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Towards historical roots of necessary conditions of optimality: Regula of Peano[†]

by

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Abstract: At the end of 19th century Peano discerned vector spaces, differentiability, convex sets, limits of families of sets, tangent cones, and many other concepts, in a modern perfect form. He applied these notions to solve numerous problems. The theorem on necessary conditions of optimality (*Regula*) is one of these. The formal language of logic that he developed, enabled him to perceive mathematics with great precision and depth. Actually he built mathematics axiomatically based exclusively on logical and set-theoretic primitive terms and properties, which was a revolutionary turning point in the development of mathematics.

Keywords: differentiability, convex set, Kuratowski limit of sets, tangent cones, condition of optimality, mean value theorem, Peano remainder, Peano's regula, strict derivability, Peano's Formulario Mathematico, Peano's Geometric Calculus, Grassmann's Ausdehnungslehre.

1. Introduction

The aim of this paper is to trace back the evolution of mathematical concepts in the work of Giuseppe Peano (1858-1932) that are constituents of *Regula*, that is, Peano's theorem on necessary conditions of optimality.

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Well-known necessary conditions of maximality of a function at a point, are formulated in terms of derivative of the function and of tangent cone of the constraint at that point. Consider a real-valued function $f : X \to \mathbb{R}$, where X is a Euclidean affine space, and a subset A of X

Regula (of Optimality) *If f is differentiable at $x \in A$ and $f(x) = \max\{f(y) : y \in A\}$, then*

$$\langle Df(x), y - x\rangle \leq 0 \text{ for every } y \in \mathrm{Tang}(A, x). \tag{1}$$

The *derivative* $Df(x)$ is defined to be *the* vector $Df(x)$ such that

$$\lim_{y\to x} \frac{f(y) - f(x) - \langle Df(x), y - x\rangle}{|y - x|} = 0. \tag{2}$$

The *affine tangent cone* $\mathrm{Tang}(A, x)$ of A at x (for arbitrary $x \in X$) is given by

$$\mathrm{Tang}(A, x) := \mathrm{Ls}_{\lambda\to+\infty} (x + \lambda(A - x)), \tag{3}$$

where the *upper limit* $\mathrm{Ls}_{\lambda\to+\infty} A_\lambda$ of sets A_λ (as λ tends to $+\infty$) is defined by

$$\mathrm{Ls}_{\lambda\to+\infty} A_\lambda := \{y \in X : \liminf_{\lambda\to+\infty} \mathrm{d}(y, A_\lambda) = 0\}. \tag{4}$$

Of course, d is the distance and in (3), $A_\lambda := x + \lambda(A - x) := \{x + \lambda(a - x) : a \in A\}$.

It is generally admitted among those who study optimization, that modern definition of differentiability was introduced by Fréchet (1911a), of tangent cone by Bouligand (1932), and of limit of sets by Painlevé (see Zoretti, 1905, p. 8)[1]. So we were very surprised to discover that *Regula* was already known by Peano in 1887. Indeed, in order to formulate it, one needed to possess the notions of differentiability and of affine tangent cone, hence also that of limit of sets.

But our surprise was even greater, because not only all these notions were familiar to Peano at the end of the 19$^{\text{th}}$ century, but they were formulated in a rigorous, mature way of today mathematics, in contrast with the approximated imprecise style that dominated in mathematical writings of those times, and often persisted during several next decades.

Impressed by the so early emergence of these notions, we started to peruse the work of Peano in order to understand the evolution of the ideas that lead to *Regula*.

[1] A Painlevé's student, Zoretti (1880-1948), attests in Zoretti (1912, p. 145) that Painlevé introduced both upper and lower limit of a family of sets. Following Zoretti, Hausdorff (1927, p. 280) and Kuratowski (1928, p. 169) reiterate this attribution to Painlevé. More clearly, Zoretti calls "*set-limit*" of a sequence of sets the today's upper limit; while by "*point-limit*" he means "point belonging to the lower limit" of the sequence. Painlevé's use of the notion of "*set-limit*" is dated 1902 in Zoretti (1905, p. 8); on the other hand, one finds in Zoretti (1909, p. 8) the first occurrence of the notion of "*point-limit*", without any reference to Painlevé. Both "*set-limit*" and "*point-limit*" are present also in Zoretti (1912, p. 145).

All the citations of Peano's activity, related to the concepts involved in the optimality conditions, prevalently concern the span of time between the first appearance of *Regula* in "**Applicazioni Geometriche**"(1887) and its ultimate form in "**Formulario Mathematico**"[2] (1908, p. 335), where *Regula* is stated exactly as above.

Tracing back the development and applications of differentiability, tangency, limit and other concepts, in the work of Peano over the years, we see evolution and enrichment of their facets. Peano built mathematics axiomatically, based exclusively on logical and set-theoretic primitive terms and properties. This was a revolutionary turning point in the development of mathematics. The reduction of every mathematical object to the founding concept of "set" (*genus supremum*) of Cantor, enabled the emergence of new concepts related to properties of sets, unconceivable otherwise. Early and illuminating examples of the fecundity of Cantor's views are in the books *Fondamenti per la teorica delle funzioni di variabili reali* of Dini (1878), *Calcolo differenziale e integrale* of Genocchi and Peano (1884), and the second edition of *Cours d'Analyse* of Jordan (1893-96).

To appreciate the novelty of Cantor's approach to mathematics, we should remember the opposition of some luminaries of mathematics that existed at the beginning of the twentieth century. For example, in the address to the Congress of Mathematicians in Rome in 1908, Poincaré (1909, p. 182) said

> Quel que soit le remède adopté [contre le "cantorisme"], nous pouvons nous promettre la joie du médecin appelé à suivre un beau cas pathologique.

This paper is not a definitive word on historical roots of conditions of optimality. For instance, a comparison of (1) with the virtual work principle has still to be addressed[3]. We have found no evidence of this relationship in the work of Peano, but it is plausible that he was aware of it (remark that *Regula* is placed in *Formulario Mathematico* within the context of mechanics).

This article concerns several historical aspects. From a methodological point of view, we are focused on primary sources, and not on secondary founts, that is, on mathematical facts, and not on opinions or interpretations of other scholars of history of mathematics. On the other hand, we will avoid to mention, if not necessary, historical facts that are well-known among those who study optimization (see, for example, Rockafellar and Wets, 1997; Borwein and Lewis, 2000;

[2]The previous four editions of *Formulario Mathematico* are *Formulaire Mathématique* tome 1 (1895), tome 2 (1899), tome 3 (1901) and tome 4 (1903). The first half of the fifth edition was printed in 1905; the other half in 1908. The "Index and Vocabulary" to *Formulario Mathematico* of 1908 was published separately in 1906.

[3]If a force acting on a material point in equilibrium x, has a potential f, then the virtual work principle states

$$\delta L := \langle Df(x), \delta x\rangle \leq 0 \text{ for each } \delta x,$$

where δx is a virtual displacement of that point with respect to an *ideal constraint* A (either bilateral or unilateral) independent of the time (see, for example, Banach, 1951).

Aubin, 2000; Aubin and Frankowska, 1990; Hirriart-Urruty and Lemaréchal, 1996; Pallaschke and Rolewicz, 1997).

2. Affine and vector spaces

Applicazioni Geometriche is based on the extension theory (*Ausdehnungslehre*, 1844 edition) of Grassmann (1894-1911), presented in detail in Peano (1888), where, forgoing the philosophical aura founding the work of Grassmann, Peano introduces the modern notion of vector space.

In Grassmann's work, points and vectors coexist distinctly in a common structure, together with other objects, like exterior products of points and vectors (see Greco and Pagani, 2007). This subtle distinction was very demanding in comparison with today habits of mathematicians. Peano maintains the distinction. For instance, a difference $y - x$ is a vector if both y and x are either points or vectors; otherwise, it is a point (if x is a vector and y is a point) or a point of mass -1 (if y is vector and x is a point).

Moreover, Peano follows Grassmann in construction of metric concepts from the scalar product of vectors (introduced by Grassmann, 1847). Following Grassmann and Hamilton, he conceives the gradient of a function as a vector, differently from a common habit (of using the norm of the gradient) that prevailed at the pre-vectorial epoch [4].

In several papers Peano applies the geometric calculus of Grassmann, for instance, to define area of a surface (Peano, 1887, p. 164, and 1890a) and to give in Peano (1897-8) an axiomatic refoundation (today standard) of Euclidean geometry, based on the primitive notions of point, vector and scalar product.

Peano's approach to the definition of linear map was slightly different from (but equivalent to) that commonly adopted nowadays. Peano says that a map g between spaces is *linear* if it is additive and bounded, that is, $g(x + y) = g(x) + g(y)$ for all x and y, and if $\sup\{|g(x)| : |x| < 1\}$ is finite. The reader has certainly observed that the today condition of *homogeneity* is substituted by that of *boundedness* [5]. For Peano, the interest of employing boundedness in the definition, was to obtain simultaneously a concept of *norm* (*module* in his terminology) on spaces of linear maps.

The norm was useful in his study of systems of linear differential equations, Peano (1888b); to give a formula for a solution in terms of resolvent, he defines the exponential of matrix and proves its convergence using the norm (see also Peano, 1894, and the English translation of Peano, 1888, in Birkhoff, 1973).

[4] These observations are relevant for the understanding of Peano's interpretation of the formula $\langle Df(x), y - x\rangle \leq 0$ that appears in *Regula*.

[5] In other moments (for example, in 4$^{\text{th}}$ edition of *Formulaire Mathématique* (Peano, 1903, p. 203) Peano adopts a different (but equivalent) definition of linearity, replacing *boundedness* with *continuity*. All these variants are related to the following fundamental lemma (see a proof in Peano, 1908, pp. 117-118, where Peano quotes Darboux, 1880, footnote of p. 56):"*For an additive function $f : \mathbb{R}^n \to \mathbb{R}$ are equivalent: (1) homogeneity, (2) continuity and (3) boundedness on bounded sets*".

As other new theories, the theory of vector spaces was contested by many prominent mathematicians. Even those (few) who adopted the vector approach, were not always entirely acquainted with its achievements. To perceive the atmosphere of that time, we give an excerpt from the introduction of Goursat to *Leçons de géométrie vectorielle* of Bouligand (1924):

> Si le calcul vectoriel a été un peu lent à pénétrer en France, il est bien certain que la multiplicité des notations et l'abus du symbolisme ont justifié en partie la défiance de nos étudiants. Or, dans le livre de M. Bouligand, le symbolisme est réduit au minimum, et l'auteur n'hésite pas à revenir aux procédés habituels du calcul quand les méthodes lui paraissent plus directes. [...]
>
> M. Bouligand a devisé son ouvrage en trois parties, consacrées respectivement aux opérations vectorielles en géométrie linéaire, en géométrie métrique et aux opérations infinitésimales.

