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New formula for the failure/repair frequency of multi-state monotone systems and its applications

by

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Abstract: The paper deals with calculation methods for failure and repair frequencies of multi-state monotone systems, both for the instantaneous and steady state cases. First, the fixed demand is considered. Based on the binary representation of multi-state structure, we have obtained a new formula for the failure/repair frequency. This formula is used to derive simple rules for the calculation of failure/repair frequency. These rules show how to convert an expression for the system availability into appropriate expression for the system failure/repair frequency. The applicability of the rules is illustrated by some examples. Finally, expressions for the failure/repair frequency for systems operating under random demand are obtained. Since both input and output data have similar format, the rules can be used in a recursive manner, allowing easy analysis of complex systems with hierarchical structure.

Keywords: multi-state monotone systems, availability, failure frequency, repair frequency, binary representation, Shannon decomposition, random demand.

1. Introduction

The failure frequency, called also the rate of occurrence of failures (ROCOF) or the failure intensity, is defined as the mean number of failures per unit time. Let us first consider a binary item (an element or system). Let $W(t)$ be the mean number of failures of the item in time-interval $(0, t]$. When $W(t)$ is an absolutely continuous function in any finite time interval, then the failure frequency $w(t)$ is defined as the density of $W(t)$ with respect to the Lebesgue measure on the real line, i.e.

$$W(t) = \int_0^t w(s)ds, \quad w(t) = \mathrm{d}W(t)/\mathrm{d}t \quad \text{(a.e.)}.$$

The limiting (or steady-state) failure frequency $w(\infty)$ is defined as the limiting value of $w(t)$ when t tends to infinity. The failure frequency is an important reliability measure of repairable items, since it may be used to compute the expected number of failures in a given interval. Furthermore, $w(\infty)$ is equal to the reciprocal of the mean time between failures, and the following well known expressions hold true:

$$\mathrm{MUT} = A(\infty)/w(\infty), \quad \mathrm{MDT} = (1 - A(\infty))/w(\infty),$$

where MUT = mean up-time, MDT = mean down time and $A(\infty)$ is the limiting (or steady state) availability of the item.

The repair (or restoration) frequency $v(t)$ of an item is defined similarly as the failure frequency, by replacing failures with restorations (i.e. completion of repairs) of the item. That is, by integrating $v(t)$ over given time-interval $[a, b]$, we obtain the mean number of restorations of the item in that interval.

For these reasons, considerable efforts have been devoted to the problem of finding the efficient calculation methods for the failure/repair frequency of binary monotone systems composed of independent binary components. See Amari (2000, 2002), Chang et al. (2004), Schneeweiss (1999) and the references given therein. The main objective of these studies was to obtain simple rules for transforming expressions of system availability/unavailability given in terms of element availabilities and unavailabilities into an expression for system failure frequency, and system repair frequency as well, both for time-dependent and steady-state cases.

In many real-life situations, however, the systems and their elements are capable of assuming a whole range of performance levels, varying from perfect functioning to complete failure. The examples of multi-state systems (MSS) are power systems, complex electronic/computing systems, systems of public transportation, oil/gas production and transportation, electric power transmission, water supply, and so on. An MSS fails if its performance level is less than the desired performance level (demand). The demand may be fixed or variable in time, deterministic or random. Beginning from the middle of the 1970s, the theory of binary systems is being replaced by the theory of MSS. The present state-of-art of the theory and practice of MSS may be found in recent monographs by Kołowrocki (2004), Kuo and Zuo (2003), Levitin (2005), and Lisnianski and Levitin (2003).

Contrary to the binary case, rather little attention has been devoted to finding practical methods for computation of the frequency-type indices for MSS. Main results have been obtained by Murchland (1975), where very general relations for the computation of failure frequency and related indices were given. Similar relations were considered in Aven and Jensen (1999), Natvig and Streller (1984) and Franken et al. (1984) for the steady-state case of multi-state monotone systems (MMS). However, the expressions obtained are stated in a general form, which is not very convenient for practical purpose, due to its complexity. Another approach, based on the inclusion-exclusion principle applied to the set of prime implicants of an MSS was suggested by Bossche (1984, 1986), see also Sherwin and Bossche (1993). None of the results mentioned so far has the form of rules converting availability expression to failure frequency expression, as in binary case.

The main aim of this paper is to show how to calculate the failure/repair frequency of multi-state systems using conversion rules being generalizations of the rules known from the binary systems. These multi-state conversion rules are obtained using a new general formula for the failure/repair frequency of MMS, which has very simple form, is easy to remember and to apply for fairly complex systems. All the results of the paper apply to both time-specific and steady state cases. The conversion rules may be easily included into existing algorithms for availability evaluation, extending its applicability.

The paper is organized as follows. Section 2 introduces basic terminology, notation and assumptions. In particular a concept of the binary representation of an MMS is described. Section 3 introduces definitions of basic reliability indices of an MMS and its elements. Furthermore, a brief overview of availability calculation methods for an MMS is given. Section 4 contains main results of the paper:

- new general formula for the failure/repair frequency of an MMS
- rules for converting the availability expressions into appropriate failure/repair frequency expression, assuming that the demand (or acceptable system's performance level) is fixed
- factoring (or Shannon decomposition) formulae for the failure frequency.

Section 5 contains some examples of application of the results to basic types of multi-state structures. Section 6 deals with the case of randomly changing demand. It is shown how the basic results of Section 4 can be applied to this case. Some conclusions are given in Section 7.

2. Basic definitions and assumptions

Let $< \boldsymbol{C}, \mathbf{K}, \mathbf{K}_1, \ldots, \mathbf{K}_n, \varphi >$ be a multi-state system consisting of n multi-state elements with the index set $\boldsymbol{C} = \{1, 2, ..., n\}$, where $\mathbf{K} = \{g(0), g(1), ..., g(M)\} \subseteq [0, +\infty)$ is the set of the system states, $\mathbf{K}_i = \{g_i(0), g_i(1), \ldots, g_i(M_i)\} \subseteq [0, +\infty)$ is the set of the states of element $i \in \boldsymbol{C}$, and $\varphi : \mathbf{V} \to \mathbf{K}$ is the system structure function, where $\mathbf{V} = \mathbf{K}_1 \times \mathbf{K}_2 \times \ldots \times \mathbf{K}_n$ is the space of element state vectors. We assume that the states of the system [element i] represent successive performance rates ranging from the perfect functioning level $g(M)$ $[g_i(M_i)]$ down to the complete failure level $g(0)$ $[g_i(0)]$, that is $0 \leqslant g(0) < g(1) < ... < g(M)$ and $0 \leqslant g_i(0) < g_i(1) < ... < g_i(M_i)$. The system is a *multi-state monotone system* (MMS) if its structure function φ is nondecreasing in each argument, $\varphi(\mathbf{g}(\mathbf{0})) = g(0)$ and $\varphi(\mathbf{g}(\mathbf{M})) = g(M)$, where $\mathbf{g}(\mathbf{0}) = (g_1(0), g_2(0), ..., g_n(0))$, $\mathbf{g}(\mathbf{M}) = (g_1(M_1), g_2(M_2), ..., g_n(M_n))$. We refer to Kuo and Zuo (2003), Levitin

(2005) and Lisnianski and Levitin (2003) for detailed description and numerous examples of MMS. Throughout the paper, we will consider MMS only.

A vector $\mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbf{V}$ is said to be a *path vector to level* $c \in \mathbf{K}$ of an MMS iff $\varphi(\mathbf{y}) \geqslant c$. It is called a *minimal path vector to level* c if, in addition, $\mathbf{x} < \mathbf{y}$ implies $\varphi(\mathbf{x}) < c$ where $\mathbf{x} < \mathbf{y}$ means $x_i \leqslant y_i$ for $i = 1, ..., n$, and $x_i < y_i$ for some i. The set of all minimal path vectors to level c is denoted by $\mathbf{U}_c$, where $\mathbf{U}_{g(0)} = \{\mathbf{g(0)}\}$. A vector $\mathbf{z} = (z_1, z_2, ..., z_n) \in \mathbf{V}$ is said to be a *cut vector to level* c of an MMS iff $\varphi(\mathbf{z}) < c$. It is called a *minimal cut vector to level* c if, in addition, $\mathbf{z} < \mathbf{x}$ implies $\varphi(\mathbf{x}) \geqslant c$. The set of all minimal cut vectors to level c is denoted by $\mathbf{L}_c$, where $\mathbf{L}_{g(0)} = \emptyset$.

