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φ-regular functions in Asplund spaces

by

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Abstract: We introduce in the context of Asplund spaces, a new class of φ-regular functions. This new concept generalizes the one of prox-regularity introduced by Poliquin & Rockafellar (2000) in $\mathbb{R}^n$ and extended to Banach spaces by Bernard & Thibault (2004). In particular, the class of φ-regular functions includes all lower semicontinuous convex functions, all lower-$\mathcal{C}^2$ functions, and convexly $\mathcal{C}^{1,0}-$composite functions as well. Geometrical and subdifferential characterizations for this new class of functions are investigated.

Keywords: φ-regularity, Fréchet-subdifferential, limiting subdifferential, prox-regularity, lower-$\mathcal{C}^2$ function, convexly composite function, Moreau envelope, regularization.

1. Introduction and preliminaries

The class of convex functions enjoys many nice properties. However, the convexity is often a too strong assumption for the needs of applications. Recently, much attention has been given to special classes of nonsmooth functions having interesting properties which could serve as substitute to these assumptions. Among these classes, the class of prox-regular functions was introduced and studied by Poliquin & Rockafellar (2000) in finite dimension. It extends the class of so-called lower nice functions introduced by Poliquin (1991). This notion was motivated by the strong connection between functions and their Moreau envelopes as defined in Moreau (1965). The class of prox-regular functions contains lower semicontinuous convex functions, lower-$\mathcal{C}^2$ functions (i.e., functions which are expressible locally as a difference between a finite convex function and a positive multiple of $\|\cdot\|^2$) and strongly amenable functions (i.e., functions which are obtained by composing extended-real valued convex functions with $\mathcal{C}^2$-mappings under a constraint qualification). Among the characterizations of prox-regular functions, one of them is of particular importance, since it links prox-regularity of lower semicontinuous functions to hypomonotonicity of a localization of their subdifferential. Recently, Bernard & Thibault (2005) have identified and extensively studied the prox-regularity in infinite dimension.

In another direction, Ngai-Luc & Théra (2000), have introduced the class of approximately convex functions. This concept has also been considered in Aussel, Daniilidis & Thibault (2005), Daniilidis & Georgiev (2004), Ngai & Penot (2007). In the finite dimensional setting Daniilidis & Georgiev (2004) have shown that the class of locally Lipschitz approximately convex functions coincides with the one of lower-$\mathcal{C}^1$ functions. Following Spingarn (1981) we recall that a locally Lipschitz real-valued function f defined on an open set Ω of $\mathbb{R}^n$ is lower-$\mathcal{C}^1$, if for every $x_0 \in \Omega$, there exist a neighborhood V of x_0, a compact set S and a jointly continuous function $g : V \times S \to \mathbb{R}$, such that for all $x \in V$ we have $f(x) = \max_{s \in S} g(x, s)$ and the derivative with respect to x exists and is jointly continuous. We also refer to Rolewicz (2000, 2001) and the references therein, showing that $\alpha(\cdot)$- paraconvex functions can be viewed as a uniformization of the concept of approximately convex functions.

Motivated by the calmness property of inequality systems of lower semicontinuous functions, Jourani (2006) has recently introduced the notion of weak-regularity in the context of Asplund spaces, i.e., in Banach spaces where every convex continuous function is generically Fréchet differentiable. In $\mathbb{R}^n$, this class of functions coincides with the one of the so-called Fréchet regular functions (see Theorem 4.2, Jourani, 2006).

The purpose of this paper is to consider the class of φ-regular functions for a suitable class of convex functions $\varphi : \mathbb{R}_+ := [0, +\infty[\to \mathbb{R}_+$. We show that this class includes all prox-regular functions as well as uniformly approximately convex functions. We also observe that nice characterizations for prox-regular functions can be extended naturally to φ-regular functions.

The paper is organized as follows. Below, we recall basic definitions and preliminaries used throughout the paper. In Section 2, we introduce the concept of φ-regular functions and sets. The equivalence between φ-regularity of a function and φ-regularity of its epigraph is investigated. We also establish a characterization of φ-regularity of sets using distance functions. We prove in Section 2 that a lower semicontinuous function is φ-regular if and only if there is a localization of its subdifferential (this notion was considered by Poloquin & Rockafellar) which is φ-submonotone in a sense which is precised in the paper. In the final section, we present a result on Moreau-envelopes of φ-regular functions, which generalizes the works by Poliquin & Rockafellar (2000) and by Bernard & Thibault (2004, 2005) for prox-regular functions.

Let X be a Banach space with closed unit ball B_X and topological dual $X^\star$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous extended-real-valued function defined on X. As usual

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}, \ \operatorname{epi} f := \{(x, \alpha) \in X \times \mathbb{R} : \alpha \geq f(x)\}$$

denote the *domain,* the *epigraph* of f, respectively. Recall that the *Fréchet* *subdifferential* of f at $x \in \operatorname{dom} f$ is defined by

$$\partial^F f(x) := \Big\{ x^\star \in X^\star \ : \ \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^\star, h \rangle}{\|h\|} \geq 0 \Big\}.$$

If $x \notin \operatorname{dom} f$, we set $\partial^F f(x) = \emptyset$. Most of the definitions below are standard and can be found in Mordukhovich & Shao (1996) or in the new book by Mordukhovich (2006). The *limiting Fréchet subdifferential* at $x \in \operatorname{dom} f$ is defined by

$$\partial f(x) = w^\star - \limsup_{(u, f(u)) \to (x, f(x))} \partial^F f(u),$$

where the $w^\star - \limsup$ is the set of weak* cluster points of sequences $(u_n^\star)$ with $u_n^\star \in \partial^F f(u_n)$, $(u_n) \to x$ and $(f(u_n)) \to f(x)$. The *Fréchet normal cone and the limiting normal cone* to a closed subset C of X at $x \in C$ are defined by

$$N^F(C, x) := \partial^F \delta_C(x) = \Big\{ x^\star \in X^\star : \limsup_{y \xrightarrow{C} x} \frac{\langle x^\star, y - x \rangle}{\|y - x\|} \leq 0 \Big\}$$

and $N(C, x) := \partial \delta_C(x)$, respectively. As usual $\delta_C(\cdot)$ stands for the *indicator function* of C, that is, $\delta_C(x) = 0$ if $x \in C$ and $+\infty$, otherwise, and the notation $y \overset{C}{\to} x$ means that $y \to x$ and $y \in C$. The function f (the set C, respectively) is said to be Fréchet regular at $x \in \operatorname{dom} f$ ($x \in C$) if $\partial f(x) = \partial^F f(x)$ ($N(C, x) = N^F(C, x)$). Note that the normal cone can be also represented by the subdifferential of the distance function:

$$N^F(C, x) = \bigcup_{\lambda > 0} \lambda \partial^F d_C(x), \text{ and } \partial^F d_C(x) = N^F(C, x) \cap B_{X^\star} \text{ for } x \in C. \quad (1)$$

The Fréchet subdifferential and the limiting Fréchet subdifferential of a lower semicontinuous extended-real-valued function f can be defined by means of the Fréchet and the limiting Fréchet normal cone to the epigraph of f as follows:

$$\partial^F f(x) = \{ x^\star \in X^\star : \ (x^\star, -1) \in N^F(\operatorname{epi} f(x, f(x)) \}.$$
$$\partial f(x) = \{ x^\star \in X^\star : \ (x^\star, -1) \in N(\operatorname{epi} f(x, f(x)) \}.$$

The history of these constructions can be found in Mordukhovich & Shao (1996) and in Mordukhovich (2006).

An important property of the Fréchet subdifferential of the distance function will be needed in the sequel (see, e.g., Borwein & Fitzpatrick, 1989; Jourani & Thibault, 1995; Ngai & Théra, 2001).