A decisive role in the dissemination of vector spaces had a book *Space-Time-Matter* (Weyl, 1918, for details see Zaddach, 1988, 1994).

3. Differentiability

In *Applicazioni Geometriche* Peano (1887, p. 131) says that a vector $\mathbf{u}$ is a *derivative* at a point x of a real-valued function f defined on a finite-dimensional Euclidean affine space X, if there exists a vector $\varepsilon(y)$ such that

$$f(y) - f(x) = \langle y - x, \mathbf{u} + \varepsilon(y)\rangle \text{ with } \lim_{y\to x} \varepsilon(y) = 0. \tag{5}$$

The reader recognizes in (5) the Taylor formula of order 1 and, on the other hand, the characterization of derivability, given in Carathéodory (1964, p. 119): "f is derivable at x if there is a function φ continuous at x such that $f(y) - f(x) = \langle y - x, \varphi(y)\rangle$ for every y" [6].

In *Formulario Mathematico* (Peano, 1908, p. 334 and 330) the derivative $\mathbf{u}$ is denoted by $Df(x)$ and is defined by (2) and, more generally, one finds a definition of differential of map between finite-dimensional Euclidean vector spaces, namely if $f : \mathbb{R}^m \to \mathbb{R}^n$ then a *derivative* of f at x is *the* linear map $Df(x) : \mathbb{R}^m \to \mathbb{R}^n$ (referred to as *Jacobi-Grassmann derivative* by Peano, 1908, p. 455, and called nowadays the *Fréchet derivative* of f at x) such that

$$\lim_{y\to x} \frac{f(y) - f(x) - Df(x)(y-x)}{|y-x|} = 0. \tag{6}$$

In giving this definition, Peano refers to the second edition of *Ausdehnungslehre* of Grassmann (1894-1911, vol. 2, p. 295) and to an article *De determinantibus functionalibus* (1841) of Jacobi (1881-91, vol. 3, p. 421). Actually, the citation of Jacobi refers to the concept of *Jacobian.*

[6] Remarkably, this Carathéodory reformulation "leads to some sharp, concise proofs of important theorems: chain rule, inverse function theorem, ..." (see Kuhn, 1991) and "makes perfect sense in general linear topological spaces" (see Acosta and Delgado, 1994).

With respect to Peano's quotation of Grassmann, our verification of the source leads the following facts. For a map $f : \mathbb{R}^m \to \mathbb{R}^n$, Grassmann defines a *differential* $df(x)$ at x, as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$, by

$$df(x)(v) := \lim_{q\to 0} \frac{f(x+q\,v)-f(x)}{q}. \tag{7}$$

Grassmann proves that if the differential $df(x)$ exists at every x and is radially continous[7] in variable x for each fixed $v \in V$, then f is radially continuous, the differential is linear in v, and that (7) becomes the partial derivative when v is an element of the canonical base. Moreover, he claims that the chain rule holds. In contrast to the comments of Kannenberg (author of English translation of *Ausdehnungslehre*, see Grassman, 2000, p. 398), this claim is, however, false, as can be seen from Acker and Dickstein example (1983, Ex. 2.5, p. 26)[8].

In various moments of his activity, Peano studied the concept of derivative. His contributions are very rich and diversified, and concern

(1) strict derivative (see Peano, 1884a,b, and 1892)[9],
(2) Taylor formula with infinitesimal remainder (called *Peano remainder*) (see Genocchi, 1884 p. XIX, or Stolz, 1893, pp. 90-91),
(3) asymptotic development and Taylor formula (Peano, 1891)[10],
(4) derivation of measures (Peano, 1887, p. 169, see also Greco, 2007) and
(5) mean value theorem.

Here is an excerpt concerning the latter. In *Calcolo Geometrico* Peano (1888a) gives a *mean value theorem* for vector-valued functions f of one variable[11], that is, if f has an $(n+1)$-derivative $f^{(n+1)}$ on $[t, t+h]$, then there exists an element k of the closed convex hull of the image of interval $[t, t+h]$ by $f^{(n+1)}$ such that

$$f(t+h) = f(t) + hf'(t) + \cdots + \frac{h^n}{n!} f^{(n)}(t) + \frac{h^{n+1}}{(n+1)!} k. \tag{8}$$

[7] A map g is *radially continuous at* x if for every vector v the map $h \longmapsto g(x+hv)$ is continuous at 0. Grassmann adopts the term *continuous* to denote "radially continuous".

[8] A counterexample to Grassmann's claim is given by the functions f and g of two real variables defined by $g(x,y) := (x, y^2)$, $f(x,y) := \frac{x^3 y}{x^4+y^2}$ and $f(0,0) := 0$. These functions were used in Acker and Dickstein (1983) to invalidate a similar claim for Gateaux differentiability.

[9] Peano (1884a) observes the equivalence between continuity of derivative f' of f at x and "strict derivability of f at x" (that is, $\lim_{a,b\to x} \frac{f(b)-f(a)}{b-a} = f'(x)$). Peano (1884b) notices that the uniform convergence of the difference quotient function $\frac{f(x+h)-f(x)}{h}$ in variable x to $f'(x)$ (as h tends to 0) amounts to continuity of derivative f' in variable x. As observed in Mawhin (1997, p. 430), Peano formulates an approximation property of primitives equivalent to Kurzweil integrability of all the functions having a primitive.

[10] If f is a function, and $P(h) = a_0 + a_1 h + \cdots + a_n h^n$ is a polynomial function such that $f(x+h) - P(h) = h^n \eta(h)$ where $\eta(h)$ tends to 0 with h, then the *Peano derivative of order* n is $n! a_n$ (see, for example, Weil, 1995, Svetic and Volkmer, 1998). Peano gives an example of function that is discontinuous in every neighborhood of x, and for which the Peano derivatives of all order exist.

[11] Peano gives a proof (by scalarization) in Peano (1895-6, p. 975).

Here is another surprise in terms of the first appearance of modern notions of *convex set*[12] and *convex hull*, as we thought that it was Minkowski (1896) who introduced these concepts for the first time.

Among the first who studied the modern notion of differentiability of functions of several variables, were Stolz (1893, §.IV 8, p. 130), Pierpont (1905, p. 269), W.H. Young (1910, p. 21). Besides, Maurice Fréchet (1911) [13] defined the *differential* (of a function of two variables) as the linear part of the approximation

$$f(a+h, b+k) - f(a, b) = hp + kq + h\rho(h, k) + k\sigma(h, k), \tag{9}$$

where ρ and σ tend to 0 with h and k.

Peano's definition liberates the concept of derivative from the coordinate system and from partial derivatives. The definitions (5)-(6) are coordinate-free in contrast with the predominant habit of the epoch.

We have found no clear evidence in the mathematical literature of an acknowledgement of Peano's definition of derivative, with the exception of the paper by Wilkosz (1921), where Peano is cited jointly with Stolz.

4. Limits of variable sets

In *Applicazioni Geometriche*, Peano (1887, p. 30) introduces a notion of limit of straight lines, planes, circles and spheres (that depend on parameter). He considers these objects as sets, which leads him to extend the definition of limit to variable figure (in particular, curves and surfaces).

A *variable figure* (or *set*) is a family, indexed by the reals, of subsets A_λ of an affine Euclidean space X. Peano (1887, p. 302) defines the *lower limit of a variable figure* by

$$\mathrm{Li}_{\lambda\to+\infty} A_\lambda := \{y \in X : \lim_{\lambda\to+\infty} \mathrm{d}(y, A_\lambda) = 0\}. \tag{10}$$

In the edition of *Formulario Mathematico* (Peano, 1908, p. 237) we find the lower limit together with a definition of *upper limit of a variable figure*:

$$\mathrm{Ls}_{\lambda\to+\infty} A_\lambda := \{y \in X : \liminf_{\lambda\to+\infty} \mathrm{d}(y, A_\lambda) = 0\}, \tag{11}$$

which we have already seen in (4). Besides, he writes down (p. 413) the upper limit as

$$\mathrm{Ls}_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \mathrm{cl} \bigcup_{k\geq n} A_k, \tag{12}$$

[12] Peano employs the concept of convex set for the first time in axiomatic foundation of geometry (Peano, 1889, p. 90, Axioma XVII); more precisely, his *Axiom XVII of continuity* states: *Let A be a convex set of points, and let x, y points such that $x \in A$ and $y \notin A$. Then there exists a point $w \in xy$ (the open segment between x and y) such that $xw \subset A$ and $wy \cap A = \varnothing$.*

[13] A month after the publication of Fréchet (1911a), in which he presented the concept of differentiability, he publishes the note Fréchet (1911b) in order to recognize the priority of Young.

where the *closure* $\mathrm{cl}\, A$ of a set A, is defined (p. 177) by[14]

$$\mathrm{cl}\, A := \{y \in X : \mathrm{d}(y, A) = 0\}. \tag{13}$$

In several papers, Peano analyzes the meanings that are given in mathematics to the word *limit* (see, for example, Peano, 1894b): least upper bound, greatest lower bound of a set, (usual) limit and adherence of sequences and functions.

Peano conceives the "upper limit of variable sets" as a natural extension of the *adherence* of functions. He attributes to Cauchy the introduction of adherence, see Peano (1894b, p. 37) where he says:

> Selon la définition de la limite, aujourd'hui adoptée dans tous les traités, toute fonction a une limite seule, ou n'a pas de limite. [...]
>
> Cette idée plus générale de la limite [*the adherence*] est clairement énoncée par Cauchy; on lit en effet dans son Cours d'Analyse algébrique, 1821, p. 13: ≪Quelquefois...une expression converge à-la-fois vers plusieurs limites différentes les unes des autres≫, et à la page 14 il trouve que les valeurs limites de $\sin \frac{1}{x}$, pour $x = 0$, constituent l'intervalle de -1 à $+1$. Les auteurs qui ont suivi Cauchy, en cherchant de préciser sa définition un peu vague, se sont mis dans un cas particulier.

Peano studies the notion of "lower limit of variable sets", in particular, in a celebrated article on existence of solutions of a system of ordinary differential equations (see Peano, 1888). Peano carries on the proof of existence in a framework of logical and set-theoretic ideography, thanks to which he is able to detect the *axiom of choice*[15].

The awareness of the problem of "limits of variable sets" was present at the turn of the 20$^{\text{th}}$ century (for example see Manheim, 1964).[16] The book of Kuratowski (1948, p. 241)[17] has definitely propagated the concept of limit of variable sets.

[14] Peano defined closure, interior and boundary earlier (1887, pp. 152-158); later, these notions were introduced by Jordan (1893-96). Peano relates the closure with the concept of *closed set* of Cantor: *the closure of A is the least closed set including A.*

[15] Peano proves the existence of a solution with the aid of approximated solutions. In order to obtain a solution, he is confronted with a problem of non-emptiness of the lower limit of a sequence of subsets of a finite-dimensional Euclidean space. To this end, he needs to select an element from every set of the sequence. At that point he realizes that he would need to make *infinite arbitrary choices*, which, starting from the paper of Zermelo (1904) is called *Axiom of choice*. He avoids to apply a new axiom, which is not present in mathematical literature and, consequently, the tradition does not grant it. Instead, using the lexicographic order, he is able to construct a particular element of every set, because the sets of the sequence are compact.

