Control and Cybernetics

vol. **36** (2007) No. 3

Error bounds for convex constrained systems in Banach spaces[1]

by

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Abstract: In this paper, we first establish both primal (involving directional derivatives and tangent cones) and dual characterizations (involving subdifferential and normal cones) for the local (global) error bounds of constrained set-valued systems; as an application, we then derive both primal and dual characterizations for the local (global) error bounds of the constrained convex inequality systems in a general Banach space and also some sufficient conditions. The obtained results improve or generalize some known results.

Keywords: error bounds, constrained convex system, set-valued mapping, Banach space.

1. Introduction

Let X and Y be normed spaces, let $C \subset X$ be a closed convex set and let $S\colon X \rightrightarrows Y$ be a convex set-valued mapping.

We consider a set-valued constrained system of the type

$$M(y) = C \cap S^{-1}(y). \tag{1}$$

We say that M has a local error bound at $(\bar{y}, \bar{x}) \in \mathrm{gr}M$ if there exist $\delta, \gamma > 0$ such that

$$d(x, M(\bar{y})) \leq \gamma \max\{d(x, C), d(\bar{y}, S(x))\} \text{ for all } x \in B(\bar{x}, \delta) \tag{2}$$

(the set-valued mapping S^{-1} is considered instead of S in Li and Singer, 1998; Song, 2006); if the inequality (2) is satisfied for all $x \in X$, i.e., there exists $\gamma > 0$ such that

$$d(x, M(\bar{y})) \leq \gamma \max\{d(x, C), d(\bar{y}, S(x))\} \text{ for all } x \in X, \tag{3}$$

we say that M has a global error bound at $\bar{y}$.

[1]This work was partially supported by the National Natural Sciences Grant (No. 10671050) and the Excellent Young Teachers Program of MOE, P.R.C

Clearly, when $S(x) = [f(x), \infty)$, where $f\colon X \to \mathbb{R}\cup\{+\infty\}$ is a proper convex function, $\bar{y} = 0$ and

$$M = M(\bar{y}) = \{x \in C \mid f(x) \leq 0\}, \tag{4}$$

the above definitions reduce to the following: there exist $\gamma, \delta > 0$ such that

$$d(x, M) \leq \gamma \max\{f(x), d(x, C)\} \text{ for all } x \in B(\bar{x}, \delta), \tag{5}$$

and

$$d(x, M) \leq \gamma \max\{f(x), d(x, C)\} \text{ for all } x \in X. \tag{6}$$

In the last decade, the study of error bounds of convex systems has received a growing interest in mathematical programming literature. There are both theoretical and practical reasons for this, since error bounds are closely related to the Lipschitz stability of feasible/optimal solution set and the sensitivity analysis of convex programming and complementarity, and also the convergence analysis of some descent methods.

When C is the whole space of $\mathbb{R}^n$ or a general Banach space X, the existence of global (local) error bounds for (4) has been widely discussed in literature (see Hoffman, 1952; Ioffe, 1975; Robinson, 1976; Burke and Tseng, 1996; Deng, 1998; Lewis and Pang, 1998; Klatte and Li, 1999; Ng and Zheng, 2001; Ng and Yang, 2002; Zălinescu, 2002; Zheng and Ng, 2003; Wu and Ye, 2003).

When C is a proper subset of $\mathbb{R}^n$, Lewis and Pang (1998) established a primal characterization involving directional derivatives and tangent cones, and derived several sufficient conditions for the existence of a global error bound for system (4). Zălinescu (2002) gave an infinite-dimensional version of Lewis and Pang's result in a reflexive Banach space. Recently, Zheng and Ng (2003), Burke and Deng (2002) and Song (2006) considered the existence of local (global) error bounds for system (4) in a general Banach space.

For system (1), Robinson (1976), Li and Singer (1998) and Zheng (2003) proposed some (regular point type) sufficient conditions for the existence of a local (global) error bound, and Ng and Zheng (2004) proposed several characterizations for a global error bound in terms of the contingent derivative of the set-valued mapping in the case where $C = X$. Recently, in Song (2006) we have presented a dual sufficient condition for the existence of a local error bound for system (1) in terms of the normal cones and the coderivative of the set-valued mapping.

As observed by Lewis and Pang in (1998), the inequality system (4) has a global error bound if and only if M is the set of weak sharp minima of the function $\phi(x) := f_+(x) + d(x, C)$, (where $f_+(x) = \max\{f(x), 0\}$), i.e., there exists some $\tau > 0$ such that

$$\phi(x) \geq \phi(\bar{x}) + \tau d(x, M), \ \forall x \in X, \ \forall \bar{x} \in M.$$

The notion of weak sharp minima was introduced by Ferris (1998). Since then the weak sharp minima have been extensively studied both in finite dimensional and infinite dimensional spaces along with their connection to error bounds (see Burke and Ferris, 1993; Studniarski and Ward, 1999; Burke and Deng, 2002, 2005; Zălinescu, 2002).

Another concept, which is closely related to error bounds and weak sharp minima, is calmness. A set-valued mapping $M\colon Y \rightrightarrows X$ is said to be calm at $(\bar{y}, \bar{x})$ (see Henrion and Outrata, 2001), if there exist neighborhoods $\mathcal{V}, \mathcal{U}$ of $\bar{y}, \bar{x}$, respectively, and some $\gamma > 0$ such that $d(x, M(\bar{y})) \leq \gamma d(y, \bar{y}) \quad \forall x \in M(y) \cap \mathcal{U}, \forall y \in \mathcal{V}$, or equivalently, $M(y) \cap \mathcal{U} \subset M(\bar{y}) + \gamma d(y, \bar{y}) B_X,\ \forall y \in \mathcal{V}$.

It has been proved in Song (2006) (see also Dontchev and Rockafellar, 2004, under some additional assumption) that $M(y) = C \cap S^{-1}(y)$ is calm at $(\bar{y}, \bar{x})$ if and only if there exist $\delta, \gamma > 0$ such that

$$d(x, M(\bar{y})) \leq \gamma d(\bar{y}, S(x)) \text{ for all } x \in C \cap B(\bar{x}, \delta).$$

This shows that the calmness of M at $(\bar{y}, \bar{x})$ amounts to the existence of a local error bound of M at the same point whenever $C = X$. A similar observation for the inequality system (4) was given in Henrion and Outrata (2005). The criteria of calmness and its application in optimization problems have been discussed in Henrion and Jourani (2002), Henrion, Jourani and Outrata (2002), Henrion and Outrata (2001, 2005), Song (2006).

In this paper, we first establish both primal (involving directional derivatives and tangent cones) and dual characterizations (involving subdifferentials and normal cones) for the local (global) error bounds of system (1); as an application, we then derive both primal and dual characterizations for the local (global) error bounds of system (4) in a general Banach space as well as some sufficient conditions. The obtained results improve or generalize the corresponding results of Lewis and Pang (1998), Ng and Zheng (2004), Zălinescu (2002), to a general case.