Proposition 1 *Suppose that C is a closed nonempty subset of an Asplund space X and that $x^\star \in \partial^F d(x, C)$ with $x \notin C$. Then, $\|x^\star\| = 1$ and there exists a minimizing sequence $\{z_n\}_{n \in \mathbb{N}}$ of $d(x, C)$ in C and $z_n^\star \in N^F(C, z_n)$ such that*

$$d(x, C) = \lim_{n \to \infty} \langle z_n^\star, x - z_n \rangle, \ \|z_n^\star - x^\star\| \to 0 \ \textit{as}\ n \to \infty.$$

In the framework of Asplund spaces, the Fréchet subdifferential enjoys a fuzzy sum rule which was proved by Fabian (1989).

THEOREM 1 (Fabian, 1989) *Assume that X is Asplund and consider a finite family (f_i), $(i = 1, \cdots, n)$ of extended-real-valued mappings which are lower semicontinuous and such that all but one of them is Lipschitzian around $\bar{x} \in \operatorname{dom} f_1 \cap \cdots \cap \operatorname{dom} f_n$. Then, for each $\varepsilon > 0$, one has*

$$\partial^F(f_1 + \cdots + f_n)(\bar{x}) \subseteq$$
$$\bigcup\{\partial^F f_1(x_1) + \cdots + \partial^F f_n(x_n) + \varepsilon B_{X^\star} : (x_i, f_i(x_i)) \in (\bar{x}, f_i(\bar{x})) + \varepsilon B_{X\times\mathbb{R}}\}.$$

2. φ-regular functions and φ-regular sets

In what follows, X is supposed to be Asplund and $\mathcal{C}$ to be the set of all continuously differentiable convex functions $\varphi : \mathbb{R}_+ := [0, +\infty) \to \mathbb{R}_+$, that are strictly increasing on $\mathbb{R}_+$ and satisfy $\varphi(0) = \varphi'(0) = 0$; $\lim_{t\to+\infty} \varphi(t) = +\infty$.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function defined on X.

DEFINITION 1 *Let $\varphi \in \mathcal{C}$ be given. The function f is said to be φ-regular at $\bar{x} \in \operatorname{dom} f$ with respect to $\bar{x}^* \in \partial^F f(\bar{x})$, if there exist reals $t, \delta > 0$ such that*

$$\langle x^*, y - x\rangle \le f(y) - f(x) + \varphi(t\|y - x\|) \tag{2}$$

for all x and y satisfying $(x, f(x)) \in B((\bar{x}, f(\bar{x})), \delta)$, $y \in B(\bar{x}, \delta)$ and all $x^* \in B(\bar{x}^*, \delta) \cap \partial^F f(x)$.

DEFINITION 2 *Let $C \subseteq X$ be a closed subset of X, $\bar{x} \in C$ and $\bar{x}^* \in N(C, \bar{x})$ be given. If the indicator function δ_C is φ-regular at $\bar{x}$ with respect to $\bar{x}^*$, then we will say that C is φ-regular at $\bar{x} \in C$ with respect to $\bar{x}^*$. When a function f (respectively a set C) is φ-regular at $\bar{x}$ with respect to all $x^* \in \partial^F f(\bar{x})$ ($x^* \in N^F(C, \bar{x})$, respectively), then f (respectively C) is said to be φ-regular at $\bar{x}$.*

The definition above is inspired by the notion of weak regularity introduced in Jourani (2006), which generalizes the one of prox-regularity (when $\varphi(t) := t^2$) introduced by Poliquin & Rockafellar (2000) in $\mathbb{R}^n$, then extended to the infinite dimensional setting and studied by Bernard & Thibault (2004, 2005). The following proposition characterizes φ-regularity:

PROPOSITION 2 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $\bar{x} \in \operatorname{dom} f$, $\bar{x}^* \in \partial^F f(\bar{x})$ be given. Then the following two statements are equivalent:*

(i) There exists $\varphi \in \mathcal{C}$ such that f is φ-regular at $\bar{x}$ with respect to $\bar{x}^*$;

(ii) There exists a real $\gamma > 0$ such that for all $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\langle x^*, y - x\rangle \le f(y) - f(x) + \varepsilon\|y - x\|$$

for all $x, y \in B(\bar{x}, \gamma)$; $\|x - y\| < \delta$; $|f(x) - f(\bar{x})| < \gamma$; $x^* \in B(\bar{x}^*, \gamma)$ with $x^* \in \partial^F f(x)$.

Proof. $(i) \Rightarrow (ii)$ is obvious from the definition of $\mathcal{C}$. The proof of $(ii) \Rightarrow (i)$ is inspired by Spingarn (1981). Let $\sigma : X \times X \times X^* \to \mathbb{R} \cup \{+\infty\}$ and $\xi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ be the functions defined by

$$\sigma(x, y, x^*) := \begin{cases} 0 & \text{if } x = y \\ \dfrac{\langle x^*, y - x\rangle - f(y) + f(x)}{\|y - x\|} & \text{otherwise} \end{cases}$$

and $\xi(t) := \sup\{\sigma(x, y, x^*) : x, y \in B(\bar{x}, \gamma),\ \|x - y\| \le t,\ |f(x) - f(\bar{x}| < \gamma,\ x^* \in \partial^F f(x) \cap B(\bar{x}^*, \gamma)\}$ if $t \ne 0$, $\xi(0) := 0$. By (ii), for all $\varepsilon > 0$, there exists $\delta > 0$ such that $(0 \le)\xi(t) \le \varepsilon$, for all $t \in (0, \delta)$. As $\lim_{t\to 0_+} \xi(t) = 0$, by virtue of Spingarn (1981), Lemma 3.7, there is a continuously differentiable function $\alpha(\cdot)$ defined on $[0, s]$ for some $s > 0$ such that $\alpha(0) = \alpha'(0) = 0$, and $\alpha(t) \ge t\xi(t)$ for all $t \in [0, s]$. Also, similarly to the proof of Lemma 3.7 in Spingarn (1981), let $s_n = s/2^n$, $n \in \mathbb{N}$ and set $\beta(\cdot)$ be the infimum of all the affine functions $l : \mathbb{R} \to \mathbb{R}$ satisfying $l(s_n) \ge \alpha'(t)$ for all $t \in (0, s_n),\ n \in \mathbb{N}$. Then β is continuous, nondecreasing on $[0, s]$, satisfies $\beta(0) = 0$ and $\beta \ge \alpha'$ on $[0, s]$. Let $\bar{\beta} : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined by $\bar{\beta}(t) = \beta(t)$ for $t \in [0, s]$ and $\bar{\beta}(t) := \beta(s)$, otherwise. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\varphi(t) := \int_0^t \bar{\beta}(r)dr + t^2.$$

Then, obviously $\varphi \in \mathcal{C}$ and $\varphi \ge \alpha$ on $[0, s]$. Furthermore, for all $x, y \in B(\bar{x}, s/2)$ with $x \ne y$, $|f(x) - f(\bar{x})| < \gamma$, $x^* \in \partial^F f(x) \cap B(\bar{x}^*, \gamma)$, one has $(t := \|x - y\| \in (0, s))$

$$\sigma(x, y, x^*) := \frac{\langle x^*, y - x\rangle - f(y) + f(x)}{\|y - x\|} \le \xi(t) \le \frac{\varphi(t)}{t} = \frac{\varphi(\|x - y\|)}{\|x - y\|},$$

that is, $\langle x^*, y - x\rangle \le f(y) - f(x) + \varphi(\|x - y\|)$, establishing the proof. ∎

According to Ngai-Luc & Théra (2000), we recall that a function f is approximately convex around $\bar{x} \in \operatorname{dom} f$ if for each $\varepsilon > 0$, there is some $\delta > 0$ such that for all $x, y \in B(\bar{x}, \delta)$ and $\lambda \in [0, 1]$, one has

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y) + \varepsilon\lambda(1 - \lambda)\|x - y\|. \tag{3}$$

We will say that a function f is *uniformly approximately convex* around $\bar{x} \in \operatorname{dom} f$ if there exists a real $\gamma > 0$ such that for each $\varepsilon > 0$, there is some $\delta > 0$ such that (3) holds for all $x, y \in B(\bar{x}, \gamma)$ with $\|x - y\| < \delta$, $\lambda \in [0, 1]$.

