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A weak* approximation of subgradient of convex function

by

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Abstract: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of proper lower semicontinuous convex functions on a weakly compactly generated Banach space. Conditions ensuring the weak* convergence of their subgradients are given.

Keywords: subdifferentials, Attouch's theorem, Kronecker's lemma, Mosco convergence, convex functions,weakly compactly generated Banach spaces.

1. Introduction

In the reflexive Banach setting the Attouch theorem gives the equivalence of Mosco convergence of convex lower semicontinuous functions to the Kuratowski-Painlevè graph convergence of its subdifferentials, see Attouch (1984) for example. Unfortunately, this result is no longer true in nonreflexive Banach spaces, see Proposition 4.1 of Attouch and Beer (1993) for a general couterexample or Example 1.1 of Zagrodny (2005) where it is shown that even if the limit of Mosco converging sequence is the norm of l_1, then its subgradients can not be suitably approximated. There are results extending the Attouch theorem to arbitrary Banach space, but in order to obtain this, additional assumptions are needed, see Attouch and Beer (1993), Beer and Théra (1994), Combari and Thibault (1998) for more details. It seems that the main obstacle in extending the theorem to a larger class of spaces is the strong topology, which is involved in the graph convergence of subdifferentials. Recently, it has been shown that for each subgradient x^* of f, the Mosco convergence of $\{f_n\}_{n=1}^{\infty}$ to f with the additional assumption that the sequence of functions is uniformly bounded from above on some open set, ensures the existence of a sequence of subgradients $x_n^* \in \partial f_n(x_n)$, which is weakly* convergent to x^*, see Zagrodny (2005a,b). Of course, the uniform boundedness assumption is a restrictive one. Herein, it is shown that it can be relaxed to (we consider the case $f(0) = 0$, $0 \in \partial f(0)$)

$$\exists \{\epsilon_n\}_{n=1}^{\infty} \subset [0,\infty), \epsilon_n \searrow 0 : \exists M \geq 0 : \forall n \in \mathbb{N}, \forall x \in X, \qquad f_n(x) + M d_Q(x) + \epsilon_n \geq 0 \quad (1)$$

and the result still holds true if Q is convex weakly compact and generates the space, where d_Q stands for the distance from Q. This result is applied to provide an extension of the Kronecker lemma, see Moricz (1982) for example.

2. Preliminaries

In the sequel E will be a real Banach space which is weakly compactly generated (WCG). We recall that a Banach space is WCG if there exists a weakly compact subset Q of E that spans a dense linear space in E, see Phelps (1984) for more details. We may assume that $0 \in Q$ and Q is symmetric, i.e. $Q = -Q$, then, equivalently, E is WCG if $E = cl \bigcup_{n=1}^{\infty} nQ$, where "cl" stands for the topological closure. By E^* we denote the dual space to E.

For any convex function $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ finite at x and $\epsilon \geq 0$, by $\partial_\epsilon f(x)$ we denote its ϵ-subdifferential i.e.

$$\partial_\epsilon f(x) := \{x^* \in E^* \mid < x^*, h > \leq \epsilon + f(x+h) - f(x) \text{ for every } h \in E\}.$$

When $\epsilon = 0$ then ϵ-subdifferential is called subdifferential and it is denoted by $\partial f(x)$, see Attouch (1984), Phelps (1984) for properties.

Let us remind that for any sequence $\{q_i\}_{i=1}^{\infty} \subset Q$ there is a subsequence $\{q_{i_k}\}_{k=1}^{\infty} \subset \{q_i\}_{i=1}^{\infty} \subset Q$ which is weakly convergent to an element of Q. This is a consequence of the Eberlein-Smulian theorem, see Theorem 5.3.1 of Rolewicz (1984).

Below we recall a powerful existence result, the Brondsted-Rockafellar theorem, see Theorem 3.18 of Phelps (1984), for example.

THEOREM 1 *Suppose that f is a convex proper lower semicontinuos function on the Banach space E. Then, given any point $x_0 \in dom(f)$, $\epsilon > 0$ and any $x_0^* \in \partial_\epsilon f(x_0)$ there exists x_ϵ in $dom(f)$ and x_ϵ^* in E^* such that*

$$x_\epsilon^* \in \partial f(x_\epsilon), \|x_\epsilon - x_0\| \leq \sqrt{\epsilon} \text{ and } \|x_\epsilon^* - x_0^*\| \leq \sqrt{\epsilon}.$$

In particular, the domain of ∂f is dense in $dom(f)$.