[16] Borel (1903) (see also Manheim, 1964, p. 114) suggests a "promising" notion of limit of straight lines and of planes, that is, 16 years after the introduction of the limit of arbitrary sets in Peano(1887).

[17] Kuratowski, by his work, establishes the use of upper and lower limits in mathematics, that are called today *upper and lower Kuratowski limits.*

Among the first mathematicians who studied the limits of variable sets are Burali-Forti (1895)[18], Zoretti (1905, p. 8, 1909), Janiszewski (1911), Hausdorff (1914, p. 234)[19], Vietoris (1922)[20], Vasilesco (1925), Cassina (1926-27) and Kuratowski (1928).

5. Tangent cones

In *Applicazioni Geometriche*, Peano (1887, pp. 58, 116) gives a metric definition of tangent straight line and tangent plane, then reaches, in a natural way, a unifying notion: that of *affine tangent cone*:

$$\operatorname{tang}(A,x) := \operatorname{Li}_{\lambda\to+\infty}(x+\lambda(A-x)). \tag{14}$$

Later, in *Formulario Mathematico* (Peano, 1908, p. 331), he introduces another type of tangent cone, namely

$$\operatorname{Tang}(A,x) := \operatorname{Ls}_{\lambda\to+\infty}(x+\lambda(A-x)). \tag{15}$$

To distinguish the two notions above, we shall call the first *lower affine tangent cone* and the second *upper affine tangent cone* [21]. Peano lists several properties of the upper tangent cone. If A is a subset of a Euclidean affine space X, then

(1) If $x \notin \operatorname{cl} A$ then $\operatorname{Tang}(A,x) = \varnothing$;
(2) If x is isolated in A then $\operatorname{Tang}(A,x) = \{x\}$;
(3) If $x \in \operatorname{cl}(A\setminus\{x\})$ then $\operatorname{Tang}(A,x) \neq \varnothing$;
(4) If $x \in \operatorname{int} A$ then $\operatorname{Tang}(A,x) = X$;
(5) If $y \in \operatorname{Tang}(A,x)\setminus\{x\}$ then $x+\mathbb{R}_+(y-x) \subset \operatorname{Tang}(A,x)$;

[18] Burali-Forti studies only lower limits.

[19] In the celebrated *Grundzüge der Mengelehre* Hausdorff (1914) studies both upper and lower limits. Moreover, he defines a metric on the set of bounded subsets of a metric space X (*Hausdorff distance*) and proves that the related convergence of bounded subsets $\{A_\lambda\}_\lambda$ to A (as $\lambda\to+\infty$) is equivalent to $\operatorname{Ls}_{\lambda\to+\infty} A_\lambda \subset A \subset \operatorname{Li}_{\lambda\to+\infty} A_\lambda$, if X is compact.

[20] Kuratowski limits, Hausdorff distance and Vietoris's topology (see Reitberger, 2002, p. 1234) are milestones in the search of notions of limit of variable sets.

[21] Of course, $\operatorname{tang}(A,x)$ is defined with the aid of lower limit, while $\operatorname{Tang}(A,x)$ with the aid of the upper limit of the same homothetic sets. Hence, $\operatorname{tang}(A,x) \subset \operatorname{Tang}(A,x)$. For the covenience of the reader, in order to compare the two definitions, we give their alternative descriptions in terms of limits of sequences:

$$\operatorname{tang}(A,x) = x + \left\{ v : \exists \{x_n\}_n \subset A \text{ such that } x = \lim_{n\to\infty} x_n \text{ and } v = \lim_{n\to\infty} \frac{x_n - x}{1/n} \right\}; \tag{16}$$

$$\operatorname{Tang}(A,x) = x + \left\{ v : \exists \{\lambda_n\}_n \to 0^+,\ \exists \{x_n\}_n \subset A \text{ such that } x = \lim_{n\to\infty} x_n \text{ and } v = \lim_{n\to\infty} \frac{x_n - x}{\lambda_n} \right\}.$$

The second formula is standard, while we have never seen in the literature the first one, (16). We have not found in Peano's papers any example of set A, for which the cones above are different. Here is another, perhaps most intuitive, formula for the "lower" affine tangent cone:

$$\operatorname{tang}(A,x) = x + \{ v : \exists\gamma : [0,1] \to A \text{ such that } x = \gamma(0), \gamma'(0) \text{ exists and } v = \gamma'(0) \}. \tag{17}$$

Notice that $\operatorname{tang}(A,x) = \operatorname{Tang}(A,x)$ in case of differential manifold A (at x).

(6) If $A \subset B$ then $\mathrm{Tang}(A,x) \subset \mathrm{Tang}(B,x)$;
(7) $\mathrm{Tang}(A \cup B, x) = \mathrm{Tang}(A,x) \cup \mathrm{Tang}(B,x)$;
(8) $\mathrm{Tang}(\mathrm{Tang}(A,x),x) = \mathrm{Tang}(A,x)$.[22]

As usual, after abstract investigation of a notion, Peano considers significant special cases; he calculates the upper affine tangent cone in several basic figures (closed ball, curves and surfaces parametrized in a regular way).

Various types of tangent cones have been studied in the literature. Their definitions depend on variants of the limiting process. The most known contribution to the investigation of tangent cones is due to Bouligand (1932, p. 60). One can find a mention about other contributors in a paper of Fréchet (1937, p. 241):

> Cette théorie des "contingents et paratingents" dont l'utilité a été signalée d'abord par M. Beppo Levi, puis par M. Severi, mais dont on doit à M. Bouligand et à ses élèves d'avoir entrepris l'étude systématique.

The diffusion of the concept of tangent cone was due mainly to Saks (1937, pp. 262–263), who adopted the definition of Bouligand, and to Federer (1959, p. 433), who introduced it in a modern vector version: if $x \in A$ then define the *upper vector tangent cone*:

$$\mathrm{Tan}(A,x) := \{0\} \cup \left\{ u \neq 0 : \forall \varepsilon > 0,\ \exists y \in A,\ 0 < |y-x| < \varepsilon \text{ and } \left| \frac{y-x}{|y-x|} - \frac{u}{|u|} \right| < \varepsilon \right\}. \tag{18}$$

Federer does not give any reference of the origin of the definition (18)[23]. Notice that

$$\mathrm{Tang}(A,x) = x + \mathrm{Tan}(A,x). \tag{19}$$

Neither Whitney (1972, chap. 7), intruducing six variants of vector tangent cone among which one recognizes the upper vector tangent cone discussed above (18), cites anybody.

We can say that, as far as tangent cones are concerned, main references are, respectively, Bouligand in optimization theory, Ferderer in geometric measure theory and calculus of variations and Whitney in differential geometry. A rare

[22] One finds in the 4th edition of *Formulaire Mathématique*, Peano (1903, p. 296) the same definition of upper affine tangent cone and, besides, the eight properties (1)–(8). Besides, one finds both lower and upper limit of variable sets, Peano (1903, p. 289).

We have not found in any of five editions of *Formulario Mathematico* (= collection of logical and set-teoretical formulas) any other property on tangent cones. Today other fundamental properties are well-known: (1) $x \in \mathrm{cl}\, A \iff \mathrm{Tang}(A,x) \neq \emptyset \iff x \in \mathrm{Tang}(A,x)$; (2) $x \in \mathrm{cl}(A \setminus \{x\}) \iff \mathrm{Tang}(A,x) \setminus \{x\} \neq \emptyset$; (3) $\mathrm{Tang}(A,x) = \mathrm{Tang}(A \cap B, x)$, if $x \in \mathrm{int}\, B$; (4) $\mathrm{Tang}(A,x) = \mathrm{Tang}\big(\mathrm{cl}(A), x\big)$; finally, (5) $\mathrm{Tang}(A,x)$ is closed (because it is an upper limit of variable sets).

[23] The book of Saks (1937) is among bibliographic references in Federer 1959.

direct reference to Peano's definition is that of Guido Ascoli[24] (1952), who writes about Peano's work in Ascoli (1955, pp. 26-27):

> [...] il merito maggiore [...] specialmente delle *Applicazioni* [*Geometriche*], non sta tanto nel metodo usato, quanto nel contenuto; ché vi sono profusi, in forma così semplice da parere definitiva, idee e risultati divenuti poi classici, come quelli sulla misura degli insiemi, sulla rettificazione delle curve, sulla definizione dell'area di una superfice, sull'integrazione di campo, sulle funzioni additive d'insieme; ed altri che sono tuttora poco noti o poco studiati. Ci basti indicare tra questi il concetto di limite di una figura variabile, destinato a ricomparire, con altro nome di autore, quarant'anni dopo presso la scuola di "geometria infinitesimale diretta" del Bouligand, e l'originalissima definizione di "figura tangente ad un insieme in un punto", che ha fornito a chi scrive, or è qualche anno, la chiave di una difficile questione asintotica.

The *contingent cone* of Bouligand (1932) is defined by Saks (1937, p. 262) as follows: if x is an accumulation point of A, then the *contingent cone* of A at x is given by

$$\mathrm{Cont}(A, x) := \{l : l \text{ tangent half-line to } A \text{ at } x\} \tag{20}$$

where a *half-line* l issued from x is said to be *tangent to* A at x if there exist a sequence $\{x_n\}_n \subset A$ and a sequence of half-lines $\{l_n\}_n$ issued from x such that $x \neq x_n \in l_n, x = \lim x_n$ and the angle between l_n and l tends to 0.

Peano's upper affine tangent cone, Federer's upper vector tangent cone and Bouligand's contingent cone describe the same intuitive concept in terms, respectively, of points of affine space (via *blow-up*), of vectors (via *directions* of tangent half-lines) and of half-lines (via *limits* of half-lines)[25]. Finally, observe that the tangent cone is built on the notion of distance by Peano, of norm by Federer and of angle (consequently, of scalar product) by Bouligand.

---

[24] One should not confound Guido Ascoli(1887-1957) with Giulio Ascoli (1843-1896), the latter known because of the Ascoli-Arzelà theorem. Guido proved the geometric version, Ascoli (1933), of the Hahn-Banach theorem for separable normed spaces; a year later, Mazur (1933) proved it for arbitrary normed spaces.