The state (performance level) of element i at time t is represented by a (random) variable $X_i(t)$, which takes values in $\mathbf{K}_i$. The state (performance level) $X(t)$ of the system at time t is fully determined by the states of the elements through the multi-state structure function φ, i.e., $X(t) = \varphi(\mathbf{X}(t))$, where $\mathbf{X}(t) = (X_1(t), X_2(t), ..., X_n(t))$.

Let $X_i(e,t) = \mathbf{1}(X_i(t) \geqslant e)$ and $X(d,t) = \mathbf{1}(X(t) \geqslant d)$, $e, d \geqslant 0$, where $\mathbf{1}(.)$ is the indicator function. It is clear that $X_i(e,t) = X_i(g_i(j),t)$ and $X(d,t) = X(g(k),t)$ for $g_i(j-1) < e \leqslant g_i(j)$ and $g(k-1) < d \leqslant g(k)$, $X_i(e,t) \equiv X(d,t) \equiv 1$ for $e \leqslant g_i(0)$ and $d \leqslant g(0)$, and $X_i(e,t) \equiv X(d,t) \equiv 0$ for $e > g_i(M_i)$ and $d > g(M)$. The following relations hold ($r, s \in \mathbf{K}_i$):

$$X_i(t) = g_i(0) + \sum_{k=1}^{M_i} (g_i(k) - g_i(k-1)) X_i(g_i(k), t),$$

$$X_i(r,t) X_i(s,t) = X_i(r,t) \wedge X_i(s,t) = X_i(\max(r,s), t), \tag{1}$$

$$X_i(r,t) \vee X_i(s,t) = X_i(\min(r,s), t), \tag{2}$$

$$\overline{X}_i(r,t) X_i(s,t) = X_i(s,t) - X_i(\max(r,s), t), \tag{3}$$

where as usual, $a \wedge b = \min(a,b) = ab$, $a \vee b = \max(a,b) = a + b - ab$ and $\bar{a} = 1 - a$ for any binary a and b. Similar relations hold true for system's level indicator processes $\{X(d,t)\}$.

Let $\varphi_d = \mathbf{1}(\varphi \geqslant d)$, $d \in \mathbf{K} - \{0\}$, be the system level indicators. They can be considered as functions of vector of binary variables $\underline{\mathbf{X}}(t) = [X_i(r,t) : i \in \boldsymbol{C}, r \in \mathbf{K}_i - \{g_i(0)\}]$, so that $\varphi_d(\underline{\mathbf{X}}(t)) = X(d,t)$, resulting in the binary representation of MMS; see Block and Savits (1982), Korczak (1993, 2005) and Lisnianski and Levitin (2003) for more details.

From the definition of minimal path and minimal cut vectors, we obtain the so-called *min-path* and *min-cut forms*:

$$\varphi_d(\underline{\mathbf{X}}(t)) = \max_{\mathbf{y} \in \mathbf{U}_d} \min_{\substack{i \in \boldsymbol{C}: \\ y_i > g_i(0)}} X_i(y_i, t), \tag{4}$$

$$\varphi_d(\underline{\mathbf{X}}(t)) = \min_{\mathbf{z} \in \mathbf{L}_d} \max_{\substack{i \in \boldsymbol{C}: \\ z_i < g_i(M_i)}} X_i(z_i \oplus_i 1, t) = 1 - \max_{\mathbf{z} \in \mathbf{L}_d} \min_{\substack{i \in \boldsymbol{C}: \\ z_i < g_i(M_i)}} \overline{X}_i(z_i \oplus_i 1, t), \tag{5}$$

where $r \oplus_i 1 = \min(\mathbf{K}_i \cap (r, \infty))$, for $r \in \mathbf{K}_i - \{g_i(M_i)\}$, is the next state in $\mathbf{K}_i$ better than state r.

By applying the inclusion-exclusion principle to (4) and (5) and then using the rules (1)-(3) to each term, we obtain the so-called *Sylvester-Poincaré's forms* of an MMS:

$$\varphi_d(\underline{\mathbf{X}}(t)) = \sum_{\emptyset\neq\mathbf{D}\subseteq\mathbf{U}_d} (-1)^{|\mathbf{D}|+1} \prod_{\substack{i\in\boldsymbol{C}:\\ u_i(\mathbf{D})>g_i(0)}} X_i(u_i(\mathbf{D}),t)$$

$$= \sum_{k=1}^{|\mathbf{U}_d|} (-1)^{k+1} \sum_{\mathbf{D}\subseteq\mathbf{U}_d:|\mathbf{D}|=k} \prod_{\substack{i\in\boldsymbol{C}:\\ u_i(\mathbf{D})>g_i(0)}} X_i(u_i(\mathbf{D}),t), \tag{6}$$

$$\varphi_d(\underline{\mathbf{X}}(t)) = 1 - \sum_{\emptyset\neq\mathbf{D}\subseteq\mathbf{L}_d} (-1)^{|\mathbf{D}|+1} \prod_{\substack{i\in\boldsymbol{C}:\\ l_i(\mathbf{D})<g_i(M_i)}} \overline{X}_i(l_i(\mathbf{D})\oplus_i 1,t)$$

$$= 1 - \sum_{k=1}^{|\mathbf{L}_d|} (-1)^{k+1} \sum_{\mathbf{D}\subseteq\mathbf{L}_d:|\mathbf{D}|=k} \prod_{\substack{i\in\boldsymbol{C}:\\ l_i(\mathbf{D})<g_i(M_i)}} \overline{X}_i(l_i(\mathbf{D})\oplus_i 1,t), \tag{7}$$

where for any $\emptyset\neq\mathbf{D}\subseteq\mathbf{V}$, $u_i(\mathbf{D})=\max\{x_i:\mathbf{x}\in\mathbf{D}\}$, $l_i(\mathbf{D})=\min\{x_i:\mathbf{x}\in\mathbf{D}\}$, $i\in\boldsymbol{C}$, $|\mathbf{D}|=\operatorname{card}(\mathbf{D})$.

Among many other forms of φ_d, there is the pseudo-polynomial form:

$$\varphi_d(\underline{\mathbf{X}}(t)) = \beta_0 + \sum_{k=1}^{m} \beta_k B_k(\underline{\mathbf{X}}(t)), \tag{8}$$

where

$$B_k(\underline{\mathbf{X}}(t)) = \prod_{i\in\boldsymbol{C}} X_i(a(k,i),t)\overline{X}_i(b(k,i),t)$$

$$= \prod_{i\in\boldsymbol{C}} (X_i(a(k,i),t) - X_i(b(k,i),t))$$

$$= \left(\prod_{i\in\boldsymbol{C}_{k,1}} X_i(a(k,i),t)\right)\left(\prod_{i\in\boldsymbol{C}_{k,2}} (1-X_i(b(k,i),t))\right)\times$$

$$\times\left(\prod_{i\in\boldsymbol{C}_{k,3}} [X_i(a(k,i),t)-X_i(b(k,i),t)]\right), \tag{9}$$

β_k are integer coefficients, $a(k,i),b(k,i)\in\mathbf{K}_i\cup\{g_i(M_i)+1\}$, $a(k,i)<b(k,i)$ for all i and k, and the products B_k are non-trivial. The term $X_i(a(k,i),t)-X_i(b(k,i),t)$ reduces to $X_i(a(k,i),t)$ if $b(k,i)=g_i(M_i)+1$, to $1-X_i(b(k,i),t)$ if $a(k,i)=g_i(0)$ and to 1, if $b(k,i)=g_i(M_i)+1$ and $a(k,i)=g_i(0)$. Here the sets $\boldsymbol{C}_{k,b}$ are defined as follows: $\boldsymbol{C}_{k,1}=\{i:a(k,i)>g_i(0),b(k,i)=g_i(M_i)+1\}$, $\boldsymbol{C}_{k,2}=\{i:a(k,i)=g_i(0),b(k,i)\leqslant g_i(M_i)\}$, $\boldsymbol{C}_{k,3}=\{i:a(k,i)>g_i(0),b(k,i)\leqslant g_i(M_i)\}$, with $\boldsymbol{C}_{k,1}\cup\boldsymbol{C}_{k,2}\cup\boldsymbol{C}_{k,3}\neq\emptyset$ (as the products are non-trivial).