Finally let us recall the notion of Mosco convergence. Attouch and Beer (1993) give a very convenient condition characterizing the notion. Following them we say that a function f is the Mosco limit of a sequence of proper lower semicontinuous functions $\{f_n\}_{n=1}^{\infty}$, we write $f = Mosco - \lim_{n\longrightarrow\infty} f_n$, if the two following conditions are satisfied:

S1: whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence weakly convergent to x, then $f(x) \leq \liminf_{n\longrightarrow\infty} f_n(x_n)$;

S2: for each $x \in E$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ converging in norm to x, for which $f(x) = \lim_{n\longrightarrow\infty} f_n(x_n)$.

3. Results

In this section we show that if for some $y_n \longrightarrow 0$ with $f_n(y_n) \longrightarrow 0$ and (1) holds true, then there are sequences $\{x_n\}_{n=1}^{\infty} \subset E$, $\{x_n^*\}_{n=1}^{\infty} \subset E^*$ such that $x_n \longrightarrow 0$, $f_n(x_n) \longrightarrow 0$, $x_n^* \stackrel{weak^*}{\longrightarrow} 0$ with $x_n^* \in \partial f_n(x_n)$. Moreover, for some sequence $\{k_n\}_{n=1}^{\infty} \subset (0,\infty)$, $k_n \longrightarrow \infty$ we get

$$\sup_{q \in k_n Q} |< x_n^*, q >| \longrightarrow 0. \tag{2}$$

Since $E = cl \bigcup_{n=1}^{\infty} k_n Q$, so we can say that (2) gives "almost" strong convergence of $\{x_n^*\}_{n=1}^{\infty}$ to 0. If, additionally, the sequence of functionals is bounded, then (2) implies that $x_n^* \stackrel{weak^*}{\longrightarrow} 0$. Of course, (2) alone does not ensure the weak* convergence to 0, see Example 2, below.

Theorem 2 *Let E be a real WCG Banach space (i.e. for some symmetric convex weakly compact subset $Q \subset E$, $E = cl \bigcup_{n=1}^{\infty} nQ$), $\{\epsilon_n\}_{n=1}^{\infty} \subset (0,\infty)$, $\epsilon_n \longrightarrow 0$ and $M > 0$ be given. Let $f, f_1, f_2, \ldots E \longrightarrow \mathbb{R} \cup \{\infty\}$ be lower semi-continuous functions such that:*

i: *for some $\{y_n\}_{n=1}^{\infty} \subset E$ we have*

$$\lim_{n\longrightarrow\infty} y_n = 0 \text{ and } \lim_{n\longrightarrow\infty} f_n(y_n) = 0;$$

ii: $\forall n \in \mathbb{N}, \forall x \in E, f_n(x) + Md_Q(x) + \epsilon_n \geq 0.$

Then there are $\{x_n\}_{n=1}^{\infty} \subset E$, $\{x_n^*\}_{n=1}^{\infty} \subset E^*$ such that:

a: $\lim_{n\longrightarrow\infty} x_n = 0$, $\lim_{n\longrightarrow\infty} f_n(x_n) = 0$;

b: $\forall n \in \mathbb{N}, \quad x_n^* \in \partial f_n(x_n)$;

c: $x_n^* \stackrel{weak^*}{\longrightarrow} 0$

d: $\exists \{r_n\}_{n=1}^{\infty} \subset (0,\infty)$, $r_n \longrightarrow 0$: $\exists \{k_n\}_{n=1}^{\infty} \subset (0,\infty)$, $k_n \longrightarrow \infty$: $\forall q \in Q$, $\forall n \in \mathbb{N}$, $|< x_n^*, k_n q >| \leq r_n$.

Proof. Assume that $f_n(0) = 0$. By (ii) we have $\forall n \in \mathbb{N}, \forall x \in E,\ f_n(x) + Md_Q(x) + \epsilon_n \geq 0$; thus $\forall n \in \mathbb{N}, 0 \in \partial_{\epsilon_n}\big(f_n(\cdot) + Md_Q(\cdot)\big)(0)$.

