)y(t) \equiv \mathrm{const}$ on Δ.*

Proof. We have $\frac{d}{dt}(\psi y) = \dot\psi y + \psi \dot y = -\psi f_x y + \psi f_x y = 0.$ ■

It follows from this proposition and (21) that for $i = 1, \dots, s$

$$\psi(\hat t_1)y^i(\hat t_1) = \psi(\hat\tau_i)y^i(\hat\tau_i + 0) = \psi(\hat\tau_i)[y^i]^i = -\psi(\hat\tau_i)[\dot{\hat x}]^i = -[\psi\dot{\hat x}]^i = [\psi_0]^i.$$

Therefore, (28) is equivalent to the conditions

$$[\psi_0]^i = 0, \quad i = 1, \dots, s,$$

which means that ψ_0 is continuous at each point $\hat\tau_i$, $i = 1, \dots, s$, and hence absolutely continuous on $\hat\Delta = [\hat t_0, \hat t_1]$. Moreover, it follows from $0 = [\psi_0]^i = -\psi(\hat\tau_i)[\dot{\hat x}]^i$ that

$$\sigma(\hat\tau_i)[\hat u]^i = 0, \tag{29}$$

where $\sigma(t) = \psi(t)B(t, \hat{x}(t))$ denotes the switching function.

Finally, differentiating (24) with respect to t, we get

$$-\psi f_x \dot{\hat{x}} + \psi f_t + \psi f_x \dot{\hat{x}} + \dot{\psi}_0 = 0, \quad \text{i.e.,} \quad -\dot{\psi}_0 = \psi f_t.$$

Thus, we have proved that $\lambda = (\alpha_0, \alpha, \beta, \psi, \psi_0) \in \Lambda$.

Vice versa, if $(\alpha_0, \alpha, \beta, \psi) \in \Lambda$, then one can show similarly that $(\alpha_0, \alpha, \beta) \in \Lambda_0$. Moreover, it is obvious that the projector in Part 1, (41), is injective on Λ_0, because ψ and ψ_0 are defined uniquely by conditions (23) and (24), respectively.

5. Critical cones

Take an element $\bar{\zeta} = (\bar{t}_0, \bar{t}_1, \bar{x}_0, \bar{\tau})$ of the critical cone $\mathcal{K}_0$ (see Part 1, equation (45)) at the point $\hat{\zeta}$ in the induced problem:

$$\mathcal{F}_0'(\hat{\zeta})\bar{\zeta} \le 0, \quad \mathcal{F}_i'(\hat{\zeta})\bar{\zeta} \le 0, \; i \in I, \quad \mathcal{G}'(\hat{\zeta})\bar{\zeta} = 0,$$

Define $\bar{\xi} := -\bar{\tau}$ and $\bar{x}$ by formulas (49) of Part 1,

$$\bar{\xi} = -\bar{\tau}, \quad \bar{x}(t) = V(t)\left(\bar{x}_0 - \dot{\hat{x}}(\hat{t}_0)\bar{t}_0\right) + \sum_{i=1}^{s} y^i(t)\bar{\tau}_i\,, \tag{30}$$

and put $\bar{z} = (\bar{t}_0, \bar{t}_1, \bar{\xi}, \bar{x})$. We shall show that $\bar{z}$ is an element of the critical cone $\mathcal{K}$ (Part 1, (20) and (21)) for the trajectory $\hat{\mathcal{T}} = \{(\hat{x}(t), \hat{u}(t) \mid t \in [\hat{t}_0, \hat{t}_1]\}$ in the main problem. Consider the first inequality $\mathcal{F}_0'(\hat{\zeta})\bar{\zeta} \le 0$ where $\mathcal{F}_0(\zeta) := J(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau))$. We obviously have

$$\begin{aligned}\mathcal{F}_0'(\hat{\zeta})\bar{\zeta} \;&=\; (J_{t_0}(\hat{p}) + J_{x_1}(\hat{p})g_{t_0}(\hat{\zeta}))\bar{t}_0 + (J_{t_1}(\hat{p}) + J_{x_1}(\hat{p})g_{t_1}(\hat{\zeta}))\bar{t}_1 \\ &\quad + (J_{x_0}(\hat{p}) + J_{x_1}(\hat{p})g_{x_0}(\hat{\zeta}))\bar{x}_0 + \sum_{i=1}^{s} J_{x_1}(\hat{p})g_{\tau_i}(\hat{\zeta})\bar{\tau}_i\,.\end{aligned}$$

Using formulas (19), (20), (17), (22) for the derivatives of g with respect to t_0, t_1, x_0, τ_i, respectively, at the point $\hat{\zeta}$, we get

$$\begin{aligned}\mathcal{F}_0'(\hat{\zeta})\bar{\zeta} \;&=\; (J_{t_0}(\hat{p}) - J_{x_1}(\hat{p})V(t_1)\dot{\hat{x}}(\hat{t}_0))\bar{t}_0 + (J_{t_1}(\hat{p}) + J_{x_1}(\hat{p})\dot{\hat{x}}(\hat{t}_1))\bar{t}_1 \\ &\quad + (J_{x_0}(\hat{p}) + J_{x_1}(\hat{p})V(\hat{t}_1))\bar{x}_0 + \sum_{i=1}^{s} J_{x_1}(\hat{p})y^i(\hat{t}_1)\bar{\tau}_i\,.\end{aligned}$$

Hence, inequality $\mathcal{F}_0'(\hat{\zeta})\bar{\zeta} \le 0$ is equivalent to inequality

$$\begin{aligned}&J_{t_0}(\hat{p})\bar{t}_0 + J_{t_1}(\hat{p})\bar{t}_1 + J_{x_0}(\hat{p})\bar{x}_0 \\ &+ J_{x_1}(\hat{p})\left(V(t_1)(\bar{x}_0 - \dot{\hat{x}}(\hat{t}_0)\bar{t}_0) + \sum_{i=1}^{s} y^i(\hat{t}_1)\bar{\tau}_i + \dot{\hat{x}}(\hat{t}_1)\bar{t}_1\right) \le 0.\end{aligned}$$

It follows from the definition (30) of $\bar{x}$ that

$$\tilde{x}_0 := \bar{x}(\hat{t}_0) + \dot{\hat{x}}(\hat{t}_0)\bar{t}_0 = \bar{x}_0, \tag{31}$$

since $V(\hat{t}_0) = E$, and $y^i(\hat{t}_0) = 0$, $i = 1, \dots, s$. Moreover, using the same definition, we get

$$\tilde{x}_1 := \bar{x}(\hat{t}_1) + \dot{\hat{x}}(\hat{t}_1)\bar{t}_1 = V(t_1)(\bar{x}_0 - \dot{\hat{x}}(\hat{t}_0)\bar{t}_0) + \sum_{i=1}^{s} y^i(\hat{t}_1)\bar{\tau}_i + \dot{\hat{x}}(\hat{t}_1)\bar{t}_1. \tag{32}$$

Thus, inequality $\mathcal{F}_0'(\hat{\zeta})\bar{\zeta} \le 0$ is equivalent to inequality

$$J_{t_0}(\hat{p})\bar{t}_0 + J_{t_1}(\hat{p})\bar{t}_1 + J_{x_0}(\hat{p})\tilde{x}_0 + J_{x_1}(\hat{p})\tilde{x}_1 \le 0,$$

or shortly,

$$J'(\hat{p})\tilde{p} \le 0,$$

where $\tilde{p} = (\bar{t}_0, \tilde{x}_0, \bar{t}_1, \tilde{x}_1)$; see definition (19) in Part 1.

Similarly, the inequalities $\mathcal{F}_i'(\hat{\zeta})\bar{\zeta} \le 0$ $\forall\, i \in I$ and the equality $\mathcal{G}'(\hat{\zeta})\bar{\zeta} = 0$ in the definition of $\mathcal{K}_0$ are equivalent to the inequalities, respectively, equality

$$F_i'(\hat{p})\tilde{p} \le 0, \quad i \in I, \quad K'(\hat{p})\tilde{p} = 0,$$

in the definition of $\mathcal{K}$; see (20) in Part 1.

Since $\dot{V} = f_x(t, \hat{x}(t), \hat{u}(t))V$ and $\dot{y}^i = f_x(t, \hat{x}(t), \hat{u}(t))y^i$, $i = 1, \dots, s$, it follows from definition (30) that $\bar{x}$ is a solution to the same linear system

$$\dot{\bar{x}} = f_x(t, \hat{x}(t), \hat{u}(t))\bar{x}.$$

Finally, recall from (21) that for each $i = 1, \dots, s$ the function $y^i(t)$ is piecewise continuous with only one jump $[y^i]^i = -[\dot{\hat{x}}]^i$ at the point $\hat{\tau}_i$ and absolutely continuous on each of the half-open intervals $[\hat{t}_0, \hat{\tau}_i)$ and $(\hat{\tau}_i, \hat{t}_1]$. Moreover, the function $V(t)$ is absolutely continuous in $[\hat{t}_0, \hat{t}_1]$. Hence, $\bar{x}(t)$ is a piecewise continuous function which is absolutely continuous on each interval of the set $[\hat{t}_0, \hat{t}_1] \setminus \hat{\theta}$ and satisfies the jump conditions

$$[\bar{x}]^i = [\dot{\hat{x}}]^i \bar{\xi}_i, \quad \bar{\xi}_i = -\bar{\tau}_i, \quad i = 1, \dots, s.$$

Thus, we have proved that $\bar{z} = (\bar{t}_0, \bar{t}_1, \bar{\xi}, \bar{x})$ is an element of the critical cone $\mathcal{K}$. Similarly, one can show that if $\bar{z} = (\bar{t}_0, \bar{t}_1, \bar{\xi}, \bar{x}) \in \mathcal{K}$, then putting $\bar{x}_0 = \bar{x}(\hat{t}_0)$ and $\bar{\tau} = -\bar{\xi}$, we obtain the element $\bar{\zeta} = (\bar{t}_0, \bar{t}_1, \bar{x}_0, \bar{\tau})$ of the critical cone $\mathcal{K}_0$.

6. Second order derivatives of $x(t_1; t_0, x_0, \tau)$ with respect to t_0, t_1, x_0, and τ

In this section we shall give formulas for all second order partial derivatives of the functions

$$x(t; t_0, x_0, \tau) \quad \text{and} \quad g(\zeta) = g(t_0, t_1, x_0, \tau) := x(t_1; t_0, x_0, \tau)$$

at the point $\hat{\zeta}$. We are not sure that all of them are known, therefore we shall also sketch the proofs. Here $x(t; t_0, x_0, \tau)$ is the solution to IVP (8). Denote by $g_k(\zeta) := x_k(t_1; t_0, x_0, \tau)$ the k-th component of the function g.

6.1. Derivatives $(g_k)_{x_0x_0}$

Let $x(t;x_0)$ be the solution to the IVP (8) with fixed t_0 and τ and $x_k(t;x_0)$ be its k-th component. For $k=1,\dots,n$, we define the $n\times n$ matrix

$$W^k(t;x_0)=\frac{\partial^2 x_k(t;x_0)}{\partial x_0\partial x_0}\quad\text{with entries}\quad w^k_{ij}(t;x_0)=\frac{\partial^2 x_k(t;x_0)}{\partial x_{0i}\partial x_{0j}}\,,$$

where x_{0i} is the i-th component of the column-vector $x_0\in\mathbb{R}^n$.

Proposition 6.1 *The matrix-valued functions $W^k(t;x_0)$, $k=1,\dots,n$, satisfy the IVPs*

$$\dot W^k=V^Tf_{kxx}V+\sum_{r=1}^n f_{kx_r}W^r,\qquad W^k\,|_{t=t_0}=O,\quad k=1,\dots,n,\tag{33}$$

where $\dot W^k=\frac{\partial W^k}{\partial t}$, O is the zero matrix, f_k is the k-th component of the vector function f, and

$$f_{kx_r}=\frac{\partial f_k(t,x(t;x_0),u(t))}{\partial x_r}\,,\qquad f_{kxx}=\frac{\partial^2 f_k(t,x(t;x_0),u(t))}{\partial x\partial x}$$

are its partial derivatives at the point $(t,x(t;x_0))$ for $t\in[t_0,t_1]$.

Proof. For notational convenience, we use the function $\varphi(t,x):=f(t,x,u(t))$. By Proposition 3.1, the matrix-valued function $V(t;x_0)=\frac{\partial x(t;x_0)}{\partial x_0}$ with entries $v_{ij}(t;x_0)=\frac{\partial x_i(t;x_0)}{\partial x_{0j}}$ is the solution to the IVP (9). Consequently, its entries satisfy the equations

$$\begin{aligned}\frac{\partial \dot x_k(t;x_0)}{\partial x_{0i}}&=\sum_r\varphi_{kx_r}(t,x(t;x_0))\frac{\partial x_r(t;x_0)}{\partial x_{0i}}\\ \frac{\partial x_k(t_0;x_0)}{\partial x_{0i}}&=e_{ki},\quad k,i=1,\dots,n,\end{aligned}$$

where e_{ki} are the elements of the identity matrix E. By differentiating these equations with respect to x_{0j}, we get

$$\begin{aligned}\frac{\partial^2 \dot x_k(t;x_0)}{\partial x_{0i}\partial x_{0j}}&=\sum_r\left(\varphi_{kx_r}(t,x(t;x_0))\right)_{x_{0j}}\frac{\partial x_r(t;x_0)}{\partial x_{0i}}\\ &\quad+\sum_r\varphi_{kx_r}(t,x(t;x_0))\frac{\partial^2 x_r(t;x_0)}{\partial x_{0i}\partial x_{0j}}\,,\end{aligned}\tag{34}$$

$$\frac{\partial^2 x_k(t_0;x_0)}{\partial x_{0i}\partial x_{0j}}=0,\quad k,i,j=1,\dots,n.\tag{35}$$

Transforming the first sum in the right hand side of equation (34), we get

$$\begin{aligned}
&\sum_r (\varphi_{kx_r}(t,x(t;x_0)))_{x_{0j}} \frac{\partial x_r(t;x_0)}{\partial x_{0i}} \\
=\ & \sum_r \sum_s \varphi_{kx_r x_s}(t,x(t;x_0)) \frac{\partial x_s(t;x_0)}{\partial x_{0j}} \cdot \frac{\partial x_r(t;x_0)}{\partial x_{0i}} \\
=\ & \left(V^T \varphi_{kxx}(t,x(t;x_0))V\right)_{ij}, \quad k,i,j = 1,\ldots,n,
\end{aligned}$$

where $(A)_{ij}$ denotes the element a_{ij} of a matrix A and A^T denotes the transposed matrix. Thus, (34) and (35) imply (33). ■

It follows from Proposition 6.1 that

$$(g_k)_{x_0x_0}(\hat{\zeta}) := \frac{\partial^2 x_k(\hat{t}_1;\hat{t}_0,\hat{x}_0,\hat{\tau})}{\partial x_0 \partial x_0} = W^k(\hat{t}_1), \quad k = 1,\ldots,n, \tag{36}$$

where the matrix-valued functions $W^k(t)$, $k = 1,\ldots,n$, satisfy the IVPs (33) along the reference trajectory $(\hat{x}(t),\hat{u}(t))$.

6.2. Mixed derivatives $g_{x_0\tau_i}$

Let $s = 1$ for notational convenience. Fix t_0 and consider the functions

$$\begin{aligned}
V(t;x_0,\tau) &= \frac{\partial x(t;x_0,\tau)}{\partial x_0}, \quad y(t;x_0,\tau) = \frac{\partial x(t;x_0,\tau)}{\partial \tau}, \\
R(t;x_0,\tau) &= \frac{\partial V(t;x_0,\tau)}{\partial \tau} = \frac{\partial^2 x(t;x_0,\tau)}{\partial x_0 \partial \tau}, \\
\dot{V}(t;x_0,\tau) &= \frac{\partial V(t;x_0,\tau)}{\partial t}, \quad \dot{R}(t;x_0,\tau) = \frac{\partial R(t;x_0,\tau)}{\partial t}.
\end{aligned}$$

Then, V, $\dot{V}$ and R, $\dot{R}$ are $n \times n$ matrix-valued functions and y is a vector function of dimension n.