[25] Bouligand, in spite of his knowledge of vector spaces (see, for example, Bouligand, 1924) and his introduction to the French translation of Weyl (1918), does not use vectors while defining the contingent cone. In the preface to Félix (1957) Bouligand appraises Peano's (1888a) *Calcolo Geometrico*: ≪Pour être moins incomplet, il faudrait encore citer l'exposé repris par Peano en 1886 [sic!] *du calcul extensif* de Grassmann, [et] l'article fondamental malgré sa brièveté paru en 1900 [sic!] dans l'Enseignement Mathématique au sujet des relations d'équivalence, rédigé par Burali-Forti (Sur quelques notions dérivées de la notion d'égalité et leurs applications dans la science).≫ Burali-Forti was Peano's assistant and friend; the article quoted by Bouligand is Burali-Forti (1899).

6. Maxima and minima

In *Applicazioni Geometriche* (1887, pp. 143-144) Peano analyzes the variation of a real-valued function around a point in a particular direction $\mathbf{p}$ in terms of the scalar product of the derivative at that point with $\mathbf{p}$.

THEOREM 6.1 *Let f be a real-valued function such that $Df(\bar{x}) \neq 0$. Let $\mathbf{p}$ be a unit vector and $\{x_n\}_n$ be a sequence so that*

$$\bar{x} = \lim_{n\to\infty} x_n \qquad \textit{and} \qquad \mathbf{p} = \lim_{n\to\infty} \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}. \tag{21}$$

$$\textit{If} \quad \langle Df(\bar{x}), \mathbf{p}\rangle > 0, \textit{ then } f(x_n) > f(x_0) \textit{ for almost all } n; \tag{22}$$

$$\textit{if} \quad \langle Df(\bar{x}), \mathbf{p}\rangle < 0 \textit{ then } f(x_n) < f(x_0) \textit{ for almost all } n. \tag{23}$$

Peano specifies that $\{x_n\}_n$ in Theorem 6.1 can be taken either arbitrarily or constrained by some conditions, for example, lying on a line or on a surface.

If $\{x_n\}_n$ is included in a set A, then $\mathbf{p}$ is one of the directions (unitary vectors) of the upper vector tangent cone of A at x. By taking all such sequences, we get all the directions of the upper vector tangent cone (18) of A at x. Hence, by relating (21) to upper vector tangent cone, Theorem 6.1 implies

THEOREM 6.2 (Regula **o**f Maximality) *If f is differentiable at $x \in A$ and $f(x) = \max\{f(y) : y \in A\}$, then$\langle Df(x), y - x\rangle \leq 0$ for every $y \in \mathrm{Tang}(A, x)$.*

THEOREM 6.3 (Regula **o**f Minimality) *If f is differentiable at $x \in A$ and $f(x) = \min\{f(y) : y \in A\}$, then$\langle Df(x), y - x\rangle \geq 0$ for every $y \in \mathrm{Tang}(A, x)$.*

One finds both Theorems 6.2 and 6.3 in Peano (1908, p. 335). It is worthwhile to note that Peano's use of *Regula* exhibits the normality of gradient with respect to the constraint.

Optimization problems were among principal interests of Peano. His research with regard to these problems was intense, continual and influential. The precision with which Peano studied maxima and minima was notorious.

Hancock, student of Weierstrass, is the author of a booklet: *Lectures on the theory of maxima and minima of functions of several variables. Weierstrass' theory* (1903). In the second edition of this book he says (Hancock, pp. iv-v):

> In the preface to the German translation by Bohlmann and Schepp of Peano's of *Calcolo differenziale e principii di calcolo integrale*, Professor A. Mayer [editor of Math. Annalen together with Felix Klein] writes that this book of Peano not only is a model of precise presentation and rigorous deduction, whose propitious influence has been unmistakably felt upon almost every calculus that has appeared (in Germany) since that time (1884), but by calling attention to old and deeply rooted errors, it has given an impulse to new and fruitful development.

> The important objection contained in this book [*Calcolo differenziale e principii di calcolo integrale*] (Nos. 133-136) showed unquestionably that the entire former theory of maxima and minima needed a thorough renovation; and in the main Peano's book is the original source of the beautiful and to a great degree fundamental works of Scheeffer, Stolz, Victor v. Dantscher, and others, who have developed new and strenuous theories for *extreme* values of functions. Speaking for the Germans, Professor A. Mayer, in the introduction to the above-mentioned book, declares that there has been a long-felt need of a work which, for the first time, not only is free from mistakes and inaccuracies that have been so long in vogue but which, besides, so incisively penetrates an important field that hitherto has been considered quite elementary.

7. Appendix

All articles of Peano are collected in *Opera Omnia* (Peano, 2002), a CD-ROM, edited by S. Roero. Selected works of Peano were assembled and commented in *Opere scelte* (Peano, 1957-9) by Cassina, a student of Peano. A few have English translations in *Selected Works* (Peano, 1973). Regrettably, even fewer Peano's articles have a public URL and are freely downloadable.

One finds the following articles of Peano:

in *Opere scelte*, vol. 1:	(1884a), (1884b), (1888b), (1890a), (1890b), (1891), (1892), (1894a), (1894b)
in *Opere scelte*, vol. 2:	(1889)
in *Opere scelte*, vol. 3:	(1896), (1898)
in *Selected Works*:	(1887, pp. 152–160, 185–7), (1888a, pp. 1–32), (1890a), (1894a), (1896).

Along with the bibliography we attach several pages of *Applicazioni Geometriche* (Peano, 1887) and of *Formulario Mathematico* (Peano, 1908) corresponding to Regula, limits of variable sets, derivatives and tangent cones.

For reader's convenience, we provide a chronological list of some mathematicians mentioned in the paper, together with biographical sources.

The `html` files with biographies of methematicians listed below with an asterisk (*) can be attained at University of St Andrews's web-page
`http://www-history.mcs.st-and.ac.uk/history/Biographies/{Name.html}`

Jacobi, Carl (1804-1851)*
Hamilton, William R. (1805-1865)*
Grassmann, Hermann (1809-1877)*
Weierstrass, Karl (1815-1897)*
Genocchi, Angelo (1817-1889)*
Jordan, Camille (1838-1922)*
Mayer, Adolph (1839-1907)*
Darboux, Gaston (1842-1917)*
Stolz, Otto (1842-1905)*
Ascoli, Giulio (1843-1896), see May (1973, p. 62)
Cantor, Georg (1845-1918)*
Dini, Ulisse (1845-1918)*
Dantscher von Kollesberg, Victor (1847-1921), see *A. M. Monthly*, 29 (1922)
Klein, Felix (1849-1925)*
Poincaré, Henri (1854-1912)*
Goursat, Edouard (1858-1936)*
Peano, Giuseppe (1858-1932), see Kennedy (1980)
Scheeffer, Ludwig (1859-1885), see *Math. Annalen (1886)* 26, p. 197
Burali-Forti, Cesare (1861-1931)*
Young, William H. (1863-1942)*
Painlevé, Paul (1863-1933)*
Pierpont, James (1866-1938), see *Bull. A.M.S.* **45** (1939), p. 481
Hancock, Harris (1867-1944), see May (1973, p. 185)
Hausdorf, Felix (1868-1942)*
Borel, Emile (1871-1956)*
Carathéodory, Constantin (1873-1950)*
Levi, Beppo (1875-1961), see May (1973, p. 238)
Fréchet, Maurice (1878-1973)*
Severi, Francesco (1879-1961)*
Zoretti, Ludovic (1880-1948), see `http://catalogue.bnf.fr`
Weyl, Hermann (1885-1955)*
Ascoli, Guido (1887-1957), see May (1973, p. 63)
Janiszewski, Zygmunt (1888-1920)*
Bouligand, Georges (1889-1979), see `http://catalogue.bnf.fr`
Vietoris, Leopold (1891-2002), see Reitberger (2002)
Wilkosz, Witold (1891-1941), see `http://www.wiw.pl/matematyka/Biogramy/Biogramy_21.Asp`
Banach, Stefan (1892-1945)*
Kuratowski, Kazimierz (1896-1980)*
Cassina, Ugo (1897-1964), see Kennedy (1980, p. 222)
Vasilesco, Florin (1897-1958), see May (1973, p. 368)
Saks, Stanislaw (1897-1942)*
Mazur, Stanislaw (1905-1981)*
Whitney, Hassler (1907-1989)*
Birkhoff, Garrett (1911-1996)*

References

Acosta, E.G. and Delgado, C.G. (1994) Frechet vs. Caratheodory. *Amer. Math. Monthly*, **101**, 332-338. `http://www.jstor.org`.

Acker, F. and Dickstein, F. (1983) Uma introdução à análise convexa. *14° Colóquio Brasileiro de Matemática*, Poços de Caldas.

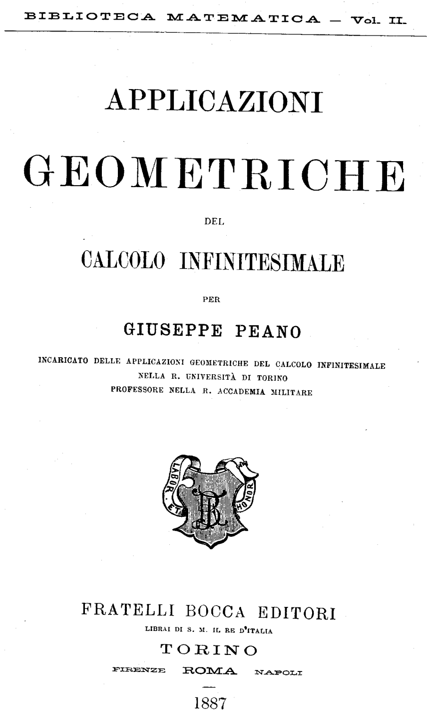
Ascoli, G. (1933) Sugli spazi lineari metrici e le loro varietà. *Ann. Mat. Pura Appl.* **10**, 33-81, 203-232. `http://www.springerlink.com`

Ascoli, G. (1952) Sopra un'estensione di una formula asintotica di Laplace agli integrali multipli. *Rend. Sem. Mat. Pad.* **21**, 209-227. `http://www.numdam.org`

Ascoli, G. (1955) I motivi fondamentali dell'opera di Giuseppe Peano. In: A. Terracini, ed., *In memoria di Giuseppe Peano*, Liceo Sc. Cuneo, 23-30.

Aubin, J.-P. (2000) *Applied Functional Analysis.* Wiley.

Aubin, J.-P. and Frankowska, H. (1990) *Set-Valued Analysis.* Birkhäuser.



BIBLIOTECA MATEMATICA — Vol. II.

APPLICAZIONI

GEOMETRICHE

DEL

CALCOLO INFINITESIMALE

PER

GIUSEPPE PEANO

INCARICATO DELLE APPLICAZIONI GEOMETRICHE DEL CALCOLO INFINITESIMALE
NELLA R. UNIVERSITÀ DI TORINO
PROFESSORE NELLA R. ACCADEMIA MILITARE

FRATELLI BOCCA EDITORI
LIBRAI DI S. M. IL RE D'ITALIA
TORINO
FIRENZE ROMA NAPOLI
—
1887

Figure 1. Peano (1887)

BANACH, S. (1951) *Mechanics.* Polish Mathematical Society. http://matwbn.icm.edu.pl/kstresc.php?tom=24&wyd=10&jez=

BIRKHOFF, G. (1973) *A Sourcebook in Classical Analysis.* Harvard Univ. Press.