By the Brondsted-Rockafellar theorem there are $x_n \in dom(f_n)$, $y_n^* \in E^*$ such that

$$\|x_n\| \leq \sqrt{\epsilon_n}, \quad \|y_n^*\| \leq \sqrt{\epsilon_n},$$
$$y_n^* \in \partial\big(f_n(\cdot) + Md_Q(\cdot)\big)(x_n) = \partial\big(f_n\big)(x_n) + \partial\big(Md_Q\big)(x_n),$$

which implies that $f_n(x_n) + Md_Q(x_n) - < y_n^*, x_n > \leq f_n(0) = 0$.
Again by (ii) and the following inequalities: $f_n(x_n) + Md_Q(x_n) \leq f_n(0)$ $+ < y_n^*, x_n > \leq \epsilon_n$, we get $\forall n \in \mathbb{N}, 0 \leq f_n(x_n) + Md_Q(x_n) + \epsilon_n \leq 2\epsilon_n$, hence $f_n(x_n) \longrightarrow 0$, which gives (a). There are $x_n^* \in \partial f_n(x_n)$ and $z_n^* \in M\partial d_Q(x_n)$ such that $\|x_n^* + z_n^*\| \leq \sqrt{\epsilon_n}$, thus $\|x_n^*\| \leq 2M$ for n large enough and

$$\forall n \in \mathbb{N}, \forall x \in E, < z_n^*, x - x_n > \leq M\Bigg(d_Q(x) - d_Q(x_n)\Bigg).$$

Let us choose $\{k_n\}_{n=1}^{\infty} \subset (0, \infty)$ such that $k_n \longrightarrow \infty$ and

$$k_n\Bigg(|< z_n^*, x_n >| + M d_Q(x_n) + \sqrt{\epsilon}_n diam(Q)\Bigg) \longrightarrow 0,$$

where $diam(Q) := \sup\{\|q_1 - q_2\| \mid q_1, q_2 \in Q\}$. For every $n \in \mathbb{N}$ put

$$r_n := k_n\big(|< z_n^*, x_n >| + M d_Q(x_n) + \sqrt{\epsilon}_n diam(Q)\big).$$

For every $q \in Q = -Q$ we get

$$\begin{aligned} < z_n^*, k_n q >= k_n < z_n^*, q > &\leq k_n(< z_n^*, x_n > - M d_Q(x_n)) \\ &\leq k_n\big(|< z_n^*, x_n >| + M d_Q(x_n)\big), \end{aligned}$$

thus $|< z_n^*, k_n q >| \leq k_n\big(|< z_n^*, x_n >| + M d_Q(x_n)\big)$, which implies

$$|< x_n^*, k_n q >| \leq k_n\big(|< z_n^*, x_n >| + M d_Q(x_n) + \sqrt{\epsilon}_n diam(Q)\big),$$

so (d) is fulfilled. In order to get (c), let us observe that for every $x \in E$ there are $q_n \in Q$ such that $k_n q_n \longrightarrow x$ and

$$|< x_n^*, x >| \leq |< x_n^*, x - k_n q_n >| + |< x_n^*, k_n q_n >| \leq 2M\|x - k_n q_n\| + r_n,$$

so $< x_n^*, x > \longrightarrow 0$. In order to finish the proof let us notice that if $f_n(y_n) \longrightarrow 0$ then by (ii) we get

$$\forall n \in \mathbb{N}, \forall x \in E, 0 \leq f_n(x) - f_n(y_n) + M d_Q(x) + \epsilon_n + \mid f_n(y_n) \mid .$$

Putting $g_n(x) := f_n(y_n + x) - f_n(y_n)$, $\tilde{\epsilon_n} := \epsilon_n + M d_Q(y_n) + \mid f_n(y_n) \mid$, we get

$$\forall n \in \mathbb{N}, \forall x \in E, 0 \leq g_n(x) + M d_Q(x) + \tilde{\epsilon_n},$$

so we can apply the first part of the proof to g_n. By the equality $\partial g_n(x_n) = \partial f_n(y_n + x_n)$ we get the statements. ■

Remark 1 Let us observe that (a)-(d) imply (i),(ii) with $M := \sup_{n\in\mathbb{N}} \|x_n^*\|$ and $\epsilon_n := 2\big(n^{-1} + \mid f_n(x_n) \mid + \sup_{q\in Q} |< x_n^*, q >| + M\|x_n\|\big)$.