PROPOSITION 6.2 *For $t \geq \tau$, the function $R(t;x_0,\tau)$ is the solution to the IVP*

$$\dot{R} = (y^T f_{xx})V + f_x R, \quad R(\tau;x_0,\tau) = -[f_x]V(\tau;x_0,\tau), \tag{37}$$

where f_x and f_{xx} are taken along the trajectory $(t,x(t;x_0,\tau),u(t,\tau))$, $t \in [t_0,t_1]$. Here, by definition, $(y^T f_{xx})$ is a $n \times n$ matrix with entries

$$(y^T f_{xx})_{k,j} = \sum_{i=1}^{n} \frac{\partial^2 f_k}{\partial x_i \partial x_j} y_i \tag{38}$$

in the k-th row and j-th column, and

$$[f_x] = f_x(\tau,x(\tau;x_0,\tau),u^2) - f_x(\tau,x(\tau;x_0,\tau),u^1)$$

is the jump of the function $f_x(\cdot,x(\cdot;x_0,\tau),u(\cdot,\tau))$ at the point τ. For $t < \tau$ we have $R(t;x_0,\tau) = 0$.

Proof. According to Proposition 3.1, the matrix-valued function V is the solution to the system

$$\dot{V}(t;x_0,\tau) = f_x(t,x(t;x_0,\tau),u(t;\tau))V(t;x_0,\tau). \tag{39}$$

By differentiating this equality with respect to τ, we get the equation

$$\frac{\partial \dot{V}}{\partial \tau} = \sum_i (f_x V)'_{x_i} \frac{\partial x_i}{\partial \tau} + f_x \frac{\partial V}{\partial \tau}\,,$$

which is equivalent to

$$\dot{R} = \sum_i (f_x V)_{x_i} y_i + f_x R. \tag{40}$$

Upon defining

$$A = \sum_i (f_x V)_{x_i} y_i,$$

the element in the r-th row and s-th column of the matrix A is equal to

$$\begin{aligned} a_{rs} &= \sum_i \left((f_x V)_{rs}\right)_{x_i} y_i = \sum_i \left(\sum_j f_{rx_j} v_{js} \right)_{x_i} y_i \\ &= \sum_i \sum_j y_i f_{rx_i x_j} v_{js} = \sum_j \left(\sum_i y_i f_{rx_i x_j} \right) v_{js} \\ &= \sum_j \left(y^T f_{xx}\right)_{rj} v_{js} = \left((y^T f_{xx})V\right)_{rs}, \end{aligned}$$

where v_{js} is the element in the j-th row and s-th column of the matrix V. Hence, we have $A = (y^T f_{xx})V$ and see that equation (40) is equivalent to equation (37). The initial condition in (37), which is similar to the initial condition (21) in Proposition 3.3, follows from (39) (see the proof of Proposition 3.3). The condition $R(t;x_0,\tau) = 0$ for $t < \tau$ is obvious. ∎

Proposition 6.2 yields the formula

$$g_{x_0\tau_i}(\hat{\zeta}) := \frac{\partial^2 x(\hat{t}_1;\hat{t}_0,\hat{x}_0,\hat{\tau})}{\partial x_0 \partial \tau_i} = R^i(\hat{t}_1), \tag{41}$$

where the matrix-valued function $R^i(t)$ satisfies the IVP

$$\begin{aligned} &\dot{R}^i(t) = \left(y^i(t)^T f_{xx}(t,\hat{x}(t),\hat{u}(t))\right) V(t) + f_x(t,\hat{x}(t),\hat{u}(t))R^i(t), \quad t \in [\hat{\tau}_i,\hat{t}_1], \\ &R^i(\hat{\tau}_i) = -[f_x]^i V(\hat{\tau}_i). \end{aligned} \tag{42}$$

Here, $V(t)$ is the solution to the IVP (9), $y^i(t)$ is the solution to the IVP (21) (for $t_0 = \hat{t}_0,\ x_0 = \hat{x}_0,\ \tau = \hat{\tau}$) and $[f_x]^i = f(\hat{\tau}_i,\hat{x}(\hat{\tau}_i),\hat{u}(\hat{\tau}_i+0)) - f(\hat{\tau}_i,\hat{x}(\hat{\tau}_i),\hat{u}(\hat{\tau}_i-0))$, $i = 1,\ldots,s$.

6.3. Derivatives $g_{\tau_i\tau_i}$

Again, let $s = 1$ for simplicity. Fix t_0 and x_0 and put

$$y(t;\tau) = \frac{\partial x(t;\tau)}{\partial \tau}, \quad z(t;\tau) = \frac{\partial y(t;\tau)}{\partial \tau} = \frac{\partial^2 x(t;\tau)}{\partial \tau^2},$$
$$\dot{y}(t;\tau) = \frac{\partial y(t;\tau)}{\partial t}, \quad \dot{z}(t;\tau) = \frac{\partial z(t;\tau)}{\partial t}.$$

Then y, $\dot{y}$ and z, $\dot{z}$ are vector functions of dimension n.

PROPOSITION 6.3 *For $t \geq \tau$ the function $z(t;\tau)$ is the solution to the system*

$$\dot{z} = f_x z + y^T f_{xx} y \tag{43}$$

with the initial condition at the point $t = \tau$

$$z(\tau;\tau) + \dot{y}(\tau + 0;\tau) = -[f_t] - [f_x](\dot{x}(\tau + 0;\tau) + y(\tau;\tau)). \tag{44}$$

In (43), f_x and f_{xx} are taken along the trajectory $(t, x(t;\tau), u(t;\tau))$, $t \in [t_0, t_1]$, and $y^T f_{xx} y$ is a vector with elements

$$(y^T f_{xx} y)_k = y^T f_{kxx} y = \sum_{i,j=1}^{n} \frac{\partial^2 f_k}{\partial x_i \partial x_j} y_i y_j, \quad k = 1, \dots, n.$$

In (44), the expressions

$$[f_t] = f_t(\tau, x(\tau;\tau), u^2) - f_t(\tau, x(\tau;\tau), u^1),$$
$$[f_x] = f_x(\tau, x(\tau;\tau), u^2) - f_x(\tau, x(\tau;\tau), u^1)$$

are the jumps of the derivatives $f_t(t, x(t;\tau), u(t;\tau))$ and $f_x(t, x(t;\tau), u(t;\tau))$ at the point τ ($u^2 = u(\tau + 0;\tau)$, $u^1 = u(\tau - 0;\tau)$). For $t < \tau$ we have $z(t;\tau) = 0$.

Proof. By Proposition 3.3, for $t \geq \tau$ the function $y(t;\tau)$ is the solution to the IVP

$$\dot{y}(t;\tau) = f_x(t, x(t;\tau), u(t;\tau)) y(t;\tau),$$
$$y(\tau;\tau) = -(f(\tau, x(\tau;\tau), u^2) - f(\tau, x(\tau;\tau), u^1)).$$

By differentiating these equalities with respect to τ at the points τ and $\tau + 0$, respectively, we obtain (43) and (44), respectively. For $t < \tau$ we have $y = 0$ and hence $z = 0$. ∎

For the solution $x(t; t_0, x_0, \tau)$ to the IVP (8) with an arbitrary s, it follows from Proposition 6.3 that

$$g_{\tau_i\tau_i}(\hat{\zeta}) := \frac{\partial^2 x(\hat{t}_1; \hat{t}_0, \hat{x}_0, \hat{\tau})}{\partial \tau_i \partial \tau_i} = z^{ii}(\hat{t}_1), \quad i = 1, \dots, s, \tag{45}$$

where for $t \geq \hat{\tau}_i$ the vector function $z^{ii}(t)$ satisfies the equation

$$\dot{z}^{ii}(t) = f_x(t, \hat{x}(t), \hat{u}(t))z^{ii}(t) + y^i(t)^T f_{xx}(t, \hat{x}(t), \hat{u}(t))y^i(t) \tag{46}$$

with the initial condition at the point $t = \hat{\tau}_i$:

$$z^{ii}(\hat{\tau}_i) + \dot{y}^i(\hat{\tau}_i + 0) = -[f_t]^i - [f_x]^i(\dot{\hat{x}}(\hat{\tau}_i + 0) + y^i(\hat{\tau}_i)). \tag{47}$$

Here, for $t \geq \hat{\tau}_i$, the function $y^i(t)$ is the solution to the IVP (21), and $y^i(t) = 0$ for $t < \hat{\tau}_i$, $i = 1, \ldots, s$. Furthermore, by definition, $[f_t]^i = f_t(\hat{\tau}_i, \hat{x}(\hat{\tau}_i), \hat{u}(\hat{\tau}_i+0)) - f_t(\hat{\tau}_i, \hat{x}(\hat{\tau}_i), \hat{u}(\hat{\tau}_i - 0))$ and $[f_x]^i = f_x(\hat{\tau}_i, \hat{x}(\hat{\tau}_i), \hat{u}(\hat{\tau}_i + 0)) - f_x(\hat{\tau}_i, \hat{x}(\hat{\tau}_i), \hat{u}(\hat{\tau}_i - 0))$ are the jumps of the derivatives $f_t(t, \hat{x}(t), \hat{u}(t))$ and $f_x(t, \hat{x}(t), \hat{u}(t))$ at the point $\hat{\tau}_i$. For $t < \hat{\tau}_i$ we put $z^{ii}(t) = 0$, $i = 1, \ldots, s$.

6.4. Mixed derivatives $g_{\tau_i \tau_j}$

For simplicity, let $s = 2$, $\tau = (\tau_1, \tau_2)$ and $t_0 < \tau_1 < \tau_2 < t_1$. Fix x_0 and t_0 and put

$$y^i(t;\tau) = \frac{\partial x(t;\tau)}{\partial \tau_i}, \quad i = 1, 2, \quad z^{12}(t;\tau) = \frac{\partial y^1(t;\tau)}{\partial \tau_2} = \frac{\partial^2 x(t;\tau)}{\partial \tau_1 \partial \tau_2},$$
$$\dot{y}^i(t;\tau) = \frac{\partial y^i(t;\tau)}{\partial t}, \quad i = 1, 2, \quad \dot{z}^{12}(t;\tau) = \frac{\partial z^{12}(t;\tau)}{\partial t}.$$

Then y^i, $\dot{y}^i$, $i = 1, 2$, and z^{12}, $\dot{z}^{12}$ are vector functions of dimension n.

Proposition 6.4 *For $t \geq \tau_2$ the function $z^{12}(t;\tau)$ is the solution to the system*

$$\dot{z}^{12} = f_x z^{12} + (y^1)^T f_{xx} y^2 \tag{48}$$

with the initial condition at the point $t = \tau_2$,

$$z^{12}(\tau_2;\tau) = -[\dot{y}^1]^2. \tag{49}$$

In (48), f_x and f_{xx} are taken along the trajectory $(t, x(t;\tau), u(t;\tau))$, $t \in [t_0, t_1]$, and $(y^1)^T f_{xx} y^2$ is a vector with elements

$$((y^1)^T f_{xx} y^2)_k = (y^1)^T f_{kxx} y^2 = \sum_{i,j=1}^{n} \frac{\partial^2 f_k}{\partial x_i \partial x_j} y_i^1 y_j^2, \quad k = 1, \ldots, n.$$

In (49) we have $[\dot{y}^1]^2 = [f_x]^2 y^1(\tau_2;\tau)$, where

$$[f_x]^2 = f_x(\tau_2, x(\tau_2;\tau), u^3) - f_x(\tau_2, x(\tau_2;\tau), u^2).$$

For $t < \tau_2$ we have $z^{12}(t;\tau) = 0$.

Proof. By Proposition 3.3, for $t \geq \tau_1$ the function $y^1(t;\tau)$ is a solution to the equation

$$\dot{y}^1(t;\tau) = f_x(t, x(t;\tau), u(t;\tau))y^1(t;\tau),$$

where $y^1(t;\tau) = 0$ for $t < \tau_1$. Differentiating this equation w.r.t. τ_2 we see that for $t \geq \tau_2$ the function $z^{12}(t;\tau) = \frac{\partial y^1(t;\tau)}{\partial \tau_2}$ is a solution to system (48). The initial condition (49) is similar to the initial condition (21) in Proposition 3.3. For $t < \tau_2$ we obviously have $z^{12}(t;\tau) = 0$. ■

For the solution $x(t; t_0, x_0, \tau)$ of IVP (8) and for $\tau_i < \tau_j$ $(i, j = 1, \ldots, s)$, it follows from Proposition 6.4 that

$$g_{\tau_i\tau_j}(\hat{\zeta}) := \frac{\partial^2 x(\hat{t}_1; \hat{t}_0, \hat{x}_0, \hat{\tau})}{\partial \tau_i \partial \tau_j} = z^{ij}(\hat{t}_1), \tag{50}$$

where for $t \geq \hat{\tau}_j$ the vector function $z^{ij}(t)$ is the solution to the equation

$$\dot{z}^{ij}(t) = f_x(t, \hat{x}(t), \hat{u}(t))z^{ij}(t) + y^i(t)^T f_{xx}(t, \hat{x}(t), \hat{u}(t))y^j(t) \tag{51}$$

satisfying the initial condition

$$z^{ij}(\hat{\tau}_j) = -[\dot{y}^i]^j = -[f_x]^j y^i(\hat{\tau}_j). \tag{52}$$

Here, for $t \geq \hat{\tau}_i$, the function $y^i(t)$ is the solution to the IVP (21), while $y^i(t) = 0$ holds for $t < \hat{\tau}_i$, $i = 1, \ldots, s$. By definition, $[\dot{y}^i]^j = \dot{y}^i(\hat{\tau}_j + 0) - \dot{y}^i(\hat{\tau}_j - 0)$ and $[f_x]^j = f_x(\hat{\tau}_j, \hat{x}(\hat{\tau}_j), \hat{u}(\hat{\tau}_j + 0)) - f_x(\hat{\tau}_j, \hat{x}(\hat{\tau}_j), \hat{u}(\hat{\tau}_j - 0))$ are the jumps of the derivatives $\dot{y}^i(t)$ and $f_x(t, \hat{x}(t), \hat{u}(t))$, respectively, at the point $\hat{\tau}_j$. For $t < \hat{\tau}_j$ we put $z^{ij}(t) = 0$.

