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Existence conditions in symmetric multivalued vector quasiequilibrium problems*

by

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Abstract: We consider symmetric multivalued vector quasiequilibrium problems in topological vector spaces. Sufficient conditions for the solution existence are established under relaxed assumptions, which are shown by examples to be essential, easy to check and more advantageous than recent known results. Applications to lower and upper bounded quasiequilibrium problems and to coincidence point problems are given.

Keywords: symmetric quasiequilibrium problems, upper semicontinuity, 0-level C-quasiconvexity relative to a set, lower and upper bounded quasiequilibrium problems, coincidence points.

1. Introduction

The equilibrium problem was proposed in Blum and Oettli (1994) and has been intensively studied since then. This problem setting proves to be general and convenient for applying various mathematical tools in investigation. It contains many optimization-related problems such as variational inequalities, complementarity problems, vector optimization, fixed point and coincidence point problems, the Nash equilibrium problem, etc. As usual for various fields of research, solution existence is one of the most important issues and so is the aim of numerous papers (see e.g., Bianchi et al., 1997; Ansari et al., 2001; Lin and Chen, 2005; Hai and Khanh, 2007a) and the references therein. To include more practical problems in a unified framework, a number of extended problem settings have been considered: variational inclusion problems (Luc and Tan, 2004;

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Tan, 2004; Hai and Khanh, 2006), systems of equilibrium or quasiequilibrium problems (Ansari et al., 2000 and 2002; Hai and Khanh, 2006; Lin, 2006), systems of variational inclusion problems (Hai and Khanh, 2007b). Noor and Oettli (1994) introduced a symmetric quasiequilibrium problem, which proved to be more suitable in modeling several practical situations. For instance, problems of finding equilibria of constrained non-cooperative games are conveniently expressed as special cases of symmetric quasiequilibrium problems, compare, e.g., Aubin (1979), p. 282. In Fu (2003) this result was extended from the scalar case to the vector case in Hausdorff locally convex spaces. Farajzadeh (2006) supplied a further extension to Hausdorff topological vector spaces with several assumptions being relaxed.

Our goal is to extend the problem considered in Noor and Oettli (1994), Fu (2003) and Farajzadeh (2006) from the single−valued case to the multivalued case, in Hausdorff topological vector spaces. Since we use mathematical tools other than that employed in Noor and Oettli (1994), Fu (2003) and Farajzadeh (2006), the results here for this more general problems are different from the ones in these references, when applied to their particular cases. However, our several assumptions are more relaxed than the corresponding ones in Noor and Oettli (1994), Fu (2003) and Farajzadeh (2006).

In the sequel, if not otherwise specified, let X and Y be Hausdorff topological vector spaces, Z be a topological vector space. Let K, D and C be nonempty closed convex subsets of X, Y and Z, respectively, with the interior $\operatorname{int} C$ being nonempty. Let $S, A : K \times D \to 2^K$, $T, B : K \times D \to 2^D$, $F : K \times D \times K \to 2^Z$ and $G : D \times K \times D \to 2^Z$ be multivalued mappings , with $S(x, y)$ and $T(x, y)$ being nonempty and convex, $\forall (x, y) \in K \times D$. The two symmetric quasiequilibrium problems under our consideration are as follows

(SVQEP$_1$) find $(\bar{x}, \bar{y}) \in K \times D$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}, \bar{y})$, $\bar{y} \in \operatorname{cl} T(\bar{x}, \bar{y})$ and
$F(x, \bar{y}, x^*) \cap (Z \backslash - \operatorname{int} C) \neq \emptyset, \forall x \in S(\bar{x}, \bar{y}), \forall x^* \in A(\bar{x}, \bar{y})$,
$G(y, \bar{x}, y^*) \cap (Z \backslash - \operatorname{int} C) \neq \emptyset, \forall y \in T(\bar{x}, \bar{y}), \forall y^* \in B(\bar{x}, \bar{y})$;

(SVQEP$_2$) find $(\bar{x}, \bar{y}) \in K \times D$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}, \bar{y})$, $\bar{y} \in \operatorname{cl} T(\bar{x}, \bar{y})$ and
$F(x, \bar{y}, x^*) \subseteq Z \backslash - \operatorname{int} C, \forall x \in S(\bar{x}, \bar{y}), \forall x^* \in A(\bar{x}, \bar{y})$,
$G(y, \bar{x}, y^*) \subseteq Z \backslash - \operatorname{int} C, \forall y \in T(\bar{x}, \bar{y}), \forall y^* \in B(\bar{x}, \bar{y})$.