Observe that $\mathbf{1}(B_k(\underline{\mathbf{X}}(t)) = 1) = \mathbf{1}(\mathbf{X}(t) \in [\mathbf{a}(k), \mathbf{b}(k)))$, where $\mathbf{a}(k) = (a(k,1), ..., a(k,n))$ and $\mathbf{b}(k) = (b(k,1), ..., b(k,n))$, and $[\mathbf{a}(k), \mathbf{b}(k)) = \{\mathbf{x} \in \mathbf{V} : \mathbf{a}(k) \leqslant x < \mathbf{b}(k)\}$. Two products B_k and B_l are disjoint (or orthogonal) if the corresponding intervals are disjoint. If in (8), any two products are disjoint, $\beta_0 = 0$ and $\beta_k = 1$ $(k = 1, ..., m)$, then form (8) is called the *orthogonal form*, or *SDP form*. It corresponds to a division of $\{\mathbf{x} \in \mathbf{V} : \varphi(\mathbf{x}) \geqslant d\}$ into disjoint intervals. A particular case of the orthogonal form is the *canonical disjunctive normal form*, in which the products correspond to single-point intervals.

When reliability structure of a binary system is complex, the Shannon decomposition is frequently used to simplify the structure. The corresponding multi-state Shannon decomposition (or pivotal decomposition, or factoring) formulae are:

$$\varphi(\mathbf{X}(t)) = \sum_{r \in \mathbf{K}_i} \mathbf{1}(X_i(t) = r)\varphi((r)_i, \mathbf{X}(t)),$$

$$\varphi_d(\underline{\mathbf{X}}(t)) = \sum_{r \in \mathbf{K}_i} (X_i(r,t) - X_i(r \oplus_i 1, t))\varphi_d(\underline{\mathbf{e}}_i(r), \underline{\mathbf{X}}(t))$$

$$= \varphi_d(\underline{\mathbf{e}}_i(g_i(0)), \underline{\mathbf{X}}(t)) + \sum_{r \in \mathbf{K}_i} X_i(r,t)[\varphi_d(\underline{\mathbf{e}}_i(r), \underline{\mathbf{X}}(t)) - \varphi_d(\underline{\mathbf{e}}_i(r -_i 1), \underline{\mathbf{X}}(t))] \quad (10)$$

where $((r)_i, \mathbf{X}(t)) = (X_1(t), ..., X_{i-1}(t), r, X_{i+1}(t), ..., X_n(t))$, $\underline{\mathbf{e}}_i(r) = (\mathbf{1}(u \leqslant r) : u \in \mathbf{K}_i - \{g_i(0)\})$, $r \in \mathbf{K}_i$, $g_i(M_i) \oplus_i 1 = g_i(M_i) + 1$, and $r -_i 1 = \max(\mathbf{K}_i \cap (-\infty, r))$, for $r \in \mathbf{K}_i - \{g_i(0)\}$, is the best state preceding state r, and $g_i(0) -_i 1 = g_i(0)$, so that $\underline{\mathbf{e}}_i(g_i(0) -_i 1) = \underline{\mathbf{e}}_i(g_i(0))$. Observe that for $r = g_i(k)$, $\underline{\mathbf{e}}_i(r) = \underline{\mathbf{e}}_i(g_i(k)) = (\underbrace{1, ..., 1}_{k}, \underbrace{0, ..., 0}_{M_i - k})$, $\underline{\mathbf{e}}_i(g_i(0)) = (\underbrace{0, ..., 0}_{M_i})$, $\underline{\mathbf{e}}_i(g_i(M_i)) = (\underbrace{1, ..., 1}_{M_i})$.

Note that $\varphi((r)_i, \mathbf{x})$ is an extended structure function (i.e. it can be degenerated), taking its values in the set $\{\varphi((r)_i, \mathbf{g}(\mathbf{0})), ..., \varphi((r)_i, \mathbf{g}(\mathbf{M}))\}$. When $r > g_i(0)$, it may happen that $\varphi((r)_i, \mathbf{g}(\mathbf{0})) > g(0) = \varphi(\mathbf{g}(\mathbf{0}))$. However, if $\varphi((r)_i, \mathbf{g}(\mathbf{0})) < \varphi((r)_i, \mathbf{g}(\mathbf{M}))$, then $\varphi((r)_i, \mathbf{x})$ fits our definition of MMS with the lowest performance levels being not necessarily 0.

Unless otherwise stated, we make the following assumptions regarding stochastic properties of the elements of an MMS.

ASSUMPTION 1 *The system's elements, that is, the stochastic processes* $\{X_i(t)\}$, $i \in \boldsymbol{C}$, *are mutually independent.*

ASSUMPTION 2 $\{X_i(t)\}$, $i \in \boldsymbol{C}$, *are regular jump processes, i.e.: have jump right-continuous sample paths with left-side limits, and have finite expected number of jumps in bounded intervals.*

Before formulating subsequent assumption, we need some additional notation. Let $N_i(t) = \sum_{n \geqslant 1} \mathbf{1}(S_{i,n} \leqslant t)$ be the number of jumps of $\{X_i(t)\}$ in $(0, t]$, where $\{S_{i,n}\}$ are the successive jump times, with $S_{i,0} = 0$. Let $W_i(t) = E[N_i(t)]$ be the expected number of jumps of $\{X_i(t)\}$ in $(0, t]$. For any $r, s \in \mathbf{K}_i$, $r \neq s$, let $N_i^{r\to s}(t)$ be the number of transitions of element i from its state r to its state s in time interval $(0, t]$. Its expected value is denoted by $W_i^{r\to s}(t) = E[N_i^{r\to s}(t)]$. Functions $W_i(t)$ and $W_i^{r\to s}(t)$ induce Lebesgue-Stieltjes measures on Borel sets of $[0, \infty)$. These measures, being σ-finite according to Assumption 2, will be denoted by the same letter as the corresponding mean function, e.g. $W_i(a, b] = W_i(b) - W_i(a)$.

Assumption 3 *For any $i \in \boldsymbol{C}$, the function $W_i(t)$ is locally absolutely continuous on $[0, \infty)$, i.e. absolutely continuous in any bounded interval. Or, equivalently, for any $i \in \boldsymbol{C}$, all the functions $W_i^{r\to s}(t)$, $r,s \in \boldsymbol{K}_i$, $r \neq s$, are locally absolutely continuous on $[0, \infty)$.*

The equivalence in the formulation of Assumption 3 follows from known properties of absolute continuity and from the facts that $W_i(t) = \sum_{r\neq s} W_i^{r\to s}(t)$ and $W_i^{r\to s}(a, b] \leqslant W_i(a, b]$ for $0 \leqslant a < b$. Similarly, since $\Pr\{S_{i,n} \leqslant b\} - \Pr\{S_{i,n} \leqslant a\} = E[\mathbf{1}(a < S_{i,n} \leqslant b)] \leqslant W_i(a, b]$, the assumption implies that the jump times $S_{i,n}$ have absolutely continuous distributions. This, in turn, leads to the conclusion that under Assumption 1, the processes $\{X_i(t)\}$, $i \in \boldsymbol{C}$, have no common jump times with probability 1. And in consequence, any change of the system's state is caused, with probability one, by a jump of exactly one element.

3. Basic reliability measures of MMS

For an MMS, one can define various reliability and performance measures, see Aven and Jensen (1999), Korczak (1997), Levitin (2005), and Lisnianski and Levitin (2003). In this paper we will consider basic reliability indices only, which, however, may be used to calculate many other measures.

