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A characterization of stability and sensitivity properties for state-constrained optimal control

by

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Abstract: In a series of the recent papers of the author, it was shown that the solutions and Lagrange multipliers of state-constrained optimal control problems are locally Lipschitz continuous and directionally differentiable functions of the parameter, under usual constraint qualifications and weakened second order conditions. In this paper, it is shown that those conditions are not only sufficient, but also necessary. Thus, they consitute a characterization of Lipschitz stability and sensitivity properties for state-constrained optimal control problems.

Keywords: optimal control, nonlinear ODEs, state constraints, parametric problems, stability and sensitivity analysis.

1. Introduction

In stability and sensitivity analysis for optimal control problems, conditions are investigated, under which the solutions and the associated Lagrange multipliers are locally Lipschitz continuous and directionally differentiable functions of the parameters. It is known that these conditions consist of constraint qualifications and second order sufficient optimality conditions, which should be satisfied at the reference point.

For control constrained problems a complete characterization of the Lipschitz stability was derived (see Dontchev and Malanowski, 2000, and Malanowski, 2001). The situation was different for problems with first order state constraints, where strong second order sufficient optimality conditions were used (Malanowski, 1995; Dontchev and Hager, 1998). However, in a series of the recent papers of the author (see Malanowski, 2007 a,b,c,d), it was shown that the stability and sensitivity results can be obtained under weakened second order conditions. These conditions are weaker in the sense that they take into account the strongly active state constraints. In this paper, we show that the new weakened second order conditions, together with the constraint qualifications, constitute not only sufficient, but also necessary conditions of Lipschitz stability and directional differentiability of the solutions and Lagrange multipliers for optimal control problems subject to first order state constraints. Thus, they constitute a characterization of these properties.

The organization of the paper is the following. In Section 2 the model optimal control problem is introduced and the basic assumptions are formulated. In Section 3, the sufficient conditions of Lipschitz stability and directional differentiability of the solutions and Lagrange multipliers, derived in Malanowski (2007d), are recalled. It is shown in Section 4, that those conditions are also necessary, and so they constitute a characterization of the stability and sensitivity properties.

2. Optimal control problem

In this section our model optimal control problem is formulated. Let us introduce the following spaces:

$$\begin{cases} Z = L^2(0,1;\mathbb{R}^n) \times L^2(0,1;\mathbb{R}^m) \times L^2(0,1;\mathbb{R}^n) \times W^{1,2}(0,1;\mathbb{R}), \\ H = W^{1,\infty}(0,1;\mathbb{R}^n) \times W^{1,\infty}(0,1;\mathbb{R}^m) \times W^{1,\infty}(0,1;\mathbb{R}^n) \times W^{2,\infty}(0,1;\mathbb{R}), \\ X^p = W_0^{1,p}(0,1;\mathbb{R}^n) \times L^p(0,1;\mathbb{R}^m), \quad p \in [1,\infty]. \end{cases} \tag{1}$$

$H \subset Z$ and X^2 will be the spaces of parameters and arguments, respectively. Consider the family of the following optimal control problems depending on $h \in H$:

$(\mathrm{O})_h$ Find $(x_h, u_h) \in X^2$ such that

$$F(x_h, u_h, h) = \min \left\{ F(x,u,h) := \int_0^1 \varphi(x(t), u(t), h(t)) dt \right\} \tag{2}$$

subject to

$$\dot{x}(t) - f(x(t), u(t), h(t)) = 0 \quad \text{for a.a. } t \in [0,1], \tag{3}$$

$$x(0) = 0, \tag{4}$$

$$\vartheta(x(t), h(t)) \leq 0, \qquad \text{for all } t \in [0,1]\ , \tag{5}$$

where

$$\begin{cases} \varphi(x(t), u(t), h(t) = \varphi_0(x(t), u(t)) - \langle h_1(t), x(t) \rangle - \langle h_2(t), u(t) \rangle, \\ f(x(t), u(t), h(t)) = f_0(x(t), u(t)) - h_3(t), \\ \vartheta(x(t), h(t)) = \vartheta_0(x(t)) - h_4(t), \\ \varphi_0 :\ \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad f_0 :\ \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \quad \vartheta_0 :\ \mathbb{R}^n \to \mathbb{R}. \end{cases} \tag{6}$$

REMARK 1 To minimize technicalities, we consider the homogeneous initial condition (4) of the state equation and the scalar-valued state constraints. However, the results can be extended to general the two-point boundary conditions and vector-valued state constraints.

The following standing assumptions are supposed to be satisfied throughout the paper:

(I) There exist open sets $\mathcal{R}^n \subset \mathbb{R}^n$ and $\mathcal{R}^m \subset \mathbb{R}^m$ such that the functions $\varphi_0(\cdot,\cdot)$ and $f_0(\cdot,\cdot)$ are twice Fréchet differentiable in (x,u) on $\mathcal{R}^n \times \mathcal{R}^m$, whereas $\vartheta_0(\cdot)$ is three times differentiable in x on $\mathcal{R}^n$.

(II) For a given reference value $\widehat{h} \in H$ of the parameter there exists a reference solution $(\widehat{x},\widehat{u})$ of $(\mathrm{O})_{\widehat{h}}$, where $\widehat{u} \in C(0,1;\mathbb{R}^m)$ and $(\widehat{x}(t),\widehat{u}(t)) \in \mathcal{R}^n \times \mathcal{R}^m$ for all $t \in [0,1]$.

To simplify notation, the functions evaluated at the reference solution will be denoted by "hat", e.g., $\widehat{\varphi} := \varphi(\widehat{x},\widehat{u},\widehat{h})$, $\widehat{\vartheta} := \vartheta(\widehat{x},\widehat{h})$. Let us define the following spaces of multipliers

$$Y^p = L^p(0,1;\mathbb{R}^n) \times W^{1,p}(0,1;\mathbb{R}), \qquad p \in [1,\infty], \tag{7}$$

and introduce the Lagrangian and Hamiltonian for $(\mathrm{O})_h$:

$$\mathcal{L} : X^2 \times Y^2 \times H \to \mathbb{R}, \quad \mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n+m+n+2} \to \mathbb{R},$$

$$\begin{cases} \mathcal{L}(x,u,p,\mu;h) = F(x,u,h) - (p, \dot{x} - f(x,u,h)) \\ \qquad\qquad + \mu(0)\vartheta(x(0),h(0)) + (\dot{\mu}, D_x\vartheta_0(x) f(x,u,h) - \dot{h}_4), \end{cases} \tag{8}$$

$$\begin{cases} \mathcal{H}(x(t),u(t),p(t),\dot{\mu}(t);h(t),\dot{h}(t)) = \varphi(x(t),u(t),h(t)) \\ \quad + \langle p(t), f(x(t),u(t),h(t))\rangle + \dot{\mu}(t)(D_x\vartheta_0(x(t)) f(x(t),u(t),h(t)) - \dot{h}_4(t)). \end{cases} \tag{9}$$

REMARK 2 Lagrangian (8) is in the so called *indirect* or Pontryagin form (see Section 5 in Hartl, Sethi and Vickson, 1995, as well as Hager, 1979 and Neustadt, 1976), where the state constraints are considered in the space $W^{1,2}(0,1;\mathbb{R})$. The general form of linear functionals, in that space, is given by $\mu(0)y(0) + (\dot{\mu},\dot{y})$, with $\mu \in W^{1,2}(0,1;\mathbb{R})$. Hence, using the state equation, we get

$$\mu(0)\vartheta(x(0),h(0)) + (\dot{\mu}, \tfrac{d}{dt}\vartheta(x,h)) = \\ \mu(0)\vartheta(x(0),h(0)) + (\dot{\mu}, D_x\vartheta_0(x) f(x,u,h) - \dot{h}_4).$$