6.5. Derivatives $g_{t_0t_0}$, $g_{t_0t_1}$ and $g_{t_1t_1}$

Here, we fix x_0 and τ and study the functions

$$w(t;t_0) = \frac{\partial x(t;t_0)}{\partial t_0}, \quad q(t;t_0) = \frac{\partial w(t;t_0)}{\partial t_0} = \frac{\partial^2 x(t;t_0)}{\partial t_0^2},$$
$$\dot{w}(t;t_0) = \frac{\partial w(t;t_0)}{\partial t}, \quad \dot{q}(t;t_0) = \frac{\partial q(t;t_0)}{\partial t}, \quad \ddot{x}(t;t_0) = \frac{\partial^2 x(t;t_0,)}{\partial t^2}.$$

Proposition 6.5 *The function $q(t;t_0)$ is the solution to the system*

$$\dot{q} = f_x q + w^T f_{xx} w, \quad t \in [t_0, t_1] \tag{53}$$

satisfying the initial condition at the point $t = t_0$,

$$\ddot{x}(t_0;t_0) + 2\dot{w}(t_0;t_0) + q(t_0;t_0) = 0. \tag{54}$$

In (53), f_x and f_{xx} are taken along the trajectory $(t, x(t; t_0), u(t))$, $t \in [t_0, t_1]$, and $w^T f_{xx} w$ is a vector with elements

$$(w^T f_{xx} w)_k = w^T f_{kxx} w = \sum_{i,j=1}^{n} \frac{\partial^2 f_k}{\partial x_i \partial x_j} w_i w_j, \quad k = 1, \ldots, n.$$

Proof. By Proposition 3.2 we have

$$\dot{w}(t; t_0) = f_x(t, x(t; t_0)) w(t; t_0), \quad \dot{x}(t_0; t_0) + w(t_0; t_0) = 0.$$

Differentiating these equalities with respect to t_0, we obtain (53) and (54). ■

From Proposition 6.5 it follows that

$$g_{t_0 t_0}(\hat{\zeta}) := \frac{\partial^2 x(\hat{t}_1; \hat{t}_0, \hat{x}_0, \hat{\tau})}{\partial t_0^2} = q(\hat{t}_1), \tag{55}$$

where the vector function $q(t)$ is the solution to the equation

$$\dot{q}(t) = f_x(t, \hat{x}(t), \hat{u}(t)) q(t) + w^T(t) f_{xx}(t, \hat{x}(t), \hat{u}(t)) w(t) \tag{56}$$

satisfying the initial condition

$$\ddot{\hat{x}}(\hat{t}_0) + 2\dot{w}(\hat{t}_0) + q(\hat{t}_0) = 0. \tag{57}$$

Since $w(t) = -V(t)\dot{\hat{x}}(\hat{t}_0)$ in view of Proposition 3.2, $\dot{V} = f_x V$ and $V(\hat{t}_0) = E$, we obtain

$$\dot{w}(\hat{t}_0) = -\dot{V}(\hat{t}_0)\dot{\hat{x}}(\hat{t}_0) = -f_x(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{u}(\hat{t}_0))\dot{\hat{x}}(\hat{t}_0).$$

Thus, the initial condition (57) is equivalent to

$$\ddot{\hat{x}}(\hat{t}_0) - 2 f_x(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{u}(\hat{t}_0))\dot{\hat{x}}(\hat{t}_0) + q(\hat{t}_0) = 0. \tag{58}$$

From (19) it follows that

$$\begin{aligned} g_{t_0 t_1}(\hat{\zeta}) &:= \frac{\partial^2 x(\hat{t}_1; \hat{t}_0, \hat{x}_0, \hat{\tau})}{\partial t_0 \partial t_1} \\ &= -\dot{V}(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0) = -f_x(\hat{t}_1, \hat{x}(\hat{t}_1), \hat{u}(\hat{t}_1)) V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0). \end{aligned} \tag{59}$$

Formula (20) implies that

$$g_{t_1 t_1}(\hat{\zeta}) := \frac{\partial^2 x(\hat{t}_1; \hat{t}_0, \hat{x}_0, \hat{\tau})}{\partial t_1^2} = \ddot{\hat{x}}(\hat{t}_1). \tag{60}$$

6.6. Derivatives $g_{x_0 t_1}$ and $g_{\tau_i t_1}$

Formula (17) implies that

$$g_{x_0 t_1}(\hat{\zeta}) := \frac{\partial^2 x(\hat{t}_1; \hat{t}_0, \hat{x}_0, \hat{\tau})}{\partial x_0 \partial t_1} = \dot{V}(\hat{t}_1), \tag{61}$$

where $V(t)$ is the solution to the IVP (9). From (22) it follows that

$$g_{\tau_i t_1}(\hat{\zeta}) := \frac{\partial^2 x(\hat{t}_1; \hat{t}_0, \hat{x}_0, \hat{\tau})}{\partial \tau_i \partial t_1} = \dot{y}^i(\hat{t}_1), \quad i = 1, \ldots, s, \tag{62}$$

where $y^i(t)$ is the solution to the IVP (21).

6.7. Derivative $g_{x_0 t_0}$

Let us fix τ and consider

$$V(t; t_0, x_0) = \frac{\partial x(t; t_0, x_0)}{\partial x_0}, \quad S(t; t_0, x_0) = \frac{\partial V(t; t_0, x_0)}{\partial t_0} = \frac{\partial^2 x(t; t_0, x_0)}{\partial x_0 \partial t_0},$$
$$\dot{V}(t; t_0, x_0) = \frac{\partial V(t; t_0, x_0)}{\partial t}, \quad \dot{S}(t; t_0, x_0) = \frac{\partial S(t; t_0, x_0)}{\partial t}.$$

Proposition 6.6 *The elements $s_{ij}(t; t_0, x_0)$ of the matrix $S(t; t_0, x_0)$ satisfy the system*

$$\dot{s}_{ij} = -e_j^T V^T (f_i)_{xx} V \dot{x}(t_0) + f_{ix} S e_j, \quad i, j = 1, \ldots, n, \tag{63}$$

and the matrix S itself satisfies the initial condition at the point $t = t_0$,

$$S(t_0; t_0, x_0) + \dot{V}(t_0; t_0, x_0) = 0. \tag{64}$$

In (63), the derivatives f_x and f_{xx} are taken along the trajectory $(t, x(t; t_0, x_0), u(t))$, $t \in [t_0, t_1]$, e_j is the j-th column of the identity matrix E, and, by definition, $\dot{x}(t_0) = \dot{x}(t_0; t_0, x_0)$.

Proof. By Proposition 3.1,

$$\dot{V}(t; t_0, x_0) = f_x(t, x(t; t_0, x_0), u(t)) V(t; t_0, x_0), \quad V(t_0; t_0, x_0) = E. \tag{65}$$

The first equality in (65) is equivalent to

$$\dot{v}_{ij}(t; t_0, x_0) = f_{ix}(t, x(t; t_0, x_0), u(t)) V(t, t_0) e_j, \quad i, j = 1, \ldots, n.$$

By differentiating these equalities with respect to t_0 and using Proposition 3.2, we obtain (63). Differentiating the second equality in (65) with respect to t_0, yields (64). ■

Proposition 6.6 implies that

$$g''_{x_0t_0}(\hat{\zeta}) := \frac{\partial^2 x(\hat{t}_1;\hat{t}_0,\hat{x}_0,\hat{\tau})}{\partial x_0 \partial t_0} = S(\hat{t}_1), \tag{66}$$

where the elements $s_{ij}(t)$ of the matrix $S(t)$ satisfy the system

$$\begin{aligned} \dot{s}_{ij}(t) = e_j^T V^T(t)(f_i)_{xx}(t,\hat{x}(t),\hat{u}(t))V(t)\dot{\hat{x}}(\hat{t}_0) + f_{ix}(t,\hat{x}(t),\hat{u}(t))S(t)e_j, \\ i,j = 1,\dots,n. \end{aligned} \tag{67}$$

Here, $V(t)$ is the solution to the IVP (9) and the matrix $S(t)$ itself satisfies the initial condition at the point $t = \hat{t}_0$,

$$S(\hat{t}_0) + \dot{V}(\hat{t}_0) = 0. \tag{68}$$

6.8. Derivative $g_{\tau_i t_0}$

Consider again the case $s = 1$ and define

$$\begin{aligned} y(t;t_0,\tau) &= \frac{\partial x(t;t_0,\tau)}{\partial \tau}, \quad r(t;t_0,\tau) = \frac{\partial y(t;t_0,\tau)}{\partial t_0} = \frac{\partial^2 x(t;t_0,\tau)}{\partial t_0 \partial \tau}, \\ \dot{y}(t;t_0,\tau) &= \frac{\partial y(t;t_0,\tau)}{\partial t}, \quad \dot{r}(t;t_0,\tau) = \frac{\partial r(t,t_0,\tau)}{\partial t}, \\ \dot{x}(t;t_0,\tau) &= \frac{\partial x(t;t_0,\tau)}{\partial t}, \quad V(t;t_0,\tau) = \frac{\partial x(t;t_0,\tau)}{\partial x_0}. \end{aligned}$$

PROPOSITION 6.7 *For $t \geq \tau$ the function $r(t;t_0,\tau)$ is the solution to the IVP*

$$\dot{r} = f_x r - y^T f_{xx} V \dot{x}(t_0), \quad r\,|_{t=\tau} = [f_x]V(\tau)\dot{x}(t_0), \tag{69}$$

where $y^T f_{xx} V\dot{x}(t_0)$ is the vector with elements $(y^T f_{xx} V\dot{x}(t_0))_i = y^T f_{ixx} V\dot{x}(t_0)$, $i = 1,\dots,n$, $V(\tau) = V(\tau;t_0,\tau)$ and

$$[f_x] = f_x(\tau, x(\tau;t_0,\tau), u^2) - f_x(\tau, x(\tau;t_0,\tau), u^1)$$

is the jump of the derivative $f_x(t, x(t;t_0,\tau), u(t;\tau))$ at the point τ. The derivatives f_x and f_{xx} are taken along the trajectory $(t, x(t;t_0,\tau), u(t;\tau))$, $t \in [\tau, t_1]$. For $t < \tau$ we have $r(t;t_0,\tau) = 0$. Then the jump of the function $r(t;t_0,\tau)$ at the point $t = \tau$ is given by $[r] = [f_x]V(\tau)\dot{x}(t_0)$.

Proof. By Proposition 3.3 we have $y(t;t_0,\tau) = 0$ for $t < \tau$ and hence $r(t;t_0,\tau) = 0$ for $t < \tau$. According to the same proposition, for $t \geq \tau$ the function $y(t;t_0,\tau)$ satisfies the equation

$$\dot{y}(t;t_0,\tau) = f_x(t, x(t;t_0,\tau), u(t;\tau))y(t;t_0,\tau).$$

Differentiating this equation w.r.t. t_0, we get

$$\dot{r} = f_x r + y^T f_{xx} \frac{\partial x}{\partial t_0}.$$

According to Proposition 3.2,

$$\frac{\partial x(t;t_0,\tau)}{\partial t_0} = -V(t;t_0,\tau)\dot{x}(t_0),$$

where $\dot{x}(t_0) = \dot{x}(t_0;t_0,\tau)$. This yields

$$\dot{r} = f_x r - y^T f_{xx} V \dot{x}(t_0).$$

By Proposition 3.3, the following initial condition holds at the point $t = \tau$:

$$y(\tau;t_0,\tau) = -(f(\tau, x(\tau;t_0,\tau), u^2) - f(\tau, x(\tau;t_0,\tau), u^1)).$$

Differentiating this condition w.r.t. t_0, we get

$$r\,|_{t=\tau} = -[f_x]\frac{\partial x}{\partial t_0}\,|_{t=\tau} = [f_x]V(\tau)\dot{x}(t_0),$$

where $V(\tau) = V(\tau;t_0,\tau)$. ■

It follows from Proposition 6.7 that for each $i = 1,\ldots,s$

$$g_{\tau_i t_0}(\hat{\zeta}) := \frac{\partial^2 x(\hat{t}_1;\hat{t}_0,\hat{x}_0,\hat{\tau})}{\partial \tau_i \partial t_0} = r^i(\hat{t}_1), \tag{70}$$

where the function $r^i(t)$ is the solution to the system

$$\dot{r}^i(t) = f_x(t,\hat{x}(t),\hat{u}(t))r^i(t) - (y^i(t))^T f_{xx}(t,\hat{x}(t),\hat{u}(t))V(t)\dot{\hat{x}}(\hat{t}_0), \tag{71}$$

and satisfies the initial condition at the point $t = \hat{\tau}_i$,

$$r^i(\hat{\tau}_i) = [f_x]^i V(\hat{\tau}_i)\dot{\hat{x}}(\hat{t}_0). \tag{72}$$

Here $V(t)$ is the solution to the IVP (9) and $y^i(t)$ is the solution to the IVP (21). The vector $(y^i)^T f_{xx} V \dot{\hat{x}}(\hat{t}_0)$ has components

$$((y^i)^T f_{xx} V \dot{\hat{x}}(\hat{t}_0))_k = (y^i)^T f_{kxx} V \dot{\hat{x}}(\hat{t}_0), \quad k = 1,\ldots,n.$$

7. Explicit representation of the quadratic form for the induced optimization problem

Let the Lagrange multipliers

$$\mu = (\alpha_0,\alpha,\beta) \in \Lambda_0, \quad \lambda = (\alpha_0,\alpha,\beta,\psi,\psi_0) \in \Lambda$$

correspond to each other, i.e, let $\pi_0\lambda = \mu$ hold; see Proposition 4.1 in Part 1. For any $\bar{\zeta} = (\bar{t}_0,\bar{t}_1,\bar{x}_0,\bar{\tau}) \in \mathbb{R}^{2+n+s}$, let us find an explicit representation for the quadratic form $\langle L_{\zeta\zeta}(\mu,\hat{\zeta})\bar{\zeta},\bar{\zeta}\rangle$. By definition,

$$\begin{aligned}\langle L_{\zeta\zeta}(\mu,\hat{\zeta})\bar{\zeta},\bar{\zeta}\rangle \;=\;& \langle L_{x_0x_0}\bar{x}_0,\bar{x}_0\rangle + 2\sum_{i=1}^{s} L_{x_0\tau_i}\bar{x}_0\bar{\tau}_i + \sum_{i,j=1}^{s} L_{\tau_i\tau_j}\bar{\tau}_i\bar{\tau}_j \\ & +2L_{x_0t_1}\bar{x}_0\bar{t}_1 + 2\sum_{i=1}^{s} L_{\tau_it_1}\bar{\tau}_i\bar{t}_1 + L_{t_1t_1}\bar{t}_1^2 \\ & +2L_{x_0t_0}\bar{x}_0\bar{t}_0 + 2\sum_{i=1}^{s} L_{t_0\tau_i}\bar{t}_0\bar{\tau}_i + 2L_{t_0t_1}\bar{t}_0\bar{t}_1 + L_{t_0t_0}\bar{t}_0^2. \end{aligned} \tag{73}$$

All derivatives in formula (73) are taken at the point $(\mu,\hat{\zeta})$. Now we shall calculate these derivatives. Recall the definition (11) of the Lagrangian,

$$L(\mu,\zeta) = L(\mu,t_0,t_1,x_0,\tau) = l(\mu,t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\,. \tag{74}$$

Note that all functions V, W^k, y^i, z^{ij}, S, R^i, q, w, r^i, introduced in Sections 3 and 6 depend now on t, t_0, x_0, and τ. For simplicity, we put $V(t) = V(t;\hat{t}_0,\hat{x}_0,\hat{\tau})$, etc.