After completion of the present paper we have learned that Zheng and Ng (2007) investigated subregularity of a constrained set-valued mapping which is defined in a similar way as (2). Some characterizations for the local (global) error bounds of system (1) obtained in this paper have been already proved in Zheng and Ng (2007). Since the proofs given in this paper in terms of Lemma 1 and 2 are much simpler than those given in Zheng and Ng (2007), for completeness, we still keep them in this paper.

2. Notations and basic results

Let X be a normed space with topological dual space X^*. Denote by B_X and B_{X^*} the closed unit balls of X and X^*, respectively. We write $B(x, \delta)$ for $x + \delta B_X$, where $x \in X$ and $\delta > 0$.

Let C be a nonempty subset of X. Denote by $\bar{C}$(or cl$C$), int$C$, bd$C$, and core$C$, respectively, the closure, the interior, the boundary of C, and the algebraic interior, and denote by cone$C$ the cone generated by C. The distance function to C is defined by $d_C(x) = d(x, C) \colon = \inf\{\|x - y\| \mid y \in C\}$.

For a nonempty convex set C in X and $x \in C$, we recall that the tangent cone to the set C at x is $T_C(x) = \overline{\text{cone}}(C - x)$ and the normal cone to C at x is $N_C(x) = T_C(x)^0 = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq 0 \text{ for all } y \in C\}$.

Let $f \colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. By epi$f$, f^*, $\partial f(\bar{x})$, and $\partial^\infty f(\bar{x})$ we denote the epigraph, conjugate function, the usual, and the singular subdifferentials of f, respectively, in the sense of convex analysis. It is well known that

$$\partial f(\bar{x}) = \{x^* \in X^* \mid (x^*, -1) \in N_{\text{epi}f}(\bar{x}, f(\bar{x}))\},$$
$$\partial^\infty f(\bar{x}) = \{x^* \in X^* \mid (x^*, 0) \in N_{\text{epi}f}(\bar{x}, f(\bar{x}))\}.$$

Let $f \colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function taking finite value at x. The directional derivative of f at x in the direction h is defined by

$$f^{'}(x, h) = \inf_{t>0} \frac{f(x + th) - f(x)}{t}.$$

It is well known (see Theorem 1 in Burke, Ferris and Qian, 1992) that $\partial d_C(x) = B_{X^*} \cap N_C(x) \ \ \forall x \in C$.

Let $I_C(\cdot)$ and $\sigma_C(\cdot)$ be the indicator and the support functions of C, respectively, i.e.,

$$I_C(x) = \begin{cases} 0 \text{ if } x \in C, \\ +\infty \text{ if } x \notin C \end{cases}$$

and $\sigma_C(x^*) = (I_C)^*(x^*) = \sup_{x \in C}\langle x^*, x\rangle$. It is obvious that $N_C(x) = \partial I_C(x)$.

As in Ng and Zheng (2004), for $x \in \text{bd}C$ and $\sigma \in (0, 1]$, we let

$$\mathcal{N}_C^1(x, \sigma) = \{h \in X \mid \|h\| = 1, \exists x^* \in N_C(x) \text{ with } \|x^*\| = 1 \text{ such that } \langle x^*, h\rangle \geq \sigma\}.$$

Let $F \colon X \rightrightarrows Y$ be a set-valued mapping. We say that F is convex (closed) if its graph, gr$F = \{(x, y) \in X \times Y \mid y \in F(x)\}$, is convex (closed).

Let F be a convex set-valued mapping from X to Y and let $(\bar{x}, \bar{y}) \in \text{gr}F$. A set-valued mapping $DF(\bar{x}, \bar{y}) \colon X \rightrightarrows Y$ whose graph is the tangent cone to the graph of F at $(\bar{x}, \bar{y})$, i.e.,

$$\text{gr}(DF(\bar{x}, \bar{y})) = T_{\text{gr}(F)}((\bar{x}, \bar{y})),$$

is called the contingent derivative of F at $(\bar{x}, \bar{y})$.

The coderivative $D^*F(\bar{x}, \bar{y}) \colon Y^* \rightrightarrows X^*$ is defined by

$$D^*F(\bar{x}, \bar{y})(v^*) \colon = \{u^* \in X^* \mid (u^*, -v^*) \in N_{\text{gr}F}((\bar{x}, \bar{y}))\}.$$

Observe that $u^* \in D^*F(\bar{x}, \bar{y})(v^*)$ if and only if $\langle u^*, u\rangle \leq \langle v^*, v\rangle$ for all $v \in DF(\bar{x}, \bar{y})(u)$.

Another kind of derivative is $\hat{D}F(\bar{x},\bar{y})$, which is defined by

$$\hat{D}F(\bar{x},\bar{y})(u):\ = \{v \in Y \mid \lim_{t\to 0} d(v, \frac{F(\bar{x}+tu)-\bar{y}}{t}) = 0\}.$$

Clearly, $\hat{D}F(\bar{x},\bar{y})(u) \subset DF(\bar{x},\bar{y})(u)$.

In the following we assume that X is a Banach space. The following result was demonstrated in Song (2006), which can be proved by Lemma 1.1 and Proposition 1.3 in Ng and Yang (2002).

LEMMA 1 *Let C be a closed convex subset of X and $\bar{x} \in C$. Then, for every $\delta > 0$, $0 < \sigma < 1$ and every $x \in B(\bar{x}, \delta/3) \setminus C$, there exist $z \in \mathrm{bd}C \cap B(\bar{x},\delta)$ and $x^* \in N_C(z)$ with $\|x^*\| = 1$ such that*

$$\sigma d(x,C) \le \sigma\|x-z\| \le \langle x^*, x-z\rangle \le d(x-z, T_C(z)).$$

The following subdifferential formula of marginal functions will be useful in the sequel. Similar results in more general settings can be found in Truong (2005), Mordukhovich, Nam and Yen (2007). Since our result cannot be derived directly from the corresponding results in Truong (2005), Mordukhovich, Nam and Yen (2007), we include a simple proof here.

LEMMA 2 *Let $F\colon X \rightrightarrows Y$ be a convex set-valued mapping and let $(\bar{x},\bar{y}) \in \mathrm{gr}F$. Then the function*

$$v(x):\ = \inf_{y\in F(x)} \|y-\bar{y}\|$$

is a convex function and $\partial v(\bar{x}) = D^*F(\bar{x},\bar{y})(B_{Y^*})$.

Proof. It is easy to verify that v is a convex function. Let $x^* \in \partial v(\bar{x})$. Then $\langle x^*, x-\bar{x}\rangle \le \|y-\bar{y}\|$ for all $(x,y) \in \mathrm{gr}F$.

Define $f\colon X\times Y \to \mathbb{R}$ by $f(x,y) = \|y-\bar{y}\| - \langle x^*, x-\bar{x}\rangle$. Then f attains its minimum on the set $\mathrm{gr}F$ at $(\bar{x},\bar{y})$. Hence $(0,0) \in \partial f(\bar{x},\bar{y}) + N_{\mathrm{gr}F}(\bar{x},\bar{y})$. It follows that there exists $y^* \in B_{Y^*}$ such that $(x^*, -y^*) \in N_{\mathrm{gr}F}(\bar{x},\bar{y})$, i.e. $x^* \in D^*F(\bar{x},\bar{y})(B_{Y^*})$. Hence $\partial v(\bar{x}) \subset D^*F(\bar{x},\bar{y})(B_{Y^*})$. The converse inclusion can be proved similarly. ■

3. Main results

We first present some primal and dual characterizations for the existence of local (global) error bounds for the system (1).