BOREL, E. (1903) Quelques remarques sur les ensembles de droites et de plans. *Bull. Soc. Math. France* **31**, 272-275. http://www.numdam.org/item?id=BSMF_1903__31__272_0

BORWEIN, J. M. and LEWIS, A. S. (2000) *Convex Analysis and Nonlinear Optimization.* Springer-Verlag.

BOULIGAND, G. (1924) *Leçons de la Géométrie Vectorielle.* Vuibert, Paris.

BOULIGAND, G. (1932) *Introduction à la géométrie infinitésimale directe.* Gauthier-Villars, Paris.

BURALI-FORTI, C. (1895) Sur quelques propriété des ensembles d'ensembles et leurs applications à la limite d'un ensemble variable. *Math. Annalen* **47**, 20-32. http://gdzdoc.sub.uni-goettingen.de/sub/digbib/loader?did=D77340

BURALI-FORTI, C. (1899) Sur l'égalité et sur l'introduction des éléments dérivés dans la sciences. *Enseignement Math.* **1**, 246-261. http://retro.seals.ch/cntmng?type=pdf&aid=c1:431211&subp=lores

Figure 2. Peano (1887, pp. 131-132): derivative

Carathéodory, C. (1964) *Theory of functions of a complex variable*, vol. 1. Chelsea Publ. Company, New York.

Cassina, U. (1926-27) Limiti delle funzioni plurivoche. *Atti R. Acc. Scienze Torino* **62**, 4-21.

Darboux, G. (1880) Sur le théorème fondamental de la géométrie projective. (Extrait d'une lettre à M. Klein) *Math. Annalen* **17**, 55–61. http://dz-srv1.sub.uni-goettingen.de/sub/digbib/loader?did=D26213

Dini, U. (1878) *Fondamenti per la teorica delle funzioni di variabili reali.* Nistri e C. Pisa.

Federer, H. (1959) Curvature measures. *Trans. Amer. Math. Soc.* **93**, 418-491.

Félix, L. (1957) *L'aspect moderne des mathématiques.* Librairie Blanchard, Paris.

Fréchet, M. (1911a) Sur la notion de différentielle. *C.R.A.Sc. Paris* **152**, 845-847, 27 March 1911. http://gallica.bnf.fr/ark:/12148/bpt6k3105c

Fréchet, M. (1911a) Sur la notion de différentielle. *C.R.A.Sc. Paris* **152**, 1950-1951, 18 April 1911. http://gallica.bnf.fr/ark:/12148/bpt6k3105c

Figure 3. Peano (1887, pp. 143-144): Regula

Fréchet, M. (1937) Sur la notion de différentielle. *J. Math. Pures Appl.* **16**, 233-250.

Genocchi, A. (1884) *Calcolo differenziale e principii di calcolo integrale pubblicato con aggiunte dal Dr. Giuseppe Peano.* Fratelli Bocca, Torino. `http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=02840002&seq=1`

Grassmann, H.G. (1847) *Geometrische Analyse.* Leipzig.

Figure 4. Peano (1887, p. 302): lower limit of variable sets

Figure 5. Peano (1887, p. 305): lower affine tangent cone

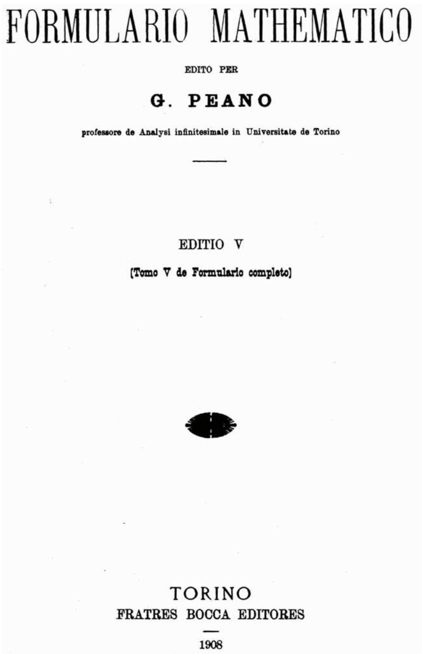
GRASSMANN, H.G. (1894-1911) *Gesammelte Werke*. Teubner, Leipzig. http://quod.lib.umich.edu/cgi/t/text/text-idx?c=umhistmath&idno=ABW0785

GRASSMANN, H.G. (2000) *Extension Theory*. American Mathematical Society.

GRECO, G.H. AND PAGANI, E.M. (2007) *Reworking on Affine Exterior Algebra of Grassmann: Peano and its School.* Preprint.

GRECO, G.H. (2007) *Reworking on Derivation of Measures: Cauchy and Peano.* Forthcoming.

HANCOCK, H. (1903) *Lectures on the Theory of Maxima and Minima of Functions of Several Variables. Weierstrass' Theory.* Cincinnati, University Press. http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=02120001&seq=1



FORMULARIO MATHEMATICO

EDITO PER

G. PEANO

professore de Analysi infinitesimale in Universitate de Torino

EDITIO V

[Tomo V de Formulario completo]

TORINO

FRATRES BOCCA EDITORES

1908

Figure 6. Peano (1908)

HANCOCK, H. (1917) *Theory of maxima and minima.* Dover Publications, 1960. `http://www.archive.org/details/theorymaxima00hancrich`

HAUSDORFF, F. (1914) *Grundzüge der Mengelehre.* Chelsea Publishing Co, New York.

HAUSDORFF, F. (1927) *Mengelehre.* Berlin.

HIRRIART-URRUTY, J.-B. and LEMARÉCHAL, C. (1996) *Convex Analysis and Minimization Algorithms.* Springer-Verlag, Berlin.

JACOBI, C. (1881-1891) *Gesammelte Werke.* Reimer, Berlin. `http://quod.lib.umich.edu/cgi/t/text/text-idx?c=umhistmath&idno=ABR8803`

JANISZEWSKI, Z. (1911) *Les continus irréductibles entre deux points* (Thèse, 1911). In: *Oeuvres Choisies*, P.W.N. (Polish Scientific Publishers), 1962, 31-125.

JORDAN, C. (1893-96) *Cours d'Analyse de l'École Polytechnique.* 2nd edition, Gauthier-Villars, Paris.

KENNEDY, H.C. (1980) *Life and Works of Giuseppe Peano.* D.Reidel, Dordrecht.

KUHN, S. (1991) The Derivative à la Carathéodory. *Amer. Math Monthly,* **98**, 40-44. `http://www.jstor.org`

KURATOWSKI, K. (1928) Sur les décompositions semi-continues d'espaces métriques compacts. *Fund. Math.* **11**, 167-85.

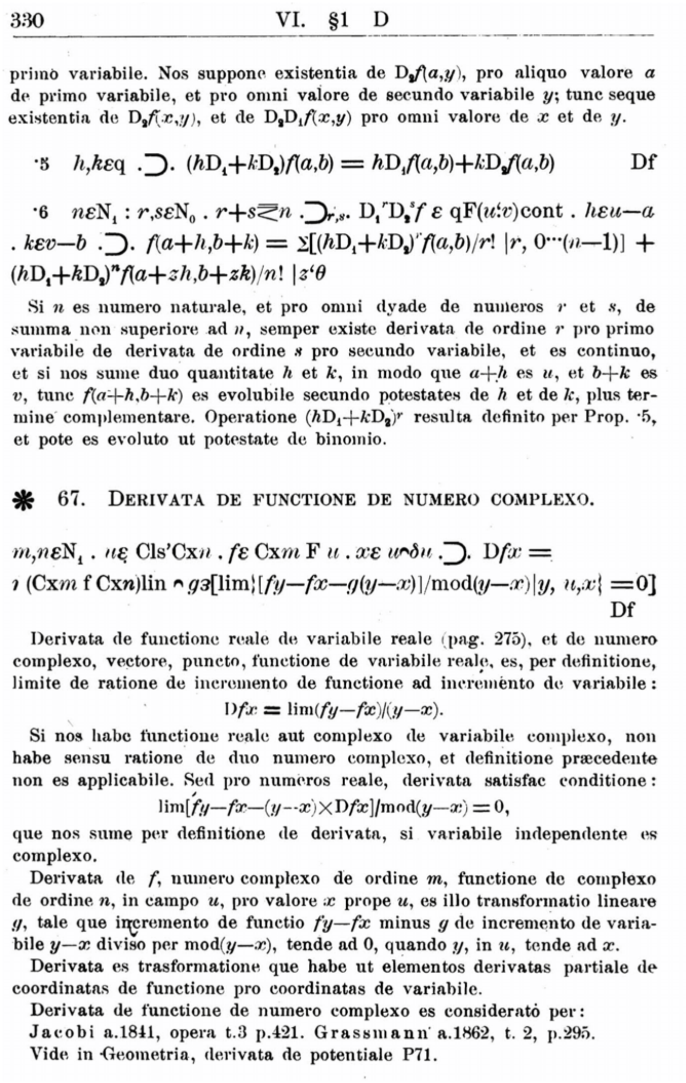
Figure 7. Peano (1908, p. 237): lower (n. 71.1) and upper (71.4) limit of sets

KURATOWSKI, K. (1948) *Topologie*, vol. I. 4th edition, Monografie Matematyczne, Warszawa. `http://matwbn.icm.edu.pl/kstresc.php?tom=20&wyd=10&jez=`

MANHEIM, J.H. (1964) *The Genesis of Point Set Topology.* Pergamon Press, Oxford.

MAWHIN, J. (1997) *Analyse. Fondaments, techniques, évolution.* De Boeck, Bruxelles.

MAZUR, S. (1933) Über konvexe Mengen in lineare normierte Räumen. Studia Math. **4**, 70-84. `http://matwbn.icm.edu.pl/ksiazki/sm/sm4/sm4113.pdf`



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primo variabile. Nos suppone existentia de $D_2 f(a,y)$, pro aliquo valore a de primo variabile, et pro omni valore de secundo variabile y; tunc seque existentia de $D_2 f(x,y)$, et de $D_2 D_1 f(x,y)$ pro omni valore de x et de y.