7.1. Derivative $L_{x_0x_0}$

Using Proposition 3.1, we get

$$\begin{aligned}\left(\frac{\partial}{\partial x_0} l(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\right)\bar{x}_0 &= l_{x_0}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\bar{x}_0 \\ &\quad + l_{x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))V(t_1;t_0,x_0,\tau)\bar{x}_0. \end{aligned} \tag{75}$$

Let us find the derivative of this function with respect to x_0. We have

$$\begin{aligned}\frac{\partial}{\partial x_0}\Big(l_{x_0}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\bar{x}_0\Big) &= \bar{x}_0^T l_{x_0x_0}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) \\ &\quad + \bar{x}_0^T l_{x_0x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))V(t_1;t_0,x_0,\tau), \end{aligned} \tag{76}$$

and

$$\begin{aligned}&\frac{\partial}{\partial x_0}\Big(l_{x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))V(t_1;t_0,x_0,\tau))\bar{x}_0\Big) \\ &\quad = \bar{x}_0^T V^T(t;t_0,x_0,\tau)\Big(l_{x_1x_0}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) \\ &\qquad + l_{x_1x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))V(t;t_0,x_0,\tau)\Big) \\ &\qquad + l_{x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\frac{\partial}{\partial x_0}V(t;t_0,x_0,\tau)\bar{x}_0. \end{aligned} \tag{77}$$

From (75)–(77) and the transversality condition $l_{x_1}(\hat{p}) = \psi(\hat{t}_1)$ it follows that at the point $\hat{\zeta}$ we have

$$\begin{aligned}\langle L_{x_0x_0}\bar{x}_0, \bar{x}_0\rangle &= \bar{x}_0^T l_{x_0x_0}(\hat{p})\bar{x}_0 + 2\bar{x}_0^T l_{x_0x_1}(\hat{p})V(\hat{t}_1)\bar{x}_0 \\ &+\bar{x}_0^T V^T(\hat{t}_1) l_{x_1x_1}(\hat{p})V(\hat{t}_1)\bar{x}_0 \\ &+\{\psi(t_1)\frac{\partial}{\partial x_0}(V(t_1;t_0,x_0,\tau)\bar{x}_0)\bar{x}_0\}\mid_{\zeta=\hat{\zeta}} . \end{aligned} \tag{78}$$

Let us calculate the last term in this formula.

Proposition 7.1 *The following equality holds*

$$\psi(t_1)\frac{\partial}{\partial x_0}\left(V(t_1;t_0,x_0,\tau)\bar{x}_0\right)\bar{x}_0 = \bar{x}_0^T\left(\sum_k \psi_k(t_1)W^k(t_1;t_0,x_0,\tau)\right)\bar{x}_0. \tag{79}$$

Proof. For brevity, put $\psi(t_1) = \psi$, $V(t_1;t_0,x_0,\tau) = V$, $W(t_1;t_0,x_0,\tau) = W$. Then we have

$$\begin{aligned}&\psi\tfrac{\partial}{\partial x_0}\left(V\bar{x}_0\right)\bar{x}_0 = \psi\tfrac{\partial}{\partial x_0}\left(\tfrac{\partial x}{\partial x_0}\bar{x}_0\right)\bar{x}_0 = \psi\tfrac{\partial}{\partial x_0}\left(\sum_i \tfrac{\partial x}{\partial x_{0i}}\bar{x}_{0i}\right)\bar{x}_0 \\ &= \psi\sum_j\sum_i \tfrac{\partial^2 x}{\partial x_{0i}\partial x_{0j}}\bar{x}_{0i}\bar{x}_{0j} = \sum_k\sum_j\sum_i \psi_k\tfrac{\partial^2 x_k}{\partial x_{0i}\partial x_{0j}}\bar{x}_{0i}\bar{x}_{0j} \\ &= \sum_i\sum_j\left(\sum_k \psi_k\tfrac{\partial^2 x_k}{\partial x_{0i}\partial x_{0j}}\right)\bar{x}_{0i}\bar{x}_{0j} = \bar{x}_0^T\left(\sum_k \psi_k(t_1)W^k\right)\bar{x}_0. \end{aligned}$$ ■

Proposition 7.2 *For $\zeta = \hat{\zeta}$, the following equality holds*

$$\frac{d}{dt}\left(\sum_k \psi_k W^k\right) = V^T H_{xx}V, \tag{80}$$

where $H = \psi f(t,x,u)$, $H_{xx} = H_{xx}(t,\hat{x}(t),\psi(t),\hat{u}(t))$.

Proof. According to Proposition 6.1, we have

$$\dot{W}^k = V^T f_{kxx}V + \sum_r f_{kx_r}W^r, \quad k = 1\ldots,n. \tag{81}$$

Using these equations together with the adjoint equation $-\dot{\psi} = \psi f_x$, we obtain

$$\begin{aligned}&\tfrac{d}{dt}\left(\sum_k \psi_k W^k\right) = \sum_k \dot{\psi}_k W^k + \sum_k \psi_k \dot{W}^k \\ &= -\sum_k \psi f_{x_k}W^k + \sum_k \psi_k\left(V^T f_{kxx}V + \sum_r f_{kx_r}W^r\right) \\ &= -\sum_k \psi f_{x_k}W^k + \sum_k V^T\left(\psi_k f_{kxx}\right)V + \sum_k \psi_k\sum_r f_{kx_r}W^r \\ &= -\sum_r \psi f_{x_r}W^r + V^T\left(\sum_k \psi_k f_{kxx}\right)V + \sum_r\left(\sum_k \psi_k f_{kx_r}\right)W^r \\ &= -\sum_r \psi f_{x_r}W^r + V^T(\psi f_{xx})V + \sum_r \psi f_{x_r}W^r = V^T H_{xx}V. \end{aligned}$$ ■

Now we can prove the following assertion.

PROPOSITION 7.3 *The following formula holds*

$$\{\psi(t_1)\frac{\partial}{\partial x_0}\left(V(t_1;t_0,x_0,\tau)\bar{x}_0\right)\bar{x}_0\}\,|_{\hat{\zeta}}$$
$$=\int_{\hat{t}_0}^{\hat{t}_1}\left(V(t)\bar{x}_0\right)^T H_{xx}(t,\hat{x}(t),\hat{u}(t),\psi(t))V(t)\bar{x}_0\,dt. \tag{82}$$

Proof. Using Propositions 7.1 and 7.2 and the initial conditions $W^k(\hat{t}_0)=0$ for $k=1.\dots,n$, we get

$$\{\psi(t_1)\tfrac{\partial}{\partial x_0}\left(V(t_1;t_0,x_0,\tau)\bar{x}_0\right)\bar{x}_0\}\,|_{\hat{\zeta}}$$
$$=\bar{x}_0^T\left(\sum_k\psi_k(\hat{t}_1)W^k(\hat{t}_1)\right)\bar{x}_0=\bar{x}_0^T\left(\sum_k\psi_k(t)W^k(t)\right)\bar{x}_0\,|_{\hat{t}_0}^{\hat{t}_1}$$
$$=\int_{\hat{t}_0}^{\hat{t}_1}\bar{x}_0^T\tfrac{d}{dt}\left(\sum_k\psi_kW^k\right)\bar{x}_0\,dt=\int_{\hat{t}_0}^{\hat{t}_1}\bar{x}_0^TV^TH_{xx}V\bar{x}_0\,dt$$
$$=\int_{\hat{t}_0}^{\hat{t}_1}\left(V\bar{x}_0\right)^TH_{xx}\left(V\bar{x}_0\right)\,dt.$$
■

In view of formulas (78) and (82), we obtain

$$\langle L_{x_0x_0}\bar{x}_0,\bar{x}_0\rangle$$
$$=\bar{x}_0^Tl_{x_0x_0}(\hat{p})\bar{x}_0+2\bar{x}_0^Tl_{x_0x_1}(\hat{p})V(\hat{t}_1)\bar{x}_0+(V(\hat{t}_1)\bar{x}_0)^Tl_{x_1x_1}(\hat{p})V(\hat{t}_1)\bar{x}_0$$
$$+\int_{\hat{t}_0}^{\hat{t}_1}\left(V(t)\bar{x}_0\right)^TH_{xx}(t,\hat{x}(t),\psi(t),\hat{u}(t))V(t)\bar{x}_0\,dt. \tag{83}$$

7.2. Derivative $L_{x_0\tau_i}$

Differentiating (75) with respect to τ_i and using Propositions 3.3 and 6.2, we get

$$\frac{\partial^2}{\partial x_0\partial\tau_i}l(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\bar{x}_0$$
$$=\frac{\partial}{\partial\tau_i}l_{x_0}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\bar{x}_0$$
$$+\frac{\partial}{\partial\tau_i}\left(l_{x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))V(t_1;t_0,x_0,\tau)\bar{x}_0\right)$$
$$=\bar{x}_0^Tl_{x_0x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\frac{\partial x(t_1;t_0,x_0,\tau)}{\partial\tau_i}$$

$$
\begin{aligned}
&+\left(\frac{\partial}{\partial \tau_i} l_{x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\right) V(t_1;t_0,x_0,\tau)\bar{x}_0 \\
&+l_{x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\frac{\partial V(t_1;t_0,x_0,\tau)}{\partial \tau_i}\bar{x}_0 \\
= {}& \bar{x}_0^T l_{x_0x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))y^i(t_1;t_0,x_0,\tau) \\
&+(V(t_1;t_0,x_0,\tau)\bar{x}_0)^T l_{x_1x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))y^i(t_1;t_0,x_0,\tau) \\
&+l_{x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))R^i(t_1;t_0,x_0,\tau))\bar{x}_0. \qquad (84)
\end{aligned}
$$

Hence at the point $\zeta = \hat{\zeta}$ we have

$$
\begin{aligned}
L_{x_0\tau_i}\bar{x}_0\bar{\tau}_i \;=\;& \bar{x}_0^T l_{x_0x_1}(\hat{p})y^i(\hat{t}_1)\bar{\tau}_i + \left(V(\hat{t}_1)\bar{x}_0\right)^T l_{x_1x_1}(\hat{p})y^i(\hat{t}_1)\bar{\tau}_i \\
&+\psi(\hat{t}_1)R^i(\hat{t}_1)\bar{x}_0\bar{\tau}_i\,.
\end{aligned} \qquad (85)
$$

Let us transform the last term.

PROPOSITION 7.4 *The following formula holds*

$$
\psi(\hat{t}_1)R^i(\hat{t}_1)\bar{x}_0\bar{\tau}_i = -[H_x]^i V(\hat{\tau}_i)\bar{x}_0\bar{\tau}_i + \int_{\hat{\tau}_i}^{\hat{t}_1} \langle H_{xx}y^i\bar{\tau}_i, V\bar{x}_0\rangle\, dt. \qquad (86)
$$

Proof Using equation (42) and the adjoint equation $-\dot{\psi} = \psi f_x$, we get for $t \in [\hat{\tau}_i, \hat{t}_1]$:

$$
\begin{aligned}
\tfrac{d}{dt}(\psi R^i) &= \dot{\psi}R^i + \psi\dot{R}^i = -\psi f_x R^i + \psi\left(((y^i)^T f_{xx})V + f_x R^i\right) \\
&= \sum_k \psi_k (y^i)^T f_{kxx} V = (y^i)^T \sum_k \psi_k f_{kxx} V = (y^i)^T H_{xx} V,
\end{aligned}
$$

where H_{xx} is taken along the trajectory $(t, \hat{x}(t), \psi(t), \hat{u}(t))$. Consequently,

$$
\psi(\hat{t}_1)R^i(\hat{t}_1) = \psi(\hat{\tau}_i)R^i(\hat{\tau}_i) + \int_{\hat{\tau}_i}^{\hat{t}_1} (y^i)^T H_{xx} V\, dt.
$$

Using the initial condition (42) for R^i at $\hat{\tau}_i$, we get

$$
\psi(\hat{t}_1)R^i(\hat{t}_1) = -\psi(\hat{\tau}_i)[f_x]^i V(\hat{\tau}_i) + \int_{\hat{\tau}_i}^{\hat{t}_1} (y^i)^T H_{xx} V\, dt.
$$

Hence,

$$
\psi(\hat{t}_1)R^i(\hat{t}_1)\bar{x}_0\bar{\tau}_i = -[H_x]^i V(\hat{\tau}_i)\bar{x}_0\bar{\tau}_i + \int_{\hat{\tau}_i}^{\hat{t}_1} \langle H_{xx}y^i\bar{\tau}_i, V\bar{x}_0\rangle\, dt. \qquad \blacksquare
$$

Formulas (85), (86) and the condition $y^i(t) = 0$ for $t < \hat{\tau}_i$ imply the equality

$$\begin{aligned} L_{x_0\tau_i}\bar{x}_0\bar{\tau}_i &= \bar{x}_0^T l_{x_0x_1}(\hat{p})y^i(\hat{t}_1)\bar{\tau}_i + (V(\hat{t}_1)\bar{x}_0)^T l_{x_1x_1}(\hat{p})y^i(\hat{t}_1)\bar{\tau}_i \\ &\quad -[H_x]^i V(\hat{\tau}_i)\bar{x}_0\bar{\tau}_i + \int_{\hat{t}_0}^{\hat{t}_1} \langle H_{xx}y^i\bar{\tau}_i, V\bar{x}_0\rangle\, dt. \end{aligned} \tag{87}$$

7.3. Derivative $L_{\tau_i\tau_i}$

Using the notation $\frac{\partial x}{\partial \tau_i} = y^i$ from Proposition 3.3, we get

$$\begin{aligned} &\frac{\partial}{\partial \tau_i} l(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau)) \\ &= l_{x_1}(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau))y^i(t_1; t_0, x_0, \tau)). \end{aligned} \tag{88}$$

Now, using the notation $\frac{\partial y^i}{\partial \tau_i} = z^{ii}$ as in Proposition 6.3, we obtain

$$\begin{aligned} &\frac{\partial^2}{\partial \tau_i^2} l(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau)) \\ &= \left(\frac{\partial}{\partial \tau_i} l_{x_1}(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau))\right) y^i(t_1; t_0, x_0, \tau) \\ &\quad + l_{x_1}(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau)) \frac{\partial}{\partial \tau_i} y^i(t_1; t_0, x_0, \tau) \\ &= \langle l_{x_1x_1}(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau))y^i(t_1; t_0, x_0, \tau), y^i(t_1; t_0, x_0, \tau)\rangle \\ &\quad + l_{x_1}(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau))z^{ii}(t_1; t_0, x_0, \tau), \end{aligned} \tag{89}$$

and thus,

$$\begin{aligned} L_{\tau_i\tau_i} &= \frac{\partial^2}{\partial \tau_i^2} l(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau)) \mid_{\zeta = \hat{\zeta}} \\ &= \langle l_{x_1x_1}(\hat{p})y^i(\hat{t}_1), y^i(\hat{t}_1)\rangle + l_{x_1}(\hat{p})z^{ii}(\hat{t}_1). \end{aligned} \tag{90}$$

Let us rewrite the last term in this formula. The transversality condition $l_{x_1} = \psi(\hat{t}_1)$ implies

$$l_{x_1}(\hat{p})z^{ii}(\hat{t}_1) = \psi(\hat{t}_1)z^{ii}(\hat{t}_1) = \int_{\hat{\tau}_i}^{\hat{t}_1} \frac{d}{dt}(\psi z^{ii})\, dt + \psi(\hat{\tau}_i)z^{ii}(\hat{\tau}_i). \tag{91}$$

By formula (46), we have

$$\dot{z}^{ii} = f_x z^{ii} + (y^i)^T f_{xx} y^i, \qquad t \geq \tau_i.$$

Using this equation together with the adjoint equation $-\dot{\psi} = \psi f_x$, we get

$$\begin{aligned} \frac{d}{dt}(\psi z^{ii}) = \dot{\psi} z^{ii} + \psi \dot{z}^{ii} &= -\psi f_x z^{ii} + \psi f_x z^{ii} + \sum_k \psi_k((y^i)^T f_{kxx} y^i) \\ &= (y^i)^T H_{xx} y^i, \end{aligned} \tag{92}$$

and thus

$$l_{x_1}(\hat{p})z^{ii}(\hat{t}_1) = \int\limits_{\hat{\tau}_i}^{\hat{t}_1} (y^i)^T H_{xx} y^i \, dt + \psi(\hat{\tau}_i) z^{ii}(\hat{\tau}_i). \tag{93}$$

We shall transform the last term in (93) using the relations

$$\begin{aligned} &(\Delta_i H)(t) = H(t, \hat{x}(t), \psi(t), \hat{u}^{i+}) - H(t, \hat{x}(t), \psi(t), \hat{u}^{i-}) \\ &D^i(H) = -\frac{d}{dt}(\Delta_i H)\big|_{t=\tau_i+0} = -[H_t]^i - [H_x]^i \dot{\hat{x}}(\hat{\tau}_i + 0) - \dot{\psi}(\hat{\tau}_i + 0)[H_\psi]^i, \end{aligned} \tag{94}$$

see (12) of Part 1.