THEOREM 1 *Consider the set-valued mapping $M\colon Y \rightrightarrows X$ defined as $M(y) = C \cap S^{-1}(y)$, where $S\colon X \rightrightarrows Y$ is a convex set-valued mapping and C is a closed convex set of X. Let $(\bar{y},\bar{x}) \in \mathrm{gr}M$ be such that $S^{-1}(\bar{y})$ is closed. Then the following statements are equivalent:*

(i) *M has a local error bound at* $(\bar{y}, \bar{x})$;
(ii) *there exist* $\gamma, \delta > 0$ *such that*

$$B_{X^*} \cap N_{M(\bar{y})}(u) \subset \gamma[B_{X^*} \cap N_C(u) + D^*S(u, \bar{y})(B_{Y^*})] \tag{7}$$

for all $u \in B(\bar{x}, \delta) \cap \mathrm{bd}M(\bar{y})$;
(iii) *there exist* $\delta, \gamma > 0$ *such that*

$$d(h, T_{M(\bar{y})}(u)) \leq \gamma \max\{d(h, T_C(u)), d(0, DS(u, \bar{y})(h))\} \tag{8}$$

for all $u \in B(\bar{x}, \delta) \cap \mathrm{bd}M(\bar{y}),\ h \in X$;
(iii)$'$ *there exist* $\delta, \gamma > 0$ *such that*

$$d(h, T_{M(\bar{y})}(u)) \leq \gamma \max\{d(h, T_C(u)), d(0, \hat{D}S(u, \bar{y})(h))\} \tag{9}$$

for all $u \in B(\bar{x}, \delta) \cap \mathrm{bd}M(\bar{y}),\ h \in X$;
(iv) *there exist* $\sigma \in (0, 1)$ *and* $\delta, \eta > 0$ *such that*

$$\eta \leq \max\{d(h, T_C(u)), d(0, DS(u, \bar{y})(h))\} \tag{10}$$

for all $u \in B(\bar{x}, \delta) \cap \mathrm{bd}M(\bar{y}),\ h \in \mathcal{N}^1_{M(\bar{y})}(u, \sigma)$;
(iv)$'$ *there exist* $\sigma \in (0, 1)$ *and* $\eta > 0$ *such that*

$$\eta \leq \max\{d(h, T_C(u)), d(0, \hat{D}S(u, \bar{y})(h)) \tag{11}$$

for all $u \in B(\bar{x}, \delta) \cap \mathrm{bd}M(\bar{y})$ *and* $h \in \mathcal{N}^1_{M(\bar{y})}(u, \sigma)$.

Proof. (i) $\Rightarrow$ (ii). Suppose that M has a local error bound at $(\bar{y}, \bar{x})$. Then there exist positive scalars γ and δ_1 such that

$$d(x, M(\bar{y})) \leq \gamma(d(x, C) + d(\bar{y}, S(x)) \quad \text{for all } x \in B(\bar{x}, \delta_1).$$

Hence, $d(x, M(\bar{y})) \leq \gamma[d(x, C) + d(\bar{y}, S(x)) + I_{B(\bar{x}, \delta_1)}(x)]$ for all $x \in X$. As both functions on the two sides of the above inequality are 0 at each $u \in \mathrm{bd}M(\bar{y}) \cap \mathrm{int}B(\bar{x}, \delta_1)$, by Lemma 2 and a well known subdifferential formula (see Theorem 2.8.7 in Zălinescu, 2002), we have that

$$\begin{aligned} N_{M(\bar{y})}(u) \cap B_{X^*} &= \partial d(\cdot, M(\bar{y}))(u) \subset \gamma \partial[d(\cdot, C) + d(\bar{y}, S(\cdot)) + I_{B(\bar{x}, \delta_1)}(\cdot)](u) \\ &= \gamma[\partial d(\cdot, C)(u) + \partial d(\bar{y}, S(\cdot))(u) + \partial I_{B(\bar{x}, \delta_1)}(u)] \\ &= \gamma[B_{X^*} \cap N_C(u) + D^*S(u, \bar{y})(B_{Y^*})]. \end{aligned}$$

By taking $\delta > 0$ such that $B(\bar{x}, \delta) \subset \mathrm{int}B(\bar{x}, \delta_1)$, we obtain the desired result.

(ii) $\Rightarrow$ (iii). Let $h \in X$ and $u \in B(\bar{x}, \delta) \cap \mathrm{bd}M(\bar{y})$. For every $x^* \in N_{M(\bar{y})}(u) \cap B_{X^*}$, according to (7) there exist $x_1^* \in B_{X^*} \cap N_C(u)$, $y^* \in B_{Y^*}$ and $x_2^* \in D^*S(u, \bar{y})(y^*)$ such that $x^* = \gamma(x_1^* + x_2^*)$. Since

$$\langle x_1^*, h \rangle = \frac{1}{t}\langle x_1^*, u + th - u \rangle \leq \frac{1}{t}(d(u + th, C) - d(u, C)) \text{ for all } t \geq 0,$$

we have $\langle x_1^*, h \rangle \leq d'(\cdot, C)(u, h) = d(h, T_C(u))$.

On the other hand, $\langle x_2^*, h\rangle \leq \langle y^*, y\rangle \leq \|y\|$ for every $y \in DS(u,\bar{y})(h)$.
It follows that $\langle x^*, h\rangle \leq \gamma[d(h, T_C(u)) + d(0, DS(u,\bar{y})(h))]$.
Since $d(h, T_{M(\bar{y})}(u)) = \sup\{\langle x^*, h\rangle \mid x^* \in N_{M(\bar{y})}(u) \cap B_{X^*}\}$, we have (8).

(iii) $\Rightarrow$ (iv). For every $\sigma \in (0,1)$, fixed $u \in B(\bar{x},\delta) \cap \mathrm{bd}M(\bar{y})$ and $h \in \mathcal{N}^1_{M(\bar{y})}(u,\sigma)$, there exists $x^* \in N_{M(\bar{y})}(u)$ with $\|x^*\| = 1$ such that $\langle x^*, h\rangle \geq \sigma$. Hence

$$\sigma \leq \langle x^*, h\rangle \leq d(h, T_{M(\bar{y})}(u)),$$

and so $\sigma \leq \gamma[d(h, T_C(u)) + d(0, DS(u,\bar{y})(h))$. Set $\eta = \frac{\sigma}{\gamma}$, and we obtain (10).

(iii)$'$ $\Rightarrow$ (iv)$'$. It can be proved similarly.

(iii) $\Rightarrow$ (iii)$'$ and (iv) $\Rightarrow$ (iv)$'$. Both implications follow from $\hat{D}S(u,\bar{y})(h) \subset DS(u,\bar{y})(h)$.