The class of approximately convex functions contains all convex functions as well as strictly differentiable functions (see Ngai-Luc & Théra, 2000). Obviously, the class of uniformly approximately convex functions includes all lower C^1-functions. Note that in finite dimension, by a compact argument, it is easy to observe that a function f is uniformly approximately convex around x_0 if and only if f is approximately convex at all points in a neighborhood of x_0.

Corollary 1 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. If f is uniformly approximately convex at $\bar{x} \in \operatorname{dom} f$, then f is φ-regular at $\bar{x}$ for some $\varphi \in \mathcal{C}$.*

Proof. The proof follows directly from Proposition 2 and Theorem 10 in Ngai & Penot (2007). ■

We say that a mapping h is convexly $C^{1,0}-$composite over an open subset U of X if h is of the form $h := f \circ F$ where $f : Z \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex function defined on a Banach space Y and $F : X \to Z$ is uniformly continuously differentiable over U.

The following corollary shows that convexly $C^{1,0}-$composite functions are φ-regular under the Robinson qualification condition. Prox-regularity of convexly $C^{1,1}-$composite functions (i.e., instead of the uniformly continuous differentiability of F, we suppose that ∇F is locally Lipschitz) has been established by Bernard & Thibault (2004).

Corollary 2 *Let $h = f \circ F$ be a convexly $C^{1,0}-$composite function over an open subset $U \subseteq X$. Suppose that the following Robinson qualification condition is satisfied at $\bar{x} \in \operatorname{dom} h$:*

$$\mathbb{R}_+(\operatorname{dom} f - F(\bar{x})) - \nabla F(\bar{x})(X) = Y.$$

Then for all $\bar{x}^* \in \partial^F h(\bar{x})$, there is $\varphi \in \mathcal{C}$ such that h is φ-regular at $\bar{x}$ with respect to $\bar{x}^*$.

Proof. It suffices to show that statement (ii) in Proposition 2 is verified. It is well known (see, for example, Combari, Poliquin & Thibault, 1999, or Theorem 10.6 (basic chain rule) in Rockafellar & Wets, 2002) that under the Robinson qualification condition, for all x sufficiently close to $\bar{x}$, say $x \in B(\bar{x}, \gamma)$, one has

$$\partial^F h(x) = \{z^* \circ \nabla F(x) : \quad z^* \in \partial f(F(x))\}.$$

Let $\bar{x}^* \in \partial^F h(\bar{x})$. Moreover, observe from the proof of Proposition 2.4 in Bernard & Thibault (2005) that there are $c > 0$, $\gamma_1 \in (0, \gamma)$ such that

$$\|z^*\| \le c(1 + \|x^*\|) \ \text{ for all } x \in B(\bar{x}, \gamma_1),\ x^* = z^* \circ \nabla F(x),\ z^* \in \partial f(F(x)).$$

Let $\varepsilon > 0$ be given. Since F is uniformly continuously differentiable on U, there exists $\delta > 0$ such that

$$\|F(y) - F(x) - \nabla F(x)(y - x)\| \le \varepsilon \|x - y\| \quad \text{for all } x,\ y \in U,\ \|x - y\| < \delta. \tag{4}$$

Let $x, y \in B(\bar{x}, \gamma_1)$ with $\|x - y\| < \delta$, $x^* = z^* \circ \nabla F(x) \in \partial^F h(x) \cap B(\bar{x}^*, \gamma)$ with $z^* \in \partial f(F(x))$. By relation (4), one has

$$\begin{aligned}
\langle x^*, y - x\rangle &= \langle z^*, \nabla F(x)(y - x)\rangle \\
&= \langle z^*, F(y) - F(x)\rangle + \langle z^*, \nabla F(x)(y - x) - F(y) + F(x)\rangle \\
&\le f \circ F(y) - f \circ F(x) + c(1 + \|\bar{x}^*\| + \gamma)\varepsilon \|y - x\|.
\end{aligned}$$

This shows that statement (ii) in Proposition 2 is verified, and the proof is complete. ■

The next theorem characterizes the φ-regularity of sets using the distance function.

THEOREM 2 *Let C be a closed subset of X and $\bar{x} \in C$. Let $\varphi \in \mathcal{C}$ be given. Then the following statements are equivalent:*

(i) C is φ-regular at $\bar{x}$ with respect to $\bar{x}^* \in N^F(C, \bar{x}) \cap B_{X^*}$, where B_{X^*} stands for the closed unit ball in X^*;

(ii) There exist $t, \delta > 0$ such that for all $x, y \in B(\bar{x}, \delta)$ with either $x \in C$ or $y \in C$, $x^* \in \partial^F d_C(x) \cap B(\bar{x}^*, \delta)$, one has $\langle x^*, y-x \rangle \le d_C(y) - d_C(x) + \varphi(t\|y-x\|)$. As result, if d_C is φ-regular at $\bar{x}$ with respect to $\bar{x}^*$, then so is C.

Proof. $(ii) \Rightarrow (i)$. If (ii) is satisfied, there exist $t, \delta_0 > 0$ such that whenever $x, y \in B(\bar{x}, \delta_0)$, $y \in C$ or $x \in C$, $x^* \in \partial^F d_C(x) \cap B(\bar{x}^*, \delta_0)$, one has

$$\langle x^*, y - x \rangle \le d_C(y) - d_C(x) + \varphi(t\|x - y\|).$$

If $\|\bar{x}^*\| < 1$ by picking $\delta = \min\{\delta_0, 1 - \|\bar{x}^*\|\}$, then for all $x, y \in C \cap B(\bar{x}, \delta)$, $x^* \in N(C, x) \cap B(\bar{x}^*, \delta)$, one has $x^* \in \partial^F d_C(x) \cap B(\bar{x}^*, \delta_0)$ and therefore

$$\langle x^*, y - x \rangle \le \varphi(t\|x - y\|),$$

showing that C is φ-regular at $\bar{x}$ with respect to $\bar{x}^*$.

Assume now that $\|\bar{x}^*\| = 1$. Pick a $\delta \in (0, \delta_0)$ such that $\delta(2+\delta)(1+\delta)^{-1} < \delta_0$. Let $x, y \in B(\bar{x}, \delta)$, $x^* \in N(C, x) \cap B(\bar{x}^*, \delta)$. Then $x^*/(1 + \delta) \in \partial^F d_C(x)$ since $\|x^*\| \le 1 + \delta$. Moreover,

$$\|(1+\delta)^{-1}x^* - \bar{x}^*\| \le (1+\delta)^{-1}\delta\|x^*\| + \|x^* - \bar{x}^*\| < (1+\delta)^{-1}\delta(2+\delta) < \delta_0.$$

That is, $(1 + \delta)^{-1}x^* \in \partial^F d_C(x) \cap B(\bar{x}^*, \delta_0)$. Hence,

$$\langle (1+\delta)^{-1}x^*, y - x \rangle \le \varphi(t\|x - y\|).$$

By the convexity of φ, $(1+\delta)\varphi(t\|x-y\|) \le \varphi((1+\delta)t\|x-y\|)$, and consequently, $\langle x^*, y-x \rangle \le \varphi((1+\delta)t\|x-y\|)$. This shows that C is φ-regular at $\bar{x}$ with respect to x^*.