PROPOSITION 7.5 *The following equality holds,*

$$\psi(\hat{\tau}_i) z^{ii}(\hat{\tau}_i) = D^i(H) - [H_x]^i [y^i]^i. \tag{95}$$

Proof. Multiplying the initial condition (47) for z^{ii} at the point $t = \hat{\tau}_i$ by $\psi(\hat{\tau}_i)$, we get

$$\psi(\hat{\tau}_i) z^{ii}(\hat{\tau}_i) + \psi(\hat{\tau}_i)\dot{y}^i(\hat{\tau}_i + 0) = -\psi(\hat{\tau}_i)[f_t]^i - \psi(\hat{\tau}_i)[f_x]^i \left(\dot{\hat{x}}(\hat{\tau}_i + 0) + y^i(\hat{\tau}_i)\right). \tag{96}$$

Here, we obviously have the relations $\psi(\hat{\tau}_i)[f_t]^i = [H_t]^i$, $\psi(\hat{\tau}_i)[f_x]^i = [H_x]^i$ and $y^i(\hat{\tau}_i) = [y^i]^i$. Moreover, equation (21) for y^i together with the adjoint equation $-\dot{\psi} = \psi f_x$ imply that $\psi \dot{y}^i = \psi f_x y^i = -\dot{\psi} y$. Hence, in view of the initial condition (21) for y^i we find

$$\psi(\hat{\tau}_i)\dot{y}^i(\hat{\tau}_i + 0) = -\dot{\psi}(\hat{\tau}_i + 0) y^i(\hat{\tau}_i) = \dot{\psi}(\hat{\tau}_i + 0)[f]^i = \dot{\psi}(\hat{\tau}_i + 0)[H_\psi]^i.$$

Thus, (96) and (94) imply (95). ■

From the relations (90), (93), (95) and the equality $y^i(t) = 0$ for $t < \hat{\tau}_i$, it follows that

$$\begin{aligned} L_{\tau_i\tau_i}\bar{\tau}_i^2 \quad = \quad & \langle l_{x_1x_1}(\hat{p}) y^i(\hat{t}_1)\bar{\tau}_i, y^i(\hat{t}_1)\bar{\tau}_i\rangle + \int\limits_{\hat{t}_0}^{\hat{t}_1} (y^i\bar{\tau}_i)^T H_{xx} y^i \bar{\tau}_i \, dt \\ & + D^i(H)\bar{\tau}_i^2 - [H_x]^i [y^i]^i \bar{\tau}_i^2, \quad i = 1, \ldots, s. \end{aligned} \tag{97}$$

7.4. Derivative $L_{\tau_i\tau_j}$

Note that $L_{\tau_i\tau_j} = L_{\tau_j\tau_i}$ for all i, j. Therefore,

$$\sum_{i,j=1}^{s} L_{\tau_i\tau_j}\bar{\tau}_i\bar{\tau}_j = \sum_{i=1}^{s} L_{\tau_i\tau_i}\bar{\tau}_i^2 + 2\sum_{i<j} L_{\tau_i\tau_j}\bar{\tau}_i\bar{\tau}_j. \tag{98}$$

Let us calculate $L_{\tau_i\tau_j}$ for $i<j$. Differentiating (88) w.r.t. τ_j, we get

$$\begin{aligned}
&\frac{\partial^2}{\partial\tau_i\partial\tau_j} l(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) \\
&= \left(\frac{\partial}{\partial\tau_j} l_{x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\right) y^i(t_1;t_0,x_0,\tau) \\
&\qquad + l_{x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))\frac{\partial}{\partial\tau_j} y^i(t_1;t_0,x_0,\tau) \\
&= \langle l_{x_1x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))y^i(t_1;t_0,x_0,\tau), y^j(t_1;t_0,x_0,\tau)\rangle \\
&\quad + l_{x_1}(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau))z^{ij}(t_1;t_0,x_0,\tau). \qquad (99)
\end{aligned}$$

Thus,

$$\begin{aligned}
L_{\tau_i\tau_j} &= \frac{\partial^2}{\partial\tau_i\partial\tau_j} l(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) \mid_{\zeta=\hat\zeta} \\
&= \langle l_{x_1x_1}(\hat p)y^i(\hat t_1), y^j(\hat t_1)\rangle + l_{x_1}(\hat p)z^{ij}(\hat t_1). \qquad (100)
\end{aligned}$$

We can rewrite the last term in this formula as

$$l_{x_1}(\hat p)z^{ij}(\hat t_1) = \psi(\hat t_1)z^{ij}(\hat t_1) = \int_{\hat\tau_j}^{\hat t_1} \frac{d}{dt}(\psi z^{ij})\,dt + \psi(\hat\tau_j)z^{ij}(\hat\tau_j).$$

By formula (51), $\dot z^{ii} = f_x z^{ii} + (y^i)^T f_{xx} y^j$ for $t \ge \hat\tau_j$. Similarly to (92), we get $\frac{d}{dt}(\psi z^{ij}) = (y^i)^T H_{xx} y^j$, and thus obtain

$$l_{x_1}(\hat p)z^{ij}(\hat t_1) = \int_{\hat\tau_j}^{\hat t_1} (y^i)^T H_{xx} y^j\,dt + \psi(\hat\tau_j)z^{ij}(\hat\tau_j). \qquad (101)$$

Since $y^j(t) = 0$ for $t < \hat\tau_j$, we have

$$\int_{\hat\tau_j}^{\hat t_1} (y^i)^T H_{xx} y^j\,dt = \int_{\hat t_0}^{\hat t_1} (y^i)^T H_{xx} y^j\,dt. \qquad (102)$$

Using the initial condition (52) for z^{ij} at the point $\hat\tau^j$, we get

$$\psi(\hat\tau_j)z^{ij}(\hat\tau_j) = -\psi(\hat\tau_j)[f_x]^j y^i(\hat\tau_j) = -[H_x]^j y^i(\hat\tau_j). \qquad (103)$$

Formulas (100)–(103) imply the following representation for all $i<j$,

$$\begin{aligned}
L_{\tau_i\tau_j}\bar\tau_i\bar\tau_j &= \langle l_{x_1x_1}(\hat p)y^i(\hat t_1)\bar\tau_i, y^j(\hat t_1)\bar\tau_j\rangle \\
&\quad + \int_{\hat t_0}^{\hat t_1} (y^i\bar\tau_i)^T H_{xx} y^j\bar\tau_j\,dt - [H_x]^j y^i(\hat\tau_j)\bar\tau_i\bar\tau_j\,. \qquad (104)
\end{aligned}$$

7.5. Derivative $L_{x_0 t_1}$

Using Proposition 3.1, we get

$$\frac{\partial^2}{\partial x_0 \partial t_1} l(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau)) = \frac{\partial}{\partial t_1}\{l_{x_0} + l_{x_1} V\} \mid_{t=t_1}$$
$$= (l_{x_0 t_1} + l_{x_0 x_1}\dot{x} + \frac{\partial l_{x_1}}{\partial t_1} V + l_{x_1}\dot{V}) \mid_{t=t_1}$$
$$= (l_{x_0 t_1} + l_{x_0 x_1}\dot{x} + (l_{x_1 t_1} + l_{x_1 x_1}\dot{x})V + l_{x_1} f_x V) \mid_{t=t_1} .$$

Again, we transform the last term in this formula at the point $\zeta = \hat{\zeta}$. Using the adjoint equation $-\dot{\psi} = \psi f_x$ and the transversality condition $\psi(t_1) = l_{x_1}$, we get

$$l_{x_1} f_x V \mid_{t=\hat{t}_1} = \psi f_x V \mid_{t=\hat{t}_1} = -\dot{\psi}(\hat{t}_1) V(\hat{t}_1).$$

Consequently,

$$\begin{aligned} L_{x_0 t_1}\bar{x}_0 \bar{t}_1 = l_{x_0 t_1}\bar{x}_0\bar{t}_1 + \langle l_{x_0 x_1}\dot{\hat{x}}(\hat{t}_1)\bar{t}_1, \bar{x}_0\rangle \\ + l_{x_1 t_1} V(\hat{t}_1)\bar{x}_0\bar{t}_1 + \langle l_{x_1 x_1}\dot{\hat{x}}(\hat{t}_1)\bar{t}_1, V(\hat{t}_1)\bar{x}_0\rangle - \dot{\psi}(\hat{t}_1) V(\hat{t}_1)\bar{x}_0\bar{t}_1 . \end{aligned} \tag{105}$$

7.6. Derivative $L_{\tau_i t_1}$

Using the notation $\frac{\partial x}{\partial \tau_i} = y^i$ and Proposition 3.3, we get

$$\frac{\partial^2}{\partial \tau_i \partial t_1} l(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau)) = \frac{\partial}{\partial t_1}\{l_{x_1} y^i\} \mid_{t=t_1}$$
$$= \{(l_{x_1 x_1}\dot{x} + l_{x_1 t_1}) y^i + l_{x_1}\dot{y}^i\} \mid_{t=t_1} = \{(l_{x_1 x_1}\dot{x} y^i + l_{x_1 t_1} y^i + l_{x_1} f_x y^i\} \mid_{t=t_1} .$$

We evaluate the last term in this formula at the point $\zeta = \hat{\zeta}$ using the adjoint equation $-\dot{\psi} = \psi f_x$ and the transversality condition $\psi(\hat{t}_1) = l_{x_1}$:

$$l_{x_1} f_x y^i \mid_{t=\hat{t}_1} = \psi f_x y^i \mid_{t=\hat{t}_1} = -\dot{\psi}(\hat{t}_1) y^i(\hat{t}_1).$$

Therefore,

$$L_{\tau_i t_1}\bar{\tau}_i\bar{t}_1 = \langle l_{x_1 x_1}\dot{x}(\hat{t}_1)\bar{t}_1, y^i(\hat{t}_1)\bar{\tau}_i\rangle + l_{x_1 t_1} y^i(\hat{t}_1)\bar{\tau}_i\bar{t}_1 - \dot{\psi}(\hat{t}_1) y^i(\hat{t}_1)\bar{\tau}_i\bar{t}_1. \tag{106}$$

7.7. Derivative $L_{t_1 t_1}$

We have

$$\frac{\partial^2}{\partial t_1^2} l(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau)) = \frac{\partial}{\partial t_1}\{l_{t_1} + l_{x_1}\dot{x}\} \mid_{t=t_1}$$
$$= \{(l_{t_1 t_1} + l_{t_1 x_1}\dot{x}) + (l_{x_1 t_1} + l_{x_1 x_1}\dot{x})\dot{x} + l_{x_1}\ddot{x}\} \mid_{t=t_1}$$

which gives

$$L_{t_1 t_1} = l_{t_1 t_1} + 2 l_{t_1 x_1}\dot{x}(\hat{t}_1) + \langle l_{x_1 x_1}\dot{x}(\hat{t}_1), \dot{x}(\hat{t}_1)\rangle + \psi(\hat{t}_1)\ddot{\hat{x}}(\hat{t}_1). \tag{107}$$

Let us transform the last term. Equation (24) in the definition of M_0 is equivalent to the relation $\psi\dot{\hat{x}} + \psi_0 = 0$. Differentiating this equation with respect to t, we get

$$\dot{\psi}\dot{\hat{x}} + \psi\ddot{\hat{x}} + \dot{\psi}_0 = 0. \tag{108}$$

Hence, formula (107) implies the following equality

$$\begin{aligned} L_{t_1t_1}\bar{t}_1^2 &= l_{t_1t_1}\bar{t}_1^2 + 2l_{t_1x_1}\dot{\hat{x}}(\hat{t}_1)\bar{t}_1^2 + \langle l_{x_1x_1}\dot{\hat{x}}(\hat{t}_1)\bar{t}_1, \dot{\hat{x}}(\hat{t}_1)\bar{t}_1\rangle \\ &\quad -(\dot{\psi}(\hat{t}_1)\dot{\hat{x}}(\hat{t}_1) + \dot{\psi}_0(\hat{t}_1))\bar{t}_1^2. \end{aligned} \tag{109}$$

7.8. Derivative $L_{x_0t_0}$

In view of the relation $\frac{\partial x}{\partial x_0} = V$, we obtain

$$\frac{\partial^2}{\partial x_0 \partial t_0} l(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau)) = \frac{\partial}{\partial t_0}\{l_{x_0} + l_{x_1}V\}\mid_{t=t_1}$$
$$= \left\{ l_{x_0t_0} + l_{x_0x_1}\frac{\partial x}{\partial t_0} + (l_{x_1t_0} + l_{x_1x_1}\frac{\partial x}{\partial t_0})V + l_{x_1}\frac{\partial V}{\partial t_0} \right\}\mid_{t=t_1}.$$

Now, using the transversality condition $l_{x_1} = \psi(\hat{t}_1)$, formula (19), and the notation $\frac{\partial V}{\partial t_0} = S$, we get

$$\begin{aligned} L_{x_0t_0} = l_{x_0t_0} &- l_{x_0x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0) + l_{x_1t_0}V(\hat{t}_1) - \dot{\hat{x}}(\hat{t}_0)^T V(\hat{t}_1)^T l_{x_1x_1}V(\hat{t}_1) \\ &+\psi(\hat{t}_1)S(\hat{t}_1). \end{aligned} \tag{110}$$

The transformation of the last term in this formula proceeds as follows. Using the adjoint equation for ψ and the system (67) for S, we obtain the equation

$$\begin{aligned} \tfrac{d}{dt}(\psi S) &= \dot{\psi}S + \psi\dot{S} = -\psi f_x S + \psi f_x S - \dot{\hat{x}}(\hat{t}_0)^T V^T \textstyle\sum_i \psi_i f_{ixx} V \\ &= -\dot{\hat{x}}(\hat{t}_0)^T V^T H_{xx} V, \end{aligned} \tag{111}$$

which yields

$$\psi(\hat{t}_1)S(\hat{t}_1) = -\int_{\hat{t}_0}^{\hat{t}_1} \dot{\hat{x}}(\hat{t}_0)^T V^T H_{xx} V\,dt + \psi(\hat{t}_0)S(\hat{t}_0). \tag{112}$$

Using now the initial condition (68) for S at the point $t = \hat{t}_0$ and the equation $\dot{V} = f_x V$, we get

$$(\psi S)\mid_{\hat{t}_0} = -(\psi\dot{V})\mid_{\hat{t}_0} = -(\psi f_x V)\mid_{\hat{t}_0} = (\dot{\psi}V)\mid_{\hat{t}_0} = \dot{\psi}(\hat{t}_0), \tag{113}$$

since $V(\hat{t}_0) = E$. Formulas (110), (112) and (113) then imply the equality

$$\begin{aligned} L_{x_0t_0} &= l_{x_0t_0} - l_{x_0x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0) + l_{x_1t_0}V(\hat{t}_1) - \dot{\hat{x}}(\hat{t}_0)^T V(\hat{t}_1)^T l_{x_1x_1}V(\hat{t}_1) \\ &\quad - \int_{\hat{t}_0}^{\hat{t}_1} \dot{\hat{x}}(\hat{t}_0)^T V^T H_{xx} V\,dt + \dot{\psi}(\hat{t}_0). \end{aligned} \tag{114}$$