(iv)$'$ $\Rightarrow$ (i). Let $x \in B(\bar{x}, \frac{\delta}{3}) \setminus M(\bar{y})$ and $\sigma \in (0,1)$. By Lemma 1, there exist $u \in \mathrm{bd}M(\bar{y}) \cap B(\bar{x},\delta)$ and $x^* \in N_{M(\bar{y})}(u)$ with $\|x^*\| = 1$ such that

$$\sigma\|x-u\| \leq \langle x^*, x-u\rangle.$$

This implies that $\frac{x-u}{\|x-u\|} \in \mathcal{N}^1_{M(\bar{y})}(u,\sigma)$. It follows from (iv)$'$ that

$$\eta\|x-u\| \leq \max\{d(x-u, T_C(u)), d(0, \hat{D}S(u,\bar{y})(x-u))\}.$$

Since C and $\mathrm{gr}S$ are convex, we have

$$d(x-u, T_C(u)) \leq d(x,C), \quad S(x) - \bar{y} \subset \hat{D}S(u,\bar{y})(x-u).$$

Hence, $\eta\|x-u\| \leq \max\{d(x,C), d(\bar{y}, S(x))\}$, and consequently

$$\eta d(x, M(\bar{y}) \leq \max\{d(x,C), d(\bar{y}, S(x))\}.$$

Let $\gamma = \frac{1}{\eta}$, and we get that

$$d(x, M(\bar{y}) \leq \gamma \max\{d(x,C), d(\bar{y}, S(x))\}. \qquad \blacksquare$$

Remark 1 In Song (2006), we have proved that (ii) implies (i). The equivalence among (i)-(iii) was proved recently by Zheng and Ng (2007)(see Theorems 3.1 and 3.3 in Zheng and Ng, 2007) under the assumption that $S\colon X \rightrightarrows Y$ is a closed set-valued mapping instead of the closedness of $S^{-1}(y)$. The proof given here is much simpler than those given in Zheng and Ng (2007).

Set $S(x) = x - D$ for some closed convex subset D of X. In that case $M(0) = C \cap D$, and Theorem 1 reduces to the following result (see Song and Zang, 2006).

Corollary 1 *Let $C, D \subset X$ be closed convex sets, let $M = C \cap D$, and let $\bar{x} \in M$. Then the following statements are equivalent:*

(i) *there exist* $\delta, \gamma > 0$ *such that*

$$d(x, M) \leq \gamma \max\{d(x, C), d(x, D)\} \text{ for all } x \in B(\bar{x}, \delta);$$

(ii) *there exist* $\delta, \gamma > 0$ *such that*

$$N_M(x) \cap B_{X^*} \subset \gamma(N_C(x) \cap B_{X^*} + N_D(x) \cap B_{X^*}) \text{ for all } x \in B(\bar{x}, \delta) \cap M;$$

(iii) *there exist* $\delta, \gamma > 0$ *such that*

$$d(h, T_M(x)) \leq \gamma \max\{d(h, T_C(x)), d(h, T_D(x))\} \text{ for all } x \in B(\bar{x}, \delta) \cap M,\ h \in X.$$

In the following theorem, we shall present some characterizations of global error bounds for a set-valued mapping.

THEOREM 2 *Consider the set-valued mapping* $M\colon Y \rightrightarrows X$ *defined as* $M(y) = C \cap S^{-1}(y)$, *where* $S\colon X \rightrightarrows Y$ *is a convex set-valued mapping and* C *is a closed convex set of* X. *Let* $\bar{y} \in Y$ *be such that* $M(\bar{y})$ *is nonempty and* $S^{-1}(\bar{y})$ *is closed. Then the following statements are equivalent:*
(i) M *has a global error bound at* $\bar{y}$;
(ii) *there exists* $\gamma > 0$ *such that*

$$B_{X^*} \cap N_{M(\bar{y})}(u) \subset \gamma[B_{X^*} \cap N_C(u) + D^*S(u, \bar{y})(B_{Y^*})] \tag{12}$$

for all $u \in \mathrm{bd}M(\bar{y})$;
(iii) *there exists* $\gamma > 0$ *such that*

$$d(h, T_{M(\bar{y})}(u)) \leq \gamma \max\{d(h, T_C(u)), d(0, DS(u, \bar{y})(h))\} \tag{13}$$

for all $u \in \mathrm{bd}M(\bar{y}),\ h \in X$;
(iii)$'$ *there exists* $\gamma > 0$ *such that*

$$d(h, T_{M(\bar{y})}(u)) \leq \gamma \max\{d(h, T_C(u)), d(0, \hat{D}S(u, \bar{y})(h))\} \tag{14}$$

for all $u \in \mathrm{bd}M(\bar{y}),\ h \in X$;
(iv) *there exist* $\sigma \in (0, 1)$ *and* $\eta > 0$ *such that*

$$\eta \leq \max\{d(h, T_C(u)), d(0, DS(u, \bar{y})(h))\} \tag{15}$$

for all $u \in \mathrm{bd}M(\bar{y})$ *and* $h \in \mathcal{N}^1_{M(\bar{y})}(u, \sigma)$;
(iv)$'$ *there exist* $\sigma \in (0, 1)$ *and* $\eta > 0$ *such that*

$$\eta \leq \max\{d(h, T_C(u)), d(0, \hat{D}S(u, \bar{y})(h))\} \tag{16}$$

for all $u \in \mathrm{bd}M(\bar{y})$ *and* $h \in \mathcal{N}^1_{M(\bar{y})}(u, \sigma)$.
Moreover, if $M(\bar{y})$ *can be reproduced as* $M(\bar{y}) = A + \mathrm{rec}(M(\bar{y}))$, *where* A *is a convex subset of* $M(\bar{y})$ *and* $\mathrm{rec}(M(\bar{y})) = \{d \in X \mid x + td \in M(\bar{y}),\ \ \forall\ x \in M(\bar{y}),\ \ \forall\ t > 0\}$ *is the recession cone of* $M(\bar{y})$, *then the above equivalent conditions are also equivalent to each of the following conditions:*

(ii)* *there exists* $\gamma > 0$ *such that*

$$B_{X^*} \cap N_A(u) \cap (\mathrm{rec}(M(\bar{y})))^0 \subset \gamma[B_{X^*} \cap N_C(u) + D^*S(u,\bar{y})(B_{Y^*})] \quad (17)$$

for all $u \in A \cap \mathrm{bd}M(\bar{y})$*;*

(iii)* *there exists* $\gamma > 0$ *such that*

$$d(h, T_{M(\bar{y})}(u)) \leq \gamma \max\{d(h, T_C(u)), d(0, DS(u,\bar{y})(h))\} \quad (18)$$

for all $u \in A \cap \mathrm{bd}M(\bar{y}),\ h \in X$*;*

(iii)*′ *there exists* $\gamma > 0$ *such that*

$$d(h, T_{M(\bar{y})}(u)) \leq \gamma \max\{d(h, T_C(u)), d(0, \hat{D}S(u,\bar{y})(h))\} \quad (19)$$

for all $u \in A \cap \mathrm{bd}M(\bar{y}),\ h \in X$*.*

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). These implications follow from the proof of Theorem 1.