If F and G are single-valued, C is a closed convex cone with $\operatorname{int} C \neq \emptyset$ and $A(x, y) = \{x\}$ and $B(x, y) = \{y\}$, for all $(x, y) \in K \times D$, then our two problems reduce to problem (SVQEP) investigated in Farajzadeh (2006). If, in addition, X and Y are locally convex and C and D are compact, the two problems reduce to problem (SVQEP) of Fu (2003). If, furthermore, $Z = R$ and $C = R_+$, these problems coincide with the scalar problem studied in Noor and Oettli (1994).

If $Y = X, G(y, \bar{x}, y^*) \equiv C, B(x, y) \equiv D$ and $T(x, y) = S(x, y)$ then these problems are reduced to multivalued vector quasiequilibrium problems considered by many authors. If, more specifically, $F(x, y, x^*) = \big(H(y), x - x^*\big)$ where

$H : X \to 2^{L(X,Z)}$ and (h, x) is the value of linear mapping h at x, then the two problems become a multivalued vector quasivariational inequality.

The layout of this paper is as follows. In the remaining part of this section we recall some definitions and preliminaries needed in the sequel. Section 2 is devoted to the main existence results for our problems. Examples are also provided there to see that the imposed assumptions are essential, relaxed and not hard to be checked, and hence the results are more advantageous than those of recent works in many situations. In Section 3, applications of the main results in some typical situations are presented.

Recall first some notions. Let X and Y be topological spaces and $G : X \to 2^Y$ be a multifunction. G is called upper semicontinuous (usc) at x_0 if for each open set $U \supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq G(N)$. We say that G satisfies a certain property in a subset $A \subseteq X$ if G satisfies it at every point of A. If $A = X$ we omit "in X" in the statement.

Recall that a point $x \in X$ is called a maximal element of $F : X \to 2^Y$, where X and Y are topological spaces, if $F(x) = \emptyset$. The main machinery for proving existence results in this paper is the following result which is a slightly weaker form of a theorem in Deguire et al. (1999).

Lemma 1 *Let $i = 1, 2$, let X_i be a Hausdorff topological vector space, $K_i \subseteq X_i$ be nonempty convex subset and let $Q_i : K = K_1 \times K_2 \to 2^{K_i}$ have convex values. Assume that the following conditions hold*

- (i) $Q_i^{-1}(x_i)$ *is open in K for all $x_i \in K_i$ and $i = 1, 2$;*
- (ii) $x_i \notin Q_i(x)$ *for each $x = (x_1, x_2)$ and $i = 1, 2$;*
- (iii) *if K is not compact, then there exists a nonempty compact subset $\overline{K}$ of K and, $\forall i = 1, 2$, a nonempty compact convex subset B_i of A_i such that, for each $x \in K \setminus \overline{K}$, there exists $i \in \{1, 2\}$ such that $B_i \cap Q_i(x) \neq \emptyset$.*

Then, there exists $\bar{x} \in K$ such that $Q_i(\bar{x}) = \emptyset$ for all $i = 1, 2$.

2. Main results

The following relaxed quasiconvexity will be assumed in our main existence theorems.

Definition 1 *Let X and Z be vector spaces, let $B \subseteq X$ and $C \subseteq Z$ be nonempty and convex, with* $\operatorname{int} C \neq \emptyset$ *and let $F : X \times B \to 2^Z$.*

- (i) *F is said to be* $0-$level $C-$quasiconvex relative to B of type 1 *if for any $x_1, x_2 \in X$, any $t \in [0, 1]$,*

$$\big[\exists x_i^* \in B, i = 1, 2 : F(x_i, x_i^*) \subseteq -\operatorname{int} C\big]$$
$$\Longrightarrow \big[\exists x^* \in B : F\big((1-t)x_1 + tx_2, x^*\big) \subseteq -\operatorname{int} C\big]. \qquad (1)$$

(ii) *F is called* 0−level C−quasiconvex relative to B of type 2 *if (1) is replaced by*

$$\begin{aligned}&\big[\exists x_i^* \in B, i=1,2: F(x_i,x_i^*)\cap -\operatorname{int} C \neq \emptyset\big]\\ &\Longrightarrow \big[\exists x^* \in B: F\big((1-t)x_1+tx_2,x^*\big)\cap -\operatorname{int} C\neq\emptyset\big].\end{aligned}$$

To see the nature of the above generalized convexity, let us consider the simplest case, where B is a singleton, $Z=R$ and F is single-valued depending only on $x\in X$. Then (i) and (ii) coincide and become: if $F(x_i)<0, i=1,2$, then $\forall t\in[0,1]$, $F\left((1-t)x_1+tx_2\right)<0$. This property is a relaxed 0-level quasiconvexity, since F is called quasiconvex if $F\left((1-t)x_1+tx_2\right)\leq \max\limits_{i=1,2} F(x_i)$.