Denote by $K = \{d \in W^{1,2}(0,1;\mathbb{R}) \mid d(t) \le 0\}$ the cone of nonpositive functions in $W^{1,2}(0,1;\mathbb{R})$. The cone polar to K is given (see e.g., Outrata and Schindler, 1980) by:

$$K^+ = \{W^{1,2}(0,1;\mathbb{R}) \mid \mu(0) - \dot{\mu}(0+) \ge 0,\ \dot{\mu}(t) \ge 0 \text{ and } \dot{\mu}(\cdot) \text{ is nonincreasing}\}. \tag{10}$$

Clearly, if $\mu \in W^{2,2}(0,1;\mathbb{R})$, the last condition in (10) reduces to $\ddot{\mu}(t) \leq 0$ for almost all $t \in [0,1]$. The normal cone to K^+ at μ is denoted by

$$\mathcal{N}_{K^+}(\mu) := \begin{cases} \{y \in W^{1,2}(0,1;\mathbb{R}) \mid (y, \nu - \mu)_{1,2} \leq 0 \ \forall \nu \in K^+\}, & \text{if } \mu \in K^+, \\ \emptyset & \text{if } \mu \notin K^+. \end{cases} \tag{11}$$

In view of (6), the stationarity conditions of Lagrangian (8) can be expressed by the following system:

$$\begin{cases} \dot{p} + D_x\mathcal{H}(x,u,p,\dot{\mu};h,\dot{h}) \\ \quad := \dot{p} + D_x f_0^*(x,u)p + D_x\varphi_0(x,u) + (D_x f_0^*(x,u)D_x\vartheta_0^*(x) \\ \quad +(f_0(x,u) - h_3)^* D_{xx}^2\vartheta_0^*(x))\dot{\mu} - h_1 = 0, \\ p(1) = 0, \end{cases} \tag{12}$$

$$\begin{cases} D_u\mathcal{H}(x,u,p,\dot{\mu};h,\dot{h}) \\ \quad := D_u\varphi_0(x,u) + D_u f_0^*(x,u)p + D_u f_0^*(x,u,h)D_x\vartheta_0^*(x)\dot{\mu} - h_2 = 0, \end{cases} \tag{13}$$

$$\vartheta_0(x) - h_4 \in \mathcal{N}_{K^+}(\mu). \tag{14}$$

simplicity, we will denote $\xi = (x,u) \in X^2$, $\lambda = (p,\mu) \in Y^2$.

In Malanowski (2007d) conditions were studied, under which the solutions and Lagrange multipliers of $(\mathrm{O})_h$ are locally Lipschitz continuous and directionally differentiable functions of the parameter, in a neighborhood of the reference point. The purpose of this paper is to investigate how far the sufficient conditions derived in Malanowski (2007d) are from the conditions necessary for those properties.

3. Basic stability and sensitivity results

The assumptions needed in stability and sensitivity analysis in Malanowski (2007d) consist of some constraint qualifications and coercivity conditions, satisfied at the reference point. To formulate constraint qualifications, for a fixed $\alpha \geq 0$, introduce the set of α-active constraints:

$$M_\alpha = \{t \in [0,1] \mid \vartheta(\widehat{x}(t), \widehat{h}(t)) \geq -\alpha\}. \tag{15}$$

Assume:

(A1) There exists $\alpha > 0$ such that $0 \notin M_\alpha$.

(A2) (*Linear independence*) There exist $\alpha > 0$ and $\chi > 0$ such that

$$|D_u\widehat{f}^*(t)D_x\widehat{\vartheta}^*(t)| \geq \chi \quad \text{for all } t \in M_\alpha.$$

Note that by (A2) the analysis is restricted to the so called *first order state constraints*, Hartl, Sethi and Vickson (1995). By Theorem 4.3 in Malanowski (2003) we get:

Lemma 1 *If assumptions* (A1) *and* (A2) *are satisfied, then there exists a unique Lagrange multiplier* $\widehat{\lambda} = (\widehat{p}, \widehat{\mu}) \in Y^2$ *such that the first order optimality conditions* (12)-(14) *hold at* $(\widehat{x}, \widehat{u}, \widehat{p}, \widehat{\mu})$.

In addition to the constraint qualifications, we will need some coercivity conditions. Assume:

(A3) (*Legendre-Clebsch condition*) there exists $\bar{\gamma} > 0$ such that

$$\langle v, D^2_{uu}\widehat{\mathcal{L}}(t)v\rangle \geq \bar{\gamma}|v|^2 \qquad \text{for all } v \in \mathbb{R}^m \text{ and all } t \in [0,1].$$

The following regularity result is a special case of Theorem 2.1 in Hager, 1979 (see Proposition 6.6 in Malanowski, 1995):

Lemma 2 *If assumptions* (A1) – (A3) *are satisfied, then* $\widehat{u}, \dot{\widehat{x}}, \dot{\widehat{p}}, \dot{\widehat{\mu}}$ *are Lipschitz continuous on* $[0,1]$.

Let us denote by $\widehat{\sigma} > 0$ the Lipschitz modulus in Lemma 2. For a fixed $\sigma > \widehat{\sigma}$ introduce the following sets:

$$\Xi = \{(x,u) \in X^2 \mid \|\ddot{x}\|_\infty, \|\dot{u}\|_\infty \leq \sigma\}, \quad \Lambda = \{(p,\mu) \in Y^2 \mid \|\ddot{p}\|_\infty, \|\ddot{\mu}\|_\infty \leq \sigma\}. \tag{16}$$

In view of the uniqueness and regularity of $\widehat{\mu}$, we can introduce the following sets, open in $[0,1]$, which depend on the parameter $\alpha > 0$:

$$N_\alpha = [0,1] \setminus \overline{\{t \in [0,1] \mid -\ddot{\widehat{\mu}}(t) \leq \alpha\}}, \quad \text{as well as } N_0 = \bigcup_{\alpha > 0} N_\alpha. \tag{17}$$

Define

$$\mathcal{E}_\alpha = \left\{ (y,v) \in X^2 \mid \left\{ \begin{array}{l} \dot{y}(t) - D_x\widehat{f}(t)y(t) - D_u\widehat{f}(t)v(t) = 0, \\ \langle D_x\widehat{\vartheta}(t), y(t)\rangle = 0 \quad \forall t \in N_\alpha, \\ \langle D_x\widehat{\vartheta}(1), y(1)\rangle = 0 \;\text{ if } \dot{\widehat{\mu}}(1) > 0. \end{array} \right. \right\}. \tag{18}$$

For the sake of simplicity we will denote

$$\begin{aligned} D^2\widehat{\mathcal{L}} &:= D^2_{(x,u)(x,u)}\mathcal{L}(\widehat{x},\widehat{u},\widehat{p},\widehat{\mu};\widehat{h}) \\ &= \begin{pmatrix} D^2_{xx}\mathcal{H}(\widehat{x},\widehat{u},\widehat{p},\widehat{\mu};\widehat{h},\dot{\widehat{h}}) & D^2_{xu}\mathcal{H}(\widehat{x},\widehat{u},\widehat{p},\widehat{\mu};\widehat{h},\dot{\widehat{h}}) \\ D^2_{ux}\mathcal{H}(\widehat{x},\widehat{u},\widehat{p},\widehat{\mu};\widehat{h},\dot{\widehat{h}}) & D^2_{uu}\mathcal{H}(\widehat{x},\widehat{u},\widehat{p},\widehat{\mu};\widehat{h},\dot{\widehat{h}}) \end{pmatrix}. \end{aligned} \tag{19}$$

Assume:

(A4) (*Coercivity*) There exist constants $\alpha > 0$ and $\gamma > 0$ such that

$$\Big((y,v), D^2\widehat{\mathcal{L}}(y,v)\Big) \geq \gamma(\|y\|^2_{1,2} + \|v\|^2_2) \quad \text{for all } (y,v) \in \mathcal{E}_\alpha. \tag{20}$$

REMARK 3 The coercivity condition (A4) takes into account strongly active state constraints. It is weaker than the strong coercivity condition, where the active inequality constraints are ignored. The latter condition was used in stability analysis in Malanowski (1995) and in Dontchev and Hager (1998). The application of the weaker condition (A4) is the main new contribution of Malanowski (2007a,b,c,d). In this paper, necessity of conditions (A1), (A2), (A3) and (A4) will be investigated.