(iii) $\Rightarrow$ (iii)′ and (iv) $\Rightarrow$ (iv)′. Both implications follow from $\hat{D}S(u,\bar{y})(h) \subset DS(u,\bar{y})(h)$.

(iii)′ $\Rightarrow$ (iv)′. It can be proved similarly to the implication (iii) $\Rightarrow$ (iv) in Theorem 1.

(iv)′ $\Rightarrow$ (i). Let $x \in X \setminus M(\bar{y})$ and $\sigma \in (0,1)$. Take $\delta > 3d(x, M(\bar{y})$ and $\bar{x} \in M(\bar{y})$ satisfying $\|x - \bar{x}\| \leq \delta/3$. By Lemma 1, there exists $u \in \mathrm{bd}M(\bar{y}) \cap B(\bar{x}, \delta)$ and $x^* \in N_{M(\bar{y})}(u)$ with $\|x^*\| = 1$ such that

$$\sigma\|x - u\| \leq \langle x^*, x - u\rangle.$$

This implies that $\frac{x-u}{\|x-u\|} \in \mathcal{N}^1_{M(\bar{y})}(u,\sigma)$. It follows from (iv)′ that

$$\eta\|x - u\| \leq \max\{d(x-u, T_C(u)), d(0, \hat{D}S(u,\bar{y})(x-u))\}.$$

Since C and $\mathrm{gr}S$ are convex, we have

$$d(x-u, T_C(u)) \leq d(x, C), \quad S(x) - \bar{y} \subset \hat{D}S(u,\bar{y})(x-u).$$

Hence

$$\eta\|x - u\| \leq \max\{d(x,C), d(\bar{y}, S(x))\},$$

and consequently,

$$\eta d(x, M(\bar{y}) \leq \max\{d(x,C), d(\bar{y}, S(x))\}.$$

Let $\gamma = \frac{1}{\eta}$, and we get that

$$d(x, M(\bar{y}) \leq \gamma \max\{d(x,C), d(\bar{y}, S(x))\}.$$

(ii) $\Longrightarrow$ (ii)*. Suppose that $M(\bar{y}) = A + \mathrm{rec}(M(\bar{y}))$. We claim that

$$N_{M(\bar{y})}(u) = N_A(u) \cap (\mathrm{rec}(M(\bar{y})))^0 \text{ for all } u \in A \cap \mathrm{bd}M(\bar{y}).$$

Indeed, if $x^* \in N_{M(\bar{y})}(u)$, then

$$\langle x^*, a + c - u\rangle \le 0 \text{ for all } a \in A,\ c \in \mathrm{rec}(M(\bar{y})).$$

It follows by taking $c = 0$ or $a = u$ that $x^* \in N_A(u) \cap (\mathrm{rec}(M(\bar{y})))^0$. It is easy to see that the converse is also true. Hence, (ii)* is a special case of (ii).

(ii)* $\Longrightarrow$ (iii)* and (iii)* $\Longrightarrow$ (iii)$^{*'}$. They follow from the previous part.

(iii)$^{*'}$ $\Longrightarrow$ (i). Let $x \in X \setminus M(\bar{y})$ and $\sigma \in (0, 1)$. By Lemma 1, there exist $u \in \mathrm{bd}M(\bar{y})$ and $x^* \in N_{M(\bar{y})}(u)$ with $\|x^*\| = 1$ such that

$$\sigma\|x - u\| \le \langle x^*, x - u\rangle.$$

Since $M(\bar{y}) = A + \mathrm{rec}(M(\bar{y}))$, there exist $a \in A$ and $c \in \mathrm{rec}(M(\bar{y}))$ such that $u = a + c$. It follows that

$$\langle x^*, z - a - c\rangle \le 0 \text{ for all } z \in M(\bar{y}).$$

By taking $z = a$ and $z = a + 2c$, respectively, we can deduce that $\langle x^*, c\rangle = 0$, and hence

$$\langle x^*, z - a\rangle \le 0 \text{ for all } z \in M(\bar{y}).$$

This implies that $x^* \in N_{M(\bar{y})}(a)$, $a \in A \cap \mathrm{bd}M(\bar{y})$, and

$$\sigma\|x - u\| \le \langle x^*, x - a\rangle.$$

Note that $T_{M(\bar{y})}(a) \subset \{z \in X \mid \langle x^*, z\rangle \le 0\} =: D$. Hence, we can deduce that

$$\sigma\|x - u\| \le \langle x^*, x - a\rangle = d(x - a, D) \le d(x - a, T_{M(\bar{y})}(a)).$$

It follows from (iii)$^{*'}$ that

$$\sigma\|x - u\| \le \gamma \max\{d(x - a, T_C(a)), d(0, \hat{D}S(u, \bar{y})(x - a))\}.$$

Since C and $\mathrm{gr}S$ are convex, we have

$$d(x - a, T_C(a)) \le d(x, C), \quad S(x) - \bar{y} \subset \hat{D}S(a, \bar{y})(x - a).$$

Hence

$$\sigma\|x - u\| \le \gamma \max\{d(x, C), d(\bar{y}, S(x))\},$$

and consequently,

$$\sigma d(x, M(\bar{y}) \le \gamma \max\{d(x, C), d(\bar{y}, S(x))\}.$$

Let $\sigma \to 1$, and we get that

$$d(x, M(\bar{y}) \le \gamma \max\{d(x, C), d(\bar{y}, S(x))\}. \qquad \blacksquare$$

REMARK 2 When $C = X$, Ng and Zheng (2004) proved the equivalence between (i) and (iv), and (i) and (iv)$'$ in the case when Y is reflexive. Very recently, Zheng and Ng (2007) proved that (i), (ii) and (ii)* are equivalent by using different methods.

As observed, M has a global error bound at $\bar{y}$ if and only if $M(\bar{y})$ is a weak sharp minima set for the function $\phi(x) = d(x, C) + d(\bar{y}, S(x))$. The later is equivalent to the statement that there exists some $\tau > 0$ such that

$$\tau B_{X^*} \cap N_{M(\bar{y})}(u) \subset \partial\phi(u), \forall u \in M(\bar{y}).$$

By applying the subdifferential formula (see Theorem 2.8.7 in Zălinescu, 2002) and Lemma 2, we have that

$$\partial\phi(u) = B_{X^*} \cap N_C(u) + D^*S(u, \bar{y})(B_{Y^*})\ \forall u \in \mathrm{bd}M(\bar{y}).$$

Hence we obtain the equivalence between (i) and (ii) in Theorem 2.