A sufficient condition for the solution existence of problem (SVQEP$_1$) is

THEOREM 1 *Assume that*

(i) $\forall(x,y)\in K\times D, \forall(x^*,y^*)\in A(x,y)\times B(x,y), F(x,y,x^*)\not\subseteq -\operatorname{int}C$ *and* $G(y,x,y^*)\not\subseteq -\operatorname{int}C$;

(ii) $\forall(x,y)\in K\times D$, $F(.,y,.)$ *and* $G(.,x,.)$ *are* 0−*level* C−*quasiconvex relative to* $A(x,y)$ *and* $B(x,y)$*, respectively, of type* 1;

(iii) $\forall(x,y)\in K\times D$, *the sets* $\{(\bar x,\bar y)\in K\times D \mid F(x,\bar y,x^*)\not\subseteq -\operatorname{int}C, \forall x^*\in A(\bar x,\bar y)\}$ *and* $\{(\bar x,\bar y)\in K\times D\mid G(y,\bar x,y^*)\not\subseteq -\operatorname{int}C,\forall y^*\in B(\bar x,\bar y)\}$ *are closed in* $K\times D$;

(iv) $\operatorname{cl}S(.,.)$ *and* $\operatorname{cl}T(.,.)$ *are usc in* $K\times D$ *and,* $\forall(x,y)\in K\times D$, $S^{-1}(x)$ *and* $T^{-1}(y)$ *are open in* $K\times D$;

(v) *if* $K\times D$ *is not compact, there exist a nonempty compact subset* $\overline{K}\times\overline{D}$ *of* $K\times D$ *and a nonempty compact convex subset* $K_0\times D_0$ *of* $K\times D$ *such that, for each* $(x,y)\in(K\times D)\setminus(\overline{K}\times\overline{D})$ *there are* $\bar x\in K_0\cap S(x,y)$ *and* $x^*\in A(x,y)$ *such that* $F(\bar x,y,x^*)\subseteq -\operatorname{int}C$*, or there are* $\bar y\in D_0\cap T(x,y)$ *and* $y^*\in B(x,y)$ *such that* $G(\bar y,x,y^*)\subseteq -\operatorname{int}C$.

Then problem (SVQEP$_1$) *is solvable.*

Proof. For $(x,y)\in K\times D$, set

$$\begin{aligned}&E_1=\{(x,y)\in K\times D\mid x\in \operatorname{cl}S(x,y)\},\\ &E_2=\{(x,y)\in K\times D\mid y\in\operatorname{cl}T(x,y)\},\\ &P_1(x,y)=\{\hat x\in K\mid \exists x^*\in A(x,y), F(\hat x,y,x^*)\subseteq -\operatorname{int}C\},\\ &P_2(x,y)=\{\hat y\in D\mid \exists y^*\in B(x,y), G(\hat y,x,y^*)\subseteq -\operatorname{int}C\},\\ &Q_1(x,y)=\begin{cases}S(x,y)\cap P_1(x,y) & \text{if } (x,y)\in E_1,\\ S(x,y) & \text{otherwise,}\end{cases}\\ &Q_2(x,y)=\begin{cases}T(x,y)\cap P_2(x,y) & \text{if } (x,y)\in E_2,\\ T(x,y) & \text{otherwise.}\end{cases}\end{aligned}$$

We claim that $Q_i(.,.)$ satisfies all the conditions of Lemma 1, $i = 1, 2$.
For $x \in K$ we have

$$\begin{aligned} Q_1^{-1}(x) &= \{(\bar{x},\bar{y}) \in E_1 \mid x \in S(\bar{x},\bar{y}) \cap P_1(\bar{x},\bar{y})\} \\ &\quad \cup \{(\bar{x},\bar{y}) \in (K \times D) \setminus E_1 \mid x \in S(\bar{x},\bar{y})\} \\ &= \{(\bar{x},\bar{y}) \in E_1 \mid (\bar{x},\bar{y}) \in S^{-1}(x) \cap P_1^{-1}(x)\} \\ &\quad \cup \{(\bar{x},\bar{y}) \in (K \times D) \setminus E_1 \mid (\bar{x},\bar{y}) \in S^{-1}(x)\} \\ &= \{E_1 \cap S^{-1}(x) \cap P_1^{-1}(x)\} \cup \{[(K \times D) \setminus E_1] \cap S^{-1}(x)\} \\ &= \{[(K \times D) \setminus E_1] \cup P_1^{-1}(x)\} \cap S^{-1}(x). \end{aligned}$$