The following result is proved in Malanowski (2007d) (see Lemma 3.2):

LEMMA 3 *If conditions* (A1) *and* (A2) *are satisfied, then* (A4) *implies* (A3).

We will need the following necessary optimality condition for $(\mathrm{O})_{\widehat{h}}$ (see Theorem 3.5 in Malanowski, 2007d):

PROPOSITION 1 *If* $(\widehat{x},\widehat{u})$ *is a solution of* $(\mathrm{O})_{\widehat{h}}$ *and conditions* (A1)–(A2) *are satisfied, then*

$$\begin{cases} ((y,v),D^2\widehat{\mathcal{L}}(y,v))\geq 0 \quad \text{for all } (y,v)\in X^2 \text{ such that} \\ \dot{y}(t)-D_x\widehat{f}(t)y(t)-D_u\widehat{f}(t)v(t)=0, \quad y(0)=0, \\ \langle D_x\widehat{\vartheta}(t),y(t)\rangle=0 \qquad\qquad \text{for all } t\in M_0. \end{cases} \tag{21}$$

In the sequel, by $\mathcal{B}^X_\rho(y)=\{x\in X \mid \|x-y\|_Z<\rho\}$ we will denote the open ball in a Banach space X of radius $\rho>0$, centered at y. Moreover, we will denote $\mathcal{B}^{\Xi\times\Lambda}_\tau(\widehat{\xi},\widehat{\lambda})=\mathcal{B}^{X^2\times Y^2}_\tau(\widehat{\xi},\widehat{\lambda})\cap(\Xi\times\Lambda)$.

For our purpose, the principal stability and sensitivity results obtained in Malanowski (2007d) can be formulated as follows:

THEOREM 1 *Suppose that* (A1)–(A4) *are satisfied. Then there exist constants* $\theta>0$, $\tau>0$ *and* $\ell>0$ *such that for each* $h\in\mathcal{B}^H_\theta(\widehat{h})$, *there exists a stationary point* $(\xi_h,\lambda_h):=(x_h,u_h,p_h,\mu_h)$ *of* $(\mathrm{O})_h$, *which is unique in* $\mathcal{B}^{\Xi\times\Lambda}_\tau(\widehat{\xi},\widehat{\lambda})$, *where* (x_h,u_h) *is a solution. Moreover*

$$\begin{aligned} &\|x_{h'}-x_{h''}\|_{1,2}, \|u_{h'}-u_{h''}\|_2, \|p_{h'}-p_{h''}\|_{1,2}, \|\mu_{h'}-\mu_{h''}\|_{1,2}\leq \ell\|h'-h''\|_Z \\ &\qquad \text{for all } h',h''\in\mathcal{B}^H_\theta(\widehat{h}). \end{aligned} \tag{22}$$

Finally, (ξ_h,λ_h) *is directionally differentiable on* $\mathcal{B}^H_\theta(\widehat{h})$ *and the directional differential at* $h\in\mathcal{B}^H_\theta(\widehat{h})$, *in the direction* $g=(g_1,g_2,g_3,g_4)\in H$, *is given by the stationary point of the following linear-quadratic optimal control problem:*

$$(\mathrm{DO})_{h,g} \quad \begin{aligned} &\text{Minimize } J_h(y,v;g) \text{ subject to} \\ &\dot{y}(t)-D_xf(x_h(t),u_h(t),h(t))y(t)-D_uf(x_h(t),u_h(t),h(t))v(t) \\ &\qquad -g_3(t)=0, \quad y(0)=0, \\ &\langle D_x\vartheta(x_h(t),h(t)),y(t)\rangle-g_4(t)\begin{cases} =0 & \text{for all } t\in N_{h,0}, \\ \leq 0 & \text{for all } t\in M_{h,0}\setminus N_{h,0}, \end{cases} \\ &\langle D_x\vartheta(x_h(1),h(1)),y(1)\rangle-g_4(1)=0 \quad \text{if } \dot{\mu}_h(1-)>0, \end{aligned}$$

where

$$\begin{cases} J_h(y,v;g) &= \frac{1}{2}\big((y,v), D^2\mathcal{L}(x_h,u_h,p_h,\mu_h;h)(y,v)\big) - (g_1,y) - (g_2,v), \\ M_{h,0} &= \{t \in [0,1] \mid \vartheta(x_h(t),h(t)) = 0\}, \\ N_{h,\alpha} &= [0,1] \setminus \overline{\{t \in (0,1) \mid -\ddot{\mu}_h(t) \le \alpha\}}, \quad \textit{for } \alpha > 0, \\ N_{h,0} &= \bigcup_{\alpha>0} N_{h,\alpha}. \end{cases} \tag{23}$$

REMARK 4 Note that in condition (22) two norms are involved. That estimate holds for h in a ball in H, whereas, on the right hand side of the inequality, there appears the weaker norm of the space Z, defined in (1). The fact that we have to confine ourselves to more regular perturbations, is connected with the difficulties arising in the application of the Robinson's implicit function theorem (Robinson, 1980) to state-constrained optimal control problems (see Malanowski, 1993). These difficulties are caused by the phenomenon of the so called *two-norm discrepancy* (see Maurer, 1981).

Suppose that the following *strict complementarity* conditions hold at h:

$$\overline{M}_{h,0} = \overline{N}_{h,0} \quad \text{and } \dot{\mu}_h(1-) > 0 \text{ if } \vartheta(x_h(1),h(1)) = 0, \tag{24}$$

then only the equality type constraints are present in $(\mathrm{DO})_{h,g}$ and the stationary point becomes independent of the sign of the vector g. Hence we get:

COROLLARY 1 *If, in addition to the assumptions of Theorem* 1*, the strict complementarity conditions* (24) *hold, then the stationary point* (ξ_h,λ_h) *is Gâteaux differentiable on* H *at* h*.*

4. Necessary conditions for stability and sensitivity

In this section the conditions are investigated necessary in order that the stability and sensitivity properties be sasfied for $(\mathrm{O})_h$. We assume:

(H) Conditions (I) and (II) hold and there exist constants $\theta > 0$, $\tau > 0$ and $\ell > 0$, such that, for each $\mathcal{B}^H_\theta(\widehat{h})$ there is a unique in $\mathcal{B}^{\Xi\times\Lambda}_\tau(\widehat{\xi},\widehat{\lambda})$ stationary point $(\xi_h,\lambda_h) := (x_h,u_h,p_h,\mu_h)$ of $(\mathrm{O})_h$, where ξ_h is a solution. Moreover, the estimates (22) hold and (ξ_h,λ_h) is directionally differentiable in H.