For any fixed performance level d, $g(0) < d \leqslant g(M)$, we define the system reliability measures like for the binary systems, considering the sets $\mathbf{G}(d) = \mathbf{K} \cap [d, \infty)$ and $\mathbf{F}(d) = \mathbf{K} - \mathbf{G}(d)$ as up and down states respectively. The system availability to level d (or to demand d) is defined as $A(d, t) = \Pr\{X(t) \geqslant d\} = E[X(d, t)] = E[\varphi_d(\underline{\mathbf{X}}(t))]$. The system unavailability to level d (or to demand d) is defined as $U(d, t) = \Pr\{X(t) < d\} = 1 - A(d, t)$. A transition from $\mathbf{G}(d)$ to $\mathbf{F}(d)$ is called d-failure, and the reverse transition is called d-repair. The instantaneous failure [repair] frequency to level d (shortly: d-failure [repair] frequency) is denoted by $w(d, t)$ [$v(d, t)$] and defined as the density of the function $W(d, t)$ [$V(d, t)$], the expected number of d-failures [d-repairs] in $(0, t]$, i.e.:

$$W(d, t) = \int_0^t w(d, s)ds, \quad V(d, t) = \int_0^t v(d, s)ds.$$

We set $A(g(0), t) \equiv U(u, t) \equiv 1$ and $w(g(0), t) \equiv w(u, t) \equiv A(u, t) \equiv U(g(0), t) \equiv 0$ for $u > g(M)$. Binary-like reliability indices of the system elements are defined similarly, and are denoted as $A_i(r,t)$, $U_i(r,t)$, $w_i(r,t)$ and $v_i(r,t)$ for $i \in \boldsymbol{C}$ and $g_i(0) < r \leqslant g_i(M_i)$, with $A_i(g_i(0),t) \equiv U_i(u,t) \equiv 1$, $w_i(g_i(0),t) \equiv w_i(u,t) \equiv A_i(u,t) \equiv U_i(g_i(0),t) \equiv 0$ for $u > g_i(M_i)$. The steady state (or limiting, asymptotic) reliability indices are defined as the limiting values of the corresponding instantaneous indices, by letting $t \to \infty$, if the limits exist. In steady state, the failure and repair frequencies of system, and each of its element as well, are equivalent:

$$w(d,\infty) = v(d,\infty), \quad w_i(r,\infty) = v_i(r,\infty).$$

The steady state system failure frequency is important for applications, since under rather mild assumptions, see Cocozza-Thivent (1997) and Cocozza-Thivent and Roussignol (2000) (for example when system's elements are modelled by irreducible time-continuous Markov chains, or by its functions), we have the following familiar relations:

$$\mathrm{MUT}(d) = A(d,\infty)/w(d,\infty), \quad \mathrm{MDT}(d) = U(d,\infty)/w(d,\infty),$$

where $\mathrm{MUT}(d)[\mathrm{MDT}(d)]$ is the mean up-time [down-time] to level d of the system.

Our aim is to express the system availability and failure/repair frequency in terms of availabilities and failure/repair frequencies of system's elements. Therefore we assume that the appropriate indices of elements are known (e.g. from statistical data) or may be calculated. In particular, if the interstate frequencies $w_i^{r\to s}(t)$, defined as densities of the mean functions $W_i^{r\to s}(t)$, i.e.

$$W_i^{r\to s}(t) = \int_0^t w_i^{r\to s}(s)ds,$$

are known, then we have:

$$w_i(u,t) = \sum_{\substack{r,s\in\mathbf{K}_i \\ s<u, r\geqslant u}} w_i^{r\to s}(t), \quad v_i(u,t) = \sum_{\substack{r,s\in\mathbf{K}_i \\ s<u, r\geqslant u}} w_i^{s\to r}(t), \quad u \in \mathbf{K}_i - \{g_i(0)\}. \quad (11)$$

The existence of $w_i^{r\to s}(t)$ is a consequence of Assumption 3.

Suppose that the stochastic evolution of element i is described by a homogeneous time-continuous Markov chain with transition rate matrix $[\lambda_i(r,s) : r,s \in \mathbf{K}_i]$. Then:

$$w_i^{r\to s}(t) = P_i(r,t)\lambda_i(r,s),$$

where $t \geqslant 0$ or $t = \infty$ (for the limiting case), and $P_i(r,t) = \Pr\{X_i(t) = r\}$. Formal and elementary proof of the above simple formula is given in Lam (1997), though it was widely known and used in the past, see Singh and Billinton (1977).

It is worth noting that this expression is also a consequence of the Lévy formula, see Brémaud (1981) and Cocozza-Thivent (1997). For the case of semi-Markov processes we refer to Cocozza-Thivent (1997) and Ouhbi and Limnios (2002).

The availability and failure/repair frequencies are defined in the broad sense. That is, they also include the non-repairable case. When an element (system) is non-repairable to a given performance level and is initially up with respect to this level, then the availability is equal to the reliability (or survival) function, the unavailability is equal to the unreliability function, and the failure frequency is just the probability density function of time to the first failure to the given level. The repair frequency is equal to 0 in this case.

For any fixed $i \in \boldsymbol{C}$, stochastic processes $\{X_i(e,t)\}$, $e \in \mathbf{K}_i - \{g_i(0)\}$, are dependent, as $1 \geqslant X_i(g_i(1),t) \geqslant X_i(g_i(2),t) \geqslant \ldots \geqslant X_i(g_i(M_i),t) \geqslant 0$. However, by stochastic independence of elements, the processes belonging to different elements are independent. Therefore, having $\varphi_d(\underline{\mathbf{X}}(t))$ written in a suitable form, and knowing availabilities/unavailabilities of independent elements, calculation of the system availability is very easy. For example, by equations (6)-(9), we have:

$$A(d,t) = \sum_{\emptyset \neq \mathbf{D} \subseteq \mathbf{U}_d} (-1)^{|\mathbf{D}|+1} \prod_{\substack{i \in \boldsymbol{C}: \\ u_i(\mathbf{D}) > g_i(0)}} A_i(u_i(\mathbf{D}),t) \tag{12}$$

$$U(d,t) = \sum_{\emptyset \neq \mathbf{D} \subseteq \mathbf{L}_d} (-1)^{|\mathbf{D}|+1} \prod_{\substack{i \in \boldsymbol{C}: \\ l_i(\mathbf{D}) < g_i(M_i)}} U_i(l_i(\mathbf{D}) \oplus_i 1, t), \tag{13}$$

$$A(d,t) = \beta_0 + \sum_{k=1}^{m} \beta_k B_k(\underline{\mathbf{A}}(t)), \tag{14}$$

where $A_i(g_i(0),t) \equiv 1$, $A_i(u,t) \equiv 0$ for $u > g_i(M_i+1)$, and

$$B_k(\underline{\mathbf{A}}(t)) = E[B_k(\underline{\mathbf{X}}(t))] = \prod_{i \in \boldsymbol{C}} (A_i(a(k,i),t) - A_i(b(k,i),t)),$$

with $\underline{\mathbf{A}}(t) = [A_i(r,t) : i \in \boldsymbol{C}, r \in \mathbf{K}_i - \{g_i(0)\}]$.

Applying the factoring formula (10), we get:

$$\begin{aligned} A(d,t) &= \sum_{r \in \mathbf{K}_i} (A_i(r,t) - A_i(r \oplus_i 1, t)) A^{(i,r)}(d,t) \\ &= A^{(i,g_i(0))}(d,t) + \sum_{r \in \mathbf{K}_i} A_i(r,t)[A^{(i,r)}(d,t) - A^{(i,r-_i 1)}(d,t)], \end{aligned} \tag{15}$$

where $A^{(i,r)}(d,t) = \Pr\{\varphi(\mathbf{X}(t)) \geqslant d \,|\, X_i(t) = r\} = \Pr\{\varphi((r)_i, \mathbf{X}(t)) \geqslant d\} = E[\varphi_d(\underline{\mathbf{e}}_i(r), \underline{\mathbf{X}}(t))]$ is the availability to level d of the system with indicator structure function $\varphi_d(\underline{\mathbf{e}}_i(r), \underline{\mathbf{X}}(t))$, or in other words, $A^{(i,r)}(d,t)$ is the availability of the system with structure function φ, given that element i is strapped in state r. Note that for the validity of the above expressions, only Assumption 1 is needed. By letting $t \to \infty$ in the above formulae, we obtain the steady state system availability/unavailability, provided the appropriate limits for element availabilities exist.