Therefore,

$$\begin{aligned} (K \times D) \setminus Q_1^{-1}(x) &= \{K \times D\} \setminus \{[((K \times D) \setminus E_1) \cup P_1^{-1}(x)] \cap S^{-1}(x)\} \\ &= \{[K \times D] \setminus [((K \times D) \setminus E_1) \cup P_1^{-1}(x)]\} \cup \{[K \times D] \setminus S^{-1}(x)\} \\ &= \{E_1 \cap [(K \times D) \setminus P_1^{-1}(x)]\} \cup \{[K \times D] \setminus S^{-1}(x)\}. \qquad (2) \end{aligned}$$

Since $\operatorname{cl} S(.,.)$ is usc and has closed values, E_1 is closed. We also have

$$(K \times D) \setminus P_1^{-1}(x) = \{(\bar{x},\bar{y}) \in K \times D \mid \forall \bar{x}^* \in A(\bar{x},\bar{y}), F(x,\bar{y},\bar{x}^*) \not\subseteq -\operatorname{int} C\},$$

which is closed by (iii). It follows from (2) that $(K \times D) \setminus Q_1^{-1}(x)$ is closed. Thus $Q_1^{-1}(x)$ is open in $K \times D, \forall x \in K$. Similarly, $Q_2^{-1}(y)$ is open in $K \times D, \forall y \in D$. Due to the fact that $F(.,\bar{y},.)$ is 0-level C- quasiconvex relative to $A(x,y)$ of type 1, $P_1(x,y)$ is convex and hence $Q_1(x,y)$ is convex, for all $(x,y) \in K \times D$. In a similar way we see that $Q_2(x,y)$ is convex, for all $(x,y) \in K \times D$.

Since $\forall (x,y) \in K \times D, \forall x^* \in A(x,y), F(x,y,x^*) \not\subseteq -\operatorname{int} C$, one has $x \notin P_1(x,y)$. If $(x,y) \in E_1$ then $x \notin Q_1(x,y)$. If $(x,y) \in (K \times D) \setminus E_1$, then $x \notin \operatorname{cl} S(x,y)$ and hence $x \notin Q_1(x,y)$. Similarly, one has $y \notin Q_2(x,y), \forall (x,y) \in K \times D$.

Because of (v), for each $(x,y) \in (K \times D) \setminus (\overline{K} \times \overline{D})$, there exist $\bar{x} \in K_0 \cap S(x,y)$ and $x^* \in A(x,y)$, $F(\bar{x},y,x^*) \subseteq -\operatorname{int} C$. Therefore, $K_0 \cap Q_1(x,y) \neq \emptyset$, or there exist $\bar{y} \in D_0 \cap T(x,y)$ and $y^* \in B(x,y)$, $G(\bar{y},x,y^*) \subseteq -\operatorname{int} C$ and hence $D_0 \cap Q_2(x,y) \neq \emptyset$.

Now that all the assumptions of Lemma 1 are satisfied, there exists $(\bar{x},\bar{y}) \in K \times D$ such that $Q_1(\bar{x},\bar{y}) = Q_2(\bar{x},\bar{y}) = \emptyset$. Since $S(\bar{x},\bar{y})$ and $T(\bar{x},\bar{y})$ are nonempty subsets, $(\bar{x},\bar{y})$ must be in $E_1 \times E_2$. Consequently, $\emptyset = Q_1(\bar{x},\bar{y}) = S(\bar{x},\bar{y}) \cap P_1(\bar{x},\bar{y})$ and $\emptyset = Q_2(\bar{x},\bar{y}) = T(\bar{x},\bar{y}) \cap P_2(\bar{x},\bar{y})$. Then $\forall x \in S(\bar{x},\bar{y}), \forall y \in T(\bar{x},\bar{y})$, $x \notin P_1(\bar{x},\bar{y})$ and $y \notin P_2(\bar{x},\bar{y})$, i.e., $\forall x^* \in A(\bar{x},\bar{y}), F(x,\bar{y},x^*) \not\subseteq -\operatorname{int} C$ and $\forall y^* \in B(\bar{x},\bar{y}), G(y,\bar{x},y^*) \not\subseteq -\operatorname{int} C$, which means that $(\bar{x},\bar{y})$ is a solution. ■

The following examples show that none of the assumptions of Theorem 1 can be dropped.

EXAMPLE 1 (Assumption (i) is essential). Let $X = Y = Z = R, K = D = [0,1], C = R_+, S(x,y) \equiv T(x,y) \equiv [0,1], A(x,y) = \{x\}, B(x,y) = \{y\}$, $F(x,\bar{y},x^*) = \{x^* - 2\}$ and $G(y,\bar{x},y^*) = \{y^* - 2\}$.