·5 $h,k \varepsilon \mathrm{q} \;.\supset.\; (hD_1+kD_2)f(a,b) = hD_1 f(a,b)+kD_2 f(a,b)$ Df

·6 $n \varepsilon N_1 : r,s \varepsilon N_0 \,.\, r+s \lesseqgtr n \;.\supset_{r,s}.\; D_1{}^r D_2{}^s f \;\varepsilon\; \mathrm{qF}(u{:}v)\mathrm{cont} \,.\, h \varepsilon u-a \,.\, k \varepsilon v-b \;.\supset.\; f(a+h,b+k) = \Sigma[(hD_1+kD_2)^r f(a,b)/r! \,|r, 0\cdots(n-1)] + (hD_1+kD_2)^n f(a+zh,b+zk)/n! \,|z'\theta$

Si n es numero naturale, et pro omni dyade de numeros r et s, de summa non superiore ad n, semper existe derivata de ordine r pro primo variabile de derivata de ordine s pro secundo variabile, et es continuo, et si nos sume duo quantitate h et k, in modo que $a+h$ es u, et $b+k$ es v, tunc $f(a+h,b+k)$ es evolubile secundo potestates de h et de k, plus termine complementare. Operatione $(hD_1+kD_2)^r$ resulta definito per Prop. ·5, et pote es evoluto ut potestate de binomio.

✻ 67. DERIVATA DE FUNCTIONE DE NUMERO COMPLEXO.

$m,n \varepsilon N_1 \,.\, u \varepsilon \mathrm{Cls'Cx}n \,.\, f \varepsilon \mathrm{Cx}m \,\mathrm{F}\, u \,.\, x \varepsilon u \frown \delta u \;.\supset.\; Dfx = \imath\, (\mathrm{Cx}m \,\mathrm{f}\, \mathrm{Cx}n)\mathrm{lin} \frown g \ni [\lim\{[fy-fx-g(y-x)]/\mathrm{mod}(y-x)|y, u,x\} = 0]$ Df

Derivata de functione reale de variabile reale (pag. 275), et de numero complexo, vectore, puncto, functione de variabile reale, es, per definitione, limite de ratione de incremento de functione ad incremento de variabile:

$$Dfx = \lim(fy-fx)/(y-x).$$

Si nos habe functione reale aut complexo de variabile complexo, non habe sensu ratione de duo numero complexo, et definitione præcedente non es applicabile. Sed pro numeros reale, derivata satisfac conditione:

$$\lim[fy-fx-(y-x)\times Dfx]/\mathrm{mod}(y-x) = 0,$$

que nos sume per definitione de derivata, si variabile independente es complexo.

Derivata de f, numero complexo de ordine m, functione de complexo de ordine n, in campo u, pro valore x prope u, es illo transformatio lineare g, tale que incremento de functio $fy-fx$ minus g de incremento de variabile $y-x$ diviso per $\mathrm{mod}(y-x)$, tende ad 0, quando y, in u, tende ad x.

Derivata es trasformatione que habe ut elementos derivatas partiale de coordinatas de functione pro coordinatas de variabile.

Derivata de functione de numero complexo es consideratò per:
Jacobi a.1841, opera t.3 p.421. Grassmann a.1862, t. 2, p.295.
Vide in Geometria, derivata de potentiale P71.

Figure 8. Peano (1908, p. 330): derivative (n. 67)

May, K.O. (1973) *Bibliography and Research Manual of the History of Mathematics.* Toronto.

Minkowski, H. (1896) *Geometrie der Zahlen.* Teubner, Leipzig. http://gallica.bnf.fr/ark:/12148/bpt6k99643x and bpt6k3102f

Pallaschke, D. and Rolewicz, S. (1997) *Foundations of Mathematical Optimization. Convex Analysis without Linearity.* Kluwer, Dordrecht.

Peano, G. (1884a) Extrait d'une lettre. *Nouvelles Annales de Mathématiques* **3**, 45-47.

VI. §1 D 331

✱ 68. Tang (Figura tangente).

$u \varepsilon \mathrm{Cls'p} \,.\, x \varepsilon \mathrm{p} \;.\supset.$

·0 $\mathrm{Tang}(u,x) = \mathrm{Lm}\{[x + h(u-x)] \,|\, h,\ \mathrm{Q},\ \infty\}$
$= \mathrm{Lm}[\mathrm{Homot}(x,h)u \,|\, h,\ \mathrm{Q},\ \infty]$ Df

Si u es figura, et x puncto, tunc $\mathrm{Tang}(u,x)$, lege « figura tangente ad u in puncto x », indica limite de figura homothetico de u cum centro de homothetia in x, quando ratione de homothetia h cresce ad infinito.

·1 $x \,\varepsilon\, \mathrm{ex}u \;.\supset.\; \mathrm{Tang}(u,x) = \Lambda$

Si x es puncto externo ad u, tunc figura tangente ad u in puncto es classe nullo.

·2 $x \varepsilon\, u - \delta u \;.\supset.\; \mathrm{Tang}(u,x) = \iota x$

Si x es puncto isolato de u, figura tangente contine solo puncto x.

·3 $x \varepsilon\, \delta u \;.\supset.\; \exists\, \mathrm{Tang}(u,x)$

Si x es puncto prope alios u, tunc semper existe punctos de figura tangente ad u in puncto x.

·4 $x \varepsilon\, \mathrm{in}u \;.\supset.\; \mathrm{Tang}(u,x) = \mathrm{p}$

Si x es interno ad u, figura tangente es toto spatio.

·5 $y \varepsilon\, \mathrm{Tang}(u,x) - \iota x \;.\supset.\; x + \mathrm{Q}(y-x) \supset \mathrm{Tang}(u,x)$

Si in figura tangente ad u, in puncto x, nos sume aliquo puncto y, differente de x, toto radio de origine x, et que i trans y, pertine ad figura tangente.

·6 $v \varepsilon \mathrm{Cls'p} \,.\, v \supset u \;.\supset.\; \mathrm{Tang}(v,x) \supset \mathrm{Tang}(u,x)$

Si figura v continere in u, et figura tangente ad v in puncto x continere in figura tangente ad u.

·7 $v \varepsilon \mathrm{Cls'p} \;.\supset.\; \mathrm{Tang}(u \cup v, x) = \mathrm{Tang}(u,x) \cup \mathrm{Tang}(v,x)$

Operatione « Tang » es distributivo ad « $\cup$ ».

·8 $\mathrm{Tang}[\mathrm{Tang}(u,x),x] = \mathrm{Tang}(u,x)$

✱ 69. $a,p\, \varepsilon \mathrm{p} \,.\, r \varepsilon \mathrm{Q} \,.\, \mathrm{d}(p,a) = r \;.\supset.$

·1 $\mathrm{Tang}\{\mathrm{p} \cap x \ni [\mathrm{d}(x,a) = r],\ p\} = \mathrm{plan}[p,\ \mathrm{I}(p-a)]$

Nos considera loco de punctos que dista ab puncto dato a per quantitate dato r, id es superficie de sphæra de centro a et de radio r. Tunc figura tangente ad superficie dicto, in suo puncto p, es plano per puncto p, et normale ad vectore $p-a$.

Figure 9. Peano (1908, p. 331): upper tangent affine cone (n. 68)

Peano, G. (1884b) Réponse à Ph. Gilbert. *Nouvelles Annales de Mathématiques* **3**, 252-256.

Peano, G. (1887) *Applicazioni geometriche del calcolo infinitesimale.* Fratelli Bocca, Torino. `http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=00610002&seq=1`

Peano, G. (1888a) *Calcolo geometrico secondo Ausdehnungslehre di H. Grassmann.* Fratelli Bocca, Torino.

Figure 10. Peano (1908, p. 332): Calculus of tangent lines and planes

Peano, G. (1888) Intégration par séries des équations différentielles linéaires. *Mathematische Annalen* **32**, 450-456. http://gdzdoc.sub.uni-goettingen.de/sub/digbib/loader?ht=VIEW&did=D29534

Peano, G. (1889) *I principii di geometria logicamente esposti*. Fratelli Bocca Editori, Torino. http://quod.lib.umich.edu/cgi/t/text/text-idx?c=umhistmath&idno=ABV4128

Peano, G. (1890a) Sulla definizione dell'area di una superficie. *Rend. Acc. Lincei* **6**, 54-57.

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Si, in superficie es descripto curva que habe tangente in suo puncto p, tunc tangente ad curva jace in plano tangente ad superficie. Seque de Prop. 68·6.

Plure Auctore sume ce proprietate ut definitione. « Plano tangente ad superficie in suo puncto p » ès definito ut « plano que contine recta tangente in p ad omni curva, descripto in superficie, et que i trans p ».

Tunc, si per puncto p, in superficie dato, nos duce linea sine tangente (ut spira mirabile in suo polo, loxodromia in suos polo, etc.), plano tangente contine tangente ad linea, que non habe tangente; quod es contradictorio.

Aliquo Auctore corrige definitione præcedente, et voca plano tangente « plano que contine tangente ad dicto curvas, que habe tangente ». Tunc omni plano es tangente ad superficie, que contine nullo linea cum tangente. Es tale superficie genito per rotatione de curva $y = x \sin 1/x$, circa oy, in puncto o.

Vide alio Df de plano tangente in Formul. t.4 p.295.

$$\cdot 2 \quad k \varepsilon \mathrm{Intv} \,.\, a \varepsilon \mathrm{pF}k \,.\, u \varepsilon (\mathrm{v}-\iota 0)\mathrm{F}k \,.\, x \varepsilon k \,.\, y \varepsilon \mathrm{q} \,.\, \mathrm{D}ax, \mathrm{D}ux\, \varepsilon \mathrm{v} \,.\, \mathrm{D}ax + y\mathrm{D}ux - \varepsilon\, \mathrm{q}\, ux \,.\supset.\, \mathrm{Tang}[\bigcup \mathrm{recta}(ax,ux)\,|x‘k,\ ax+yux] = \mathrm{plan}(ax,\ ux,\ \mathrm{D}ax+y\mathrm{D}ux)$$

Dem.
$$\begin{aligned} & \mathrm{plan}[\bigcup \mathrm{recta}(ax,ux)|x‘k,\ au+yux] \\ = {} & \mathrm{plan}[(ax+yux)|(x;y)‘(k\text{:}\mathrm{q}),\ ax+yux] \\ = {} & \mathrm{plan}\{ax+yux,\ \mathrm{D}_1[(ax+yux)|(x,y)](x,y),\ \mathrm{D}_2[(ax+yux)(x,y)](x,y)\} \\ = {} & \mathrm{plan}(ax+yux,\ \mathrm{D}ax+y\mathrm{D}ux,\ ux) \\ = {} & \mathrm{plan}(ax,ux,\ \mathrm{D}ax+y\mathrm{D}ux) \end{aligned}$$

Nos considera puncto a et vectore non nullo u, ambo functione dato in intervallo k.

Es dato x in intervallo k, et quantitate reale y. Nos suppone existentia de derivatas de a et de u, et que vectore $\mathrm{D}ax+y\mathrm{D}ux$ non es parallelo ad ux. Superficie loco de rectas per ax et parallelo ad ux, ubi x varia in intervallo k, es expresso per $\bigcup \mathrm{recta}(ax,ux)|x‘k$.

Theorema dice que figura tangente ad ce superficie in suo puncto $ax+yux$ es plano per ax, et parallelo ad vectores ux et $\mathrm{D}ax+y\mathrm{D}ux$.