4. Applications to the convex inequalities

We now give some characterizations of the existence of local (global) error bounds for (4), i.e.,

$$M = M(0) = \{x \in C \mid f(x) \leq 0\}, \tag{20}$$

where $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function. Assume that M is nonempty and closed. Define $S(x) = [f(x), \infty)$ for $x \in X$. Then S is a convex set-valued mapping with closed images, $d(0, S(x)) = [f(x)]_+ = \max\{f(x), 0\}$ for each $x \in X$, and $\mathrm{gr}S = \mathrm{epi}f$. From the definition of the coderivative of the set-valued mapping, we have that for every $x \in \mathrm{dom}f$

$$D^*S(x, f(x))(r) = \begin{cases} r\partial f(x) \text{ if } r > 0, \\ \partial^\infty f(x) \quad \text{if } r = 0, \\ \emptyset \qquad\quad\ \text{if } r < 0. \end{cases}$$

Hence

$$D^*S(x, f(x))([-1, 1]) = \bigcup_{\lambda \in [0,1]} \partial(\lambda f)(x) = [0, 1]\partial f(x) \cup \partial^\infty f(x). \tag{21}$$

By the definition of contingent derivative and Proposition 2.60 in Bonnans and Shapiro (2000),

$$\mathrm{gr}DS(x, f(x))(\cdot) = \mathrm{cl}T_{\mathrm{epi}f}(x, f(x)) = \mathrm{cl}[\mathrm{epi}f'(x, \cdot)].$$

Hence

$$DS(x, f(x))(h) = [\mathrm{lsc}f'(x, h), +\infty) \subset [f'(x, h), +\infty),$$

where $\mathrm{epi}(\mathrm{lsc}f^{'}(x,\cdot)) = \mathrm{cl}(\mathrm{epi}f^{'}(x,\cdot))$, and hence

$$d(0, DS(x, f(x))(h)) = [\mathrm{lsc}f^{'}(x,h)]_+. \tag{22}$$

By Proposition 2.126 in Bonnans and Shapiro (2000), $\mathrm{lsc}f^{'}(x,h) = f^{'}(x,h)$ if f is continuous at x.

It has been proved in Ng and Zheng (2004) that for each $x \in \mathrm{dom}f$,

$$\hat{D}S(x, f(x))(h) = f^{'}(x,h) + \mathbb{R}_+.$$

Hence

$$d(0, \hat{D}S(x, f(x))(h)) = [f^{'}(x,h)]_+. \tag{23}$$

THEOREM 3 *Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function, let $C \subset X$ be a closed convex set, and let $M = \{x \in C \mid f(x) \leq 0\}$ be closed. Let $\bar{x} \in C$ be such that $f(\bar{x}) = 0$. Suppose that one of the following conditions holds:*
(a) $\mathrm{bd}M \cap B(\bar{x}, \hat{\delta}) \subset f^{-1}(0)$ *for some* $\hat{\delta} > 0$;
(b) $\{x \in C \mid f(x) < 0\} \subset \mathrm{int}\{x \mid f(x) \leq 0\}$;
(c) *there exist* $\gamma, \hat{\delta} > 0$ *such that for every* $u \in B(\bar{x}, \hat{\delta}) \cap \mathrm{bd}M$
$[N_M(u) \cap B_{X^*} \subset \gamma[N_C(u) \cap B_{X^*} + N_{\{x \mid f(x) \leq 0\}}(u)]$.

Then the following statements are equivalent:
(i) *there exist* $\delta, \gamma > 0$ *such that*
$d(x, M) \leq \gamma \max\{f(x), d(x, C)\}$ *for all* $x \in B(\bar{x}, \delta)$;
(ii) *there exist* $\delta, \gamma > 0$ *such that*
$[N_M(u) \cap B_{X^*} \subset \gamma[N_C(u) \cap B_{X^*} + ([0,1]\partial f(u) \cup \partial^\infty f(u))]$ *for all* $u \in B(\bar{x}, \delta) \cap f^{-1}(0) \cap C$;
(iii) *there exist* $\delta, \gamma > 0$ *such that*
$d(h, T_M(u)) \leq \gamma \max\{f^{'}(u,h), d(h, T_C(u))\}$
for all $u \in B(\bar{x}, \delta) \cap f^{-1}(0) \cap C, h \in X$;
(iv) *there exist* $\sigma \in (0,1)$ *and* $\delta, \eta > 0$ *such that*
$\eta \leq \max\{f^{'}(u,h), d(h, T_C(u))\}$ *for all* $u \in B(\bar{x}, \delta) \cap f^{-1}(0) \cap C, h \in N^1_M(u, \sigma)$.

Proof. (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv). These implications are clear from Theorem 1 and formulas (21) and (23).

(iv) $\Rightarrow$ (i). Let $\delta_1 = \delta/3$. For $\sigma \in (0,1)$ and $x \in B(\bar{x}, \delta_1) \setminus M$, by Lemma 1, there exist $u \in \mathrm{bd}M \cap B(\bar{x}, \delta)$ and $x^* \in N_M(u)$ with $\|x^*\| = 1$ such that

$$\sigma \|x - u\| \leq \langle x^*, x - u \rangle \leq d(x - u, T_M(u)).$$

This implies that $\frac{x-u}{\|x-u\|} \in \mathcal{N}^1_{M(\bar{y})}(u, \sigma)$.

If $f(u) = 0$, then, by (iv),

$$\eta \|x - u\| \leq \max\{f^{'}(u, x - u), d(x - u, T_C(u))\}.$$

Since C and f are convex, we have

$$\eta d(x, M) \leq \eta \|x - u\| \leq \max\{f(x), d(x, C)\}.$$

Clearly, if (a) is satisfied, then (i) is true by letting $\delta_1 = \min\{\hat{\delta}, \delta\}/3$ and $\gamma = \frac{1}{\eta}$.

Suppose (b) is satisfied and $f(u) < 0$. Then, by the assumption, $u \in \operatorname{int}\{x \mid f(x) \leq 0\}$. It follows that $T_C(u) = T_M(u)$. Consequently,

$$\sigma\|x - u\| \leq d(x - u, T_C(u)) \leq \max\{f(x), d(x, C)\}.$$

Hence, we have

$$\sigma d(x, M) \leq \max\{f(x), d(x, C)\}.$$

Letting $\gamma = \max\{\frac{1}{\eta}, \frac{1}{\sigma}\}$, we get (i).

Suppose (c) is satisfied. Let $\delta_1 = \min\{\hat{\delta}, \delta\}$ and let $u \in B(\bar{x}, \delta_1) \cap \mathrm{bd}M$ be such that $f(u) < 0$. Then, it is easy to verify that $N_{\{x \mid f(x) \leq 0\}}(u) = N_{\mathrm{dom} f}(u) = \partial^\infty f(u)$ (see Proposition 3 in Burke and Deng, 2005). It follows from (c) and (ii) $\Rightarrow$ (iv) in Theorem 1 that

$$\eta \leq \max\{f^{'}(u, h), d(h, T_C(u))\} \text{ for all } h \in N^1_M(u, \sigma).$$

This, together with (iv), implies (i) due to Theorem 1 and formula (23). ■

Remark 3 When $C = X$, Condition (ii) becomes

$$N_M(x) \cap B_{X^*} \subset \begin{cases} [0, \gamma]\partial \mathrm{f(x)} + \partial^\infty \mathrm{f(x)} \text{ if } \partial f(x) \neq \emptyset, \\ \partial^\infty f(x) \text{ if } \partial f(x) = \emptyset \end{cases}$$

for all $x \in B(\bar{x}, \delta) \cap f^{-1}(0) \cap C$. In this case, the equivalence between (i) and (ii) was proved in Ng and Zheng (2001) under the assumption that $\mathrm{bd}M \subset f^{-1}(0)$.