We check assumptions (ii) - (v). To see (ii), for given $x_i, x_i^* \in A(x,y) = \{x\}$ and $y_i, y_i^* \in B(x,y) = \{y\}$, we simply take $x^* = x_i^*, y^* = y_i^*$. Assumption (iii) is satisfied since the mentioned set is empty. (iv) is clearly fulfilled and (v) is satisfied as K and D are compact. However, problem (SVQEP$_1$) has no solutions, since $\forall(\bar{x},\bar{y}) \in K \times D, \forall(x,x^*) \in S(\bar{x},\bar{y}) \times A(\bar{x},\bar{y}), \forall(y,y^*) \in T(\bar{x},\bar{y}) \times B(\bar{x},\bar{y})$,

$$F(x,\bar{y},x^*) = x^* - 2 < 0,$$
$$G(y,\bar{x},y^*) = y^* - 2 < 0.$$

The reason is that assumption (i) is violated.

EXAMPLE 2 ((ii) is essential). Let $X, Y, Z, C, A(x,y)$ and $B(x,y)$ be as in Example 1. Let $K = D = [0,2]$, $S(x,y) = T(x,y) \equiv [0,2]$ and

$$F(x,\bar{y},x^*) = \begin{cases} \{1\} & \text{if } \ x^* = x, \\ \{-1\} & \text{if } \ x^* \neq x, \end{cases}$$
$$G(y,\bar{x},y^*) = \begin{cases} \{1\} & \text{if } \ y^* = y, \\ \{-1\} & \text{if } \ y^* \neq y. \end{cases}$$

Then assumptions (i), (iii)-(v) are clearly satisfied. However, $\forall(\bar{x},\bar{y}) \in K \times D$, for $x^* \neq x, y^* \neq y$ one has

$$F(x,\bar{y},x^*) = G(y,\bar{x},y^*) = -1 < 0,$$

i.e. problem (SVQEP$_1$) is not solvable. To see the reason we check assumption (ii) by picking $x = 1, y = 1, x_1 = x_2 = y_1 = y_2 = \frac{1}{2}, \alpha_1 = \alpha_2 = \frac{1}{2}, x_1^* = x_2^* = 1 \in A(1,y), y_1^* = y_2^* = 1 \in B(y,1)$. Then, $\forall x^* \in A(1,1)$, $\forall y^* \in B(1,1)$ and $i = 1,2$,

$$F(x_i,y,x_i^*) = F(\tfrac{1}{2},1,1) = -1 < 0,$$
$$G(y_i,x,y_i^*) = G(\tfrac{1}{2},1,1) = -1 < 0,$$

but

$$F(\alpha_1 x_1 + \alpha_2 x_2, y, x^*) = F(1,1,1) = 1 > 0,$$
$$G(\alpha_1 y_1 + \alpha_2 y_2, x, y^*) = G(1,1,1) = 1 > 0,$$

i.e. assumption (ii) is not satisfied.

Example 3 ((iii) is essential). Let X, Y, Z, K, D and C be as in Example 1. Let

$$S(x,y) \equiv [0,\tfrac{1}{4}] \cup [\tfrac{3}{4},1],$$
$$T(x,y) = [0,1],$$
$$A(x,y) = \begin{cases} \{x\} & \text{if} \quad x \neq \frac{1}{2}, \\ \{\frac{x}{2}\} & \text{if} \quad x = \frac{1}{2}, \end{cases}$$
$$B(x,y) = \{y\},$$
$$F(x,\bar{y},x^*) = \begin{cases} \{-1\} & \text{if} \quad x + x^* = 1, \\ \{1\} & \text{otherwise}, \end{cases}$$
$$G(y,x,y^*) \equiv \{1\}.$$

Then, assumptions (i), (iv) and (v) are easy to check. $G(.,x,.)$ clearly satisfies (ii). Let $(x,y) \in K \times D$ and $\{x_1,...,x_n\} \subseteq K$ be arbitrary. If $x \neq \frac{1}{2}$, then $A(x,y) = \{x\}$ and if $F(x_i,y,x_i^*) = F(x_i,y,x) < 0$ then $x_i + x = 1$ and for $\alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1$, $\sum_{i=1}^n \alpha_i x_i + x = 1$. Hence $F\big(\sum_{i=1}^n \alpha_i x_i, y, x\big) = -1 < 0$. If $x = \frac{1}{2}$, then $A(x,y) = \{\frac{x}{2}\}$. From $F(x_i,y,x_i^*) = F(x_i,y,\frac{x}{2}) < 0$ it follows that $x_i + \frac{x}{2} = 1$ and, for $\alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1$, $\sum_{i=1}^n \alpha_i x_i + \frac{x}{2} = 1$. Therefore, $F\big(\sum_{i=1}^n \alpha_i x_i, y, \frac{x}{2}\big) = -1 < 0$. Thus $F(.,y,.)$ satisfies (ii). However, assumption (iii) is violated, since for $(0,0) \in K \times D$, the set

$$\{(\bar{x},\bar{y}) \in K \times D \mid F(0,\bar{y},x^*) \geq 0, x^* \in A(\bar{x},\bar{y})\} = [0,1) \times D$$

is not closed in $K \times D$.