We will show that (H) implies that conditions (A1)-(A4) hold with some $\alpha>0$, so, in view of Theorem 1, these conditions constitute a characterization of the Lipschitz stability and directional differentiability properties for the solutions and Lagrange multipliers of $(\mathrm{O})_h$. Let us start with the following lemma:

LEMMA 4 *If conditions* (A1), (A2) *and* (A4) *are satisfied for* $\alpha = 0$*, then they are also satisfied for some* $\alpha > 0$*.*

Proof. For (A1) and (A2), the proof follows immediately from the continuity of $\widehat{\vartheta}(\cdot)$ and of $D_u\widehat{f}^*(\cdot)D_x\widehat{\vartheta}^*(\cdot)$. To prove the assertion of the lemma for (A4), let us assume that there exists $\widetilde{\gamma} > 0$, such that

$$\Big((y,v), D^2\widehat{\mathcal{L}}(y,v)\Big) \ge \widetilde{\gamma}(\|y\|^2_{1,2} + \|v\|^2_2) \quad \text{for all } (y,v) \in \mathcal{E}_0. \tag{25}$$

Then, by Lemma 3, there exists $\bar{\gamma} > 0$ such that

$$\left(v, D^2_{uu}\widehat{\mathcal{L}}v\right) \geq \bar{\gamma}\|v\|_2^2 \quad \text{for all } v \in L^2(0,1;\mathbb{R}^m). \tag{26}$$

Suppose that the assertion of the lemma is not true. Then, for any $\gamma > 0$ and any $\alpha > 0$, there exists $(y_\alpha, v_\alpha) \in \mathcal{E}_\alpha$, $\|(y_\alpha, v_\alpha)\|_{X^2} = 1$, such that

$$\left((y_\alpha, v_\alpha), D^2\widehat{\mathcal{L}}(y_\alpha, v_\alpha)\right) < \gamma. \tag{27}$$

Let us choose a sequence $\{(\alpha_i, \gamma_i)\} \to (0,0)$ and let $\{(y_{\alpha_i}, v_{\alpha_i})\}$ be the corresponding sequence of normalized elements $(y_{\alpha_i}, v_{\alpha_i}) \in \mathcal{E}_{\alpha_i}$, such that (27) holds for $\gamma = \gamma_i$. We can extract a weakly convergent subsequence, still denoted $\{(y_{\alpha_i}, v_{\alpha_i})\}$, i.e., there exists an element $(\bar{y}, \bar{v}) \in X^2$ such that:

$$\begin{array}{lll} y_{\alpha_i} \rightharpoonup \bar{y} & \text{weakly in } W^{1,2}(0,1;\mathbb{R}^n), \text{ i.e., strongly in } C(0,1;\mathbb{R}^n), \\ v_{\alpha_i} \rightharpoonup \bar{v} & \text{weakly in } L^2(0,1;\mathbb{R}^m). \end{array} \tag{28}$$

Note that, in view of (26), $\left(v, D^2_{uu}\widehat{\mathcal{L}}v\right)$ is weakly lower-semicontinuous on $L^2(0,1;\mathbb{R}^m)$. Hence, passing to the limit in (27), we get

$$\begin{aligned} 0 &= \lim_{i\to\infty} \gamma_i \geq \overline{\lim}_{i\to\infty} \left((y_{\alpha_i}, v_{\alpha_i}), D^2\widehat{\mathcal{L}}(y_{\alpha_i}, v_{\alpha_i})\right) \\ &\geq \underline{\lim}_{i\to\infty} \left((y_{\alpha_i}, v_{\alpha_i}), D^2\widehat{\mathcal{L}}(y_{\alpha_i}, v_{\alpha_i})\right) \\ &= \underline{\lim}_{i\to\infty} \left(y_{\alpha_i}, D^2_{xx}\widehat{\mathcal{L}}y_{\alpha_i}\right) + 2\underline{\lim}_{i\to\infty} \left(y_{\alpha_i}, D^2_{xu}\widehat{\mathcal{L}}v_{\alpha_i}\right) \\ &\quad + \underline{\lim}_{i\to\infty} \left(v_{\alpha_i}, D^2_{uu}\widehat{\mathcal{L}}v_{\alpha_i}\right) \geq \left((\bar{y}, \bar{v}), D^2\widehat{\mathcal{L}}(\bar{y}, \bar{v})\right). \end{aligned} \tag{29}$$

On the other hand, by the strong convergence $y_{\alpha_i} \to \bar{y}$ in $C(0,1;\mathbb{R}^n)$, we get $(\bar{y}, \bar{v}) \in \mathcal{E}_0$. Thus, (25) and (29) imply that $(\bar{y}, \bar{v}) = (0,0)$ and

$$\underline{\lim}_{i\to\infty} \left((y_{\alpha_i}, v_{\alpha_i}), D^2\widehat{\mathcal{L}}(y_{\alpha_i}, v_{\alpha_i})\right) = \left((\bar{y}, \bar{v}), D^2\widehat{\mathcal{L}}(\bar{y}, \bar{v})\right) = 0.$$

In particular, we get $\lim_{i\to\infty} \left(v_{\alpha_i}, D^2_{uu}\widehat{\mathcal{L}}v_{\alpha_i}\right) = \left(\bar{v}, D^2_{uu}\widehat{\mathcal{L}}\bar{v}\right) = 0$, which, in view of (A3), is equivalent to

$$\|v_{\alpha_i}\|_2 \to \|\bar{v}\|_2 = 0. \tag{30}$$

By (28) and (30), we get $v_{\alpha_i} \to \bar{v}$, strongly in $L^2(0,1;\mathbb{R}^m)$, which, in view of (20) implies $(y_{\alpha_i}, v_{\alpha_i}) \to (\bar{y}, \bar{v}) = (0,0)$ strongly in X^2. That contradicts the assumption $\|(y_{\alpha_i}, v_{\alpha_i})\|_{X^2} = 1$ and completes the proof of the lemma. ■

It follows from Lemmas 4 and 3 that to prove that conditions (A1)–(A4) are satisfied for some $\alpha > 0$, it is enough to show that they hold for $\alpha = 0$.

We will need the following result:

PROPOSITION 2 *Suppose that* (H) *is satisfied and there exists* $h \in \mathcal{B}_\theta^H(\widehat{h})$ *such that the strict complementarity conditions* (24) *hold at* (x_h, u_h, p_h, μ_h). *Then the following conditions, similar to* (A1) – (A4) *are satisfied:*

$(\mathrm{A1})_h$ $\vartheta(x_h(0), h(0)) < 0$,

$(\mathrm{A2})_h$ *There exists* $\chi > 0$, *such that*

$$|D_u f_h^*(t) D_x \vartheta_h^*(t)| \ge \chi \quad \text{for all } t \in M_{h,0},$$

$(\mathrm{A3})_h$ *There exists* $\bar{\gamma} > 0$ *such that*

$$\langle v, D_{uu}^2 \mathcal{L}_h(t) v\rangle \ge \bar{\gamma}|v|^2 \quad \text{for all } v \in \mathbb{R}^m,$$

$(\mathrm{A4})_h$ *There exists* $\gamma > 0$ *such that*

$$\big((y,v), D^2\mathcal{L}_h(y,v)\big) \ge \gamma\|(y,v)\|_{X^2}^2 \quad \text{for all } (y,v) \in \mathcal{E}_{h,0},$$

where we denoted:

$$\mathcal{L}_h = \mathcal{L}(x_h, u_h, p_h, \mu_h; h),$$
$$f_h(t) = f(x_h(t), u_h(t), h(t)), \quad \vartheta_h(t) = \vartheta(x_h(t), h(t)),$$
$$\mathcal{E}_{h,0} = \left\{(y,v) \in X^2 \mid \left\{ \begin{array}{l} y(t) - D_x f_h(t) y(t) - D_u f_h(t) v(t) = 0, \\ \langle D_x \vartheta_h(t), y(t)\rangle = 0 \quad \forall t \in N_{h,0} \end{array}\right.\right\}.$$