Therefore,

$$\begin{aligned} L_{x_0t_0}\bar{x}_0\bar{t}_0 &= l_{x_0t_0}\bar{x}_0\bar{t}_0 - \langle l_{x_0x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0, \bar{x}_0\rangle + l_{x_1t_0}V(\hat{t}_1)\bar{x}_0\bar{t}_0 \\ &\quad -\langle l_{x_1x_1}V(\hat{t}_1)\bar{x}_0, V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\rangle + \dot{\psi}(\hat{t}_0)\bar{x}_0\bar{t}_0 \\ &\quad -\int_{\hat{t}_0}^{\hat{t}_1} \langle H_{xx}V\bar{x}_0, V\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\rangle\, dt. \end{aligned} \tag{115}$$

7.9. Derivative $L_{\tau_i t_0}$

Using the notations $\frac{\partial x}{\partial t_0} = w$, $\frac{\partial x}{\partial \tau_i} = y^i$ and $\frac{\partial y_i}{\partial t_0} = r^i$, we obtain

$$\begin{aligned} &\frac{\partial^2}{\partial \tau_i \partial t_0} l(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau)) = \frac{\partial}{\partial t_0}\{l_{x_1}y^i\}\mid_{t=t_1} \\ &= \left\{ l_{x_1t_0}y^i + (y^i)^T l_{x_1x_1}\frac{\partial x}{\partial t_0} + l_{x_1}\frac{\partial y^i}{\partial t_0}\right\}\mid_{t=t_1} \\ &= \{l_{x_1t_0}y^i + (y^i)^T l_{x_1x_1}w + l_{x_1}r^i\}\mid_{t=t_1}. \end{aligned}$$

According to condition (19) we have $w\mid_{t=\hat{t}_1} = -V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)$. Using this condition together with the transversality condition $l_{x_1} = \psi(\hat{t}_1)$, we find

$$L_{\tau_i t_0} = l_{x_1t_0}y^i(\hat{t}_1) - (y^i(\hat{t}_1))^T l_{x_1x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0) + \psi(\hat{t}_1)r^i(\hat{t}_1). \tag{116}$$

Let us transform the last term in this formula. Using the adjoint equation for ψ and the system (71) for r^i, we get for $t \geq \hat{\tau}_i$:

$$\begin{aligned} \frac{d}{dt}(\psi r^i) &= \dot{\psi}r^i + \psi\dot{r}^i = -\psi f_x r^i + \psi f_x r^i - (y_i)^T \sum_k \psi_k f_{kxx}V\dot{\hat{x}}(\hat{t}_0) \\ &= -(y_i)^T H_{xx}V\dot{\hat{x}}(\hat{t}_0). \end{aligned}$$

It follows that

$$\psi(\hat{t}_1)r^i(\hat{t}_1) = -\int_{\hat{\tau}_i}^{\hat{t}_1}(y_i)^T H_{xx}V\dot{\hat{x}}(\hat{t}_0)\, dt + \psi(\hat{\tau}_i)r^i(\hat{t}_i). \tag{117}$$

The initial condition (72) for r^i at the point $\hat{\tau}_i$ then yields

$$\psi(\hat{\tau}_i)r^i(\hat{\tau}_i) = \psi(\hat{\tau}_i)[f_x]^i V(\hat{\tau}_i)\dot{\hat{x}}(\hat{t}_0) = [H_x]^i V(\hat{\tau}_i)\dot{\hat{x}}(\hat{t}_0). \tag{118}$$

Formulas (116)–(118) and the condition $y^i(t) = 0$ for $t < \hat{\tau}_i$ then imply the equality

$$\begin{aligned} L_{\tau_i t_0} &= l_{x_1t_0}y^i(\hat{t}_1) - (y^i(\hat{t}_1))^T l_{x_1x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0) \\ &\quad +[H_x]^i V(\hat{\tau}_i)\dot{\hat{x}}(\hat{t}_0) - \int_{\hat{t}_0}^{\hat{t}_1}(y_i)^T H_{xx}V\dot{\hat{x}}(\hat{t}_0)\, dt. \end{aligned} \tag{119}$$

Hence,

$$L_{\tau_i t_0}\bar{\tau}_i\bar{t}_0 = l_{x_1 t_0} y^i(\hat{t}_1)\bar{\tau}_i\bar{t}_0 - (y^i(\hat{t}_1)\bar{\tau}_i)^T l_{x_1 x_1} V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0 + [H_x]^i V(\hat{\tau}_i)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\bar{\tau}_i - \int_{\hat{t}_0}^{\hat{t}_1} (y^i\bar{\tau}_i)^T H_{xx} V\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\, dt. \qquad (120)$$

7.10. Derivative $L_{t_1 t_0}$

We have

$$\frac{\partial^2}{\partial t_1 \partial t_0} l(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau)) = \frac{\partial}{\partial t_0}\{l_{t_1} + l_{x_1}\dot{x}\}\mid_{t=t_1}$$
$$= \left\{ l_{t_1 t_0} + l_{t_1 x_1}\frac{\partial x}{\partial t_0} + \left(\frac{\partial}{\partial t_0} l_{x_1}\right)\dot{x} + l_{x_1}\frac{\partial}{\partial t_0}\dot{x}\right\}\mid_{t=t_1}.$$

Using the equalities

$$\frac{\partial}{\partial t_0} l_{x_1} = l_{x_1 t_0} + l_{x_1 x_1}\frac{\partial x}{\partial t_0}, \quad \frac{\partial x}{\partial t_0} = -V\dot{x}(t_0),$$

we get

$$\frac{\partial^2}{\partial t_1 \partial t_0} l(t_0, x_0, t_1, x(t_1; t_0, x_0, \tau))$$
$$= l_{t_1 t_0} - l_{t_1 x_1} V\dot{x}(t_0) + l_{x_1 t_0}\dot{x}(t_1) - (V(t_1)\dot{x}(t_0))^T l_{x_1 x_1}\dot{x}(t_1) + l_{x_1}\frac{\partial}{\partial t_0}\dot{x}\mid_{t=t_1}. \qquad (121)$$

Let us calculate the last term. Differentiating the equation $\dot{x}(t; t_0, x_0, \tau) = f(t, x(t; t_0, x_0, \tau), u(t; \tau))$ w.r.t. t_0, we get

$$\frac{\partial}{\partial t_0}\dot{x} = f_x \frac{\partial}{\partial t_0} x = -f_x V\dot{x}(t_0).$$

Consequently, at the point $\zeta = \hat{\zeta}$ we obtain

$$l_{x_1}\frac{\partial}{\partial t_0}\dot{x}\mid_{t=\hat{t}_1} = \{\psi\frac{\partial}{\partial t_0}\dot{x}\}\mid_{t=\hat{t}_1} = \{-\psi f_x V\dot{x}(\hat{t}_0)\}\mid_{t=\hat{t}_1} = \dot{\psi}(\hat{t}_1)V(\hat{t}_1)\dot{x}(t_0).$$

Using this equality in (121), we get at the point $\zeta = \hat{\zeta}$

$$L_{t_1 t_0} = l_{t_1 t_0} - l_{t_1 x_1} V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0) + l_{x_1 t_0}\dot{\hat{x}}(\hat{t}_1) - \langle l_{x_1 x_1} V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0), \dot{\hat{x}}(\hat{t}_1)\rangle + \dot{\psi}(\hat{t}_1)V(\hat{t}_1)\dot{x}(\hat{t}_0), \qquad (122)$$

which yields

$$L_{t_1 t_0}\bar{t}_1\bar{t}_0 = l_{t_1 t_0}\bar{t}_1\bar{t}_0 - l_{t_1 x_1}(V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0)\bar{t}_1 + l_{x_1 t_0}(\dot{\hat{x}}(\hat{t}_1)\bar{t}_1)\bar{t}_0 - \langle l_{x_1 x_1} V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0, \dot{\hat{x}}(\hat{t}_1)\bar{t}_1\rangle + \dot{\psi}(\hat{t}_1)(V(\hat{t}_1)\dot{x}(\hat{t}_0)\bar{t}_0)\bar{t}_1. \qquad (123)$$

7.11. Derivative $L_{t_0t_0}$

We have

$$\begin{aligned}
&\frac{\partial^2}{\partial t_0^2}\, l(t_0,x_0,t_1,x(t_1;t_0,x_0,\tau)) = \frac{\partial}{\partial t_0}\left\{ l_{t_0} + l_{x_1}\frac{\partial x}{\partial t_0}\right\} |_{t=t_1} \\
&= \left\{ l_{t_0t_0} + l_{t_0x_1}\frac{\partial x}{\partial t_0} + \left(l_{x_1t_0} + l_{x_1x_1}\frac{\partial x}{\partial t_0}\right)\frac{\partial x}{\partial t_0} + l_{x_1}\frac{\partial^2 x}{\partial t_0^2}\right\} |_{t=t_1} \\
&= \left\{ l_{t_0t_0} + 2l_{t_0x_1}\frac{\partial x}{\partial t_0} + \langle l_{x_1x_1}\frac{\partial x}{\partial t_0}, \frac{\partial x}{\partial t_0}\rangle + l_{x_1}\frac{\partial^2 x}{\partial t_0^2}\right\} |_{t=t_1} \\
&= \{ l_{t_0t_0} + 2l_{t_0x_1}w + \langle l_{x_1x_1}w,w\rangle + l_{x_1}q\} |_{t=t_1}, \qquad (124)
\end{aligned}$$

where

$$w = \frac{\partial x}{\partial t_0}, \quad q = \frac{\partial w}{\partial t_0} = \frac{\partial^2 x}{\partial t_0^2}.$$

The transversality condition $l_{x_1} = \psi(\hat{t}_1)$ yields

$$L_{t_0t_0} = l_{t_0t_0} + 2l_{t_0x_1}w(\hat{t}_1) + \langle l_{x_1x_1}w(\hat{t}_1), w(\hat{t}_1)\rangle + \psi(\hat{t}_1)q(\hat{t}_1). \qquad (125)$$

Let us transform the last term using the adjoint equation for ψ and the system (56) for q :

$$\frac{d}{dt}(\psi q) = \dot{\psi}q + \psi\dot{q} = -\psi f_x q + \psi f_x q + \sum_k \psi_k(w^T f_{kxx} w) = w^T H_{xx} w.$$

Also, using the equality $w = -V\dot{\hat{x}}(\hat{t}_0)$, we obtain

$$\begin{aligned}
\psi(\hat{t}_1)q(\hat{t}_1) &= \psi(\hat{t}_0)q(\hat{t}_0) + \int_{\hat{t}_0}^{\hat{t}_1} w^T H_{xx} w\, dt \\
&= \psi(\hat{t}_0)q(\hat{t}_0) + \int_{\hat{t}_0}^{\hat{t}_1} \langle H_{xx}V\dot{\hat{x}}(\hat{t}_0), V\dot{\hat{x}}(\hat{t}_0)\rangle\, dt. \qquad (126)
\end{aligned}$$

The initial condition (57) for q then implies

$$\psi(\hat{t}_0)q(\hat{t}_0) = -\psi(\hat{t}_0)\ddot{\hat{x}}(\hat{t}_0) - 2\psi(\hat{t}_0)\dot{w}(\hat{t}_0). \qquad (127)$$

From the equation $\dot{w} = f_x w$ (see Proposition 3.2), the adjoint equation $-\dot{\psi} = \psi f_x$ and the formula $w = -V\dot{\hat{x}}(\hat{t}_0)$ it follows that

$$-\psi\dot{w} = -\psi f_x w = \dot{\psi} w = -\dot{\psi} V\dot{\hat{x}}(\hat{t}_0).$$

Since $V(\hat{t}_0) = E$, we obtain

$$\psi(\hat{t}_0)\dot{w}(\hat{t}_0) = \dot{\psi}(\hat{t}_0)\dot{\hat{x}}(\hat{t}_0). \qquad (128)$$

Moreover, by formula (108) we have

$$-\psi\ddot{\hat{x}} = \dot{\psi}\dot{\hat{x}} + \dot{\psi}_0. \tag{129}$$

Formulas (127)–(129) imply

$$\psi(\hat{t}_0)q(\hat{t}_0) = \dot{\psi}_0(\hat{t}_0) - \dot{\psi}(\hat{t}_0)\dot{\hat{x}}(\hat{t}_0). \tag{130}$$

Combining formulas (125), (19), (126) and (130), we obtain

$$\begin{aligned} L_{t_0t_0} &= l_{t_0t_0} - 2l_{t_0x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0) + \langle l_{x_1x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0), V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\rangle \\ &\quad +\dot{\psi}_0(\hat{t}_0) - \dot{\psi}(\hat{t}_0)\dot{\hat{x}}(\hat{t}_0) + \int_{\hat{t}_0}^{\hat{t}_1} \langle H_{xx}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0), V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\rangle\, dt. \end{aligned} \tag{131}$$

Thus we have found the representation

$$\begin{aligned} L_{t_0t_0}\bar{t}_0^2 &= l_{t_0t_0}\bar{t}_0^2 - 2l_{t_0x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0^2 + \langle l_{x_1x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0, V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\rangle \\ &\quad +\dot{\psi}_0(\hat{t}_0)\bar{t}_0^2 - \dot{\psi}(\hat{t}_0)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0^2 + \int_{\hat{t}_0}^{\hat{t}_1} \langle H_{xx}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0, V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\rangle\, dt. \end{aligned} \tag{132}$$

7.12. Representation of the quadratic form $\langle L_{\zeta\zeta}\bar{\zeta}, \bar{\zeta}\rangle$

Combining all results and formulas from the preceding sections we have proved the following theorem.

THEOREM 7.1 *Let the Lagrange multipliers $\mu = (\alpha_0, \alpha, \beta) \in \Lambda_0$ and $\lambda = (\alpha_0, \alpha, \beta, \psi, \psi_0) \in \Lambda$ correspond to each other, i.e., let $\pi_0\lambda = \mu$ hold; see Proposition 4.1 in Part 1. Then, for any $\bar{\zeta} = (\bar{t}_0, \bar{t}_1, \bar{x}_0, \bar{\tau}) \in \mathbb{R}^{2+n+s}$, formulas (73), (83), (87), (97), (98), (104), (105), (106), (109), (115), (120), (123), (132) hold, where the matrix $V(t)$ is the solution to the IVP (9) and the function y^i is the solution to the IVP (21) for each $i = 1, \ldots, s$.*