We verify that problem (SVQEP$_1$) is not solvable. Indeed, $\forall(\bar{x},\bar{y}) \in S(\bar{x},\bar{y}) \times T(\bar{x},\bar{y}) = \big([0,\frac{1}{4}] \cup [\frac{3}{4},1]\big) \times [0,1]$, $\exists x \in S(\bar{x},\bar{y}), \exists x^* \in A(\bar{x},\bar{y}) = \{\bar{x}\}$, such that $x + x^* = 1$ and hence $F(x,\bar{y},x^*) = -1 < 0$.

Example 4 ((iv) cannot be dropped). Let $X, Y, Z, C, K, D, A(x,y)$ and $B(x,y)$ be as in Example 2. Let, for $x, \bar{x}, x^* \in K$ and $y, \bar{y}, y^* \in D$, $T(x,y) = [0, y+1) \cap D$, $F(x,\bar{y},x^*) = G(y,\bar{x},y^*) \equiv \{1\}$ and

$$S(x,y) = \begin{cases} [\frac{3}{2},2] & \text{if} \quad x \leq 1, \\ [0,\frac{1}{2}] & \text{otherwise}. \end{cases}$$

Then, (i) is satisfied since $\forall(x,y) \in K \times D$, $\forall(x^*,y^*) \in A(x,y) \times B(x,y) = (x,y)$, $F(x,y,x) = G(y,x,y) = \{1\}$. Assumptions (ii) is clearly satisfied for $F(.,y,.)$ and $G(.,x,.)$. To check (iii) we have, $\forall(x,y) \in K \times D$,

$$U_1 := \{(\bar{x},\bar{y}) \in K \times D \mid F(x,\bar{y},x^*) \geq 0, \forall x^* \in A(\bar{x},\bar{y})\} = K \times D.$$
$$U_2 := \{(\bar{x},\bar{y}) \in K \times D \mid G(y,\bar{x},y^*) \geq 0, \forall y^* \in B(\bar{x},\bar{y})\} = K \times D.$$

Therefore, $U_i = K \times D$ is closed in $K \times D$, $i = 1, 2$. Finally, as K and D are compact, assumption (v) is obviously fulfilled. However, problem (SVQEP$_1$)

has no solution, since $E = \emptyset$. The reason is that assumption (iv) is violated, since (although $\mathrm{cl}\, T(.,.)$ are continuous, $T^{-1}(y)$ is open in $K \times D$ for $y \in D$) $\mathrm{cl}\, S$ is not usc in $K \times D$.

EXAMPLE 5 ((v) cannot be omitted). Let $X = Y = Z = K = D = R, C = R_+, S(x,y) = T(x,y) \equiv R, A(x,y) = \{x\}, B(x,y) = \{y\}$ and

$$F(x, \bar{y}, x^*) = \{x - x^*\},$$
$$G(y, \bar{x}, y^*) = \{y - y^*\}.$$

Then, it is easy to see that assumptions (i)-(iv) are fulfilled. However, problem (SVQEP_1) has no solutions, since $\forall(\bar{x}, \bar{y}) \in K \times D, \exists(x,y) \in S(\bar{x}, \bar{y}) \times T(\bar{x}, \bar{y}), \exists(x^*, y^*) \in A(\bar{x}, \bar{y}) \times B(\bar{x}, \bar{y}) = \{(\bar{x}, \bar{y})\}$,

$$F(x, \bar{y}, x^*) = x - x^* < 0,$$
$$G(y, \bar{x}, y^*) = y - y^* < 0.$$

To see that assumption (v) is violated let $\overline{K} \times \overline{D} \subseteq K \times D$ and $K_0 \times D_0 \subseteq K \times D$ be compact. Then, there is $(x,y) \in R^2 \setminus \overline{K} \times \overline{D}$ such that $\forall(\bar{x}, \bar{y}) \in K_0 \times D_0, \forall(x^*, y^*) \in A(x,y) \times B(x,y) = \{(x,y)\}$,

$$F(\bar{x}, y, x^*) = \bar{x} - x^* = \bar{x} - x \geq 0,$$
$$G(\bar{y}, x, y^*) = \bar{y} - y^* = \bar{y} - y \geq 0.$$

i.e. (v) is not fulfilled.