$(i) \Rightarrow (ii)$. Assume that there exist $t, \delta > 0$ such that

$$\langle x^*, y-x \rangle \le \varphi(t\|x-y\|), \text{ for all } x, y \in C \cap B(\bar{x}, \delta),\ x^* \in N(C, x) \cap B(\bar{x}^*, \delta). \quad (5)$$

Let $x, y \in B(\bar{x}, \delta/2)$ with either $x \in C$ or $y \in C$, $x^* \in \partial^F d_C(x) \cap B(\bar{x}^*, \delta/2)$. Let us consider the following two cases:

Case 1. $x \in C$. Let (y_n) be a sequence with $y_n \in C$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \|y - y_n\| = d_C(y_n)$. Without loss of generality, we can assume that $y_n \in B(\bar{x}, \delta)$, $\forall n \in \mathbb{N}$. One has

$$\begin{aligned}\langle x^*, y-x\rangle = \langle x^*, y-y_n\rangle + \langle x^*, y_n-x\rangle &\leq \|y-y_n\| + \varphi(t\|y_n-x\|)\\ &\leq \|y-y_n\| + \varphi(t(\|y_n-y\| + \|y-x\|)).\end{aligned}$$

By letting $n \to \infty$ and using the the fact $d_C(y) \leq \|y - x\|$, one obtains the desired inequality $\langle x^*, y - x\rangle \leq d_C(y) + \varphi(2t\|y - x\|)$.

Case 2. $x \notin C$. By assumption, $y \in C$. By virtue of Proposition 1, we can select sequences (x_n) in C; (x_n^*) with $x_n^* \in N(C, x_n)$ such that

$$\|x - x_n\| \to d_C(x);\ \|x^* - x_n\| \to 0;\ \langle x_n^*, x - x_n\rangle \to d_C(x).$$

Without loss of generality, assume that $x_n \in B(\bar{x}, \delta)$ and $x_n^* \in B(\bar{x}^*, \delta)\ \forall n \in \mathbb{N}$. From (5), one has

$$\begin{aligned}\langle x_n^*, y - x\rangle = \ &\langle x_n^*, x_n - x\rangle + \langle x_n^*, y - x_n\rangle\\ &\leq \langle x_n^*, x_n - x\rangle + \varphi(2t\|y - x_n\|)\\ &\leq \langle x_n^*, x_n - x\rangle + \varphi(2t(\|y - x\| + \|x - x_n\|)).\end{aligned}$$

By letting $n \to \infty$, one obtains $\langle x^*, y - x\rangle \leq -d_C(x) + \varphi(4t\|y - x\|)$ as desired. ■

Remark 1 *φ-regularity of a set does not imply necessarily φ-regularity of the corresponding distance function. Indeed, consider the following example from Ngai & Penot (2006):*

Given $p \in (1, 2)$, let E be the hypograph of the function $f : r \mapsto |r|^p$ from $\mathbb{R}$ to $\mathbb{R}$:

$$E := \{(r, s) \in \mathbb{R}^2 : s \leq |r|^p\}.$$

In Ngai & Penot (2006), it was shown that E is intrinsically $p-$paraconvex around $(0, 0)$ but d_E is not p-paraconvex around $(0, 0)$. Note that the intrinsical $p-$paraconvexity implies φ-regularity for sets with $\varphi(t) := t^p$. On the other hand, observe from Theorem 8, Ngai & Penot (2006) that in the finite dimensional setting, for a locally Lipschitzian Fréchet regular function at a point, $p-$paraconvexity and t^p-regularity at this point are equivalent. Hence, there exists $x^* \in \partial^F d_E(0, 0) = N^F(E, (0, 0)) \cap B_{\mathbb{R}^2}$ such that d_E is not t^p-regular at $(0, 0)$ with respect to x^*.

The following theorem establishes the equivalence between φ-regularity of a function and its epigraph (see Theorem 3.5, Poliquin & Rockafellar, 2000, and Theorem 4.1 in Bernard & Thibault, 2004, for a similar result concerning prox-regularity).

Theorem 3 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $\varphi \in \mathcal{C}$ be given. Let $\bar{x} \in \operatorname{dom} f$, $\bar{x}^* \in \partial^F f(\bar{x})$. Then f is φ-regular at $\bar{x}$ with respect to $\bar{x}^*$ if and only if* epi f *is φ-regular at $(\bar{x}, f(\bar{x}))$ with respect to $(\bar{x}^*, -1)$.*

Proof. For the necessary part, let f be φ-regular at $\bar{x}$ with respect to $\bar{x}^*$. Then, there are $t, \delta_0 > 0$ such that whenever $x, y \in B(\bar{x}, \delta_0)$, $|f(x) - f(\bar{x})| < \delta_0$, $x^* \in \partial^F f(x) \cap B(\bar{x}^*, \delta_0)$, one has $\langle x^*, y - x\rangle \leq f(y) - f(x) + \varphi(t\|x - y\|)$.

Pick $\delta \in (0, \delta_0)$ such that $0 < \delta(1-\delta)^{-1}(1 + \|\bar{x}^*\|) < \delta_0$. Let (y, α), $(x, \beta) \in \operatorname{epi} f \cap B((\bar{x}, f(\bar{x})), \delta)$, $(x^*, -\lambda) \in N^F(\operatorname{epi} f, (x, \beta)) \cap B((\bar{x}^*, -1), \delta)$. We observe that necessarily $\lambda > 0$ since $\delta \in (0, 1)$. It implies, obviously, that $\beta = f(x)$. Moreover, $x^*/\lambda \in \partial^F f(x)$, and by the triangle inequality,

$$\|x^*/\lambda - \bar{x}^*\| \leq \|x^*\||1 - \lambda|/\lambda + \|x^* - \bar{x}^*\| < \delta(1 + \|\bar{x}^*\|)/(1-\delta) < \delta_0.$$

Hence, $\langle x^*/\lambda, y - x\rangle \leq f(y) - f(x) + \varphi(t\|x - y\|)$. Since $f(y) \leq \alpha$, this inequality implies

$$\langle x^*, y - x\rangle - \lambda(\alpha - f(x)) \leq \lambda\varphi(t\|x - y\|) \leq \varphi(t(1+\delta)\|x - y\|).$$

The necessary part is proved.

Conversely, let $t, \delta_0 \in (0, 1)$ such that

$$\langle x^*, y - x\rangle - (f(y) - f(x)) \leq \varphi(t\|y - x\| + t|f(y) - f(x)|) \tag{6}$$

whenever

$$(x, f(x)), (y, f(y)) \in B((\bar{x}, f(\bar{x})), \delta_0);$$
$$(x^*, -1) \in N^F(\operatorname{epi} f, (x, f(x))) \cap B((\bar{x}^*, -1), \delta_0).$$

Since $\lim_{s \to 0_+} \varphi(s)/s = 0$, there exists $\delta_1 \in (0, \delta_0)$ such that $\varphi(s) < s/t$, $\forall s \in (0, 4t\delta_1)$. Next, by the lower semicontinuity of f, we can pick $\delta \in (0, \delta_1/2)$ such that $2\delta(\|\bar{x}^*\| + \delta) < \delta_1/2$ and $f(y) > f(\bar{x}) - \delta_1$, $\forall y \in B(\bar{x}, \delta)$. Let now $x, y \in B(\bar{x}, \delta)$ with $|f(x) - f(\bar{x})| < \delta$ and $x^* \in \partial^F f(x) \cap B(x^*, \delta)$. We distinguish the following two cases :

Case 1. $|f(y) - f(\bar{x})| \geq \delta_1$. Then $f(y) - f(\bar{x}) \geq \delta_1$. Consequently,

$$f(y) - f(x) \geq \delta_1/2 \geq 2\delta(\|\bar{x}^*\| + \delta) > \|x^*\|\|y - x\| \geq \langle x^*, y - x\rangle.$$

Case 2. $|f(y) - f(\bar{x})| < \delta_1 (< \delta_0)$. From the relation (6) and by the convexity of φ, one has

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varphi(2t\|y - x\|)/2 + \varphi(2t|f(y) - f(x)|)/2. \tag{7}$$

On the other hand, since $\|x - y\|$, $|f(y) - f(x)| \in (0, 2\delta_1)$, then

$$\varphi(2t\|x - y\|) \leq 2\|x - y\| \text{ and } \varphi(2t|f(y) - f(x)|) \leq 2|f(y) - f(x)|.$$

Hence, if $M := (\|\bar{x}^*\| + \delta + 1)/2$, we have

$$f(y) - f(x) > (\langle x^*, y - x\rangle - \|x - y\|)/2 \geq -M\|x - y\|.$$

If $|f(y) - f(x)| > 2M\|x - y\|$, then $f(y) - f(x) > 2M\|x - y\|$ and obviously, one has $\langle x^*, y - x\rangle < f(y) - f(x)$; otherwise, from inequality (6) and using the fact that the function φ is increasing, one obtains

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varphi((2M + 1)t\|y - x\|).$$

Combining the two cases, we see that the final inequality always holds. The proof is complete. ■

When f is locally Lipschitzian, the φ-regularity of f is equivalent to the one of $d_{\text{epi}\, f}$ for some compatible norm on $X \times \mathbb{R}$ as shown in the following corollary.