Thus we have obtained the following explicit and massive representation of the quadratic form in the induced optimization problem:

$$\begin{aligned} \langle L_{\zeta\zeta}\bar{\zeta}, \bar{\zeta}\rangle &= \langle L_{\zeta\zeta}(\mu, \hat{\zeta})\bar{\zeta}, \bar{\zeta}\rangle \\ &= \langle L_{x_0x_0}\bar{x}_0, \bar{x}_0\rangle + 2\sum_{i=1}^{s} L_{x_0\tau_i}\bar{x}_0\bar{\tau}_i + \sum_{i=1}^{s} L_{\tau_i\tau_i}\bar{\tau}_i^2 + 2\sum_{i<j}^{s} L_{\tau_i\tau_j}\bar{\tau}_i\bar{\tau}_j \\ &\quad +2L_{x_0t_1}\bar{x}_0\bar{t}_1 + 2\sum_{i=1}^{s} L_{\tau_it_1}\bar{\tau}_i\bar{t}_1 + L''_{t_1t_1}\bar{t}_1^2 \\ &\quad +2L_{x_0t_0}\bar{x}_0\bar{t}_0 + 2\sum_{i=1}^{s} L_{t_0\tau_i}\bar{t}_0\bar{\tau}_i + 2L_{t_0t_1}\bar{t}_0\bar{t}_1 + L_{t_0t_0}\bar{t}_0^2 \end{aligned} \tag{133}$$

$$
\begin{aligned}
= \quad & \bar{x}_0^T l_{x_0x_0}\bar{x}_0 + 2\bar{x}_0^T l_{x_0x_1} V(\hat{t}_1)\bar{x}_0 + (V(\hat{t}_1)\bar{x}_0)^T l_{x_1x_1} V(\hat{t}_1)\bar{x}_0 \\
& + \int_{\hat{t}_0}^{\hat{t}_1} (V\bar{x}_0)^T H_{xx} V\bar{x}_0 \, dt \\
& + \sum_{i=1}^{s} 2\bar{x}_0^T l_{x_0x_1} y^i(\hat{t}_1)\bar{\tau}_i + \sum_{i=1}^{s} 2(V(\hat{t}_1)\bar{x}_0)^T l_{x_1x_1} y^i(\hat{t}_1)\bar{\tau}_i \\
& - \sum_{i=1}^{s} 2[H_x]^i V(\hat{\tau}_i)\bar{x}_0\bar{\tau}_i + \sum_{i=1}^{s} \int_{\hat{t}_0}^{\hat{t}_1} 2\langle H_{xx} y^i\bar{\tau}_i, V\bar{x}_0\rangle \, dt \\
& + \sum_{i=1}^{s} \langle l_{x_1x_1} y^i(\hat{t}_1)\bar{\tau}_i, y^i(\hat{t}_1)\bar{\tau}_i\rangle + \sum_{i=1}^{s} \int_{\hat{t}_0}^{\hat{t}_1} (y^i\bar{\tau}_i)^T H_{xx} y^i\bar{\tau}_i \, dt \\
& + \sum_{i=1}^{s} D^i(H)\bar{\tau}_i^2 - \sum_{i=1}^{s} [H_x]^i [y^i]^i \bar{\tau}_i^2 \\
& + \sum_{i<j} 2\langle l_{x_1x_1} y^i(\hat{t}_1)\bar{\tau}_i, y^j(\hat{t}_1)\bar{\tau}_j\rangle \\
& + \sum_{i<j} \int_{\hat{t}_0}^{\hat{t}_1} 2(y^i\bar{\tau}_i)^T H_{xx} y^j\bar{\tau}_j \, dt - \sum_{i<j} 2[H_x]^j y^i(\hat{\tau}_j)\bar{\tau}_i\bar{\tau}_j \\
& + 2l_{x_0t_1}\bar{x}_0\bar{t}_1 + 2\langle l_{x_0x_1}\dot{\hat{x}}(\hat{t}_1)\bar{t}_1, \bar{x}_0\rangle + 2l_{x_1t_1}V(\hat{t}_1)\bar{x}_0\bar{t}_1 \\
& + 2\langle l_{x_1x_1}\dot{\hat{x}}(\hat{t}_1)\bar{t}_1, V(\hat{t}_1)\bar{x}_0\rangle - 2\dot{\psi}(\hat{t}_1)V(\hat{t}_1)\bar{x}_0\bar{t}_1 \\
& + \sum_{i=1}^{s} 2\langle l_{x_1x_1}\dot{x}(\hat{t}_1)\bar{t}_1, y^i(\hat{t}_1)\bar{\tau}_i\rangle + \sum_{i=1}^{s} 2l_{x_1t_1} y^i(\hat{t}_1)\bar{\tau}_i\bar{t}_1 \\
& - \sum_{i=1}^{s} 2\dot{\psi}(\hat{t}_1) y^i(\hat{t}_1)\bar{\tau}_i\bar{t}_1 \\
& + l_{t_1t_1}\bar{t}_1^2 + 2l_{t_1x_1}\dot{x}(\hat{t}_1)\bar{t}_1^2 + \langle l_{x_1x_1}\dot{x}(\hat{t}_1)\bar{t}_1, \dot{x}(\hat{t}_1)\bar{t}_1\rangle \\
& - (\dot{\psi}(\hat{t}_1)\dot{\hat{x}}(\hat{t}_1) + \dot{\psi}_0(\hat{t}_1))\bar{t}_1^2 \\
& + 2l_{x_0t_0}\bar{x}_0\bar{t}_0 - 2\langle l_{x_0x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0, \bar{x}_0\rangle + 2l_{x_1t_0}V(\hat{t}_1)\bar{x}_0\bar{t}_0 \\
& - 2\langle l_{x_1x_1}V(\hat{t}_1)\bar{x}_0, V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\rangle + 2\dot{\psi}(\hat{t}_0)\bar{x}_0\bar{t}_0 \\
& - \int_{\hat{t}_0}^{\hat{t}_1} 2\langle H_{xx}V\bar{x}_0, V\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\rangle \, dt
\end{aligned}
$$

$$+\sum_{i=1}^{s} 2l_{x_1t_0}y^i(\hat{t}_1)\bar{\tau}_i\bar{t}_0 - \sum_{i=1}^{s} 2(y^i(\hat{t}_1)\bar{\tau}_i)^T l_{x_1x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0$$

$$+\sum_{i=1}^{s} 2[H_x]^i V(\hat{\tau}_i)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\bar{\tau}_i - \sum_{i=1}^{s}\int_{\hat{t}_0}^{\hat{t}_1} 2(y^i\bar{\tau}_i)^T H_{xx}V\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\,dt$$

$$+2l_{t_1t_0}\bar{t}_1\bar{t}_0 - 2l_{t_1x_1}(V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0)\bar{t}_1 + 2l_{x_1t_0}(\dot{\hat{x}}(\hat{t}_1)\bar{t}_1)\bar{t}_0$$
$$-2\langle l_{x_1x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0, \dot{\hat{x}}(\hat{t}_1)\bar{t}_1\rangle + 2\dot{\psi}(\hat{t}_1)(V(\hat{t}_1)\dot{x}(\hat{t}_0)\bar{t}_0)\bar{t}_1$$
$$+l_{t_0t_0}\bar{t}_0^2 - 2l_{t_0x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0^2 + \langle l_{x_1x_1}V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0, V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\rangle$$

$$+\dot{\psi}_0(\hat{t}_0)\bar{t}_0^2 - \dot{\psi}(\hat{t}_0)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0^2 + \int_{\hat{t}_0}^{\hat{t}_1}\langle H_{xx}V\dot{\hat{x}}(\hat{t}_0)\bar{t}_0, V\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\rangle\,dt.$$

Again, we wish to emphasize that this explicit representation involves only *first order* variations y^i and V of the trajectories $x(t; t_0, x_0, \tau)$.

8. Equivalence of the quadratic forms in the main and induced problems

In this section we shall prove Theorem 4.4 of Part 1, which is the main result of the paper. Let the Lagrange multipliers $\mu = (\alpha_0, \alpha, \beta) \in \Lambda_0$ and $\lambda = (\alpha_0, \alpha, \beta, \psi, \psi_0) \in \Lambda$ correspond to each other, and take any $\bar{\zeta} = (\bar{t}_0, \bar{t}_1, \bar{x}_0, \bar{\tau}) \in \mathbb{R}^{2+n+s}$. Consider the representation (133) of the quadratic form $\langle L_{\zeta\zeta}\bar{\zeta}, \bar{\zeta}\rangle$, which is far from revealing the equivalence of the quadratic forms for the main control problem and the induced optimization problem. However, we will show now that by a careful regrouping of the terms in (133) we shall arrive at the desired equivalence. The quadratic form (133) contains terms of the following types.

Type (a): Positive terms with coefficients $D^i(H)$ multiplied by with the variation of the switching time $\bar{\tau}_i$,

$$a := \sum_{i=1}^{s} D^i(H)\bar{\tau}_i^2. \tag{134}$$

Type (b): Mixed terms with $[H_x]^i$ connected with the variation $\bar{\tau}_i$,

$$b \;:=\; -\sum_{i=1}^{s} 2[H_x]^i V(\hat{\tau}_i)\bar{x}_0\bar{\tau}_i - \sum_{i=1}^{s}[H_x]^i[y^i]^i\bar{\tau}_i^2 - \sum_{i<j} 2[H_x]^j y^i(\hat{\tau}_j)\bar{\tau}_i\bar{\tau}_j + \sum_{i=1}^{s} 2[H_x]^i V(\hat{\tau}_i)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0\bar{\tau}_i\,. \tag{135}$$

Since

$$\sum_{i<j}[H_x]^j y^i(\hat\tau_j)\bar\tau_i\bar\tau_j = \sum_{j<i}[H_x]^i y^j(\hat\tau_i)\bar\tau_i\bar\tau_j = \sum_{i=1}^{s}\sum_{j=1}^{i-1}[H_x]^i y^j(\hat\tau_i)\bar\tau_i\bar\tau_j,$$

we get from (135)

$$\begin{aligned} b \quad = \quad & -\sum_{i=1}^{s} 2[H_x]^i\Bigg(V(\hat\tau_i)\bar x_0 + \frac{1}{2}[y^i]^i\bar\tau_i \\ & + \sum_{j=1}^{i-1} y^j(\hat\tau_i)\bar\tau_j - V(\hat\tau_i)\dot{\hat x}(\hat t_0)\bar t_0\Bigg)\bar\tau_i\,. \end{aligned} \tag{136}$$

According to (30) put

$$\bar x(t) = V(t)\bar x_0 + \sum_{i=1}^{s} y^i(t)\bar\tau_i - V(t)\dot{\hat x}(\hat t_0)\bar t_0. \tag{137}$$

Then we have

$$\bar x(\hat\tau^i - 0) = V(\hat\tau_i)\bar x_0 + \sum_{j=1}^{i-1} y^j(\hat\tau_i)\bar\tau_j - V(\hat\tau_i)\dot{\hat x}(\hat t_0)\bar t_0,$$

since $y^j(\hat\tau_i - 0) = y^j(\hat\tau_i) = 0$ for $j > i$ and $y^i(\tau_i - 0) = 0$. Moreover, the jump of $\bar x(t)$ at the point $\hat\tau_i$ is equal to the jump of $y^i(t)\bar\tau_i$ at the same point, i.e. $[\bar x]^i = [y^i]^i\bar\tau_i$. Therefore,

$$\begin{aligned} & V(\hat\tau_i)\bar x_0 + \frac{1}{2}[y^i]^i\bar\tau_i + \sum_{j=1}^{i-1} y^j(\hat\tau_i)\bar\tau_j - V(\hat\tau_i)\dot{\hat x}(\hat t_0)\bar t_0 \\ & = \bar x(\hat\tau_i - 0) + \frac{1}{2}[\bar x]^i = \frac{1}{2}(\bar x(\hat\tau_i - 0) + \bar x(\hat\tau_i + 0)) = \bar x^i_{av}\,. \end{aligned}$$

Thus, we get

$$b = -\sum_{i=1}^{s} 2[H_x]^i \bar x^i_{av}\bar\tau_i\,. \tag{138}$$

Type (c): Integral terms

$$\begin{aligned} c \quad := \quad & \int_{\hat t_0}^{\hat t_1} (V\bar x_0)^T H_{xx} V\bar x_0\,dt + \sum_{i=1}^{s}\int_{\hat t_0}^{\hat t_1} 2\langle H_{xx} y^i\bar\tau_i, V\bar x_0\rangle\,dt \\ & + \sum_{i=1}^{s}\int_{\hat t_0}^{\hat t_1} (y^i\bar\tau_i)^T H_{xx} y^i\bar\tau_i\,dt + \sum_{i<j}\int_{\hat t_0}^{\hat t_1} 2(y^i\bar\tau_i)^T H_{xx} y^j\bar\tau_j\,dt \end{aligned}$$

$$- \int_{\hat{t}_0}^{\hat{t}_1} 2\langle H_{xx} V \bar{x}_0, V \dot{\hat{x}}(\hat{t}_0)\bar{t}_0 \rangle \, dt - \sum_{i=1}^{s} \int_{\hat{t}_0}^{\hat{t}_1} 2(y^i \bar{\tau}_i)^T H_{xx} V \dot{\hat{x}}(\hat{t}_0)\bar{t}_0 \, dt$$
$$+ \int_{\hat{t}_0}^{\hat{t}_1} \langle H_{xx} V \dot{\hat{x}}(\hat{t}_0)\bar{t}_0, V \dot{\hat{x}}(\hat{t}_0)\bar{t}_0 \rangle \, dt. \tag{139}$$

Obviously, this sum can be transformed to a *perfect square*.

$$\begin{aligned} c &= \int_{\hat{t}_0}^{\hat{t}_1} \langle H_{xx}(V\bar{x}_0 + \sum_{i=1}^{s} y^i \bar{\tau}_i - V \dot{\hat{x}}(\hat{t}_0)\bar{t}_0), V\bar{x}_0 + \sum_{i=1}^{s} y^i \bar{\tau}_i - V \dot{\hat{x}}(\hat{t}_0)\bar{t}_0 \rangle \, dt \\ &= \int_{\hat{t}_0}^{\hat{t}_1} \langle H_{xx} \bar{x}, \bar{x} \rangle \, dt. \end{aligned} \tag{140}$$

Type (d): Endpoint terms. We shall divide them into several groups.

Group (d1): This group contains the terms with second order derivatives of the endpoint Lagrangian l with respect to t_0, x_0, t_1 :

$$d_1 := \bar{x}_0^T l_{x_0 x_0} \bar{x}_0 + 2 l_{x_0 t_1} \bar{x}_0 \bar{t}_1 + l_{t_1 t_1} \bar{t}_1^2 + 2 l_{x_0 t_0} \bar{x}_0 \bar{t}_0 + 2 l_{t_1 t_0} \bar{t}_1 \bar{t}_0 + l_{t_0 t_0} \bar{t}_0^2. \tag{141}$$

Group (d2): We collect the terms with $l_{t_0 x_1}$:

$$\begin{aligned} d_2 &:= 2 l_{x_1 t_0} V(\hat{t}_1)\bar{x}_0 \bar{t}_0 + \sum_{i=1}^{s} 2 l_{x_1 t_0} y^i(\hat{t}_1) \bar{\tau}_i \bar{t}_0 \\ &\quad + 2 l_{x_1 t_0} \dot{\hat{x}}(\hat{t}_1) \bar{t}_1 \bar{t}_0 - 2 l_{t_0 x_1} V(\hat{t}_1) \dot{\hat{x}}(\hat{t}_0) \bar{t}_0^2 \\ &= 2 l_{x_1 t_0} \left(V(\hat{t}_1)\bar{x}_0 + \sum_{i=1}^{s} y^i(\hat{t}_1)\bar{\tau}_i + \dot{\hat{x}}(\hat{t}_1)\bar{t}_1 - V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0 \right) \bar{t}_0 \\ &= 2 l_{x_1 t_0} \widetilde{x}_1 \bar{t}_0, \end{aligned} \tag{142}$$

where in view of (30),

$$\widetilde{x}_1 := V(\hat{t}_1)\bar{x}_0 + \sum_{i=1}^{s} y^i(\hat{t}_1)\bar{\tau}_i + \dot{\hat{x}}(\hat{t}_1)\bar{t}_1 - V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0 = \bar{x}(\hat{t}_1) + \dot{\hat{x}}(\hat{t}_1)\bar{t}_1 \, . \tag{143}$$

Group (d3): Consider the terms with $l_{x_0 x_1}$:

$$\begin{aligned} d_3 &:= 2\bar{x}_0^T l_{x_0 x_1} V(\hat{t}_1)\bar{x}_0 + \sum_{i=1}^{s} 2\bar{x}_0^T l_{x_0 x_1} y^i(\hat{t}_1)\bar{\tau}_i \\ &\quad + 2\langle l_{x_0 x_1} \dot{\hat{x}}(\hat{t}_1)\bar{t}_1, \bar{x}_0 \rangle - 2\langle l_{x_0 x_1} V(\hat{t}_1)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0, \bar{x}_0 \rangle \end{aligned}$$

$$
\begin{aligned}
&= 2\langle l_{x_0x_1}(V(\hat t_1)\bar x_0 + \sum_{i=1}^{s} y^i(\hat t_1)\bar\tau_i + \dot{\hat x}(\hat t_1)\bar t_1 - V(\hat t_1)\dot{\hat x}(\hat t_0)\bar t_0), \bar x_0\rangle \\
&= 2\langle l_{x_0x_1}\widetilde x_1, \bar x_0\rangle. \qquad (144)
\end{aligned}
$$