Passing to problem (SVQEP_2) we have

THEOREM 2 *Assume five conditions corresponding to that of Theorem 1: in* (i) *and* (iii) *"$\not\subseteq -\mathrm{int}\, C$" is replaced by "$\subseteq Z \setminus -\mathrm{int}\, C$"; in* (ii) *"type 1" is replaced by "type 2";* (iv) *remains the same; and in* (v) *"$\subseteq -\mathrm{int}\, C$" is replaced by "$\not\subseteq Z \setminus -\mathrm{int}\, C$".*
Then problem (SVQEP_2) *has solutions.*

Proof. We can adopt the same lines of proof as in Theorem 1 with new multifunctions $P_1(x,y)$ and $P_2(x,y)$ defined as:

$$P_1(x,y) = \{\hat{x} \in K : \exists\, x^* \in A(x,y,),\ F(\hat{x}, y, x^*) \not\subseteq Z \setminus -\mathrm{int}\ C\},$$
$$P_2(x,y) = \{\hat{y} \in D : \exists\, y^* \in B(x,y), G(\hat{y}, y, y^*) \not\subseteq Z \setminus -\mathrm{int}\ C\}.$$

REMARK 1 Since our two problems coincide if F and G are single-valued, Examples 1–5 indicate also that each of the five assumptions of Theorem 2 is essential. They explain also that in general it is not hard to check the assumptions. The following example shows that our assumptions are very relaxed by proving a case of the problem considered in Fu (2003) and Farajzadeh (2006) but the results there cannot be applied while ours can.

EXAMPLE 6 Let $X, Y, Z, C, K, D, S, T, A$ and B be as in Example 1. Let

$$F(x,\bar{y},x^*) = f(x,\bar{y}) - f(x^*,\bar{y}),$$
$$G(y,\bar{x},y^*) = g(\bar{x},y) - g(\bar{x},y^*),$$

where

$$f(x,y) = \begin{cases} 1 & \text{if } x < \frac{1}{2}, \\ -1 & \text{if } x \geq \frac{1}{2}, \end{cases}$$
$$g(x,y) = \begin{cases} 1 & \text{if } y < \frac{1}{2}, \\ -1 & \text{if } y \geq \frac{1}{2}. \end{cases}$$

Then assumptions (i), (iv) and (v) are clearly fulfilled (Theorems 1 and 2 coincide in this case). To check (ii), we have

$$F(x,\bar{y},x^*) = \begin{cases} 2 & \text{if } x < \frac{1}{2}, x^* \geq \frac{1}{2}; \\ 0 & \text{if } x,^* x < \frac{1}{2} \text{ or } x, x^* \geq \frac{1}{2}; \\ -2 & \text{if } x \geq \frac{1}{2}, x^* < \frac{1}{2}. \end{cases}$$

For $x, y \in [0,1], \{x_1, \cdots, x_n\} \subseteq R$ and $\{x_1^*, \cdots, x_n^*\} \subseteq A(x,y) = \{x\}$, if $F(x_i, y, x_i^*) < 0$, then $x_i \geq \frac{1}{2}$ and $x_i^* = x < \frac{1}{2}$. Hence, for $\alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1$, taking $x^* = x$ we have

$$F\left(\sum_{i=1}^n \alpha_i x_i, y, x^*\right) = -2 < 0.$$

As $\sum_{i=1}^n \alpha_i x_i \geq \frac{1}{2}$, the same argument is valid for G. Therefore, (ii) is satisfied.

To see (iii) being fulfilled consider any $x, y \in [0,1]$. If $x \geq \frac{1}{2}$, then $F(x,\bar{y},x^*) \geq 0, \forall \bar{y} \in [0,1]$ and for $x^* \geq \frac{1}{2}$. If $x < \frac{1}{2}$, then $F(x,\bar{y},x^*) \geq 0, \forall \bar{y}$ and for $x^* \in [0,1]$. The argument for G is similar. Hence, setting

$$U = \{(\bar{x},\bar{y}) \in K \times D \mid F(x,\bar{y},x^*) \geq 0, G(y,\bar{x},y^*) \geq 0, \text{ for } (x^*,y^*) = (\bar{x},\bar{y})\}$$

we see that

$$U = [\tfrac{1}{2},1] \times [\tfrac{1}{2},1], \quad \text{if} \quad x \geq \tfrac{1}{2},\ y \geq \tfrac{1}{2},$$
$$U = [\tfrac{1}{2},1] \times D, \quad \text{if} \quad x \geq \tfrac{1}{2},\ y < \tfrac{1}{2},$$
$$U = K \times [\tfrac{1}{2},1] \quad \text{if} \quad x < \tfrac{1}{2},\ y \geq \tfrac{1}{2},$$
$$U = K \times D \quad \text{if} \quad x < \tfrac{1}{2},\ y < \tfrac{1}{2}.$$