4. Failure and repair frequency calculation for fixed demand level

4.1. The main formula

According to general results obtained by Murchland (1975), we have:

$$w(d,t) = \sum_{i \in \boldsymbol{C}} \sum_{\substack{r,s \in \mathbf{K}_i \\ r \neq s}} \Pr\{\varphi((r)_i, \mathbf{X}(t)) \geqslant d, \varphi((s)_i, \mathbf{X}(t)) < d\} w_i^{r \to s}(t) \quad (16)$$

$$v(d,t) = \sum_{i \in \boldsymbol{C}} \sum_{\substack{r,s \in \mathbf{K}_i \\ r \neq s}} \Pr\{\varphi((r)_i, \mathbf{X}(t)) \geqslant d, \varphi((s)_i, \mathbf{X}(t)) < d\} w_i^{s \to r}(t) \quad (17)$$

for both monotone and non-monotone systems. For monotone systems we have:

$$\begin{aligned} &\Pr\{\varphi((r)_i, \mathbf{X}(t)) \geqslant d, \varphi((s)_i, \mathbf{X}(t)) < d\} \\ &= \mathbf{1}(r > s) \cdot (\Pr\{\varphi((r)_i, \mathbf{X}(t)) \geqslant d\} - \Pr\{\varphi((s)_i, \mathbf{X}(t)) \geqslant d\}) \\ &= \mathbf{1}(r > s) \cdot (A^{(i,r)}(d,t) - A^{(i,s)}(d,t)), \end{aligned}$$

hence the general expressions (16) and (17) reduce to the following:

$$w(d,t) = \sum_{i \in \boldsymbol{C}} \sum_{\substack{r,s \in \mathbf{K}_i \\ r > s}} (A^{(i,r)}(d,t) - A^{(i,s)}(d,t)) w_i^{r \to s}(t) \quad (18)$$

$$v(d,t) = \sum_{i \in \boldsymbol{C}} \sum_{\substack{r,s \in \mathbf{K}_i \\ r > s}} (A^{(i,r)}(d,t) - A^{(i,s)}(d,t)) w_i^{s \to r}(t)\,. \quad (19)$$

The main disadvantages of these formulae are that they depend on a number of element's inter-state transition frequencies, and that the format of input data $\{w_i^{r \to s}(t)\}$ is different from the format of output data $\{w(d,t), v(d,t)\}$. As a result, recursive application of these formulae for complex systems with hierarchical structure is difficult, or even impossible. In particular, the powerful modular decomposition technique cannot be applied. More convenient are formulae stated in terms of element's failure/repair frequencies, $w_i(r,t)$ and $v_i(r,t)$. Observe that for $r > s$, $r, s \in \mathbf{K}_i$:

$$A^{(i,r)}(d,t) - A^{(i,s)}(d,t) = \sum_{\substack{u \in \mathbf{K}_i: \\ s < u \leqslant r}} (A^{(i,u)}(d,t) - A^{(i,u-_i 1)}(d,t)). \quad (20)$$

Substituting (20) into (18) and (19), interchanging the order of summation and using relations (11), we obtain the following result:

Proposition 1

$$w(d,t) = \sum_{i\in \boldsymbol{C}} \sum_{r\in \boldsymbol{K}_i-\{g_i(0)\}} [A^{(i,r)}(d,t) - A^{(i,r-_i1)}(d,t)]w_i(r,t),$$
$$v(d,t) = \sum_{i\in \boldsymbol{C}} \sum_{r\in \boldsymbol{K}_i-\{g_i(0)\}} [A^{(i,r)}(d,t) - A^{(i,r-_i1)}(d,t)]v_i(r,t).$$

Furthermore, according to (15),

$$A^{(i,r)}(d,t) - A^{(i,r-_i1)}(d,t) = \frac{\partial A(d,t)}{\partial A_i(r,t)},$$

where we consider any $U_i(r,t)$ appearing in the expression for $A(d,t)$ as $1 - A_i(r,t)$, so that $\partial U_i(r,t)/\partial A_i(r,t) = \partial(1 - A_i(r,t))/\partial A_i(r,t) = -1$.

Thus we have proved the following main result:

Proposition 2

$$w(d,t) = \sum_{i\in \boldsymbol{C}} \sum_{r\in \boldsymbol{K}_i-\{g_i(0)\}} w_i(r,t)\frac{\partial A(d,t)}{\partial A_i(r,t)}, \tag{21}$$
$$v(d,t) = \sum_{i\in \boldsymbol{C}} \sum_{r\in \boldsymbol{K}_i-\{g_i(0)\}} v_i(r,t)\frac{\partial A(d,t)}{\partial A_i(r,t)}.$$

Now the input and output data are of the same format as the failure/repair frequencies. By comparing (18) and (21), we see that Murchland's formula may require $\sum_{i\in \boldsymbol{C}}(M_i + 1)M_i/2$ inter-state frequencies as the input data, whereas the new one requires at most $\sum_{i\in \boldsymbol{C}} M_i$ level-crossing frequencies. Moreover, the expressions obtained are easy to remember, and are very similar to the formulae known from the binary system theory. Since the expressions for $w(d,t)$ and $v(d,t)$ are similar, we will restrict our consideration to $w(d,t)$.

4.2. Conversion rules for the failure frequency calculation

Fairly general conversion rule, that is – a rule that converts an availability expression of an MMS into its failure frequency expression – can be described as follows. Suppose that the availability $A(d,t)$ is given in the following sum of products form:

$$A_\varphi(d;t) = \beta_0 + \sum_{k=1}^{L} \beta_k H_k(\underline{\mathbf{A}}(t)) \tag{22}$$

where $H_k(\underline{\mathbf{A}}(t))$ are non-trivial products of the form:

$$H_k(\underline{\mathbf{A}}(t)) = \prod_{m\in \mathbf{E}_k} G_{k,m}(\underline{\mathbf{A}}(t)), \tag{23}$$

where $\mathbf{E}_k$ is a non-empty index set, and $G_{k,m}(\underline{\mathbf{A}}(t))$, $m \in \mathbf{E}_k$, are functions having no common relevant variable belonging to the same system's element. That is, if $G_{k,m}(\underline{\mathbf{A}}(t))$ depends on the variable $A_i(r,t)$ (belonging to element i), then other functions $G_{k,l}(\underline{\mathbf{A}}(t))$, $l \neq m$, do not depend on variables $A_i(s,t)$, $s \in \mathbf{K}_i - \{g_i(0)\}$. This relevant variable disjointness property relates to each product separately. We assume that $G_{k,m}(\underline{\mathbf{A}}(t))$ are differentiable with respect to each variable (the derivatives being 0 for the non-relevant variable). Then by applying Proposition 2 to $A_\varphi(d;t)$ given in (22), and using usual algebra and calculus, we obtain:

$$\begin{aligned} w(d,t) &= \sum_{k=1}^{L} \beta_k \sum_{m \in \mathbf{E}_k} H_k^{(m)}(\underline{\mathbf{A}}(t)) \cdot w_{k,m}(t) \\ &= \sum_{k=1}^{L} \beta_k H_k(\underline{\mathbf{A}}(t)) \sum_{m \in \mathbf{E}_k} \frac{w_{k,m}(t)}{G_{k,m}(\underline{\mathbf{A}}(t))}\,, \end{aligned} \tag{24}$$

where, by convention, $a/0 = 0$ for any a, and

$$\begin{aligned} H_k^{(m)}(\underline{\mathbf{A}}(t)) &= \prod_{l \in \mathbf{E}_k - \{m\}} G_{k,l}(\underline{\mathbf{A}}(t)) \\ w_{k,m}(t) &= \sum_{i \in \boldsymbol{C}} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} \frac{\partial G_{k,m}(\underline{\mathbf{A}}(t))}{\partial A_i(r,t)} \cdot w_i(r,t)\,. \end{aligned}$$

Notice that according to (21), when $G_{k,m}(\underline{\mathbf{A}}(t))$ is the availability (unavailability) to a given level of a multi-state subsystem, then $w_{k,m}(t)$ $(-w_{k,m}(t))$ is its failure frequency to this level. By a suitable choice of functions $G_{k,m}(\underline{\mathbf{A}}(t))$ in the above general rule, we may obtain several special cases, being multi-state generalizations of conversion rules known for binary systems, see Amari (2000, 2002). Let us now consider some simpler, but important cases. By applying (24) with $A(d,t)$ given by (14), we obtain:

$$\begin{aligned} w(d,t) &= \sum_{k=1}^{m} \beta_k \sum_{z \in \boldsymbol{C}} (w_z(a(k,z),t) - w_z(b(k,z),t)) B_k^{(z)}(\underline{\mathbf{A}}(t)) \\ &= \sum_{k=1}^{m} \beta_k B_k(\underline{\mathbf{A}}(t)) \sum_{i \in \boldsymbol{C}} \frac{w_i(a(k,i),t) - w_i(b(k,i),t)}{A_i(a(k,i),t) - A_i(b(k,i),t)}, \end{aligned} \tag{25}$$

where, $a/0 = 0$ for any a, and

$$B_k^{(z)}(\underline{\mathbf{A}}(t)) = \prod_{i \in \boldsymbol{C} - \{z\}} (A_i(a(k,i),t) - A_i(b(k,i),t))\,.$$