Indeed, by (a)-(d) we get $\epsilon_n \longrightarrow 0$ (keeping in mind that the Banach-Steinhaus theorem implies that the sequence $\{\|x_n^*\|\}_{n=1}^{\infty}$ is bounded). For every $q \in Q$ and $y \in E$ we have

$$\begin{aligned} f_n(y + x_n) - f_n(x_n) &\geq < x_n^*, y - q > + < x_n^*, q > \\ &\geq -M\|y + x_n - q\| - M\|x_n\| - \sup_{q\in Q} |< x_n^*, q >|, \end{aligned}$$

thus $\forall n \in \mathbb{N}, \forall x \in E, 0 \leq f_n(x) + M d_Q(x) + \epsilon_n$.

Let us mention two important consequences of the theorem. First, it yields a new proof of the necessity part of the Attouch theorem. It also encompasses a result on weak* convergence of subdifferentials.

THEOREM 3 (The necessity part of the Attouch theorem) *Let* $f, f_1, f_2, \ldots : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be convex lower semicontinuous on a reflexive Banach space* E *and* $f = Mosco - \lim_{n\longrightarrow\infty} f_n$*. For any* $\bar{x^*} \in \partial f(\bar{x})$ *there are sequences* $\{x_n\}_{n=1}^{\infty} \subset E$*,* $\{x_n^*\}_{n=1}^{\infty} \subset E^*$ *such that*

a: $\lim_{n\longrightarrow\infty} x_n = \bar{x}$, $\lim_{n\longrightarrow\infty} f_n(x_n) = f(\bar{x})$;
b: $\forall n \in \mathbb{N}, \quad x_n^* \in \partial f_n(x_n)$;
c: $x_n^* \longrightarrow \bar{x^*}$.

Proof. Let (S1) and (S2) be satisfied, $\lim_{n\longrightarrow\infty} y_n = 0$, $\lim_{n\longrightarrow\infty} f_n(y_n + \bar{x}) = f_n(\bar{x})$, $\bar{x^*} \in \partial f(\bar{x})$ and $g_n : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be defined as follows:

$$g_n(x) := \begin{cases} f_n(x + y_n + \bar{x}) - f_n(y_n + \bar{x}) - < \bar{x^*}, x >, & \text{if } \|x\| \leq 1 \\ +\infty, & \text{otherwise .} \end{cases}$$

For any $M > 0$ we have

$$\forall n \in \mathbb{N}, \forall x \in E, 0 \leq g_n(x) + M d_Q(x) + \epsilon_n, \tag{3}$$

where $Q := \{x \in E \mid \|x\| \leq 1\}$ and

$$\epsilon_n := 2 \mid \min_{q\in Q}\big(f_n(q + y_n + \bar{x}) - f_n(y_n + \bar{x}) - < \bar{x^*}, q >\big) \mid .$$

It is enough to show that $\epsilon_n \longrightarrow 0$. We have

$$\min_{q\in Q}\left(f_n(q + y_n + \bar{x}) - f_n(y_n + \bar{x}) - < \bar{x^*}, q > \right) \leq 0.$$

If for some $q_{n_k} \in Q$ and $\delta > 0$

$$f_{n_k}(q_{n_k} + y_{n_k} + \bar{x}) - f_{n_k}(y_{n_k} + \bar{x}) - < \bar{x^*}, q_{n_k} > \leq -\delta < 0,$$

then, assuming that $q_{n_k} \overset{weak}{\longrightarrow} \bar{q}$ (we are able to do it by the Eberlein-Smulian theorem, taking a subsequence if needed), by (S1) we get

$$0 \leq f(\bar{q} + \bar{x}) - f(\bar{x}) - < \bar{x^*}, \bar{q} > \leq \liminf_{k\longrightarrow\infty}\bigg(f_{n_k}(q_{n_k} + y_{n_k} + \bar{x}) - f_{n_k}(y_{n_k} + \bar{x}) - < \bar{x^*}, q_{n_k} > \bigg) \leq -\delta < 0,$$

a contradiction. Hence, for functions g_n the assumptions (i) and (ii) of Theorem 2 are satisfied. Thus, there are sequences $\{x_n\}_{n=1}^{\infty} \subset E$, $\{x_n^*\}_{n=1}^{\infty} \subset E^*$ such that