Group (d4): This group contains all terms with $l_{t_1x_1}$:

$$
\begin{aligned}
d_4 \;:=\;& 2l_{x_1t_1}V(\hat t_1)\bar x_0\bar t_1 + \sum_{i=1}^{s} 2l_{x_1t_1}y^i(\hat t_1)\bar\tau_i\bar t_1 \\
&+2l_{t_1x_1}\dot x(\hat t_1)\bar t_1^2 - 2l_{t_1x_1}(V(\hat t_1)\dot{\hat x}(\hat t_0)\bar t_0)\bar t_1 \\
=\;& 2l_{x_1t_1}(V(\hat t_1)\bar x_0 + \sum_{i=1}^{s} y^i(\hat t_1)\bar\tau_i + \dot x(\hat t_1)\bar t_1 - V(\hat t_1)\dot{\hat x}(\hat t_0)\bar t_0)\bar t_1 \\
=\;& 2l_{x_1t_1}\widetilde x_1\bar t_1. \qquad (145)
\end{aligned}
$$

Group (d5): We collect all terms containing $l_{x_1x_1}$:

$$
\begin{aligned}
d_5 \;:=\;& (V(\hat t_1)\bar x_0)^T l_{x_1x_1}V(\hat t_1)\bar x_0 + \sum_{i=1}^{s} 2(V(\hat t_1)\bar x_0)^T l_{x_1x_1}y^i(\hat t_1)\bar\tau_i \\
&+\sum_{i=1}^{s}\langle l_{x_1x_1}y^i(\hat t_1)\bar\tau_i, y^i(\hat t_1)\bar\tau_i\rangle + \sum_{i<j} 2\langle l_{x_1x_1}y^i(\hat t_1)\bar\tau_i, y^j(\hat t_1)\bar\tau_j\rangle \\
&+2\langle l_{x_1x_1}\dot{\hat x}(\hat t_1)\bar t_1, V(\hat t_1)\bar x_0\rangle + \sum_{i=1}^{s} 2\langle l_{x_1x_1}\dot x(\hat t_1)\bar t_1, y^i(\hat t_1)\bar\tau_i\rangle \\
&+\langle l_{x_1x_1}\dot x(\hat t_1)\bar t_1, \dot x(\hat t_1)\bar t_1\rangle - 2\langle l_{x_1x_1}V(\hat t_1)\bar x_0, V(\hat t_1)\dot{\hat x}(\hat t_0)\bar t_0\rangle \\
&-\sum_{i=1}^{s} 2(y^i(\hat t_1)\bar\tau_i)^T l_{x_1x_1}V(\hat t_1)\dot{\hat x}(\hat t_0)\bar t_0 - 2\langle l_{x_1x_1}V(\hat t_1)\dot{\hat x}(\hat t_0)\bar t_0, \dot{\hat x}(\hat t_1)\bar t_1\rangle \\
&+\langle l_{x_1x_1}V(\hat t_1)\dot{\hat x}(\hat t_0)\bar t_0, V(\hat t_1)\dot{\hat x}(\hat t_0)\bar t_0\rangle. \qquad (146)
\end{aligned}
$$

One can easily check that this sum can be transformed to the *perfect square*

$$
\begin{aligned}
d_5 \;:=\;& \langle l_{x_1x_1}(V(\hat t_1)\bar x_0 + \sum_{i=1}^{s} y^i(\hat t_1)\bar\tau_i + \dot x(\hat t_1)\bar t_1 \\
&-V(\hat t_1)\dot{\hat x}(\hat t_0)\bar t_0), V(\hat t_1)\bar x_0 + \sum_{i=1}^{s} y^i(\hat t_1)\bar\tau_i + \dot x(\hat t_1)\bar t_1 - V(\hat t_1)\dot{\hat x}(\hat t_0)\bar t_0\rangle \\
=\;& \langle l_{x_1x_1}\widetilde x_1, \widetilde x_1\rangle. \qquad (147)
\end{aligned}
$$

Group (d6): Terms with $\dot\psi(\hat t_0)$ and $\dot\psi_0(\hat t_0)$:

$$
d_6 := 2\dot\psi(\hat t_0)\bar x_0\bar t_0 + (\dot\psi_0(\hat t_0) - \dot\psi(\hat t_0)\dot{\hat x}(\hat t_0))\bar t_0^2. \qquad (148)
$$

Group (d7): Terms with $\dot{\psi}(\hat{t}_1)$ and $\dot{\psi}_0(\hat{t}_1)$:

$$\begin{aligned} d_7 \quad := \quad & -2\dot{\psi}(\hat{t}_1)V(\hat{t}_1)\bar{x}_0\bar{t}_1 - \sum_{i=1}^{s} 2\dot{\psi}(\hat{t}_1)y^i(\hat{t}_1)\bar{\tau}_i\bar{t}_1 \\ & -(\dot{\psi}(\hat{t}_1)\dot{\hat{x}}(\hat{t}_1) + \dot{\psi}_0(\hat{t}_1))\bar{t}_1^2 + 2\dot{\psi}(\hat{t}_1)(V(\hat{t}_1)\dot{x}(\hat{t}_0)\bar{t}_0)\bar{t}_1 \\ = \quad & -2\dot{\psi}(\hat{t}_1)\left(V(\hat{t}_1)\bar{x}_0 + \sum_{i=1}^{s} y^i(\hat{t}_1)\bar{\tau}_i - V(\hat{t}_1)\dot{x}(\hat{t}_0)\bar{t}_0)\right)\bar{t}_1 \\ & -(\dot{\psi}(\hat{t}_1)\dot{\hat{x}}(\hat{t}_1) + \dot{\psi}_0(\hat{t}_1))\bar{t}_1^2 \\ = \quad & -2\dot{\psi}(\hat{t}_1)\bar{x}(\hat{t}_1)\bar{t}_1 - (\dot{\psi}(\hat{t}_1)\dot{\hat{x}}(\hat{t}_1) + \dot{\psi}_0(\hat{t}_1))\bar{t}_1^2. \end{aligned} \tag{149}$$

Using the equality $\widetilde{x}_1 = \bar{x}(\hat{t}_1) + \dot{\hat{x}}(\hat{t}_1)\bar{t}_1$ in (149), we obtain

$$d_7 = -2\dot{\psi}(\hat{t}_1)\widetilde{x}_1\bar{t}_1 - (\dot{\psi}_0(\hat{t}_1) - \dot{\psi}(\hat{t}_1)\dot{\hat{x}}(\hat{t}_1))\bar{t}_1^2. \tag{150}$$

This completes the whole list of all terms in the quadratic form associated with the induced problem. Hence, we have

$$\langle L_{\zeta\zeta}\bar{\zeta}, \bar{\zeta}\rangle = a + b + c + d, \quad d = \sum_{k=1}^{7} d_k\,.$$

We thus have found the following representation of this quadratic form, see formulas (134) for a, (138) for b, and (140) for c:

$$\langle L_{\zeta\zeta}\bar{\zeta}, \bar{\zeta}\rangle \quad = \textstyle\sum_{i=1}^{s} D^i(H)\bar{\tau}_i^2 - \sum\limits_{i=1}^{s} 2[H_x]^i\bar{x}_{av}^i\bar{\tau}_i + \int\limits_{\hat{t}_0}^{\hat{t}_1}\langle H_{xx}\bar{x}, \bar{x}\rangle + d, \tag{151}$$

where according to formulas (141), (142), (144), (145), (147), (148), (150) for $d_1, \ldots, d_7$, respectively,

$$\begin{aligned} d \quad = \quad & \langle l_{x_0x_0}\bar{x}_0, \bar{x}_0\rangle + 2l_{x_0t_1}\bar{x}_0\bar{t}_1 + l_{t_1t_1}\bar{t}_1^2 \\ & +2l_{x_0t_0}\bar{x}_0\bar{t}_0 + 2l_{t_1t_0}\bar{t}_1\bar{t}_0 + l_{t_0t_0}\bar{t}_0^2 + 2l_{x_1t_0}\widetilde{x}_1\bar{t}_0 \\ & +2\langle l_{x_0x_1}\widetilde{x}_1, \bar{x}_0\rangle + 2l_{x_1t_1}\widetilde{x}_1\bar{t}_1 + \langle l_{x_1x_1}\widetilde{x}_1, \widetilde{x}_1\rangle \\ & +2\dot{\psi}(\hat{t}_0)\bar{x}_0\bar{t}_0 + (\dot{\psi}_0(\hat{t}_0) - \dot{\psi}(\hat{t}_0)\dot{\hat{x}}(\hat{t}_0))\bar{t}_0^2 \\ & -2\dot{\psi}(\hat{t}_1)\widetilde{x}_1\bar{t}_1 - (\dot{\psi}_0(\hat{t}_1) - \dot{\psi}(\hat{t}_1)\dot{\hat{x}}(\hat{t}_1))\bar{t}_1^2. \end{aligned} \tag{152}$$

In (151) and (152) the function $\bar{x}(t)$ and the vector $\widetilde{x}_1$ are defined by (137) and (143), respectively. Note that in (152),

$$\begin{aligned} & \langle l_{x_0x_0}\bar{x}_0, \bar{x}_0\rangle + 2l_{x_0t_1}\bar{x}_0\bar{t}_1 + l_{t_1t_1}\bar{t}_1^2 + 2l_{x_0t_0}\bar{x}_0\bar{t}_0 + 2l_{t_1t_0}\bar{t}_1\bar{t}_0 \\ & +l_{t_0t_0}\bar{t}_0^2 + 2l_{x_1t_0}\widetilde{x}_1\bar{t}_0 + 2\langle l_{x_0x_1}\widetilde{x}_1, \bar{x}_0\rangle + 2l_{x_1t_1}\widetilde{x}_1\bar{t}_1 + \langle l_{x_1x_1}\widetilde{x}_1, \widetilde{x}_1\rangle \\ & = \langle l_{pp}\widetilde{p}, \widetilde{p}\rangle, \end{aligned} \tag{153}$$

where, by definition,

$$\widetilde{p} = (\bar{t}_0, \bar{x}_0, \bar{t}_1, \widetilde{x}_1). \tag{154}$$

Finally, we get

$$\begin{aligned} d &= \langle l_{pp}\widetilde{p},\widetilde{p}\rangle + 2\dot{\psi}(\hat{t}_0)\bar{x}_0\bar{t}_0 + (\dot{\psi}_0(\hat{t}_0) - \dot{\psi}(\hat{t}_0)\dot{\hat{x}}(\hat{t}_0))\bar{t}_0^2 \\ &\quad -2\dot{\psi}(\hat{t}_1)\widetilde{x}_1\bar{t}_1 - (\dot{\psi}_0(\hat{t}_1) - \dot{\psi}(\hat{t}_1)\dot{\hat{x}}(\hat{t}_1))\bar{t}_1^2. \end{aligned} \tag{155}$$

We thus have proved the following result.

THEOREM 8.1 *Let the Lagrange multipliers* $\mu = (\alpha_0, \alpha, \beta) \in \Lambda_0$ *and* $\lambda = (\alpha_0, \alpha, \beta, \psi, \psi_0) \in \Lambda$ *correspond to each other, i.e., let* $\pi_0\lambda = \mu$ *hold. Then for any* $\bar{\zeta} = (\bar{t}_0, \bar{t}_1, \bar{x}_0, \bar{\tau}) \in \mathbb{R}^{2+n+s}$ *the quadratic form* $\langle L_{\zeta\zeta}\bar{\zeta}, \bar{\zeta}\rangle$ *has the representation (151)–(155), where the vector function* $\bar{x}(t)$ *and the vector* $\widetilde{x}_1$ *are defined by (137) and (143). The matrix-valued function* $V(t)$ *is the solution to the IVP (9) and, for each* $i = 1, \ldots, s$*, the vector function* y^i *is the solution to the IVP (21).*

Finally, we have arrived at the main result of Part 2 i.e., the present paper. Then, Theorem 4.4 of Part 1, which is the main result in both parts immediately follows from this theorem.

THEOREM 8.2 *Let* $\lambda = (\alpha_0, \alpha, \beta, \psi, \psi_0) \in \Lambda$ *and* $\bar{\zeta} = (\bar{t}_0, \bar{t}_1, \bar{x}_0, \bar{\tau}) \in \mathbb{R}^{2+n+s}$. *Put* $\mu = (\alpha_0, \alpha, \beta)$*, i.e., let* $\pi_0\lambda = \mu \in \Lambda_0$ *hold; see Proposition 4.1 in Part 1. Define the function* $\bar{x}(t)$ *by formula (137). Put* $\bar{\xi} = -\bar{\tau}$ *and* $\bar{z} = (\bar{t}_0, \bar{t}_1, \bar{\xi}, \bar{x})$*, which means* $\pi_1\bar{z} = \bar{\zeta}$*; see Propositions 4.3 and 4.4 in Part 1. Then the following equality holds,*

$$\langle L_{\zeta\zeta}(\mu, \hat{\zeta})\bar{\zeta}, \bar{\zeta}\rangle = \Omega(\lambda, \bar{z}), \tag{156}$$

where $\Omega(\lambda, \bar{z})$ *is defined by formulas (22), (23) in Part 1.*

Proof. By Theorem 8.1, the equalities (151)–(155) hold. In view of the definition (19) in Part 1 put

$$\widetilde{x}_0 = \bar{x}(t_0) + \bar{t}_0\dot{\hat{x}}(\hat{t}_0) = \left(V(\hat{t}_0)\bar{x}_0 + \sum_{i=1}^{s} y^i(\hat{t}_0)\bar{\tau}_i - V(\hat{t}_0)\dot{\hat{x}}(\hat{t}_0)\bar{t}_0 \right) + \bar{t}_0\dot{\hat{x}}(\hat{t}_0).$$

Since $y^i(\hat{t}_0) = 0$ for $i = 1, \ldots, s$ and $V(\hat{t}_0) = E$, it follows that $\widetilde{x}_0 = \bar{x}(t_0)$. Consequently, the vector $\widetilde{p}$ which was defined in Part 1, (19) as $(\bar{t}_0, \tilde{x}_0, \bar{t}_1, \tilde{x}_1)$ coincides with the vector $\widetilde{p}$, defined in this subsection by formula (154). Hence, the endpoint quadratic form d in (155) and the endpoint quadratic form $\langle A\tilde{p}, \tilde{p}\rangle$ in (23) of Part 1 take equal values, $d = \langle A\tilde{p}, \tilde{p}\rangle$. Moreover, the integral terms $\int_{\hat{t}_0}^{\hat{t}_1} \langle H_{xx}\bar{x}, \bar{x}\rangle\, dt$ in the representation (151) of the form $\langle L_{\zeta\zeta}\bar{\zeta}, \bar{\zeta}\rangle$ and those in the representation (22), Part 1, of the form Ω coincide, and

$$\sum_{i=1}^{s} \left(D^i(H)\xi_i^2 + 2[H_x]^i\bar{x}_{\rm av}^i\xi_i \right) = \sum_{i=1}^{s} D^i(H)\bar{\tau}_i^2 - \sum_{i=1}^{s} 2[H_x]^i\bar{x}_{av}^i\bar{\tau}_i,$$

because $\bar{\xi}_i = -\bar{\tau}_i$, $i = 1, \ldots, s$. Thus, the representation (151) of the form $\langle L_{\zeta\zeta}\bar{\zeta}, \bar{\zeta}\rangle$ implies the equality (156) of both forms. ■

Theorem 4.4 of Part 1, which is the main result of both parts, then follows from Theorem 8.2.

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