Proof. By (A1) and (24), as well as by Corollary 1, the stationary point $(\mathrm{O})_h$ is Gâteaux differentiable in H at h, and the Gâteaux differential, in a direction $g \in H$, is given by the stationary point (y_g, v_g, q_g, ν_g) of the following linear-quadratic optimal control problem, with equality constraints:

$(\mathrm{DO})_{h,g}$ Minimize $J_h(y, v; g)$ subject to

$$\dot{y}(t) - D_x f(x_h(t), u_h(t), h(t)) y(t) - D_u f(x_h(t), u_h(t), h(t)) v(t)$$
$$-g_3(t) = 0, \quad y(0) = 0,$$
$$\langle D_x \vartheta(x_h(t), h(t)), y(t)\rangle - g_4(t) = 0 \quad \text{for all} \quad t \in M_{h,0}.$$

The optimality system for $(\mathrm{DO})_{h,g}$ takes the following form:

$$\left\{\begin{aligned} \dot{q}_g(t) + D_x f_h^*(t) q_g(t) + D_{xx}^2\mathcal{H}_h(t) y_g(t) + D_{xu}^2 \mathcal{H}_h(t) v_g(t) & \\ + \left(\dot{x}_h^*(t) D_{xx}^2\vartheta_h(t) + D_x f_h^*(t) D_x\vartheta_h^*(t)\right)\dot{\nu}_g(t) &= g_1(t), \\ q_g(1) &= 0, \\ D_u f_h^*(t) q_g(t) + D_{ux}^2 \mathcal{H}_h(t) y_g(t) + D_{uu}^2\mathcal{H}_h(t) v_g(t) & \\ + D_u f_h^*(t) D_x \vartheta_h^*(t) \dot{\nu}_g(t) &= g_2(t), \\ \dot{y}_g(t) - D_x f_h(t) y_g(t) - D_u f_h(t) v_g(t) &= g_3(t), \\ y_g(0) &= 0, \\ \langle D_x\vartheta_h(t), y_g(t)\rangle &= g_4(t) \ \forall t \in M_{h,0}, \end{aligned}\right. \tag{31}$$

where $\mathcal{H}_h(t) = \mathcal{H}(x_h(t), u_h(t), p_h(t), \mu_h(t), h(t), \dot{h}(t))$.
The system (31) must have a unique solution (y_g, v_g, q_g, ν_g) for any $g \in H$. Moreover, in view of (22),

$$\|y_g\|_{1,2}, \|v_g\|_2, \|q_g\|_{1,2}, \|\nu_g\|_{1,2} \le \ell\|g\|_Z \quad \text{for all } g \in H. \tag{32}$$

Since the map given by the left-hand side of (31) is continuous from $X^2 \times Y^2$ into Z and H is dense in Z, (31) must have a unique solution, satisfying (32), for any $g \in Z$. Suppose that $0 \in M_{h,0}$. Then, for any g_4, such that $g_4(0) \neq 0$, (31) has no solution. The obtained contradiction shows that $(\mathrm{A1})_h$ must be satisfied.

We will show that $(\mathrm{A2})_h$ holds with $\chi = \ell^{-1}$. Suppose that this is not true, then there exists $\epsilon > 0$ and a subinterval $M \subset M_{h,0}$ of positive measure, such that

$$\begin{aligned} &|D_u f^*(x_h(t), u_h(t), h(t)) D_x \vartheta^*(x_h(t), h(t))| \\ &\quad = |D_x \vartheta(x_h(t), h(t)) D_u f(x_h(t), u_h(t), h(t))| \leq \ell^{-1} - \epsilon \quad \text{for all } t \in M. \end{aligned} \tag{33}$$

Choose any t' and $\bar{\tau} > 0$ such that $(t' - \bar{\tau}, t' + \bar{\tau}) \subset M$, and set $\bar{g} = (0, 0, 0, \bar{g}_4)$, where

$$\bar{g}_4(t) \equiv 0 \text{ for } t \notin (t' - \tau, t' + \tau), \quad \dot{\bar{g}}_4(t) = \begin{cases} 1 & \text{for } t \in (t' - \tau, \tau), \\ -1 & \text{for } t \in (\tau, t' + \tau), \end{cases} \tag{34}$$

and $\tau \leq \bar{\tau}$. Let $(\bar{y}, \bar{v}, \bar{q}, \bar{\nu})$ be the solution of (31) corresponding to $\bar{g}$. It follows from (32) that

$$\|\bar{y}\|_\infty = o(\tau). \tag{35}$$

In view of (6), we obtain from (31)

$$\begin{aligned} \dot{\bar{g}}_4(t) &= D^2_{xx} \vartheta_0(x_h(t)) \dot{x}_h(t) \bar{y}(t) + D_x \vartheta_0(x_h(t)) \dot{\bar{y}}(t) \\ &= \left(D^2_{xx} \vartheta_0(x_h(t)) \dot{x}_h(t) + D_x \vartheta_0(x_h(t)) D_x f_0(x_h(t), u_h(t)) \right) \bar{y}(t) \\ &\quad + D_x \vartheta_0(x_h(t)) D_u f_0(x_h(t), u_h(t)) \bar{v}(t). \end{aligned}$$

Thus, using (33) and (35), we get $|\dot{\bar{g}}_4(t)| - o(\tau) \leq (\ell^{-1} - \epsilon)|\bar{v}(t)|$. Hence, in view of (34), we find that, for $t \in (t' - \tau, t' + \tau)$, $|\dot{\bar{g}}_4(t)| \leq (k(\tau))^{-1}(\ell^{-1} - \epsilon)|\bar{v}(t)|$, where $k(\tau) \uparrow 1$ as $\tau \to 0$. Choosing $\tau > 0$ so small that $k(\tau) \geq \frac{\ell^{-1} - \epsilon}{\ell^{-1} - \epsilon/2}$, we get $|\dot{\bar{g}}_4(t)| \leq (\ell^{-1} - \epsilon/2)|\bar{v}(t)|$. Integrating this inequality over $(t' - \tau, t' + \tau)$, we obtain

$$\begin{aligned} \|\bar{g}\|_Z = \|\bar{g}_4\|_{1,2} = \left(\int_{t'-\tau}^{t'+\tau} \dot{\bar{g}}_4^2(t) dt \right)^{\frac{1}{2}} &\leq (\ell^{-1} - \frac{\epsilon}{2}) \left(\int_{t'-\tau}^{t'+\tau} \dot{\bar{v}}^2(t) dt \right)^{\frac{1}{2}} \\ &\leq (\ell^{-1} - \frac{\epsilon}{2}) \|\bar{v}\|_2, \end{aligned}$$

which contradicts (32) and completes the proof of $(\mathrm{A2})_h$.

We will show that $(\mathrm{A4})_h$ holds with $\gamma = \ell^{-1}$. To this end, choose $\bar{g} = (\bar{g}_1, \bar{g}_2, 0, 0) \in Z$ and denote by $(\bar{y}, \bar{v}, \bar{p}, \bar{\nu})$ the corresponding solution of (31). Then we get

$$\begin{aligned} \left((y, v), D^2 \mathcal{L}_h(\bar{y}, \bar{v}) \right) &:= \int_0^1 \begin{pmatrix} y(t) \\ v(t) \end{pmatrix}^* \begin{pmatrix} D^2_{xx} \mathcal{H}_h & D^2_{xx} \mathcal{H}_h \\ D^2_{ux} \mathcal{H}_h & D^2_{uu} \mathcal{H}_h \end{pmatrix} \begin{pmatrix} \bar{y}(t) \\ \bar{v}(t) \end{pmatrix} dt. \\ &= (\bar{g}_1, y) + (\bar{g}_2, v) \quad \text{for all } (y, v) \in \mathcal{E}_{h,0} \end{aligned} \tag{36}$$

Note that any linear continuous functional ϕ defined on the subspace $\mathcal{E}_{h,0}$ can be expressed in the form $\phi(y,v) = (g_1,y) + (g_2,v)$, where $(g_1,g_2) \in L^2(0,1;\mathbb{R}^n) \times L^2(0,1;\mathbb{R}^m)$. Indeed, a general form of a linear continuous functional on X^2 is given by $\phi(y,v) = (\dot{\phi}_1,\dot{y}) + (\phi_2,v)$, where $(\phi_1,\phi_2) \in X^2$. Hence, using the state equation, we obtain $\phi(y,v) = (D_x f_h^* \dot{\phi}_1, y) + (\phi_2 + D_u f_h^* \dot{\phi}_1, v)$, for all $(y,v) \in \mathcal{E}_{h,0}$. Thus, we get $g_1 = D_x f_h^* \dot{\phi}_1$, $g_2 = \phi_2 + D_u f_h^* \dot{\phi}_1$.