Using (12) and (13), we have:

$$w(d,t) = \sum_{\emptyset \neq \mathbf{D} \subseteq \mathbf{U}_d} (-1)^{|\mathbf{D}|+1} \left(\prod_{\substack{i \in \boldsymbol{C}: \\ u_i(\mathbf{D}) > g_i(0)}} A_i(u_i(\mathbf{D}), t) \right) \sum_{\substack{i \in \boldsymbol{C}: \\ u_i(\mathbf{D}) > g_i(0)}} \frac{w_i(u_i(\mathbf{D}), t)}{A_i(u_i(\mathbf{D}), t)},$$

$$w(d,t) = \sum_{\emptyset \neq \mathbf{D} \subseteq \mathbf{L}_d} (-1)^{|\mathbf{D}|+1} \left(\prod_{\substack{i \in \boldsymbol{C}: \\ l_i(\mathbf{D}) < \mathbf{g_i}(\mathbf{M_i})}} U_i(l_i(\mathbf{D}) \oplus_i 1, t) \right) \times$$
$$\times \sum_{\substack{i \in \boldsymbol{C}: \\ l_i(\mathbf{D}) < g_i(M_i)}} \frac{w_i(l_i(\mathbf{D}) \oplus_i 1, t)}{U_i(l_i(\mathbf{D}) \oplus_i 1, t)},$$

being multi-state generalizations of well known inclusion-exclusion method based formulae for failure frequency of binary systems, see Schneeweiss (1999).

4.3. Shannon's decomposition formulae for the failure frequency

As an application of the above general rules, let us consider the factoring formula (12) with respect to states of element i. By applying (24) to (12), we obtain:

$$\begin{aligned}
w(d,t) &= \sum_{r \in \mathbf{K}_i} \big[(w_i(r,t) - w_i(r \oplus_i 1, t)) A^{(i,r)}(d,t) \\
&\quad + (A_i(r,t) - A_i(r \oplus_i 1, t)) w^{(i,r)}(d,t) \big] \\
&= \sum_{r \in \mathbf{K}_i} (A_i(r,t) - A_i(r \oplus_i 1, t)) A^{(i,r)}(d,t) \times \\
&\quad \times \left(\frac{w_i(r,t) - w_i(r \oplus_i 1, t)}{A_i(r,t) - A_i(r \oplus_i 1, t)} + \frac{w^{(i,r)}(d,t)}{A^{(i,r)}(d,t)} \right) \\
&= w^{(i,g_i(0))}(d,t) + \sum_{r \in \mathbf{K}_i} A_i(r,t) (A^{(i,r)}(d,t) - A^{(i,r -_i 1)}(d,t)) \times \\
&\quad \times \left(\frac{w_i(r,t)}{A_i(r,t)} + \frac{w^{(i,r)}(d,t) - w^{(i,r -_i 1)}(d,t)}{A^{(i,r)}(d,t) - A^{(i,r -_i 1)}(d,t)} \right),
\end{aligned} \tag{26}$$

where $w^{(i,r)}(d,t)$ is the failure frequency to level d of the system with indicator structure function $\varphi_d(\underline{\mathbf{e}}_i(r), \underline{\mathbf{X}}(t))$, or in other words, $w^{(i,r)}(d,t)$ is the failure frequency to level d of the system with structure function φ, given that element i is strapped in state r.

The Shannon decomposition formulae (15) and (26) are useful in recursive calculation of reliability indices of multi-state systems with complicated structure.

The rules for calculation of $w(d,t)$ in the case when we have $U(d,t)$ expressed in terms of element availabilities and unavailabilities can be easily obtained from the above rules by considering $1 - U(d,t) = A(d,t)$ and using chain rule of differentiation. For example, we have the following U-version of Proposition 2:

$$w(d,t) = \sum_{i \in \boldsymbol{C}} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} w_i(r,t) \frac{\partial U(d,t)}{\partial U_i(r,t)},$$
$$v(d,t) = \sum_{i \in \boldsymbol{C}} \sum_{r \in \mathbf{K}_i - \{g_i(0)\}} v_i(r,t) \frac{\partial U(d,t)}{\partial U_i(r,t)}.$$

5. Examples

In the examples below, for the sake of brevity, we shall omit the time parameter t. Let $k(i,d) = \min(\mathbf{K}_i \cap [d, \infty))$ if $d \leqslant g_i(M_i)$, and $k(i,d) = g_i(M_i) + 1$ if $d > g_i(M_i)$. Hence if $d \leqslant g_i(M_i)$, then $k(i,d)$ is the minimal state in $\mathbf{K}_i$ equal or greater than d. For example, if $g_i(j-1) < d \leqslant g_i(j)$, $j = 1, ..., M_i$, then $k(i,d) = g_i(j)$. If $d \leqslant g_i(0)$, then $k(i,d) = k(i, g_i(0)) = g_i(0)$. Also, $k(i, g_i(j)) = g_i(j)$ for $j = 0, 1, ..., M_i$. We start with two very simple, but important in practice, examples of MMS.

Example 1 Consider the series system with the structure function $\varphi(\mathbf{X}) = \min(X_1, X_2, ..., X_n)$. We have:

$$\varphi_d(\underline{\mathbf{X}}) = \prod_{i \in \boldsymbol{C}} X_i(d) = \prod_{i \in \boldsymbol{C}} X_i(k(i,d)),$$
$$A(d) = \prod_{i \in \boldsymbol{C}} A_i(d) = \prod_{i \in \boldsymbol{C}} A_i(k(i,d)),$$
$$w(d) = A(d) \sum_{i \in \boldsymbol{C}} \frac{w_i(d)}{A_i(d)} = A(d) \sum_{i \in \boldsymbol{C}} \frac{w_i(k(i,d))}{A_i(k(i,d))} .$$

Example 2 Consider the parallel system with the structure function $\varphi(\mathbf{X}) = \max(X_1, X_2, ..., X_n)$. We have:

$$\varphi_d(\underline{\mathbf{X}}) = 1 - \prod_{i \in \boldsymbol{C}} (1 - X_i(d)) = 1 - \prod_{i \in \boldsymbol{C}} (1 - X_i(k(i,d))),$$
$$A(d) = 1 - \prod_{i \in \boldsymbol{C}} (1 - A_i(d)) = 1 - \prod_{i \in \boldsymbol{C}} (1 - A_i(k(i,d))),$$
$$w(d) = (1 - A(d)) \sum_{i \in \boldsymbol{C}} \frac{w_i(d)}{1 - A_i(d)} = (1 - A(d)) \sum_{i \in \boldsymbol{C}} \frac{w_i(k(i,d))}{1 - A_i(k(i,d))} .$$

Example 3 An MMS has the following minimal path vectors to a level d: $\mathbf{y}_1 = (a_1, b_1, c_1, 0)$, $\mathbf{y}_2 = (0, 0, c_2, d_1)$ and $\mathbf{y}_3 = (a_2, b_2, 0, d_2)$, where $a_1 \leqslant a_2$, $b_1 \leqslant b_2$, $c_1 \leqslant c_2$ and $d_1 \leqslant d_2$. According to (4), the min-path vector representation of the binary structure φ_d is:

$$\varphi_d(\underline{\mathbf{X}}) = X_1(a_1) X_2(b_1) X_3(c_1) \vee X_3(c_2) X_4(d_1) \vee X_1(a_2) X_2(b_2) X_4(d_2).$$

By applying standard Boolean techniques together with reduction rules (1)-(3), we obtain:

$$\begin{aligned}\varphi_d(\underline{\mathbf{X}}) &= X_1(a_1)X_2(b_1)X_3(c_1) + \overline{X_1(a_1)X_2(b_1)X_3(c_1)} \cdot X_3(c_2)X_4(d_1)\\ &\quad + \overline{X_1(a_1)X_2(b_1)X_3(c_1)} \cdot \overline{X_3(c_2)X_4(d_1)} \cdot X_1(a_2)X_2(b_2)X_4(d_2)\\ &= X_1(a_1)X_2(b_1)X_3(c_1) + \overline{X_1(a_1)X_2(b_1)} \cdot X_3(c_2)X_4(d_1)\\ &\quad + \overline{X_3(c_1)} \cdot X_1(a_2)X_2(b_2)X_4(d_2).\end{aligned}$$

By element independence,

$$\begin{aligned}A(d) &= A_1(a_1)A_2(b_1)A_3(c_1) + [1 - A_1(a_1)A_2(b_1)]A_3(c_2)A_4(d_1)\\ &\quad + A_1(a_2)A_2(b_2)[1 - A_3(c_1)]A_4(d_2).\end{aligned}$$