Superficie loco de recta mobile = F. surface reglée = D. Regelfläche = I. superficie rigata.

F. règle =|| D. Regel $\subset$ L. regula.

I. riga $\subset$ Germanico : riga $\supset$ D. Reihe, A. row.

Ce superficie habe in Germania nomen Latino, et in Italia nomen Germanico.

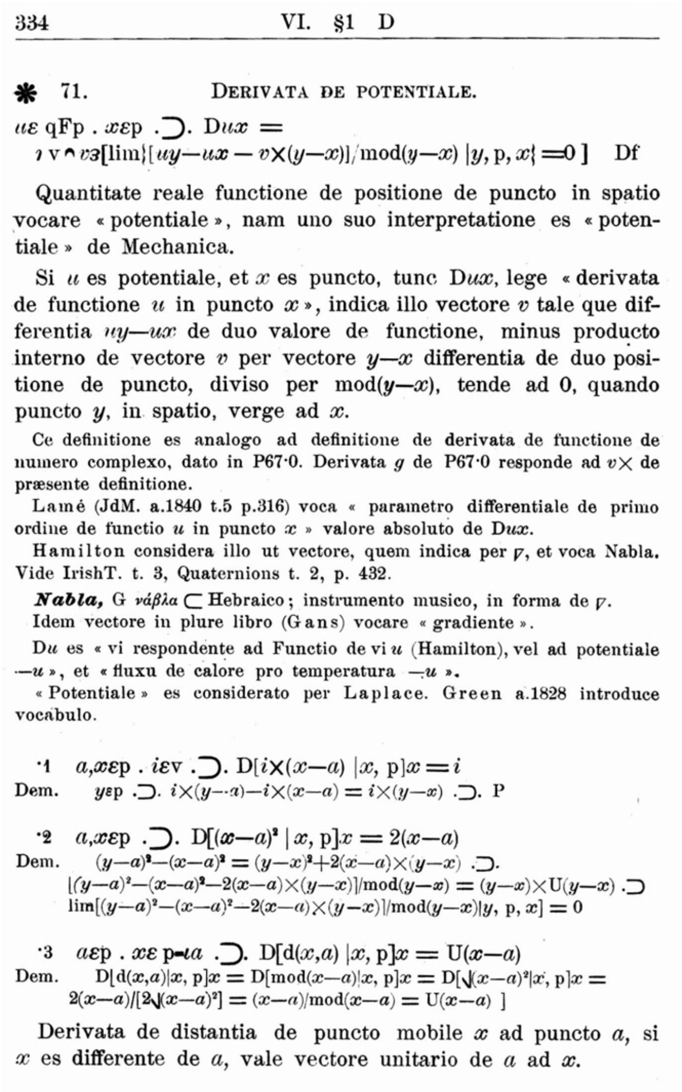
Figure 11. Peano (1908): calculus of a tangent plane

PEANO, G. (1890b) Démonstration de l'intégrabilité des équations différentielles ordinaires. *Mathematische Annalen* **37**, 182-228. http://gdzdoc.sub.uni-goettingen.de/sub/digbib/loader?ht=VIEW&did=D27538

PEANO, G. (1891) Sulla formula di Taylor. *Atti R. Accad. Scienze Torino* **27**, 40-46.

PEANO, G. (1892) Sur la définition de la dérivée. *Mathesis***2**, 12-14.

PEANO, G. (1893) *Lezioni di analisi infintesimale.* 2 vol., Candeletti, Torino.



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✱ 71. DERIVATA DE POTENTIALE.

$u \varepsilon \mathrm{qFp} \,.\, x \varepsilon \mathrm{p} \,.\supset.\, \mathrm{D}ux =$
$\quad \imath\, \mathrm{v} \frown v \ni [\lim\{[uy - ux - v\times(y-x)]/\mathrm{mod}(y-x) \,|y, \mathrm{p}, x\} = 0\,]$ Df

Quantitate reale functione de positione de puncto in spatio vocare « potentiale », nam uno suo interpretatione es « potentiale » de Mechanica.

Si u es potentiale, et x es puncto, tunc Dux, lege « derivata de functione u in puncto x », indica illo vectore v tale que differentia $uy-ux$ de duo valore de functione, minus producto interno de vectore v per vectore $y-x$ differentia de duo positione de puncto, diviso per mod$(y-x)$, tende ad 0, quando puncto y, in spatio, verge ad x.

Ce definitione es analogo ad definitione de derivata de functione de numero complexo, dato in P67·0. Derivata g de P67·0 responde ad $v\times$ de præsente definitione.

Lamé (JdM. a.1840 t.5 p.316) voca « parametro differentiale de primo ordine de functio u in puncto x » valore absoluto de Dux.

Hamilton considera illo ut vectore, quem indica per ∇, et voca Nabla. Vide IrishT. t. 3, Quaternions t. 2, p. 432.

Nabla, G νάβλα ⊂ Hebraico; instrumento musico, in forma de ∇.
Idem vectore in plure libro (Gans) vocare « gradiente ».

Du es « vi respondente ad Functio de vi u (Hamilton), vel ad potentiale $-u$ », et « fluxu de calore pro temperatura $-u$ ».

« Potentiale » es considerato per Laplace. Green a.1828 introduce vocabulo.

·1 $a, x \varepsilon \mathrm{p} \,.\, i \varepsilon \mathrm{v} \,.\supset.\, \mathrm{D}[i\times(x-a) \,|x, \mathrm{p}]x = i$
Dem. $y \varepsilon \mathrm{p} \,.\supset.\, i\times(y-a) - i\times(x-a) = i\times(y-x) \,.\supset.$ P

·2 $a, x \varepsilon \mathrm{p} \,.\supset.\, \mathrm{D}[(x-a)^2 \,|x, \mathrm{p}]x = 2(x-a)$
Dem. $(y-a)^2 - (x-a)^2 = (y-x)^2 + 2(x-a)\times(y-x) \,.\supset.$
$[(y-a)^2 - (x-a)^2 - 2(x-a)\times(y-x)]/\mathrm{mod}(y-x) = (y-x)\times \mathrm{U}(y-x) \,.\supset$
$\lim[(y-a)^2 - (x-a)^2 - 2(x-a)\times(y-x)]/\mathrm{mod}(y-x)|y, \mathrm{p}, x] = 0$

·3 $a \varepsilon \mathrm{p} \,.\, x \varepsilon \mathrm{p}{-}\iota a \,.\supset.\, \mathrm{D}[\mathrm{d}(x,a) \,|x, \mathrm{p}]x = \mathrm{U}(x-a)$
Dem. $\mathrm{D}[\mathrm{d}(x,a)|x, \mathrm{p}]x = \mathrm{D}[\mathrm{mod}(x-a)|x, \mathrm{p}]x = \mathrm{D}[\sqrt{(x-a)^2}|x, \mathrm{p}]x =$
$2(x-a)/[2\sqrt{(x-a)^2}] = (x-a)/\mathrm{mod}(x-a) = \mathrm{U}(x-a)$]

Derivata de distantia de puncto mobile x ad puncto a, si x es differente de a, vale vectore unitario de a ad x.

Figure 12. Peano (1908, p. 334): derivative and potential

PEANO, G. (1894a) Sur les systèmes linéaires. *Monatshefte für Mathematik und Physik* **5**, 136.

PEANO, G. (1894b) Sur la définition de la limite d'une fonction. Exercice de la logique mathématique. *Amer. J. Math.* **17**, 37-68.

PEANO, G. (1895-96) Saggio di calcolo geometrico. *Atti R. Accad. Scienze Torino* **31**, 952-975.

PEANO, G. (1897-98) Analisi della teoria dei vettori. *Atti R. Acc. Scienze Torino* **33**, 513-534.

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·4 $a \varepsilon\, \mathrm{p}^2 \cup \mathrm{p}^3 \,.\, x \varepsilon\, \mathrm{p} - a \;.\supset.\; \mathrm{D}[\mathrm{d}(x,a) | x, \mathrm{p}]x = \mathrm{U}[x - (\mathrm{proj}\; a)x]$

Si a indica recta vel plano, et x es puncto ex a, tunc derivata de distantia de x ad figura a es vectore unitario secundo projectione super a de x ad x.

·5 $k \varepsilon\, \mathrm{Cls'p} \,.\, k = \lambda k \,.\, x \varepsilon\, \mathrm{p} - k \,.\, a \varepsilon k \,.\, \mathrm{d}(x,a) = \mathrm{d}(x,k) : y \varepsilon\, k - \iota a \;.\supset_y.$
$\mathrm{d}(x,a) < \mathrm{d}(x,y) \;:\supset.\; \mathrm{D}[\mathrm{d}(x,k) | x, \mathrm{p}]x = \mathrm{U}(x - a)$

Derivata de distantia de puncto x ab figura k es vectore unitario in directione de distantia, in hypothesi scripto.

·6 $a \varepsilon \mathrm{p} \,.\, i \varepsilon\, \mathrm{v} - \iota 0 \,.\, x \,\varepsilon\, \mathrm{p} - (a + \mathrm{q}i) \;.\supset.\; \mathrm{D}[\mathrm{ang}(x - a, i) | x, \mathrm{p}]x =$
$\mathrm{U}\{[\mathrm{cmp}\perp(x - a)]i\}/\mathrm{d}(x,a)$

❋ 72.

·1 $u \varepsilon\, \mathrm{qFp} \,.\, x \varepsilon \mathrm{p} \,.\, ux = \max u\text{‘}\mathrm{p} \,.\, \mathrm{D}ux\, \varepsilon \mathrm{v} \;.\supset.\; \mathrm{D}ux = 0$

·11 (min | max) P·1

Si potentiale u es maximo pro aliquo puncto x, pro que existe derivata de u, tunc ce vectore derivata vale 0.
Idem pro minimo.

·2 $u \varepsilon\, \mathrm{qFp} \,.\, k \varepsilon\, \mathrm{Cls'p} \,.\, x \varepsilon k \,.\, ux = \min u\text{‘}k \,.\, \mathrm{D}ux\, \varepsilon \mathrm{v} \,.$
$y \varepsilon\, \mathrm{Tang}(k,x) \;.\supset.\; (\mathrm{D}ux) \times (y - x) \geqq 0$

Si nos considera solo punctos de aliquo figura k, et si potentiale sume pro puncto x, valore minimo inter valores respondente ad punctos de figura k, et derivata de u in x es vectore determinato, tunc producto de ce vectore per omni variatione $y - x$ de puncto in figura tangente ad k in x, es positivo aut nullo.