Since equation (36) must have a unique solution for any $(\bar{g}_1,\bar{g}_2) \in L^2(0,1;\mathbb{R}^n) \times L^2(0,1;\mathbb{R}^m)$, the operator $D^2\mathcal{L}_h$ must be invertible on $\mathcal{E}_{h,0}$. Clearly, in view of (32), the equation $\left(D^2\mathcal{L}_h - \lambda\mathcal{I}\right)\begin{pmatrix} y \\ v \end{pmatrix} = 0$ cannot have a nontrivial solution in $\mathcal{E}_{h,0}$ for any $\lambda \in (-\ell^{-1},\ell^{-1})$. Thus, $(-\ell^{-1},\ell^{-1})$ cannot belong to the spectrum σ of $D^2\mathcal{L}_h$. Since by Proposition 3 we have $\big((y,v),D^2\mathcal{L}_h(y,v)\big) \geq 0$, for all $(y,v) \in \mathcal{E}_{0,h}$, it is enough to find necessary conditions under which $\sigma \subset [\ell^{-1},\infty]$. By the well known property of the spectrum of self-adjoint operators in Hilbert spaces, we have

$$\min\{\lambda \in \mathbb{R} \mid \lambda \in \sigma\} = \inf\left\{\big((y,v),D^2\mathcal{L}_h(y,v)\big) \;\mid (y,v) \in \mathcal{E}_{h,0},\; \|(y,v)\|_{X^2} = 1\right\}.$$

Hence the condition $\sigma \subset [\ell^{-1},\infty]$ implies that $(\mathrm{A4})_h$ must hold with $\gamma = \ell^{-1}$.

To complete the proof of the proposition, let us note that, in view of Lemma 3, $(\mathrm{A4})_h$ implies $(\mathrm{A3})_h$. ∎

We are going to show that (H) implies (A1) – (A4). To prove this result for constraint qualifications (A1) and (A2), we will construct a perturbed value of the parameter $h^c = \widehat{h} + \Delta h^c$, where Δh^c can be arbitrarily small, such that, at the stationary point (x^c,u^c,p^c,μ^c) of $(\mathrm{O})_{h^c}$, the strict complementarity holds and $M_{h^c,0} = N_{h^c,0} = M_0$. Then, using Proposition 2, we show that condition (A2) holds with $\alpha = 0$, which, by Lemma 4, implies that it holds with an $\alpha > 0$.

The analogous procedure will be applied to prove coercivity conditions. We introduce $h^s = \widehat{h} + \Delta h^s$, such that, at the stationary point (x^s,u^s,p^s,μ^s) of $(\mathrm{O})_{h^s}$, the strict complementarity holds and $M_{h^s,0} = N_{h^s,0} = N_0$. Using again Proposition 2 and Lemma 4 we prove that (A4) holds.

The crucial point is that the perturbations Δh^c and Δh^s are constructed in such a way that the corresponding solutions $(\widehat{x},\widehat{u})$, as well as the Lagrange multipliers $\widehat{p}$, remain unchanged, and only the multipliers μ can differ from $\widehat{\mu}$. The construction of Δh^c and Δh^s will be presented in the following two lemmas.

Lemma 5 *If* (H) *holds, then* (A2) *is satisfied.*

Proof. Note that inclusion (14) is equivalent to

$$\vartheta_0(x) - h_4 = P_K(\vartheta_0(x) - h_4 + \mu), \quad \mu = P_{K^+}(\vartheta_0(x) - h_4 + \mu), \tag{37}$$

where P_K and P_{K^+} denote the metric projections in $W^{1,2}(0,1;\mathbb{R})$ onto K an K^+, respectively. Clearly, we have

$$\vartheta_0(\widehat{x}) - \widehat{h}_4 = P_K(\vartheta_0(\widehat{x}) - \widehat{h}_4 + \widehat{\mu}), \quad \widehat{\mu} = P_{K^+}(\vartheta_0(\widehat{x}) - \widehat{h}_4 + \widehat{\mu}).$$

Equation (37) will be satisfied by $(\widehat{x}, \widehat{h}_4 + \Delta h_4, \widehat{\mu} + \Delta\mu)$ for any perturbations Δh_4 and $\Delta\mu$ such that

$$-\Delta h_4 \in K, \quad \Delta\mu \in K^+ \text{and} \quad (\vartheta_0(\widehat{x}) - (\widehat{h}_4 + \Delta h_4), \widehat{\mu} + \Delta\mu)_{1,2} = 0. \quad (38)$$

In the special situation, where $\Delta h_4 = 0$, it follows from (10) and (15) that (38) is satisfied, provided that:

$$\begin{cases} \Delta\mu(0) = 0, \\ \Delta\dot{\mu}(\cdot) \text{ is nonincreasing and } \Delta\dot{\mu}(\cdot) = \text{const on each subinterval in } [0,1] \setminus M_0, \\ \Delta\dot{\mu}(1-) \geq 0 \quad \text{and } \Delta\dot{\mu}(1-) = 0 \text{ if } \vartheta_0(\widehat{x})(1) - \widehat{h}_4(1) < 0. \end{cases} \quad (39)$$

We will construct $\Delta\dot{\mu}^c$, satisfying (39) and such that $\Delta\dot{\mu}^c$ is decreasing on M_0 and $\Delta\dot{\mu}^c(1-) > 0$ if $\vartheta_0(\widehat{x}(1)) + \widehat{h}_4(1) = 0$. Thus, the strict complementary conditions will be satisfied at $(\widehat{x}, \widehat{h}_4, \mu^c)$ and $M_{h^c,0} = N_{h^c,0} = M_0$.

Denote by $(t_i', t_i'') \subset M_0$ a subinterval belonging to M_0. Fix any $\epsilon > 0$ and construct $\Delta\dot{\mu}^c(\cdot)$ on $[0,1]$, backward in t, as follows

$$\begin{aligned} \Delta\dot{\mu}^c(1-) &= \begin{cases} \epsilon & \text{if } \vartheta_0(\widehat{x}(1)) + \widehat{h}_4(1) = 0, \\ 0 & \text{if } \vartheta_0(\widehat{x}(1)) + \widehat{h}_4(1) < 0, \end{cases} \\ \Delta\ddot{\mu}^c(t) &= \begin{cases} \frac{2\epsilon}{(t_i''-t_i')}(t_i' - t) & \text{for } t \in [t_i', \frac{1}{2}(t_i'' + t_i')], \\ \frac{2\epsilon}{(t_i''-t_i')}(t - t_i'') & \text{for } t \in [\frac{1}{2}(t_i'' + t_i'), t_i''], \\ 0 & \text{for } t \in [0,1] \setminus M_0. \end{cases} \end{aligned}$$

Moreover, we set $\Delta\mu^c(0) = 0$. It can be easily checked that $\Delta\mu^c$ satisfies all the required conditions (39) and $\|\Delta\mu^c)\|_{2,\infty} = O(\epsilon)$.