Corollary 3 *Let $\varphi \in \mathcal{C}$ be given. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function which is locally Lipschitzian with rate c on some ball $B(\bar{x}, \rho)$. Suppose that $X \times \mathbb{R}$ is endowed with the norm given by $\|(x, r)\| = c\|x\| + |r|$. Then f is φ-regular at $\bar{x}$ with respect to $\bar{x}^* \in \partial^F f(\bar{x})$ if and only if the distance function $d_{\text{epi}\, f}$ is φ-regular at $(\bar{x}, f(\bar{x}))$ with respect to $(\bar{x}^*, -1)$.*

Proof. Denote by $E := \text{epi}\, f$. As shown in Ginsburg & Ioffe (1996), when $X \times \mathbb{R}$ is endowed with the norm described in the statement, we can find $\rho' \in (0, \rho)$ such that

$$d_E(x, r) = (f(x) - r)_+ := \max\{f(x) - r, 0\}$$

for $(x, r) \in B((\overline{x}, f(\bar{x})), \rho')$. Let f be φ-regular at $\bar{x}$ with respect to $\bar{x}^* \in \partial^F f(\bar{x})$. Then there are $t > 0$ and $\delta \in (0, \rho')$ such that whenever $x, y \in B(\bar{x}, \delta)$, $|f(x) - f(\bar{x})| < \delta$, $x^* \in B(\bar{x}^*, \delta)$ one has $\langle x^*, y - x\rangle \le f(y) - f(x) + \varphi(t\|x - y\|)$.

Let (x, α), $(y, \beta) \in B((\bar{x}, f(\bar{x})), \delta)$, $(x^*, -\lambda) \in \partial^F d_E((x, \alpha)) \cap B((\bar{x}^*, -1), \delta)$. Let us consider the following two cases.

Case 1. $(x, \alpha) \notin E$. Obviously, $\lambda = 1$ and $x^* \in \partial^F f(x)$. Therefore,

$$\begin{aligned}\langle (x^*, -1), (y, \beta) - (x, \alpha)\rangle &\le f(y) - f(x) + \varphi(t\|x - y\|) + \alpha - \beta \\ &\le d_E((y, \beta)) - d_E((x, \alpha)) + \varphi(t\|(y, \beta) - (x, \alpha)\|).\end{aligned} \tag{8}$$

Case 2. $(x, \alpha) \in E$. By Theorem 3, E is φ-regular at $(\bar{x}, f(\bar{x})$ with respect to $(x^*, -1)$. Then, from Theorem 2, we see that the inequality (8) also holds for some $t, \delta > 0$. Hence, d_E is φ-regular at $(\bar{x}, f(\bar{x}))$ with respect to $(\bar{x}^*, -1)$.

Conversely, let d_E be φ-regular at $(\bar{x}, f(\bar{x}))$ with respect to $(\bar{x}^*, -1)$. By Proposition 2, so is E. The conclusion follows directly from Theorem 3. ■

Let us recall that a closed subset $C \subseteq X$ is epi-Lipschitzian at $\bar{x} \in C$ if there exist $h \in X$ and $r > 0$ such that

$$C \cap B(\bar{x}, r) + \lambda B(h, r) \subseteq C \quad \forall \lambda \in (0, r).$$

As a result of Corollary 3, we derive

Corollary 4 *C is φ-regular at $\bar{x} \in C$ with respect to $\bar{x}^* \in N^F(C, \bar{x}) \cap B_{X^*}$ if and only if d_C is φ-regular at $\bar{x}$ with respect to $\bar{x}^*$ for some compatible norm on X.*

Proof. It suffices to notice that if C is epi-Lipschitzian at $\bar{x}$ then C has a locally Lipschitz representation at $\bar{x}$. This means that there exist ρ, $\sigma > 0$, some hyperplane H of X and some $u \in X$ with $\|u\| = 1$ such that $X = H \oplus \mathbb{R}u$ and a Lipschitzian function $f : B(\bar{x}, \rho) \cap H \to \mathbb{R}$ such that $E \cap B(\bar{x}, \sigma) = \{x + ru : x \in B(\bar{x}, \rho),\ f(x) \leq r\}$. ■

3. Subdifferential characterization of φ-regular functions

Poliquin & Rockafellar (2000) established a subdifferential characterization of the prox-regularity in the finite dimensional setting. This result has been extended to Hilbert spaces and latter to Banach spaces by Bernard & Thibault (2005). In this section, we prove that such a characterization for φ-regular functions in Asplund spaces holds. For this purpose, let us recall the notion of localization operators of the subdifferential operators (see Poliquin & Rockafellar, 2000).

Given a set-valued mapping $T : X \rightrightarrows X^*$, we use the notation $\operatorname{gph} T$ for the graph of T, that is the set of those points $(x, x^*) \in X \times X^*$ such that $x^* \in T(x)$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and suppose that $(\bar{x}, \bar{x}^*) \in \operatorname{gph} \partial^F f$ and $\delta > 0$ are given. We say that the operator $T_\delta : X \rightrightarrows X^*$ defined by

$$(x, x^*) \in \operatorname{gph} T_\delta \quad \Longleftrightarrow \quad (x, f(x)) \in B((\bar{x}, f(\bar{x})), \delta),\ x^* \in \partial^F f(x) \cap B(\bar{x}^*, \delta)$$

is the $f-$*attentive* δ-*localization* of the Fréchet subdifferential $\partial^F f$.

In order to generalize the concept of hypomonotonicity (with $\varphi(t) := t^2$) introduced by Poliquin & Rockafellar (2000), we say that $T : X \rightrightarrows X^*$ is φ-*submonotone* for some $\varphi \in \mathcal{C}$, if there exists $t > 0$ such that

$$\langle x^* - y^*, x - y \rangle \geq -\varphi(t\|x - y\|) \quad \text{for all } \ (x, x^*),\ (y, y^*) \in \operatorname{gph} T.$$

Then, we have the following characterization of φ-regularity:

Theorem 4 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous. Let $\varphi \in \mathcal{C}$ and $\bar{x} \in \operatorname{dom} f$, $\bar{x}^* \in \partial^F f(\bar{x})$. The following two statements are equivalent:*

(i) f is φ- regular at $\bar{x}$ with respect to $\bar{x}^*$;

(ii) there exist $t_0, \delta_0 > 0$ such that

$$\langle \bar{x}^*, y - \bar{x} \rangle \leq f(y) - f(\bar{x}) + \varphi(t_0\|y - \bar{x}\|)\ \forall y \in B(\bar{x}, \delta_0) \tag{9}$$

and the $f-$attentive δ_0-localization T_{δ_0} of $\partial^F f$ is φ-submonotone.