- $\lim_{n\longrightarrow\infty} x_n = \bar{x}$, $\lim_{n\longrightarrow\infty} f_n(x_n) - f(\bar{x}) - < \bar{x^*}, x_n - \bar{x} > = 0$;
- $\forall n \in \mathbb{N}, \quad x_n^* \in \partial f_n(x_n)$;
- $x_n^* \longrightarrow \bar{x^*}$ (keep in mind that $\|x_n^* - \bar{x^*}\| \leq r_n k_n^{-1}$ by (d) of Theorem 2).

■

Below we show that results concerning weak* convergence of subdiferentials, see Zagrodny (2005a,b), can be inferred from Theorem 2 as well. Namely, we show that if the assumptions of Theorem 2.4 of Zagrodny (2005b) are satisfied, then the assumptions of Theorem 2 are also satisfied.

THEOREM 4 *Let E be a WCG Banach space, $(\bar{x}, \bar{x^*}) \in E \times E^*$ be fixed and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function such that $f(\bar{x}) \in \mathbb{R}$, $\bar{x^*} \in \partial f(\bar{x})$. Assume that $f_n : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous convex functions such that:*

i: $f = Mosco - \lim_{n \longrightarrow \infty} f_n$;

ii: *there is an open nonempty subset U of E and a constant $c \in \mathbb{R}$ such that for every $u \in U$ and $n \in \mathbb{N}$ we have $f_n(u) \leq c$.*

Then there are sequences $\{x_n\}_{n=1}^{\infty} \subset E$, $\{x_n^*\}_{n=1}^{\infty} \subset E^*$ and $\{k_n\} \subset \mathbb{N}$ with $k_n \longrightarrow \infty$ such that:

a: $\lim_{n \longrightarrow \infty} x_n = \bar{x}$, $\lim_{n \longrightarrow \infty} f_n(x_n) = f(\bar{x})$;

b: $\forall n \in \mathbb{N}, \quad x_n^* \in \partial f_n(x_n)$;

c: $x_n^* \overset{weak^*}{\longrightarrow} \bar{x^*}$

d: $\limsup_{n \longrightarrow \infty} \sup_{q \in Q} < x_n^* - \bar{x^*}, {k_n}' q >= 0$ *for any sequence $\{k_n'\}_{n=1}^{\infty} \subset (0, \infty)$ such that $k_n' \longrightarrow \infty$ and $\frac{k_n'}{k_n} \longrightarrow 0$.*

Proof. Let us assume that $\bar{x} = 0$, $\bar{x^*} = 0$, $f(0) = 0$. Considering $\max\{f_n, -1\}$, if needed, we may assume that $\forall x \in E, \forall n \in \mathbb{N}, f_n(x) \geq -1$ and

$$\forall x \in B(a, r), f_n(x) \leq c$$

for some $a \in E$, $r > 0$. We may assume that $f_n(0) = 0$ (if not, then consider $\tilde{f_n}(x) := f_n(x + y_n) - f_n(y_n)$, where $y_n \longrightarrow 0$, $f_n(y_n) \longrightarrow 0$). Let us put

$$\epsilon_n := 2 \mid \min_{q \in 2Q} f_n(q) \mid .$$

Similar reasoning as in the above proof of the Attouch theorem gives $\epsilon_n \longrightarrow 0$. We have also

$$\forall x \in E, \forall n \in \mathbb{N}, \left(\|x\|^2 + f_n(x)\right) + M d_Q(x) + \epsilon_n \geq 0$$

for some $M \geq 0$. In fact, if not, then for some $\{x_{n_k}\}_{k=1}^{\infty} \subset E$ and $M_{n_k} \longrightarrow \infty$ we get

$$\left(\|x_{n_k}\|^2 + f_{n_k}(x_{n_k})\right) + M_{n_k} d_Q(x_{n_k}) + \epsilon_{n_k} < 0. \tag{4}$$