We set $h_4^c = \widehat{h}_4$ and choose the remaining components h_1^c, h_2^c, h_3^c of h^c in such a way that $(\widehat{x}, \widehat{u}, \widehat{p}, \mu^c)$ is a stationary point of $(\mathrm{O})_{h^c}$. Namely, the state equation holds if we set $h_3^c = \widehat{h}_3$, whereas (12) and (13) will be satisfied, if we set $h_1^c = \widehat{h}_1 + \Delta h_1^c$ and $h_2^c = \widehat{h}_2 + \Delta h_2^c$, where

$$\begin{cases} \Delta h_1^c = \Big(D_x f_0^*(\widehat{x}, \widehat{u}) D_x \vartheta_0^*(\widehat{x}) + (f_0(\widehat{x}, \widehat{u}) - \widehat{h}_3)^* D_{xx}^2 \vartheta_0^*(\widehat{x}) \Big) \Delta\dot{\mu}^c, \\ \Delta h_2^c = D_u f_0^*(\widehat{x}, \widehat{u}) D_x \vartheta_0^*(\widehat{x}) \Delta\dot{\mu}^c. \end{cases} \quad (40)$$

Hence, we find that $(\widehat{x}, \widehat{u}, \widehat{p}, \mu^c)$ is a stationary point of $(\mathrm{O})_{h^c}$. Moreover,

$$\|\Delta h_1^c\|_{1,\infty} = O(\epsilon), \ \|\Delta h_2^c\|_{1,\infty} = O(\epsilon), \quad \text{i.e., } \|\Delta h^c\|_H = O(\epsilon).$$

Choosing $\epsilon > 0$ sufficiently small, we get $h^c \in \mathcal{B}_\theta^H(\widehat{h})$ and $(\widehat{x}, \widehat{u}, \widehat{p}, \mu^c) \in \mathcal{B}_\tau^{\Xi \times \Lambda}(\widehat{\xi}, \widehat{\lambda})$. Thus, by Proposition 2, condition $(\mathrm{A2})_{h^c}$ is satisfied. Since $x_{h^c} = \widehat{x}$ and $h_4^c = \widehat{h}_4$, condition $(\mathrm{A2})_{h^c}$ coincides with (A2) for $\alpha = 0$. We use now Lemma 4 to find that (A2) holds for some $\alpha > 0$. ■

LEMMA 6 *If* (H) *holds, then* (A4) *is satisfied.*

Proof. As in the proof of Lemma 5, we use (38). Let us consider first the situation, where the strict complementarity condition is not violated at $t = 1$, i.e., either $\vartheta_0(\widehat{x}(1)) - \widehat{h}_4(1) < 0$ or $\vartheta_0(\widehat{x}(1)) - \widehat{h}_4(1) = 0$ and $\dot{\widehat{\mu}}(1) > 0$. In this case we introduce only a variation Δh_4, whereas $\widehat{\mu}$ remains unchanged. Thus, equation (38) will be satisfied by $(\widehat{x}, \widehat{h}_4 + \Delta h_4, \widehat{\mu})$, provided that $-\Delta h_4 \in K$, and $(\vartheta_0(\widehat{x}) - (\widehat{h}_4 + \Delta h_4), \widehat{\mu})_{1,2} = 0$. These conditions hold, if

$$\Delta h_4(t) \geq 0, \quad \text{and} \quad \Delta h_4(t) = 0 \quad \text{for } t \in N_0. \tag{41}$$

We will construct $\Delta h_4^s \in C^2(0,1;\mathbb{R})$ satisfying (41), which is positive on each subiterval (t_j', t_j'') in $[0,1] \setminus N_0$. Thus, strict complementarity will be satisfied by $(\widehat{x}, h_4^s, \widehat{\mu})$, whereas $M_{h^s,0} = N_{h^s,0} = N_0$.

Let $(t_j', t_j'') \subset [0,1] \setminus N_0$ denote a subinterval of $[0,1] \setminus N_0$. Fix any $\epsilon > 0$ and set

$$\begin{cases} \Delta h_4^s(t) = 0 \quad \text{for } t \in N_0, \\ \Delta \ddot{h}_4^s(t) \\ = \begin{cases} \frac{8\epsilon}{(t_j''-t_j')}(t - t_j') & \text{for } t \in (t_j', t_j' + \frac{1}{8}(t_j'' - t_j')), \\ \frac{8\epsilon}{(t_j''-t_j')}((t_j' + \frac{1}{4}(t_j'' - t_j')) - t) & \text{for } t \in (t_j' + \frac{1}{8}(t_j'' - t_j'), t_j' + \frac{3}{8}(t_j'' - t_j')), \\ \frac{8\epsilon}{(t_j''-t_j')}(t - \frac{1}{2}(t_j'' + t_j')) & \text{for } t \in (t_j' + \frac{3}{8}(t_j'' - t_j'), t_j' + \frac{1}{2}(t_j'' + t_j')), \\ \frac{8\epsilon}{(t_j''-t_j')}(\frac{1}{2}(t_j'' + t_j') - t) & \text{for } t \in (\frac{1}{2}(t_j'' + t_j'), t_j' + \frac{5}{8}(t_j'' - t_j')), \\ \frac{8\epsilon}{(t_j''-t_j')}(t - (t_j' + \frac{3}{4}(t_j'' - t_j'))) & \text{for } t \in (t_j' + \frac{5}{8}(t_j'' - t_j'), t_j' + \frac{7}{8}(t_j'' - t_j')), \\ \frac{8\epsilon}{(t_j''-t_j')}(t_j'' - t) & \text{for } t \in (t_j' + \frac{7}{8}(t_j'' - t_j'), t_j''). \end{cases} \end{cases} \tag{42}$$

It is easy to see that Δh_4^s satisfies all the required conditions. Moreover, $\|\Delta h_4^s\|_{2,\infty} = O(\epsilon)$. We set $\Delta h^s = (\Delta h_1^s, \Delta h_2^s, \Delta h_3^s, \Delta h_4^s)$, where $\Delta h_1^s = 0$, $\Delta h_2^s = 0$, $\Delta h_3^s = 0$. It can be easily checked that $(\widehat{x}, \widehat{u}, \widehat{p}, \widehat{\mu})$ remains a stationary point of $(\mathrm{O})_{h^s}$ and $\|\Delta h^s\|_H = O(\epsilon)$. Thus, for any $\epsilon > 0$ sufficiently small, we get $h^s \in \mathcal{B}_\theta^H(\widehat{h})$ and condition $(\mathrm{A4})_{h^s}$ follows from Proposition 2. Note that, in view of (6), the Hessian of the Lagrangian and the set $\mathcal{E}_{h,0}$ do not depend directly on h. Therefore, since $(x_{h^s}, u_{h^s}, p_{h^s}, \mu_{h^s}) = (\widehat{x}, \widehat{u}, \widehat{p}, \widehat{\mu})$, we get $\mathcal{L}_{h^s} = \widehat{\mathcal{L}}$ and $\mathcal{E}_{h^s,0} = \mathcal{E}_0$. Thus, $(\mathrm{A4})_{h^s}$ is equivalent to (A4) for $\alpha = 0$. Finally, Lemma 4 implies that (A4) holds, for some $\alpha > 0$.

Let us pass now to the situation, where the strict complementarity condition is violated at $t = 1$, i.e., if $\vartheta_0(\widehat{x}(1)) - \widehat{h}_4(1) = 0$ and $\dot{\widehat{\mu}}(1) = 0$. We consider two cases:

(i) $\dot{\widehat{\mu}}(t) = 0$ on a subinterval $(\tau, 1]$,
(ii) $\dot{\widehat{\mu}}(\cdot)$ is decreasing on a subinterval $(\tau, 1]$.