Proof. $(i) \Rightarrow (ii)$ is obvious from the definitions, while the proof of $(ii) \Rightarrow (i)$ is based on the Ekeland Variational Principle, and is inspired from Bernard & Thibault (2004). Let $t_0 > 0$ and $\delta_0 \in (0, 1)$ as in (9) and that

$$\langle x^* - y^*, x - y \rangle \geq -\varphi(t_0\|x - y\|) \quad \forall (x, x^*),\ (y, y^*) \in \operatorname{gph} T_{\delta_0}. \tag{10}$$

Let $\delta \in (0, \delta_0)$ and $t \in (4t_0, +\infty)$ (made more precise later on). Fix $(u, f(u)) \in B((\bar{x}, f(\bar{x})), \delta/4), u^* \in \partial^F f(u) \cap B(\bar{x}^*, \delta/4)$, and define the function $g : X \to \mathbb{R} \cup \{+\infty\}$ by

$$g(x) := g_t(x) = \begin{cases} f(x) + \langle u^*, u - x\rangle + \varphi(t\|x - u\|) & \text{if } x \in \overline{B}(\bar{x}, \delta), \\ +\infty & \text{otherwise} \end{cases}$$

where $\overline{B}(\bar{x}, \delta)$ stands for the closed ball with center $\bar{x}$ and radius δ. For any $x \in \overline{B}(\bar{x}, \delta) \setminus B(\bar{x}, \delta/2)$, then by the triangle inequality, $\|x - u\| \geq \delta/4$. Therefore, from (9), one has

$$\begin{aligned} g(x) &\geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x}\rangle - \varphi(t_0\|x - \bar{x}\|) + \langle u^*, x - u\rangle + \varphi(t\|x - u\|) \\ &\geq f(u) - (5\|\bar{x}^*\| + 2)\delta/4 + \varphi(t\delta/4) - \varphi(t_0\delta). \end{aligned} \tag{11}$$

On the other hand, by the convexity of φ and noting that $\varphi(0) = 0$,

$$\varphi(t\delta/4) - \varphi(t_0\delta) \geq (1 - 4t_0/t)\varphi(t\delta/4) \geq (1 - 4t_0/t)\varphi(t)\delta/4.$$

Combining this inequality and (11), we obtain

$$g(x) \geq g(u) + (1 - 4t_0/t)\varphi(t)\delta/4 - (5\|\bar{x}^*\| + 2)\delta/4 \quad \forall x \in \bar{B}(\bar{x}, \delta) \setminus B(\bar{x}, \delta/2). \tag{12}$$

Since $\lim_{s \to +\infty} \varphi(s) = +\infty$, we can fix $t > 4t_0$ such that

$$(1 - 4t_0/t)\varphi(t) > 5\|\bar{x}^*\| + 3.$$

Consequently, for this t,

$$g(x) > g(u) + \delta/4 \quad \text{for all } x \in \bar{B}(\bar{x}, \delta) \setminus B(\bar{x}, \delta/2). \tag{13}$$

Pick a positive sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $\varepsilon_n \in (0, \delta/4)$, and $\varepsilon_n \to 0$ as n goes to ∞. By the lower semicontinuity of f (so is g), g is bounded below for δ sufficiently small. Then, by virtue of the Ekeland Variational Principle, we can select a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in B(\bar{x}, \delta)$, $\forall n \in \mathbb{N}$ such that

$$g(x_n) < \inf_X g(x) + \varepsilon_n^2, \quad \text{and} \quad g(x_n) \leq g(x) + \varepsilon_n\|x - x_n\| \;\forall x \in X. \tag{14}$$

By relation (13), then, $x_n \in B(\bar{x}, \delta/2)$ for all $n \in \mathbb{N}$. Thus

$$0 \in \partial^F(g + \varepsilon_n\| \cdot - x_n\|)(x_n).$$

Applying the fuzzy sum rule, we can find sequences (u_n), (u_n^*), (v_n), (v_n^*) with u_n, $v_n \in B(\bar{x}, \delta/2)$, $u_n^* \in \partial^F f(u_n)$, $v_n^* \in \partial^F \varphi(t\| \cdot - u\|)(v_n)$ such that for all $n \in \mathbb{N}$ we have

$$\|u_n - x_n\| < \varepsilon_n; \;\; |f(u_n) - f(x_n)| < \varepsilon_n; \;\; \|v_n - x_n\| < \varepsilon_n; \;\; \|u_n^* - u^* + v_n^*\| < 2\varepsilon_n. \tag{15}$$

Since $v_n^* \in \partial^F \varphi(t\|\cdot - u\|)(v_n)$, then $v_n^* = t\varphi'(t\|v_n - u\|)z_n^*$ for some $z_n^* \in X^*$, $\|z_n^*\| \le 1$ and $\langle z_n^*, v_n - u\rangle = \|v_n - u\|$. Therefore,

$$\begin{aligned}\|u_n^* - \bar{x}^*\| &\le \|u^* - \bar{x}^*\| + \|v_n^*\| + \|u_n^* - u^* + v_n^*\| \\ &< \delta/4 + t\varphi'(t(3\delta/4 + \varepsilon_n)) + 2\varepsilon_n \\ &< 3\delta/4 + t\varphi(t\delta).\end{aligned}$$

On the other hand,

$$\begin{aligned}f(u_n) &< f(x_n) + \varepsilon_n = g(x_n) + \langle u^*, x_n - u\rangle - \varphi(t\|x_n - u\|) \\ &< f(u) + \varepsilon_n^2 + (\|\bar{x}^*\| + 1)\delta < f(\bar{x}) + (2\|\bar{x}^*\| + 3)\delta/2.\end{aligned}$$

Hence, we can find $\delta > 0$ such that (note that $\lim_{s\to 0_+} \varphi'(s) = 0$)
$3\delta/4 + t\varphi(t\delta) < \delta_0$, and $|f(u_n) - f(\bar{x})| < \delta_0$, for all $n \in \mathbb{N}$.
Thus $(u_n, u_n^*) \in \operatorname{gph} T_{\delta_0}$, for all $n \in \mathbb{N}$. According to the φ-submonotonicity (10) of T_{δ_0}, one has

$$\begin{aligned}-\varphi(t_0\|u_n - u\|) &\le \langle u^* - u_n^*, u - u_n\rangle \le \langle v_n^*, u - u_n\rangle + 2\varepsilon_n\|u - u_n\| \\ &\le \langle v_n^*, u - v_n\rangle + \delta_0\|u_n - v_n\| + 2\varepsilon_n\|u - u_n\| \\ &= -t\|u - v_n\|\varphi'(t\|v_n - u\|) + \delta_0\|u_n - v_n\| + 2\varepsilon_n\|u - u_n\|.\end{aligned}$$

By noticing that $\varphi(s) \le s\varphi'(s)$, $\forall s \in \mathbb{R}_+$, one obtains

$$\varphi(t\|v_n - u\|) - \varphi(t_0\|u_n - u\|) \le \delta_0\|u_n - v_n\| + 2\varepsilon_n\|u - u_n\|. \tag{16}$$

Let us show that $\|u_n - u\| \to 0$ as $n \to +\infty$. Indeed, if this were not the case, since $\|u_n - v_n\| \to 0$, without loss of generality, we may assume that for all $n \in \mathbb{N}, \|v_n - u\| > \|u_n - v_n\|$. By the convexity of φ, one has

$$\begin{aligned}2\varphi(t\|u_n - u\|/2) &\le \varphi(t(\|u_n - u\| - \|u_n - v_n\|)) + \varphi(t\|u_n - v_n\|) \\ &\le \varphi(t\|v_n - u\|) + \varphi(t\|u_n - v_n\|).\end{aligned}$$

This inequality, with (16), implies

$$2\varphi(t\|u_n - u\|/2) - \varphi(t_0\|u_n - u\|) \le \varphi(t\|u_n - v_n\|) + \delta_0\|u_n - v_n\| + 2\varepsilon_n\|u - u_n\|.$$