·3 $u \varepsilon\, \mathrm{qFp} \,.\, k \varepsilon\, \mathrm{Cls'p} \,.\, x \varepsilon k \,.\, ux = \max u\text{‘}k \,.\, \mathrm{D}ux\, \varepsilon \mathrm{v} \,.$
$y \varepsilon\, \mathrm{Tang}(k,x) \;.\supset.\; (\mathrm{D}ux) \times (y - x) \leqq 0$

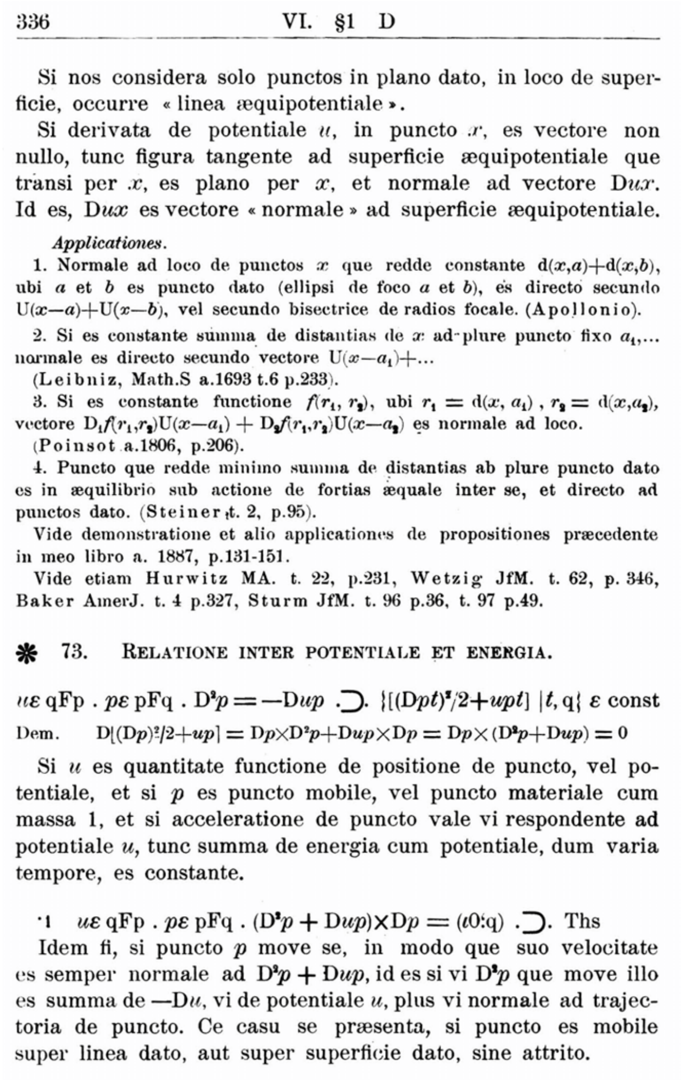
Regula simile subsiste pro maximo.

·4 $u \varepsilon\, \mathrm{qFp} \,.\, x \varepsilon \mathrm{p} \,.\, \mathrm{D}ux \,\varepsilon\, \mathrm{v} - \iota 0 \;.\supset.$
$\mathrm{Tang}[\mathrm{p} \cap y \ni (uy = ux), x] = \mathrm{plan}(x, \mathrm{ID}ux)$

Figura loco de punctos y, que satisfac conditione $uy = ux$, vocare « superficie æquipotentiale » que i trans x.

Figure 13. Peano (1908, p. 335): regula n. 72.2 (min) and n. 72.3(max)

PEANO, G. (1903) *Formulaire Mathématique (tome IV).* Fratelli Bocca, Torino.
PEANO, G. (1908) *Formulario Mathematico (Editio V).* Fratelli Bocca, Torino.
PEANO, G. (1957-9) *Opere scelte.* Edizioni Cremonese, Roma.
PEANO, G. (1960) *Formulario Mathematico.* Edizioni Cremonese, Roma 1960 (reprint of Peano, 1908).
PEANO, G. (1973) *Selected works* (H.C. Kennedy, ed.). University of Toronto Press,
PEANO, G. (2002) *Opera Omnia.* 1 CD-ROM (S. Boero, ed.). Dipartimento di Matematica, Università, Torino.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Si nos considera solo punctos in plano dato, in loco de superficie, occurre « linea æquipotentiale ».

Si derivata de potentiale u, in puncto x, es vectore non nullo, tunc figura tangente ad superficie æquipotentiale que transi per x, es plano per x, et normale ad vectore $\mathrm{D}ux$. Id es, $\mathrm{D}ux$ es vectore « normale » ad superficie æquipotentiale.

Applicationes.

1. Normale ad loco de punctos x que redde constante $\mathrm{d}(x,a)+\mathrm{d}(x,b)$, ubi a et b es puncto dato (ellipsi de foco a et b), es directo secundo $\mathrm{U}(x-a)+\mathrm{U}(x-b)$, vel secundo bisectrice de radios focale. (Apollonio).

2. Si es constante summa de distantias de x ad plure puncto fixo $a_1,\ldots$ normale es directo secundo vectore $\mathrm{U}(x-a_1)+\ldots$
(Leibniz, Math.S a.1693 t.6 p.233).

3. Si es constante functione $f(r_1, r_2)$, ubi $r_1 = \mathrm{d}(x, a_1)$, $r_2 = \mathrm{d}(x,a_2)$, vectore $\mathrm{D}_1f(r_1,r_2)\mathrm{U}(x-a_1) + \mathrm{D}_2f(r_1,r_2)\mathrm{U}(x-a_2)$ es normale ad loco.
(Poinsot a.1806, p.206).

4. Puncto que redde minimo summa de distantias ab plure puncto dato es in æquilibrio sub actione de fortias æquale inter se, et directo ad punctos dato. (Steiner t. 2, p.95).

Vide demonstratione et alio applicationes de propositiones præcedente in meo libro a. 1887, p.131-151.

Vide etiam Hurwitz MA. t. 22, p.231, Wetzig JfM. t. 62, p. 346, Baker AmerJ. t. 4 p.327, Sturm JfM. t. 96 p.36, t. 97 p.49.

✻ 73. RELATIONE INTER POTENTIALE ET ENERGIA.

$$u\varepsilon\,\mathrm{qFp}\;.\;p\varepsilon\,\mathrm{pFq}\;.\;\mathrm{D}^2p = -\mathrm{D}up\;.\supset.\;\{[(\mathrm{D}pt)^2/2+upt]\;|t,\mathrm{q}\}\;\varepsilon\;\mathrm{const}$$

Dem. $\mathrm{D}[(\mathrm{D}p)^2/2+up] = \mathrm{D}p\times\mathrm{D}^2p+\mathrm{D}up\times\mathrm{D}p = \mathrm{D}p\times(\mathrm{D}^2p+\mathrm{D}up) = 0$

Si u es quantitate functione de positione de puncto, vel potentiale, et si p es puncto mobile, vel puncto materiale cum massa 1, et si acceleratione de puncto vale vi respondente ad potentiale u, tunc summa de energia cum potentiale, dum varia tempore, es constante.

·1 $u\varepsilon\,\mathrm{qFp}\;.\;p\varepsilon\,\mathrm{pFq}\;.\;(\mathrm{D}^2p + \mathrm{D}up)\times\mathrm{D}p = (\iota 0\mathrm{:q})\;.\supset.$ Ths

Idem fi, si puncto p move se, in modo que suo velocitate es semper normale ad $\mathrm{D}^2p + \mathrm{D}up$, id es si vi D^2p que move illo es summa de $-\mathrm{D}u$, vi de potentiale u, plus vi normale ad trajectoria de puncto. Ce casu se præsenta, si puncto es mobile super linea dato, aut super superficie dato, sine attrito.

Figure 14. FM (p. 336): Applications

Pierpont, J. (1905) *The theory of functions of real variables*, vol. I. Ginn and Co., Boston.

Poincaré, H. (1908) L'avenir des mathématiques. *Atti del IV Congresso Internazionale dei Matematici* (Roma, 6-11 Aprile 1908), **I**, 167-182, Acc. Lincei, Roma 1909.

Reitberger, H. (2002) Leopold Vietoris (1891-2002). *Notices AMS* **49**, 1232-6. http://www.ams.org/notices/200210/fea-vietoris.pdf

ROCKAFELLAR, R.T. and WETS, R. (1997) *Variational Analysis.* Springer-Verlag.

SAKS, S. (1937) *Theory of the Integral.* Hafner, New York. http://matwbn.icm.edu.pl/kstresc.php?tom=7&wyd=10&jez=

STOLZ, O. (1893) *Grundzüge der Differenzial-und Integralrechnung.* Teubner, Leipzig.

SVETIC, R.E. and VOLKMER, H. (1998) On the ultimate Peano derivative. *J. Math. Anal. Appl.* **218**, 439-452.

VASILESCO, F. (1925) *Essai sur les fonctions multiformes de variables réelles* (Thèse). Gauthier-Villars, Paris.

VIETORIS, F. (1922) Bereiche zweiter Ordnung. *Monatshefte für Mathematik und Physik* **32**, 258–80.

WEIL, C. (1995) The Peano notion of higher order differentiation. *Math. Japonica* **42**, 587-600.

WEYL, H. (1918) *Raum-Zeit-Materie.* Springer, Berlin. (Engl. transl. *Space, Time, Matter*, Dover Publ. 1952).

WHITNEY, H. (1972) *Complex Analytic Varieties.* Addison-Wesley Publ. Co, Reading.

WILKOSZ, W. (1921) Sul concetto del differenziale esatto. *Fundamenta Math.* **2**, 140-144. http://matwbn.icm.edu.pl/ksiazki/fm/fm2/fm2118.pdf

YOUNG, W.H. (1910) *The fundamental theorems of differential calculus.* Cambridge Univ. Press. http://www.archive.org/details/TheFundamentalTheoremsOfTheDifferentialCalculusNo11

ZADDACH, A. (1988) *Algebra de Grassmann y Geometría Proyectiva.* Universidad de Tarapacá, Facultad de Ciencias.

ZADDACH, A. (1994) *Grassmanns Algebra in der Geometrie mit Seitenblicken auf verwandte Strukturen.* BI-Wissenschaftsverlag, Mannheim.

ZERMELO, E. (1904) Beweis, dass jede Menge wohlgeordnet werden kann. *Mathematische Annalen* **59**, 514-516. http://gdzdoc.sub.uni-goettingen.de/sub/digbib/loader?did=D28526

ZORETTI, L. (1905) Sur les fonctions analytiques uniformes qui possèdent un ensemble parfait discontinu de points singuliers. *J. Math. Pures Appl.* **1**, 1-51. http://gallica.bnf.fr/document?O=N107470

ZORETTI, L. (1909) Un théorème de la théorie des ensembles. *Bull. Soc. Math. France* **37**, 116-9. http://www.numdam.org/item?id=BSMF_1909__37__116_0

ZORETTI, L. (1912) Sur les ensembles de points. *Encyclopédie des Sciences Mathématiques*, **II** (vol. I), 113-170. http://gallica.bnf.fr/ark:/12148/bpt6k2025807