In the case (i), let us choose $(\tau, 1]$ to be the maximal subinterval on which $\dot{\widehat{\mu}}(t) = 0$. On $[\tau, 1]$, we define the variation Δh_4^s by setting

$$\Delta h_4^s(\tau) = 0, \quad \Delta \dot{h}_4^s(\tau) = 0, \quad \Delta \ddot{h}_4^s(t) = \epsilon \frac{(t-\tau)}{(1-\tau)}. \tag{43}$$

Clearly, we get $\vartheta_0(\widehat{x}(t)) - (\widehat{h}_4(t) + \Delta h_4^s(t)) < 0$ for $t \in (\tau, 1]$, so the strict complementarity condition is trivially satisfied on that subinterval. On subinterval $[0, \tau]$ we construct Δh_4^s as in (42) and the proof of (A4) follows in the same way as in the previous situation.

Finally, let us consider case (ii). Choose $\epsilon < 1-\tau$, where τ is given in (ii). On $[1-\epsilon, 1]$ define an auxiliary variation Δh_4 given by (43), where τ is substituted by $1-\epsilon$. On subinterval $[0, 1-\epsilon]$ we construct $\Delta h_4 = \Delta h_4^s$, as in (42). In contrast to the previous case, equation (38) is no longer satisfied for $\Delta\mu = 0$. Hence, we have to introduce a variation $\Delta\mu \neq 0$.

Note that on the subinterval $[1-\epsilon, 1]$ we have $\vartheta(\widehat{x}(t)) - \widehat{h}_4(t) \equiv 0$ and the function $\frac{d}{dt}(-\Delta h_4(\cdot))$ is decreasing from 0, whereas $\frac{d}{dt}\widehat{\mu}(\cdot)$ is decreasing to 0. Hence, there is a unique point $\eta \in (1-\epsilon, 1)$ such that $\frac{d}{dt}(-\Delta h_4(\eta) + \widehat{\mu}(\eta)) = 0$ and $\vartheta(\widehat{x}(\cdot)) - (\widehat{h}_4(\cdot) + \Delta h_4(\cdot)) + \widehat{\mu}(\cdot) = -\Delta h_4(\cdot) + \widehat{\mu}(\cdot)$ assumes its maximum at η. We define

$$\mu^s(t) = \begin{cases} \widehat{\mu}(t) & \text{for } t \in [0, 1-\epsilon], \\ \widehat{\mu}(t) - \Delta h_4(t) & \text{for } t \in [1-\epsilon, \eta], \\ \widehat{\mu}(\eta) - \Delta h_4(\eta) & \text{for } t \in [\eta, 1]. \end{cases} \tag{44}$$

Moreover, we set $\Delta h_4^s(t) = \Delta h_4(t)$ for $t \in [0, 1-\epsilon]$, whereas on $[1-\epsilon, 1]$ we define

$$-\Delta h_4^s(t) = \begin{cases} 0 & \text{for } t \in [1-\epsilon, \eta], \\ (\widehat{\mu}(t) - \Delta h_4(t)) - (\widehat{\mu}(\eta) - \Delta h_4(\eta)) & \text{for } t \in [\eta, 1]. \end{cases} \tag{45}$$

It can be easily seen that conditions (38) are satisfied by Δh_4^s and $\Delta\mu^s = \mu^s - \widehat{\mu}$. Moreover, the strict complementarity condition is satisfied at $(\widehat{x}, \widehat{h}_4 + \Delta h_4^s, \mu^s)$ and

$$M_{h^s,0} = N_{h^s,0} = N_0 \setminus (1-\eta, 1). \tag{46}$$

As in (40) we choose

$$\begin{cases} \Delta h_1^s = \Big(D_x f_0^*(\widehat{x}, \widehat{u}) D_x \vartheta_0^*(\widehat{x}) + (f_0(\widehat{x}, \widehat{u}) - \widehat{h}_3)^* D_{xx}^2 \vartheta_0^*(\widehat{x}) \Big) \Delta\dot{\mu}^s, \\ \Delta h_2^s = D_u f_0^*(\widehat{x}, \widehat{u}) D_x \vartheta_0^*(\widehat{x}) \Delta\dot{\mu}^s, \quad \Delta h_3^s = 0, \end{cases} \tag{47}$$

and set $\Delta h^s = (\Delta h_1^s, \Delta h_2^s, \Delta h_3^s, \Delta h_4^s)$, as well as $h^s = \widehat{h} + \Delta h^s$. It can be seen that $(\widehat{x}, \widehat{u}, \widehat{p}, \mu^s)$ is a stationary point of $(\mathrm{O})_{h^s}$.

It follows from (43) and (45) that

$$\|\Delta h_4^s\|_{2,\infty} = O(\epsilon) \quad \text{and} \quad \|\Delta \mu^s\|_{1,\infty} = O(\epsilon). \tag{48}$$

From (47) and (48), we get $\|\Delta h^s\|_H = O(\epsilon)$. Thus, choosing $\epsilon > 0$ sufficiently small, we obtain $h^s \in \mathcal{B}_\theta^H(\widehat{h})$ and $(\widehat{x}, \widehat{u}, \widehat{p}, \mu^s) \in \mathcal{B}_\tau^{X^2 \times Y^2}(\widehat{\xi}, \widehat{\lambda})$. Morover, we have

$$\operatorname{ess} \sup_{t \in [\phi, 1-\eta]} |\ddot{\mu}^s(t) - \ddot{\widehat{\mu}}(t)| = O(\varepsilon) \quad \text{and} \quad |\ddot{\mu}^s(t)| = \phi \quad \forall\, t \in [1-\eta, 1].$$

Hence, for $\varepsilon > 0$ sufficiently small, we get $||\ddot{\mu}^s||_\infty \leq \sigma$, where σ is given in (16). Thus, $(\hat{x}, \hat{u}, \hat{\rho}, \hat{\mu}^s) \in \beta_\tau^{\Xi \times \Lambda}(\hat{\xi}, \hat{\lambda})$.

Therefore, by Lemma 2, condition $(\mathrm{A4})_{h^s}$ is satisfied. Note that, in view of (8) and (48) we get $\|D^2\mathcal{L}_{h^s} - D^2\widehat{\mathcal{L}}\|_{X^2 \to X^2} = O(\epsilon)$. Hence, shrinking $\epsilon > 0$ if necessary, and using $(\mathrm{A4})_{h^s}$, we obtain

$$\Big((y, v), D^2\widehat{\mathcal{L}}(y, v)\Big) \geq \frac{1}{2}\gamma\|(y, v)\|_{X^2}^2 \quad \text{for all } (y, v) \in \mathcal{E}_{h^s,0}. \tag{49}$$

However, in view of (46), we have $\mathcal{E}_0 \subset \mathcal{E}_{0,h^s}$, so that (49) holds for all $(y, v) \in \mathcal{E}_0$. Finally, Lemma 4 implies that (A4) is satisfied for some $\alpha > 0$. That concludes the proof of the lemma. ■

Note that the necessity of (A1) follows, as in the proof of Proposition 2, whereas in view of Lemma 3, (A4) together with (A1) and (A2) implies (A3). Thus, we have shown that (H) implies (A1)–(A4). Therefore, in view of Theorem 1, we have arrived at the following characterization of the Lipschitz stability and directional differentiability properties of the solutions and Lagrange multipliers of $(\mathrm{O})_h$, which is the principal result of this paper:

THEOREM 2 *Conditions* (A1)–(A4) *are necessary and sufficient in order that* (H) *is satisfied.*

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