Since the functions are uniformly bounded from below, so $d_Q(x_{n_k}) \longrightarrow 0$, hence $x_{n_k} \overset{weak}{\longrightarrow} \tilde{x}$ for some $\tilde{x} \in E$, so by (S2) and the lower semicontinuity of $\| \cdot \|$ we get

$$0 \leq \|\tilde{x}\|^2 + f(\tilde{x}) \leq 0,$$

thus $\tilde{x} = 0$ and $M_{n_k} d_Q(x_{n_k}) \longrightarrow 0$. Let us choose a sequence $\{s_k\}_{k=1}^{\infty} \subset (0, \infty)$ such that

$$s_k \longrightarrow \infty, s_k d_Q(x_{n_k}) \longrightarrow 0 \text{ and } s_k M_{n_k} d_Q(x_{n_k}) \longrightarrow \infty.$$

We are able to choose $q_k \in Q$ such that $s_k q_k \longrightarrow a$ and $\tilde{q_k} \in Q$ for which $d_Q(x_{n_k}) = \|x_{n_k} - \tilde{q_k}\|$. We have $s_k(x_{n_k} - \tilde{q_k}) \longrightarrow 0$, so by (ii) we obtain

$$f_{n_k}(s_k q_k - s_k(1-s_k^{-1})(x_{n_k} - \tilde{q_k})) \leq c \text{ for } k \text{ large enough},$$

thus, keeping in mind (4),

$$(1-s_k^{-1})\Big(\|x_{n_k}\|^2 + f_{n_k}(x_{n_k}) + \epsilon_{n_k} + M_{n_k} d_Q(x_{n_k})\Big) + s_k^{-1} f_{n_k}(s_k q_k - s_k(1-s_k^{-1})(x_{n_k} - \tilde{q_k})) \leq c s_k^{-1}.$$

By convexity we get

$$f_{n_k}\big((1-s_k^{-1})\tilde{q_k} + q_k\big) + (1-s_k^{-1})\epsilon_{n_k} + (1-s_k^{-1}) M_{n_k} d_Q(x_{n_k}) \leq c s_k^{-1}.$$

By the definition of ϵ_n the sum of the first two terms is nonnegative, thus

$$s_k M_{n_k} d_Q(x_{n_k}) \leq c(1-s_k^{-1})^{-1} \text{ for } k \text{ large enough},$$

which is impossible, since the left hand side of the above inequality tends to infinity. Hence the assumptions of Theorem 2 are satisfied and we get the statement in this case. In general it is enough to repeat the above reasoning for the functions

$$g_n(x) := f_n(x + y_n) - f_n(y_n) - < \bar{x^*}, x > \quad \text{for all } x \in E$$

where $y_n \longrightarrow \bar{x}$, $f_n(y_n) \longrightarrow f(\bar{x})$. ■

The next two examples shed some light on assumptions of Theorem 2. The first one shows that we can not put the norm instead of the distance in (ii).

Example 1 Let us put $E := c_0$, $e_i := (0, \dots, 0, 1, 0, \dots)$ for $i = 1, 2, \dots$, where 1 is in i^{th}-position. Let us define functions f_n as $f_n(x) := x_n$, where $x = (x_1, x_2, \dots) \in E$ and the set Q as $Q := cl \quad conv\{\pm e_1, \pm e_2, \dots\}$. The set is weakly compact and $Q = -Q$, $cl \bigcup_{n=1}^{\infty} nQ = E$.

We have $\partial f_n(x) = \{e_n\}$ for every $x \in E$ and $n \in \mathbb{N}$. Moreover

$$\forall x \in E, \forall n \in \mathbb{N}, \quad f_n(x) + \|x\| \geq 0.$$

Let us observe that $f_n(e_n) = 1$ for every $n \in \mathbb{N}$, thus (c) of Theorem 2 does not hold.

In the next example we show that (a),(b),(d) of Theorem 2 do not yield (c). In order to have this implication ($(a, b, d) \Longrightarrow (c)$) some additional assumptions are needed, giving the boundedness of the sequence.