Applying (24) leads to:

$$\begin{aligned}w(d) &= A_1(a_1)A_2(b_1)A_3(c_1)\left[\frac{w_1(a_1)}{A_1(a_1)} + \frac{w_2(b_1)}{A_2(b_1)} + \frac{w_3(c_1)}{A_3(c_1)}\right]\\ &\quad + [1 - A_1(a_1)A_2(b_1)]A_3(c_2)A_4(d_1)\times\\ &\quad \times\left[\frac{-w_1(a_1)A_2(b_1) - A_1(a_1)w_2(b_1)}{1 - A_1(a_1)A_2(b_1)} + \frac{w_3(c_2)}{A_3(c_2)} + \frac{w_4(d_1)}{A_4(d_1)}\right]\\ &\quad + A_1(a_2)A_2(b_2)[1 - A_3(c_1)]A_4(d_2)\times\\ &\quad \times\left[\frac{w_1(a_2)}{A_1(a_2)} + \frac{w_2(b_2)}{A_2(b_2)} - \frac{w_3(c_1)}{1 - A_3(c_1)} + \frac{w_4(d_2)}{A_4(d_2)}\right].\end{aligned}$$

EXAMPLE 4 Consider the parallel flow transmission system with flow dispersion, Levitin (2005). Its structure function is $\varphi(\mathbf{X}) = X_1 + X_2 + ... + X_n$. In this case we consider recursive formulae. For each $m \in \boldsymbol{C}$, let $\varphi^{(m)}(\mathbf{X}^{(m)}) = X_1 + X_2 + ... + X_m$ be the structure function of a subsystem with element set $\boldsymbol{C}^{(m)} = \{1, ..., m\}$, state space $\mathbf{K}^{(m)} = \varphi^{(m)}(\mathbf{K}_l \times ... \times \mathbf{K}_m)$ and $M^{(m)} = \#\{\mathbf{K}^{(m)}\} - 1$, where $\mathbf{X}^{(m)} = (X_1, X_2, ..., X_m)$ and $\#\{\mathbf{K}^{(m)}\}$ is the cardinality of $\mathbf{K}^{(m)}$. Let $A^{(m)}(d)$ and $w^{(m)}(d)$ be the availability and the failure frequency to level d of this subsystem. For $1 < m \leqslant n$, according to Shannon decomposition formula (10) applied to last element m:

$$\begin{aligned}\varphi^{(m)}(\boldsymbol{X}^{(m)}) &= \sum_{r \in \boldsymbol{K}_m} \mathbf{1}(X_m = r)\varphi^{(m)}(r_{(m)}, \boldsymbol{X}^{(m)})\\ &= \sum_{r \in \boldsymbol{K}_m} \mathbf{1}(X_m = r)[\varphi^{(m-1)}(\mathbf{X}^{(m-1)}) + r].\end{aligned}$$

If $X_m \geqslant d$, then obviously $\varphi^{(m)}(\mathbf{X}^{(m)}) \geqslant d$. Hence for $1 < m \leqslant n$:

$$\varphi_d^{(m)}(\underline{\mathbf{X}}^{(m)}) = X_m(d) + \sum_{\substack{r\in\mathbf{K}_m:\\ r<d}} (X_m(r) - X_m(r \oplus_m 1))\varphi_{d-r}^{(m-1)}(\underline{\mathbf{X}}^{(m-1)}),$$

$$A^{(m)}(d) = A_m(d) + \sum_{\substack{r\in\mathbf{K}_m:\\ r<d}} (A_m(r) - A_m(r \oplus_m 1))A^{(m-1)}(d-r),$$

$$w^{(m)}(d) = w_m(d) + \sum_{\substack{r\in\mathbf{K}_m:\\ r<d}} (w_m(r) - w_m(r \oplus_m 1))A^{(m-1)}(d-r)$$
$$+ \sum_{\substack{r\in\mathbf{K}_m:\\ r<d}} (A_m(r) - A_m(r \oplus_m 1))w^{(m-1)}(d-r)$$

with the boundary conditions $\varphi_c^{(1)}(\underline{\mathbf{X}}^{(1)}) = X_1(k(1,c)) = X_1(c)$, $A^{(1)}(c) = A_1(c) = A_1(k(1,c))$ and $w^{(1)}(c) = w_1(c) = w_1(k(1,c))$ for any c.

As an example, let us consider a parallel system with flow dispersion composed of three elements with $\mathbf{K}_1 = \{0,2,4,6\}$, $\mathbf{K}_2 = \{0,2,6\}$, $\mathbf{K}_3 = \{0,4,8\}$. The state space of the system is $\mathbf{K} = \{0,2,4,...,20\}$. For the desired fixed system performance level $d = 10$, we have:

$$\begin{aligned}
A(10) &= A^{(3)}(10) = A_3(10) + (A_3(0) - A_3(4))A^{(2)}(10-0)\\
&\quad +(A_3(4) - A_3(8))A^{(2)}(10-4) + (A_3(8) - A_3(9))A^{(2)}(10-8)\\
&= (1 - A_3(4))A^{(2)}(10) + (A_3(4) - A_3(8))A^{(2)}(6) + A_3(8)A^{(2)}(2),\\
A^{(2)}(10) &= A_2(10) + (A_2(0) - A_2(2))A^{(1)}(10-0)\\
&\quad +(A_2(2) - A_2(6))A^{(1)}(10-2) + (A_2(6) - A_2(7))A^{(1)}(10-6)\\
&= (1 - A_2(2))A^{(1)}(10) + (A_2(2) - A_2(6))A^{(1)}(8) + A_2(6)A^{(1)}(4)\\
&= A_2(6)A_1(4),\\
A^{(2)}(6) &= A_2(6) + (1 - A_2(2))A_1(6) + (A_2(2) - A_2(6))A_1(4),\\
A^{(2)}(2) &= A_2(2) + (1 - A_2(2))A_1(2).
\end{aligned}$$

Hence:

$$\begin{aligned}
A(10) &= (1 - A_3(4))A_2(6)A_1(4)\\
&\quad +(A_3(4) - A_3(8))[A_2(6) + (1 - A_2(2))A_1(6) + (A_2(2) - A_2(6))A_1(4)]\\
&\quad +A_3(8)[A_2(2) + (1 - A_2(2))A_1(2)].
\end{aligned}$$

For the failure frequency:

$$\begin{aligned}
w(10) &= w^{(3)}(10) = w_3(10) + (w_3(0) - w_3(4))A^{(2)}(10-0)\\
&\quad +(w_3(4) - w_3(8))A^{(2)}(10-4) + (w_3(8) - w_3(9))A^{(2)}(10-8)\\
&\quad +(A_3(0) - A_3(4))w^{(2)}(10-0) + (A_3(4) - A_3(8))w^{(2)}(10-4)
\end{aligned}$$

$$
\begin{aligned}
&+(A_3(8)-A_3(9))w^{(2)}(10-8)\\
&=-w_3(4)A^{(2)}(10)+(w_3(4)-w_3(8))A^{(2)}(6)+w_3(8)A^{(2)}(2)\\
&\quad+(1-A_3(4))w^{(2)}(10)+(A_3(4)-A_3(8))w^{(2)}(6)+A_3(8)w^{(2)}(2),\\
&w^{(2)}(10)=w_2(10)+(w_2(0)-w_2(2))A^{(1)}(10-0)\\
&\quad+(w_2(2)-w_2(6))A^{(1)}(10-2)\\
&\quad+(w_2(6)-w_2(7))A^{(2)}(10-6)+(A_2(0)-A_2(2))w^{(1)}(10-0)\\
&\quad+(A_2(2)-A_2(6))w^{(1)}(10-2)+(A_2(6)-A_2(7))w^{(1)}(10-6)\\
&=-w_2(2)A^{(1)}(10)+(w_2(2)-w_2(6))A^{(1)}(8)+w_2(6)A^{(1)}(4)\\
&\quad+(1-A_2(2))w^{(1)}(10)+(A_2(2)-A_2(6))w^{(1)}(8)+A_2(6)w^{(1)}(4)\\
&=w_2(6)A^{(1)}(4)+A_2(6)w^{(1)}(4)=w_2(6)A_1(4)+A_2(6)w_1(4),\\
&w^{(2)}(6)=w_2(6)-w_2(2)A_1(6)+(w_2(2)-w_2(6))A_1(4)\\
&\quad+(1-A_2(2))w_1(6)+(A_2(2)-A_2(6))w_1(4),\\
&w^{(2)}(2)=w_2(2)(1-A_1(2))+(1-A_2(2))w_1(2).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&w(10)=-w_3(4)A_2(6)A_1(4)+(w_3(4)-w_3(8))[A_2(6)\\
&\quad+(1-A_2(2))A_1(6)+(A_2(2)-A_2(6))A_1(4)]\\
&\quad+w_3(8)[A_2(2)+(1-A_2(2))A_1(2)]\\
&\quad+(1-A_3(4))[w_2(6)A_1(4)+A_2(6)w_1(4)]+(A_3(4)-A_3(8))[w_2(6)\\
&\quad-w_2(2)A_1(6)+(w_2(2)-w_2(6))A_1(4)+(1-A_2(2))w_1(6)\\
&\quad+(A_2(2)-A_2(6))w_1(4)]+A_3(8)[w_2(2)(1-A_1(2))+(1-A_2(2))w_1(2)].
\end{aligned}
$$