The following result, concerning global error bounds, follows from Theorem 2 and formula (23), which generalizes the corresponding results from Lewis and Pang (1998) and Zălinescu (2002) to a general Banach space.

Theorem 4 *Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function, let $C \subset X$ be a closed convex set, and let $M = \{x \in C \mid f(x) \leq 0\}$ be closed. Let $\bar{x} \in C$ be such that $f(\bar{x}) = 0$. Suppose that one of the following conditions holds:*
(a) $\mathrm{bd}M \subset f^{-1}(0)$*;*
(b) $\{x \in C \mid f(x) < 0\} \subset \operatorname{int}\{x \mid f(x) \leq 0\}$;
(c) *there exists $\gamma > 0$ such that for every $u \in \mathrm{bd}M$*
$N_M(u) \cap B_{X^*} \subset \gamma[N_C(u) \cap B_{X^*} + N_{\{x \mid f(x) \leq 0\}}(u)]$.
Then the following statements are equivalent:

(i) *there exists some* $\gamma > 0$ *such that*
$d(x, M) \leq \gamma \max\{f(x), d(x, C)\}$ *for all* $x \in X$;
(ii) *there exists* $\gamma > 0$ *such that*
$N_M(x) \cap B_{X^*} \subset \gamma[N_C(x) \cap B_{X^*} + ([0,1]\partial f(x) \cup \partial^\infty f(x))]$ *for all* $x \in f^{-1}(0) \cap C$;
(iii) *there exists* $\gamma > 0$ *such that*
$d(h, T_M(x)) \leq \gamma \max\{f^{'}(x, h), d(h, T_C(x))\}$ *for all* $x \in f^{-1}(0) \cap C, h \in X$;
(iv) *there exist* $\sigma \in (0, 1)$ *and* $\eta > 0$ *such that*
$\eta \leq \max\{f^{'}(x, h), d(h, T_C(x))\}$ *for all* $x \in f^{-1}(0) \cap C, h \in N^1_M(u, \sigma)$.

Burke and Deng (2005, Theorem 9) show that each of the conditions (ii) and (iv) implies (i) under the condition of either (a) or (b).

In the next result we present two sufficient conditions for the local error bounds of the constrained system (20).

THEOREM 5 *Let* $f \colon X \to \mathbb{R} \cup \{+\infty\}$ *be a proper convex function, let* $C \subset X$ *be a closed convex set, and let* $M = \{x \in C \mid f(x) \leq 0\}$ *be a closed subset of* X. *Let* $\bar{x} \in C$ *be such that* $f(\bar{x}) = 0$. *Consider the following statements:*
(i) *there exist* $\delta, \gamma > 0$ *such that*
$d(x, M) \leq \gamma \max\{f(x), d(x, C)\}$ *for all* $x \in B(\bar{x}, \delta)$;
(ii) *there exist* $\delta, \gamma > 0$ *such that for every* $x \in B(\bar{x}, \delta) \cap C \cap f^{-1}(0)$,
$N_M(x) = N_C(x) + \bigcup_{\lambda \geq 0} \partial(\lambda f)(x)$, *and for every* $x_1^* \in N_C(x)$, $x_2^* \in \partial f(x)$ *and* $x_3^* \in \partial^\infty f(x)$, $\|x_1^*\| + 1 \leq \gamma \|x_1^* + x_2^*\|$ *and* $\|x_1^*\| \leq \gamma \|x_1^* + x_3^*\|$;
(iii) *there exists* $\delta > 0$, *such that the set* $\partial f(B(\bar{x}, \delta) \cap C \cap f^{-1}(0))$ *is bounded and*

$$0 \notin \mathrm{cl} \bigcup_{x \in B(\bar{x}, \delta) \cap C \cap f^{-1}(0)} (\partial f(x) + N_C(x)).$$

Then the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) *hold if one of the following conditions holds:*
(a) $\mathrm{bd} M \cap B(\bar{x}, \hat{\delta}) \subset f^{-1}(0)$ *for some* $\hat{\delta} > 0$;
(b) $\{x \in C \mid f(x) < 0\} \subset \mathrm{int}\{x \mid f(x) \leq 0\}$;
(c) *there exist* $\gamma, \hat{\delta} > 0$ *such that for every* $u \in B(\bar{x}, \hat{\delta}) \cap \mathrm{bd} M$
$N_M(u) \cap B_{X^*} \subset \gamma[N_C(u) \cap B_{X^*} + N_{\{x \mid f(x) \leq 0\}}(u)]$.

Proof. (ii) $\Rightarrow$ (i). Fix $x \in B(\bar{x}, \delta) \cap C \cap f^{-1}(0)$, for every $x^* \in B_{X^*} \cap N_M(x)$ ($x^* \neq 0$), then there exist $x_1^* \in N_C(x)$ and $x_2^* \in \partial(\lambda f)(x)$ for some $\lambda \geq 0$ such that $x^* = x_1^* + x_2^*$.

(1) If $\lambda = 0$, then $x_2^* \in \partial^\infty f(x)$. If $x_1^* = 0$ or $x_2^* = 0$, then it is obvious that $x^* \in N_C(x) \cap B_{X^*} + \partial^\infty f(x)$. Otherwise, since $\|x_1^*\| \leq \gamma \|x^*\| \leq \gamma$, we have $x^* = \gamma(\frac{x_1^*}{\gamma} + \frac{x_2^*}{\gamma}) \in \gamma[N_C(x) \cap B_{X^*} + \partial^\infty f(x)]$.

(2) If $\lambda > 0$, then $x_2^*/\lambda \in \partial f(x)$. Let $\mu = \|x_1^*\| + \lambda (> 0)$. It follows that

$$\frac{x^*}{\mu} = \frac{x_1^*}{\mu} + \frac{\lambda}{\mu} \frac{x_2^*}{\lambda} \in B_{X^*} \cap N_C(x) + [0, 1]\partial f(x).$$

By the assumption, $\gamma \geq \gamma\|x^*\| \geq \mu$. Hence

$$x^* \in \gamma[B_{X^*} \cap N_C(x) + [0,1]\partial f(x)].$$

We conclude that

$$N_M(x) \cap B_{X^*} \subset \gamma[N_C(x) \cap B_{X^*} + ([0,1]\partial f(x) \cup \partial^\infty f(x))]$$

for all $x \in B(\bar{x}, \delta) \cap C \cap f^{-1}(0)$. By Theorem 3, we obtain (i).