Moreover, since $t > 4t_0$, then $\varphi(t_0\|u_n - u\|) \le 2t_0/t\varphi(t\|u_n - u\|/2)$. Consequently,

$$2(1 - t_0/t)\varphi(t\|u_n - u\|/2) \le \varphi(t\|u_n - v_n\|) + \delta_0\|u_n - v_n\| + 2\varepsilon_n\|u - u_n\|.$$

This shows that $\varphi(t\|u_n - u\|) \to 0$, and therefore $\|u_n - u\| \to 0$ as $n \to \infty$ (since φ is strictly increasing), a contradiction. Consequently, (x_n) converges to u. By definition of g and relation (14) as well, one obtains

$$f(u) \le f(x) + \langle u^*, u - x\rangle + \varphi(t\|x - u\|) \quad \forall x \in B(x, \delta).$$

This means that f is φ-regular at $\bar{x}$ with respect to $\bar{x}^*$. ∎

4. Regularization of φ-regular functions

In this final section, we consider general regularizations of Moreau type for φ-regular functions. Let us recall that X is said to be uniformly convex if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that whenever

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon \text{ then } \|x + y\| \leq 2(1 - \delta(\varepsilon)).$$

Prototypes of uniformly convex spaces are the spaces ℓ^p and L^p $(1 < p < +\infty)$ and uniformly convex spaces constitute a subclass of the class of reflexive spaces. Suppose X is uniformly convex with a smooth norm. As it is well known (see Diestel, 1975, Benyamini & Lindenstrauss, 2000, for instance), the modulus of convexity of X is defined by

$$\delta(\varepsilon) := \inf\{1 - \|x - y\|/2 : \ \ \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}.$$

For $s > 1$, we note $J_s(x) := \nabla(\frac{1}{s}\| \cdot \|^s)(x)$, the $s-$duality mapping. Then, by Xu & Roach (1991), Theorem 1, there exists a real $b > 0$ such that

$$\langle J_s(x) - J_s(y), x - y\rangle \geq b \max(\|x\|, \|y\|)^s \delta\Big(\frac{\|x - y\|}{2\max(\|x\|, \|y\|)}\Big), \text{ for all } x, y \in X. \tag{17}$$

X is said to be uniformly convex of power type p $(p \geq 2)$ if for some $c > 0, \delta(\varepsilon) \geq c\varepsilon^p$, $\forall \varepsilon \in [0, 2]$ (see Diestel, 1975; Benyamini & Lindenstrauss, 2000).

For each $x^* \in X^*, s > 1$ and $t > 0$, similarly to Poliquin & Rockafellar (2000), we may define x^*-envelopes of a given lower semicontinuous function f by

$$g_t^{x^*}(w) = \inf_{x \in X} \left\{ f(x) - \langle x^*, x\rangle + \frac{t}{s}\|x - w\|^s \right\}, \quad w \in X.$$

We note $P_t^{x^*}$ the corresponding argmin.

THEOREM 5 *Let $\varphi \in \mathcal{C}$, and $s > 1$ such that for each $\tau > 0$ there exists $c := c(\tau) > 0$ such that whenever $\varepsilon \in [0, 2]$ and $\alpha \in [\varepsilon/2, 1]$, one has*

$$c\tau \min\{\epsilon^s, \alpha^s \delta(\varepsilon/\alpha)\} \geq \varphi(\tau\varepsilon).$$

Let f be φ- regular at $\bar{x} \in \operatorname{dom} f$ with respect to $\bar{x}^* \in \partial^F f(\bar{x})$. Suppose that there exists a real $m > 0$ such that the function $f(\cdot) + m\| \cdot \|^s$ is bounded below on X. Then there is $t_0 > 0$ such that for all $t > t_0$, the function $g_t := g_t^{\bar{x}^*}$ is of class $\mathcal{C}^1$ on some neighborhood U_t of $\bar{x}$ and the mapping $P_t := P_t^{\bar{x}^*}(\cdot)$ is a singleton and continuous on U_t.

Proof. First, we prove that for each t sufficiently large, there exists $\eta > 0$ such that $P_t(\cdot)$ is singleton and continuous on $B(\bar{x}, \eta)$. Let $b > 0$ as in (17). Let $t_0, \eta_0 \in (0, 2/t_0)$ with $\eta_0 < 1/2$. For all $x, y \in B(\bar{x}, \eta_0)$, $|f(x) - f(\bar{x})| < \eta_0$, $x^* \in \partial^F f(x) \cap B(\bar{x}^*, \eta_0)$ one has

$$\langle x^*, y - x\rangle \le f(y) - f(x) + \varphi(t_0\|x - y\|). \tag{18}$$

Let $c > 0$ such that $\varphi(2t_0\varepsilon) \le c\alpha^s\delta(\varepsilon/2\alpha)$, for all $\varepsilon \in [0, 2], \alpha \in [\varepsilon/2, 1]$, and $\varphi(\varepsilon) \le c\varepsilon^s$ for all $\varepsilon \in [0, 2]$. By assumption, $f(x) \ge m\|x\|^s - a \quad \forall x \in X$, for some $a \in \mathbb{R}$; obviously, we can find $t_1 = t(\eta_0) > t_0$ such that for all $w \in B(x, \eta_0/2)$,

$$\begin{aligned} g_t^{x^*}(w) & \\ &= \inf_{x\in X}\left\{f(x) - \langle x^*, x\rangle + \tfrac{t}{s}\|x - w\|^s\right\} \\ &= \inf_{x\in B(\bar{x},\eta_0)}\left\{f(x) - \langle x^*, x\rangle + \tfrac{t}{s}\|x - w\|^s\right\}. \end{aligned}$$

Let $t > \max\{sc2^{s-1}t_0, t_1, c/b\}$ be given. Let $\eta \in (0, \eta_0/2)$ with $\eta < 2/t$ (made more precise later). Take $u, v \in B(\bar{x}, \eta)$ and consider arbitrary minimizing sequences (u_n) and (v_n) of $g_t(u)$ and $g_t(v)$, respectively. Then, without loss of generality, assume that u_n, $v_n \in B(\bar{x}, \eta_0)$ for all $n \in \mathbb{N}$. Then, there exists a sequence (ε_n) of nonnegative numbers converging to zero such that

$$\begin{aligned} &f(u_n) - \langle\bar{x}^*, u_n\rangle + \frac{t}{s}\|u_n - u\|^s < g_t(u) + \varepsilon_n^2;\\ &f(v_n) - \langle\bar{x}^*, v_n\rangle + \frac{t}{s}\|v_n - v\|^s < g_t(v) + \varepsilon_n^2. \end{aligned}$$

For all $n \in \mathbb{N}$, by definition of g_t, one has

$$f(u_n) - \langle\bar{x}^*, u_n\rangle + \frac{t}{s}\|u_n - u\|^s < f(\bar{x}) - \langle\bar{x}^*, \bar{x}\rangle + \frac{t}{s}\|\bar{x} - u\|^s + \varepsilon_n^2.$$

By relation (18), and noticing that $\varphi(\varepsilon) \le c\varepsilon^s$, for each $\varepsilon \in [0, 2]$, the preceding yields

$$\frac{t}{s}\|u_n - u\|^s - ct_0^s\|u_n - \bar{x}\|^s \le \frac{t}{s}\|\bar{x} - u\|^s + \varepsilon_n^2.$$

Using the fact that $\alpha^s/2 + \beta^s/2 \ge (\alpha + \beta)^s/2^s$, $\forall\alpha,\ \beta > 0$, one gets

$$\left(\frac{t}{s2^{s-1}} - ct_0^s\right)\|u_n - \bar{x}\|^s \le 2\frac{t}{s}\|u - \bar{x}\|^s + \varepsilon_n^2 < 2t\eta^s/s + \varepsilon_n^2.$$

Equivalently, this amounts to saying that

$$\|u_n - \bar{x}\| < \gamma_n := \gamma(\eta, t, \varepsilon_n) := \left(\frac{2ts^{-1}\eta^s + \varepsilon_n^2}{ts^{-1}2^{1-s} - ct_0^s}\right)^{1/s}. \tag{19}$$