Thus, $\forall (x,y) \in K \times D, U$ is closed in $K \times D$. By Theorem 1 (or, the same, Theorem 2 problem (SVQEP$_1$) has solutions. However, since

$$f^{-1}\big([0,+\infty)\big) = (-\infty, \tfrac{1}{2}) \times R,$$
$$g^{-1}\big([0,+\infty)\big) = R \times (-\infty, \tfrac{1}{2})$$

are not closed in R^2, f and g are not demicontinuous and the results in Fu (2003) and Farajzadeh (2006) cannot be employed. Recall here that a mapping $f : X \to Z$ is said to be demicontinuous if $f^{-1}(M)$ is closed in X for each closed half space M in Z. Checking directly we see that the solution set is $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$.

3. Applications

Since our symmetric quasiequilibrium problems include many rather general problems as particular cases as mentioned in Section 1, Theorem 1 and 2 imply directly new results for these problems. In this section we present only several typical applications showing clearly the advantages of the symmetric structure of the problem setting.

3.1. A lower and upper bounded quasiequilibrium problem

Let X and K be as in Section 1. Let $S : K \to 2^K, f : K \times K \to R, \alpha, \beta \in R$. The lower and upper bounded quasiequilibrium problem consists of
(BQEP) finding $\bar{x} \in K$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}), \forall x \in S(\bar{x})$,

$$\alpha \leq f(\bar{x}, x) \leq \beta.$$

Corollary 1 *Assume that*

(i) $\forall x \in K, \alpha \leq f(x, x) \leq \beta$;

(ii) $f(.,.)$ $\big($*and* $-f(.,.)\big)$ *is* $\alpha-$*level* ($\beta-$*level, respectively*) R_+-*quasiconvex relative to* K *of type 1;*

(iii) $\forall (x, y) \in K \times K$, *the sets* $\{(\bar{x}, \bar{y}) \in K \times K \mid f(\bar{x}, x) \geq \alpha\}$ *and* $\{(\bar{x}, \bar{y}) \in K \times K \mid f(\bar{x}, x) \leq \beta\}$ *are closed in* $K \times K$;

(iv) $\operatorname{cl} S(.)$ *is usc in* K *and,* $\forall x \in K, S^{-1}(x)$ *is open in* K;

(v) *if* K *is not compact, there exist a nonempty compact subset* $\overline{K}$ *of* K *and a nonempty compact convex subset* K_0 *of* K *such that for each* $x \in K \setminus \overline{K}$, *there is* $\bar{x} \in K_0 \cap S(x)$,

$$f(x, \bar{x}) < \alpha,$$

or there is $\bar{y} \in K_0 \cap S(x)$,

$$f(x, \bar{y}) > \beta.$$

Then (BQEP) *has solutions.*

Proof. Setting $Y = X, D = K, Z = R, C = R_+, S(x, y) = T(x, y) = S(x), A(x, y) = \{x\}, B(x, y) = \{y\}, F(x, \bar{y}, x^*) = f(x^*, x) - \alpha$ and $G(y, \bar{x}, y^*) = \beta - f(\bar{x}, y)$, problem (BQEP) becomes a particular case of ($SVQEP_1$) and the corollary is a direct consequence of Theorem 1. ■

3.2. A coincidence point problem

Let X, Y, K and D be as in Section 1. Let $U : D \to 2^K$ and $V : K \to 2^D$ be multifunctions with nonempty convex images. We consider the following coincidence point problem
(CP) find $(\bar{x}, \bar{y}) \in K \times D$ such that $\bar{x} \in \mathrm{cl}\, U(\bar{y}), \bar{y} \in \mathrm{cl}\, V(\bar{x})$.

COROLLARY 2 *Assume that*

(a) $\mathrm{cl}\, U(.)$ *and* $\mathrm{cl}\, V(.)$ *are usc and,* $\forall (x, y) \in K \times D$, $V^{-1}(y)$ *and* $U^{-1}(x)$ *are open in* K *and* D*, respectively*;

(b) K *and* D *are compact.*

Then problem (CP) *has solutions.*

Proof. We set $Z = R, C = R_+, S(x, y) = U(y), T(x, y) = V(x), A(x, y) = \{x\}$, $B(x, y) = \{y\}, F(x, \bar{y}, x^*) \equiv G(y, \bar{x}, y^*) \equiv \{1\}$. Then (CP) becomes a special case of (SVQEP_1).

To apply Theorem 1 we see that assumptions (i)-(iii) are obviously satisfied. Assumption (iv) is fulfilled by (a) and (v) - by (b). Hence Theorem 1 yields the solvability of (CP). ■

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