(iii) $\Rightarrow$ (ii). The proof is same as the proof of Proposition 3.10.15 in Zălinescu (2002). By hypothesis, there exist $\alpha, \eta > 0$ such that $\|x_1^* + x_2^*\| \geq \alpha$ and $\|x_2^*\| \leq \eta$ for all $x \in B(\bar{x}, \delta) \cap C \cap f^{-1}(0)$, $x_1^* \in N_C(x)$ and $x_2^* \in \partial f(x)$. Take $\gamma = (1+\eta+\alpha)/\alpha$. Consider x, x_1^*, x_2^* as before. If $\|x_1^*\| \leq \gamma\alpha - 1$, then, obviously, $\|x_1^*\| + 1 \leq \gamma\|x_1^* + x_2^*\|$. If $\|x_1^*\| > \gamma\alpha - 1$, then $\gamma\|x_1^* + x_2^*\| \geq \gamma(\|x_1^*\| - \eta) \geq \|x_1^*\| + 1 + (\gamma - 1)(\gamma\alpha - 1) - \gamma\eta - 1 = \|x_1^*\| + 1$. ■

Remark 4 It can be proved that

$$N_M(x) = N_C(x) + \bigcup_{\lambda \geq 0} \partial(\lambda f)(x) \text{ for all } x \in M$$

provided f is lower semicontinuous and one of the following conditions holds:
(i) $C \cap f^{-1}(0) \subset \text{int}(\text{dom}) f$ and

$$\|x_1^*\| + 1 \leq \gamma\|x_1^* + x_2^*\| \text{ for all } x_1^* \in N_C(x),\ x_2^* \in \partial f(x);$$

(ii) $(\bigcup_{\lambda \geq 0} \text{epi}(\lambda f)^* + \text{epi}\sigma_C)$ is w^*-closed.

Indeed, for the case where (i) is satisfied, see proof of Proposition 3.10.15 of Zălinescu (2002).

For the case where (ii) is satisfied, let $x \in M$ and $x^* \in N_M(x)$. Then $\langle x^*, x\rangle = \sigma_M(x^*)$ and thus $(x^*, \langle x^*, x\rangle) \in \text{epi}\sigma_M$. It is well known that $\text{epi}\sigma_M = \text{cl}[\text{epi}\sigma_C + \bigcup_{\lambda \geq 0} \text{epi}(\lambda f)^*]$. By the assumption, we have that $\text{epi}\sigma_M = \text{epi}\sigma_C + \bigcup_{\lambda \geq 0}(\lambda f)^*$. Hence there exist $(x_1^*, \alpha_1) \in \text{epi}\sigma_C$, $\lambda \geq 0$ and $(x_2^*, \alpha_2) \in \text{epi}(\lambda f)^*$ such that $x^* = x_1^* + x_2^*$ and $\langle x^*, x\rangle = \alpha_1 + \alpha_2$. This, together with $\langle x_1^*, x\rangle \leq \sigma_C(x_1^*) \leq \alpha_1$ and $\langle x_2^*, x\rangle \leq \langle x_2^*, x\rangle - \lambda f(x) \leq (\lambda f)^*(x_2^*) \leq \alpha_2$, implies that $\langle x_1^*, x\rangle = \alpha_1$ and $\langle x_2^*, x\rangle = \alpha_2$. It follows that $\sigma_C(x_1^*) = \langle x_1^*, x\rangle$ and $(\lambda f)^*(x_2^*) + \lambda f(x) = \langle x_2^*, x\rangle$. This implies that $x_1^* \in N_C(x)$ and $x_2^* \in \partial(\lambda f)(x)$.

Some conditions, which ensure the closedness of the cone $(\bigcup_{\lambda \geq 0} \text{epi}(\lambda f)^* + \text{epi}\sigma_C)$ can be found in Jeyakumar et al. (2005). For instance, if $0 \in \text{int}(C - \text{dom} f)$ and there exists some $x_0 \in C$ such that $f(x_0) < 0$, then $(\bigcup_{\lambda \geq 0} \text{epi}(\lambda f)^* + \text{epi}\sigma_C)$ is w^*-closed.

Similarly, we can derive the following two sufficient conditions for the global error bounds of the constrained system (20).

THEOREM 6 *Let* $f\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a proper convex function, let* $C \subset X$ *be a closed convex set, and let* $M = \{x \in C \mid f(x) \leq 0\}$ *be closed. Let* $\bar{x} \in C$ *be such that* $f(\bar{x}) = 0$*. Consider the following statements:*
(i) *there exists some* $\gamma > 0$ *such that*

$$d(x, M) \leq \gamma \max\{f(x), d(x, C)\} \text{ for all } x \in X;$$

(ii) *there exists some* $\gamma > 0$ *such that for every* $x \in C \cap f^{-1}(0)$,

$$N_M(x) = N_C(x) + \bigcup_{\lambda \geq 0} \partial(\lambda f)(x),$$

and for every $x_1^* \in N_C(x)$, $x_2^* \in \partial f(x)$ *and* $x_3^* \in \partial^\infty f(x)$,

$$\|x_1^*\| + 1 \leq \gamma\|x_1^* + x_2^*\| \text{ and } \|x_1^*\| \leq \gamma\|x_1^* + x_3^*\|;$$

(iii) *the set* $\partial f(C \cap f^{-1}(0))$ *is bounded and*

$$0 \notin \mathrm{cl}(\bigcup_{x \in C \cap f^{-1}(0)} (\partial f(x) + N_C(x)).$$

Then the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) *hold if one of the following conditions holds:*
(a) $\mathrm{bd}M \subset f^{-1}(0)$;
(b) $\{x \in C \mid f(x) < 0\} \subset \mathrm{int}\{x \mid f(x) \leq 0\}$;
(c) *there exists* $\gamma > 0$ *such that for every* $u \in \mathrm{bd}M$
$N_M(u) \cap B_{X^*} \subset \gamma[N_C(u) \cap B_{X^*} + N_{\{x \mid f(x) \leq 0\}}(u)]$.

Theorem 6 generalizes the corresponding results from Lewis and Pang (1998), and Zălinescu (2002) to a general Banach space.

Consider now the system of inequalities

$$M = \{x \in C \mid f_i(x) \leq 0, \text{ for } i = 1, \ldots, m\}, \tag{24}$$

where $f_1, \ldots, f_m\colon X \to \bar{R}$ are lower semicontinuous proper convex functions.

Let $f\colon = \max\{f_i \mid 1 \leq i \leq m\}$. Then system (24) is equivalent to system (20). Let $I(x) = \{i \in \{1, 2 \ldots, m\} \mid f(x) = f_i(x)\}$. Assume that $[f = 0] := \{x \in X \mid f(x) = 0\} \subset \mathrm{int}(\mathrm{dom} f_i)$, $i = 1, \ldots, m$. Then it is well known that

$$\partial f(x) = \mathrm{co}(\cup_{i \in I(x)} \partial f_i(x)) \text{ for all } x \in [f = 0],$$

and

$$f'(x, h) = \max_{i \in I(x)} f_i'(x, h) \text{ for all } h \in X.$$

The characterizations for error bounds for system (24) readily follow from the previous results by simply computing the subdifferential and the directional derivatives of f.

Acknowledgement

The author would like to thank two anonymous referees for many useful suggestions.

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