Similarly, one has $\|v_n - \bar{x}\| < \gamma(\eta, t, \varepsilon_n)$. Let us pick η sufficiently small such that for all n sufficiently say big, $n \geq n_0$, $f(x) - f(\bar{x}) > -\eta_0 \ \forall x \in B(\bar{x}, \gamma_n + 2\varepsilon_n)$ and that

$$t(\gamma_n + \eta + \varepsilon_n)^{s-1} + 2\varepsilon_n < \eta_0, \ \ (\gamma_n + 2\varepsilon_n)\|\bar{x}^*\| + t\eta^s s^{-1} + \varepsilon_n + \varepsilon_n^2 < \eta_0.$$

By virtue of the Ekeland Variational Principle, we can select sequences (w_n) and (z_n) such that

$$w_n \in B(u_n, \varepsilon_n), \ f(w_n) - \langle \bar{x}^*, w_n\rangle + \frac{t}{s}\|w_n - u\|^s < g_t(u) + \varepsilon_n^2; \tag{20}$$

$$z_n \in B(v_n, \varepsilon_n), \ f(z_n) - \langle \bar{x}^*, z_n\rangle + \frac{t}{s}\|z_n - v\|^s < g_t(v) + \varepsilon_n^2 \tag{21}$$

and

$$f(w_n) - \langle \bar{x}^*, w_n\rangle + \frac{t}{s}\|w_n - u\|^s \leq f(x) - \langle \bar{x}^*, w_n\rangle + \frac{t}{s}\|x - u\|^s + \varepsilon_n\|x - w_n\| \ \ \forall x \in X;$$

$$f(z_n) - \langle \bar{x}^*, z_n\rangle + \frac{t}{s}\|z_n - v\|^s \leq f(x) - \langle \bar{x}^*, z_n\rangle + \frac{t}{s}\|x - v\|^s + \varepsilon_n\|x - z_n\| \ \ \forall x \in X.$$

Hence, applying the Fabian fuzzy sum rule (Theorem 1), we obtain the existence of sequences in X (w_n^1), (z_n^1) with $(w_n^1, f(w_n^1)) \in B((w_n, f(w_n)), \varepsilon_n)$; $(z_n^1, f(z_n^1)) \in B((z_n, f(z_n)), \varepsilon_n)$; (u_n^*), (v_n^*) with $\|u_n^*\|$, $\|v_n^*\| \leq 1$ such that

$$\bar{x}^* - tJ_s(w_n - u) + 2\varepsilon_n u_n^* \in \partial^F f(w_n^1); \ \ \bar{x}^* - tJ_s(z_n - v) + 2\varepsilon_n v_n^* \in \partial^F f(z_n^1).$$

Since $\|J_s(w_n - u)\| \leq \|w_n - u\|^{s-1} < (\gamma_n + \eta + \varepsilon_n)^{s-1}$ and

$$\|J_s(z_n - v)\| \leq \|z_n - v\|^{s-1} < (\gamma_n + \eta + \varepsilon_n)^{s-1}$$

and using the definition of η, one has

$$\bar{x}^* - tJ_s(w_n - u) + 2\varepsilon_n u_n^* \in B(\bar{x}^*, \eta_0); \ \bar{x}^* - tJ_s(z_n - v) + 2\varepsilon_n v_n^* \in B(\bar{x}^*, \eta_0).$$

Moreover, by (20), (21), we derive

$$|f(w_n^1) - f(\bar{x})|, \ |f(z_n^1) - f(\bar{x})| < \eta_0.$$

Therefore, from (18), we obtain

$$\langle -tJ_s(w_n - u) + 2\varepsilon_n u_n + tJ_s(z_n - v) - 2\varepsilon_n v_n, w_n^1 - z_n^1\rangle \geq -2\varphi(t_0\|w_n^1 - z_n^1\|).$$

On the other hand, by (17), one gets, for some constant $M > 0$ and for $n \geq n_0$

$$tb\max(\|w_n - u\|, \|z_n - v\|)^s \delta\left(\frac{\|(w_n - z_n) - (u - v)\|}{2\max(\|w_n - u\|, \|z_n - v\|)}\right) - 2\varphi(t_0\|w_n - z_n\|)$$
$$\leq M(\|u - v\| + \varepsilon_n).$$

Thus,

$$tb/c\varphi(2t_0\|(w_n - z_n) - (u - v)\|) - 2\varphi(t_0\|w_n - z_n\|) \leq M(\|u - v\| + \varepsilon_n).$$

By the convexity of φ, then,

$$\varphi(2t_0\|(w_n - z_n) - (u - v)\|) \geq 2\varphi(t_0\|w_n - z_n\|) - \varphi(2t_0\|u - v\|).$$

Hence, $2(tb/c - 1)\varphi(t_0\|w_n - z_n\|) \leq tb/c\varphi(2t_0\|u - v\|) + M(\|u - v\| + \varepsilon_n)$. Consequently,

$$\begin{aligned} 2(tb/c - 1)\varphi(t_0\|u_n - v_n\|) \leq (2tb/c - 1)\varphi(2t_0\varepsilon_n) + tb/c\varphi(2t_0\|u - v\|) \\ +M(\|u - v\| + \varepsilon_n). \end{aligned}$$

Since φ is a strictly increasing continuous function and $\varphi(0) = 0$, this inequality shows clearly that for each $u \in B(\bar{x}, \eta)$, $P_t(u)$ is nonempty, singleton, and moreover, by letting $n \to \infty$,

$$\begin{aligned} 2(tb/c - 1)\varphi(t_0\|P_t(u) - P_t(v)\|) \leq tc/b\varphi(2t_0\|u - v\|) + M\|u - v\|, \\ \text{for all } u, v \in B(\bar{x}, \eta). \end{aligned} \tag{22}$$

Consequently, $P_t(\cdot)$ is continuous on $B(\bar{x}, \eta)$.

Note that under the assumptions of the theorem, the function g_t is locally Lipschitzian (see Attouch, 1984). Let us show that g_t is continuously differentiable on $B(\bar{x}, \eta)$. For any $u \in B(\bar{x}, \eta), h \in X$, if we note $g_t^{\uparrow}(u, h)$ the Clarke generalized directional derivative of g_t, then we have,

$$\begin{aligned} g_t^{\uparrow}(u, h) &= \limsup_{(y,\lambda)\to(u,0_+)} \frac{1}{\lambda}(g_t(y + \lambda h) - g_t(y)) \\ &\leq \frac{t}{s} \limsup_{(y,\lambda)\to(u,0_+)} \frac{1}{\lambda}(\|y + \lambda h - P_t(y)\|^s - \|y - P_t(y)\|^s) \\ &\leq t \limsup_{(y,\lambda)\to(u,0_+)} \sup_{\theta\in[0,1]} \langle J_s(y - P_t(y) + \theta\lambda h), h\rangle = t\langle J_s(u - P_t(u)), h\rangle. \end{aligned}$$

Hence, $g_t^{\uparrow}(u, \cdot) = t\langle J_s(u - P_t(u)), \cdot\rangle$ and $\partial^{\uparrow} g_t(u) = tJ_s(u - P_t(u))$. Since the norm on X is Fréchet differentiable (and convex), the $s-$duality mapping $J(.)$ is continuous. Thus, $g_t(.)$ is continuously differentiable on $B(\bar{x}, \eta)$. The proof is complete. ■

When X is uniformly convex of power type p ($p \geq 2$) and $\varphi(t) := t^p$, as well as $s \in (1, p]$, then the inequality (22) tells us that $P_t(\cdot)$ is $1/p-$Hölderian continuous on $B(\bar{x}, \eta)$. Note that in this case, Theorem 5 subsumes a partial extension of Theorem 5.3 and Theorem 5.5 in Bernard & Thibault (2004) .

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