Example 2 Let us define $g : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$g(t) := \begin{cases} -\sqrt{-t}, & \text{if } t \leq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Let us put

$$f_n(x) := \max_{1\leq i\leq n}\{-i^{-1} + g(x_i)\} \text{ for } n\in\mathbb{N}, x=(x_1,x_2,\dots)\in c_0.$$

Of course, $f_n(x) \leq f_{n+1}(x)$, so there is f which is the Mosco limit of the sequence of functions, i.e. $f = Mosco - \lim_{n\longrightarrow\infty} f_n$, moreover $f(0) = 0$, $f(x) \geq 0$ for $x \in c_0$, so $0 \in \partial f(0)$. Using Theorem 2.2 of Zagrodny (2005b) we infer the existence of $\{x_n\}_{n=1}^{\infty} \subset c_0$, $\{x_n^*\}_{n=1}^{\infty} \subset l_1$ such that $x_n \longrightarrow 0$, $f_n(x_n) \longrightarrow 0$, $x_n^* \in \partial f_n(x_n)$ for every $n \in \mathbb{N}$ and for some $k_n \longrightarrow \infty$

$$\sup_{q\in Q} |< x_n^*, k_n q >| \longrightarrow 0.$$

Of course we have

$$x_n^* = \sum_{i=d_n}^{g_n} \lambda_i^n \left(2\sqrt{-x_i^n}\right)^{-1}$$

for some $d_n \leq g_n$, $d_n \longrightarrow \infty$, $\lambda_i^n \geq 0$ with $\sum_{i=d_n}^{g_n} \lambda_i^n = 1$, $x_n = (x_1^n, x_2^n, \dots) \in c_0$. Put $y_n := (\sqrt{|\, x_1^n\,|}, \sqrt{|\, x_2^n\,|}, \dots)$ and observe that $< x_n^*, y_n >= \frac{1}{2}$, so the sequence $\{\|x_n^*\|\}_{n=1}^{\infty}$ is unbounded and can not weak* converge to 0.

4. Kronecker Lemma-weak* subdifferential convergence approach

The Kronecker lemma ensures that if a series $\sum_{i=1}^{\infty} x_i$ of real numbers is convergent and $\{b_i\}_{i=1}^{\infty}$ is a nondecreasing sequence of positive numbers tending to ∞ then

$$\frac{1}{b_n}\sum_{i=1}^{\infty} b_i x_i \longrightarrow 0, \tag{5}$$

see Moricz (1982) for example. In this section we are going to show that this result pertains to much more general considerations than it appears to. Namely, in what follows a real WCG Banach space is provided such that weak* convergence of some functionals to zero corresponds to the convergence in (5). Thus Theorem 2 gives some possibilities of extending such kind of results to much more general cases. However, this is not the subject of the present paper, only the possibility is indicated here.

EXAMPLE 3 Let us put

$$E := \{x = (x_1, x_2, \dots) \mid \forall i \in \mathbb{N}, x_i \in \mathbb{R} \text{ and } \lim_{i\longrightarrow\infty} y_i = 0, \text{ where } y_i := \sup_{m\geq i} |\sum_{k=i}^{m} x_k|\}.$$

It is not difficult to show that $(E, \|\cdot\|)$ is a real WCG Banach space with $\|x\| := \sup_{n\leq m} |\sum_{k=n}^{m} x_k|$ and $Q := \{x \in E \mid \forall i \in \mathbb{N}, \quad |\, x_i \,|\leq 2^{-1}\}$.

In the sequel, E and Q stand for the space and the set given in the above example.

The corollary below is a slight modification of the Kronecker and Knopp results, see theorems A,B,C,C' of Moricz (1982).

Corollary 1 *Let a_{in}, b_{in}, $n \in \mathbb{N}, i = 1, \ldots, n$ be reals such that*

i: $\exists M \geq 0, \forall n \in \mathbb{N}, \sum_{i=1}^{n} \mid b_{in} - b_{i-1n} \mid \leq M$;

ii: $\exists \{k_n\}_{n=1}^{\infty} \subset \mathbb{N} : k_n \longrightarrow \infty$ *and* $\sum_{i=1}^{k_n} \mid a_{in} \mid \longrightarrow 0$.