6. Systems with random demand rate

Many real technical systems operate under demand randomly changing in time. Examples of such systems are power generating systems, transportation systems, distributed computer networks and production systems. We refer to Levitin (2005) and Lisnianski and Levitin (2003) for more examples and further discussion.

Let $D(t)$ be the demand rate at time t. The fixed (time-independent) demand rate $D(t) \equiv d$ was considered in Section 4. Now we consider randomly changing in time demand rate $\{D(t)\}$. We assume that the process $\{D(t)\}$ takes its state in finite set $\mathbf{D} \subseteq [0, \infty)$ and that it satisfies Assumptions 2 and 3. The system with performance process $X(t) = \varphi(\mathbf{X}(t))$ is operating at time t, if $X(t) \geqslant D(t)$. Otherwise the system is failed. It is assumed that the processes $\{X(t)\}$ and $\{D(t)\}$ are independent. We show how to apply the results of Section 4 to the case of randomly changing demand.

Let the demand states in $\mathbf{D}$ be indexed in decreasing order:

$$0 \leqslant d(m) < d(m-1) < ... < d(1), \quad m \geqslant 1.$$

Let $\mathbf{H} = \{1, 2, ..., m\}$ be the index set of demand levels and let $L(t)$ be the index of demand level at time t, so that $D(t) = d(L(t))$. Define a function $\psi : \mathbf{H} \times \mathbf{K} \to \{0, 1\}$ by:

$$\psi(k, x) = \mathbf{1}(x \geqslant d(k)).$$

Since $d(k)$ is decreasing in k, $\psi(k, x)$ is a monotone increasing binary structure, which can be considered as the structure function of a binary system consisting of two multi-state elements. The first element corresponds to the demand level index and its stochastic behaviour is described by $\{L(t)\}$. The second element corresponds to the original system with stochastic behaviour described by $\{X(t)\}$:

$$\psi(L(t), X(t)) = \mathbf{1}(X(t) \geqslant d(L(t))).$$

Applying Shannon's decomposition formula (10) with respect to the first "element" L, we have:

$$\psi(L(t), X(t)) = \sum_{k=1}^{m} \mathbf{1}(L(t) = k)\mathbf{1}(X(t) \geqslant d(k))$$
$$= \sum_{k=1}^{m} (L(k, t) - L(k+1, t))X(d(k), t)),$$

where $L(k, t) = \mathbf{1}(L(t) \geqslant k)$, $L(m+1, t) \equiv 0$ and $X(c, t) = \mathbf{1}(X(t) \geqslant c)$.

Let $w_L^{j \to l}(t)$ be the frequency of transitions of the process $\{L(t)\}$ from state j to state l at time t. Then we can apply the factoring formulae (15) and (26) to obtain availability $A(t)$, the failure frequency $w(t)$ and the repair frequency $v(t)$ of the system operating under random demand $\{D(t)\}$:

$$A(t) = \sum_{k=1}^{m} \Pr\{L(t) = k\} A(d(k), t) = \sum_{d \in \mathbf{D}} \Pr\{D(t) = d\} A(d, t),$$

$$w(t) = \sum_{k=1}^{m} (w_L(k, t) - w_L(k+1, t)) A(d(k), t) + \sum_{k=1}^{m} \Pr\{L(t) = k\} w(d(k), t), \tag{27}$$

$$v(t) = \sum_{k=1}^{m} (v_L(k, t) - v_L(k+1, t))\, A(d(k), t) + \sum_{k=1}^{m} \Pr\{L(t) = k\} v(d(k), t), \tag{28}$$

where

$$w_L(k, t) = w_L^{\{k, k+1, \ldots, m\} \to \{1, 2, \ldots, k-1\}}(t) = \sum_{j=k}^{m} \sum_{l=1}^{k-1} w_L^{j \to l}(t) \tag{29}$$

with $w_L(1,t) \equiv w_L(m+1,t) \equiv 0$;

$$v_L(k,t) = w_L^{\{1,2,\dots,k-1\}\to\{k,k+1,\dots,m\}}(t) = \sum_{j=1}^{k-1}\sum_{l=k}^{m} w_L^{j\to l}(t) \tag{30}$$

with $v_L(1,t) \equiv v_L(m+1,t) \equiv 0$.

After some algebra, using relations (29) and (30), we can express $w(t)$ and $v(t)$ by the following formulae:

$$w(t)=\sum_{j=2}^{m}\sum_{l=1}^{j-1} w_L^{j\to l}(t)[A(d(j),t)-A(d(l),t)]+\sum_{k=1}^{m}\Pr\{L(t)=k\}w(d(k),t) \tag{31}$$

$$v(t)=\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} w_L^{j\to l}(t)[A(d(l),t)-A(d(j),t)]+\sum_{k=1}^{m}\Pr\{L(t)=k\}v(d(k),t) \tag{32}$$

which in turn can also be obtained from general relations (16) and (17).

The failure [repair] frequency of system with variable demand rate has two contributors, designated by $w^{(L)}(t)$ and $w^{(X)}(t)$ [$v^{(L)}(t)$ and $v^{(X)}(t)$]:

1. $w^{(L)}(t)$ and $v^{(L)}(t)$ are related to failures caused by changes of demand rate, and correspond to the first (double) sum in equations (27), (31) and (28), (32) respectively, and
2. $w^{(X)}(t)$ and $v^{(X)}(t)$ are related to failures caused by changes of the state of the system of elements, and correspond to the second sum in these equations.

Of course, for the steady state ($t = \infty$), the failure and repair frequencies, and their two separate contributors as well, coincide:

$$w(\infty) = v(\infty), \quad w^{(L)}(\infty) = v^{(L)}(\infty), \quad w^{(X)}(\infty) = v^{(X)}(\infty).$$

Notice that the results presented in this section also include, as a special case, the random demand, which does not change in time: $D(t) \equiv D$ and consequently, $L(t) \equiv L$ (D and L are just random variables). Then all $w_L^{j\to l}(t)$ are equal to 0, and thus the demand related contributors $w^{(L)}(t) \equiv v^{(L)}(t) \equiv 0$, i.e. the first double sum in each right-hand side of each equation (27), (28), (31) and (32) disappear.

7. Conclusions

New general formula for the failure/repair frequency of a multi-state monotone system was derived in the paper. Using this formula simple conversion rules from an availability or unavailability expression into an expression for failure/repair frequency, were obtained. These rules generalise the results known from the binary system theory. All the results presented in the paper hold true for both instantaneous and steady state cases.

Since both input and output data have similar format (availabilities/unavailabilities and failure/repair frequencies), the conversion rules can be used in a recursive manner, allowing a subsystem-by-subsystem analysis of complex systems with hierarchical structure. Moreover, using the conversion rules, it is quite easy to extend some existing algorithms for availability calculation of an MMS (see Aven, 1985) in order to include failure/repair frequency.

As further investigations in the area, we may mention:

1. developing other efficient algorithms, considering, for example, application of the Universal Generating Function (UGF) technique, Levitin (2005), and Ordered Binary Decision Diagram technique, Chang et al. (2004), Rauzy (1996);
2. considering some statistical dependencies among system's elements (e.g. common-cause failures), and between demand rate and elements performance processes as well;
3. obtaining approximations useful for analysing very complex and large systems;
4. generalisation to multi-state systems which are not necessarily monotone (though in this case not so simple conversion rules are expected).

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