Then, for every $n \in \mathbb{N}$ there are $\lambda_{in} \in [0, 1]$ for $i = 1, \ldots, n$ such that

$$\forall x \in E, \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} \Big(\lambda_{in} a_{in} + (1 - \lambda_{in}) b_{in} \Big) x_i = 0.$$

Proof. If $k_n \geq n$ then put $\lambda_{1n} = 1, \ldots, \lambda_{nn} = 1$, and the statement is a simple consequence of (ii). Assume that $k_n < n$. For these n's let us consider convex functions $f_n : E \longrightarrow \mathbb{R} \cup \{\infty\}$ defined as follows

$$f_n(x) := \begin{cases} \sum_{i=1}^{n} \max\{a_{in} x_i, b_{in} x_i\}, & \text{if } \|x\| \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$

We have

$$f_n(x) \geq \sum_{i=1}^{k_n} a_{in} x_i + \sum_{i=k_n+1}^{n} b_{in} x_i \geq - \sum_{i=1}^{k_n} \mid a_{in} \mid \|x\| + \sum_{i=k_n+1}^{n} b_{in}(x_i - q_i) + \sum_{i=k_n+1}^{n} b_{in} q_i,$$

where $q \in Q$ and $d_Q(x) = \|x - q\|$. Putting $b_{0n} = 0$ we get

$$\sum_{i=k_n+1}^{n} b_{in}(x_i - q_i) = \sum_{i=k_n+1}^{n} \sum_{j=1}^{i} (b_{jn} - b_{j-1n})(x_i - q_i)$$
$$= \sum_{j=1}^{n} \Big(\sum_{i=\max\{k_n+1, j\}}^{n} (x_i - q_i) \Big) (b_{jn} - b_{j-1n}) \geq -\|x - q\| \sum_{j=1}^{n} \mid b_{jn} - b_{j-1n} \mid$$
$$\geq -M d_Q(x)$$

and

$$\sum_{i=k_n+1}^{n} b_{in} q_i = \sum_{j=1}^{n} \sum_{i=\max\{k_n+1, j\}}^{n} q_i (b_{jn} - b_{j-1n}) \geq - \sum_{i=k_n+1}^{n} 2^{-i} M \geq -M 2^{-k_n}.$$

Thus

$$f_n(x) \geq - \Big(\sum_{i-1}^{k_n} \mid a_{in} \mid + M 2^{-k_n} \Big) - M d_Q(x),$$

so, by putting $\epsilon_n := M 2^{-k_n} + \sum_{i=1}^{k_n} \mid a_{in} \mid$ we get that the assumptions of Theorem 2 are satisfied which gives the assertion. ∎

Remark 2 Let us observe that if $b_1 \leq b_2 \leq \ldots \longrightarrow \infty$, then by putting $a_{in} = b_{in} = \frac{b_i}{b_n}$ we have (i) and (ii) satisfied with $M = 1$ and k_n chosen properly, so we get the Kronecker lemma as a corollary.

Example 4 Let us consider the (h, g)-transform of a sequence $x = (x_1, x_2, \ldots)$, i.e. for some $d \geq 0$ functions $g, h : [d, \infty) \longrightarrow (d, \infty)$ are nondecreasing with $g(n) \longrightarrow \infty$, $a_{in} = b_{in} = \frac{1}{g(n)h(i)}$, $a_{0n} = b_{0n} = \frac{1}{g(n)d}$, and $f_n(x) := \sum_{i=1}^{n} a_{in} x_i$. In Jajte (2003) the (h, g)-transform is used in formulating the strong law of large numbers. There are also examples of (h, g)-transformations. Following the author, we recall them, i.e., $[h(y) = 1, g(y) = y]$, $[h(y) = y, g(y) = \log y]$, $[h(y) = 1, g(y) = y^{\frac{1}{\alpha}}, 0 < \alpha < 2, \alpha \neq 1]$.

If $x \in E$ then by the above corollary we get $f_n(x) \longrightarrow 0$. Indeed, let us observe that by the definition of a_{in} we are able to choose k_n such that $k_n \longrightarrow \infty$ and $\sum_{i=1}^{k_n} a_{in} \longrightarrow 0$. We have also

$$\sum_{i=1}^{n} | a_{in} - a_{i-1n} | = \frac{1}{g(n)} \left(\frac{1}{d} - \frac{1}{h(n)} \right) \longrightarrow 0,$$

so (i) and (ii) of the corollary are satisfied, which gives $f_n(x) \longrightarrow 0$.

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