Control and Cybernetics

vol. **36** (2007) No. 3

Condition numbers and Ritz type methods in unconstrained optimization

by

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Abstract: Condition numbers for infinite-dimensional optimization, as defined in Zolezzi (2002, 2003), are shown to exhibit a stable behavior when employing finite-dimensional solution methods of the Ritz type, in particular finite elements. The same behavior is shown to hold in connection with the so called extended Ritz method.

Keywords: condition number in optimization, Ritz method, finite elements.

Introduction

We consider scalar optimization problems without constraints in the infinite-dimensional setting. Following a general pattern, we embed the given optimization problem in a suitable family of parameterized problems. Then, we define the condition number of the given problem as a measure of sensitivity of the optimal solution with respect to small changes of the relevant parameters (within the chosen embedding). This approach makes sense for mathematical problems of a very general nature (not necessarily optimization problems) and leads to a general definition of the absolute condition number, see e.g. Demmel (1987). The chosen embedding selects those problem data (parameters) of interest, with respect to which the sensitivity of the solutions needs to be considered.

In the context of infinite-dimensional scalar optimization, definitions of condition numbers were presented in Zolezzi (2002, 2003) based on the above procedure. Such condition numbers share the following significant properties. First, they generalize to the infinite-dimensional setting the familiar notion of absolute condition number of matrix theory. Second, they are directly related with measuring the distance to ill-conditioning, thereby generalizing the distance theorem of numerical linear algebra (sometimes referred to as the Eckart-Young theorem) to the infinite-dimensional setting, as shown in Zolezzi (2002, 2003). Third, they exhibit a stable behavior under variational perturbations, as shown in Zolezzi (2007).

Such properties justify the definition of condition numbers we introduced, provided the following further crucial property is true. The behavior of these condition numbers is stable when employing standard finite-dimensional methods (of the Ritz type) for the numerical solution of the given optimization problem.

The purpose of this paper is to answer positively such a basic question by showing that several convex optimization problems fulfil the following property. When using suitable finite-dimensional numerical solution methods of the Ritz type, the condition number of the finite-dimensional approximations converges to that of the original infinite-dimensional problem as the dimension tends to infinity. Such a property means that the (appropriately defined) condition number remains stable as the finite-dimensional approximations are employed. So, if the distance to ill-conditioning of the finite-dimensional approximations remains large, i.e. the Ritz type approximate problems are (uniformly) well-conditioned, then the same is true for the original problem, and conversely.

The paper is organized as follows. In Section 1 we collect the relevant background and definitions. In Section 2 we prove the convergence of condition numbers for methods of the Ritz type and of finite elements applied to quadratic functionals of the calculus of variations. In Section 3 the same convergent behavior is proved in an abstract framework for some classes of uniformly convex functionals. In Section 4, convergence is shown to hold, within an abstract setting, of the condition numbers for uniformly convex problems by using the so called extended Ritz method, which is relevant in approximate optimization using, e.g., neural networks.

1. Preliminaries

We consider scalar optimization problems depending on a parameter $p \in X$, each of which has a unique global optimal solution $m(p) \in Y$. Both X and Y are real normed spaces. Let p^* correspond to the unperturbed problem, whose condition number we want to define. Then the (absolute) condition number of the unperturbed problem is defined as

$$\limsup_{p \to p^*} \frac{\|m(p) - m(p^*)\|}{\|p - p^*\|}. \tag{1}$$

For related definitions see Demmel (1987), Gill, Murray and Wright (1991, p. 43), Zolezzi (2003). More precisely, we deal with two real Hilbert spaces $E,\ F$ such that $E \subset F$ with continuous and dense embedding, and we identify F with its dual space F^*. Thus we have in the standard fashion $E \subset F = F^* \subset E^*$ with continuous and dense embeddings. We consider a real-valued function $f : E \to R$ and the embedding of the global optimization problem (E, f) defined by $(p, x) \to f(x) - < p, x >, p \in F, x \in E$. Here, $< \cdot, \cdot >$ denotes the duality pairing between E^* and E. We remark that, if $(\cdot, \cdot)$ denotes the scalar product of F, then $< p, x >= (p, x)$ if $p \in F, x \in E$. Throughout this paper we assume that

$$0 = \text{ arg min}(E, f) \tag{2}$$

and that for every sufficiently small $p \in F$ there exists a unique global minimizer

$$m(f, p) = \text{ arg min } [E, f(\cdot) - (p, \cdot)].$$

Taking $p^* = 0$, according to (1) and (2) the condition number of the global optimization problem (E, f) (with respect to the chosen embedding) is given by the extended real number

$$\text{cond } f = \limsup_{p} \to 0 \frac{\|m(f, p)\|_F}{\|p\|_F} \tag{3}$$

where we emphasize that the norms are taken in the (larger) space F, and the convergence of p towards 0 takes place in F. This choice is partially motivated by applications of the Ritz method to the calculus of variations and of the finite element methods to the Dirichlet problem for partial differential equations of elliptic type, as in Section 2. We shall consider sequences E_n of subsets of E, and we shall denote by f_n the restriction of f to E_n, more precisely $f_n = f +$ ind E_n, where ind denotes the indicator function. Accordingly, we shall write

$$\text{cond } f_n = \limsup_{p \to 0} \frac{\|m(f_n, p)\|_F}{\|p\|_F} \tag{4}$$

(whenever this makes sense) for the condition number of the optimization problem (E_n, f_n). The global optimization problem (E, f) is called *well-conditioned* if cond $f < +\infty$, otherwise it is called *ill-conditioned.* (However, from a practical point of view, it is well known that a finite value of cond f, which is in some sense very large, reveals numerical features of (E, f) which are close to ill-conditioning.)

The absolute condition number (3) is a modification of the Lipschitz modulus at $p^* = 0$ of the arg min map m. It differs from the relative condition number often used in linear algebra, e.g. in the error analysis of linear systems. The absolute condition number is the relevant one for measuring the distance to ill-conditioning (see Zolezzi, 2002), while the relative condition number serves different purposes, see e.g. Demmel (1987).

2. Ritz and finite element methods

We start by an abstract version of the Ritz method for quadratic functionals. We fix $A : E \to E^*$ a linear continuous symmetric coercive operator, such that there exists an orthonormal basis $\{\varphi_n\}$ of F consisting of eigenfunctions

$\varphi_n \in E, n = 1, 2, \ldots$ of A. We consider the Ritz method for minimizing the quadratic form

$$f(x) = \frac{1}{2} < Ax, x >, x \in E,$$

by employing the sequence of finite-dimensional subspaces $E_n =$ sp $\{\varphi_1, \ldots, \varphi_n\}$, $n = 1, 2, \ldots$, whose union is obviously dense in F. We assume that the sequence of eigenvalues λ_n of A fulfills

$$\sum_{n=1}^{+\infty} \frac{1}{\lambda_n^2} < +\infty. \tag{5}$$

THEOREM 1 *All problems* $(E, f), (E_n, f_n)$ *are well-conditioned. If (5) is fulfilled, then*

$$\text{cond } f_n \to \text{cond } f \text{ as } n \to +\infty.$$

Proof. Well-conditioning comes from Proposition 3.1 of Zolezzi (2002). The restriction f_n of f to each finite-dimensional subspace E_n gives rise to the strictly convex functional

$$f_n(x) - < p, x >, x \in E_n, p \in F$$

which has exactly one minimizer $u_n = m(f_n, p)$, so that definition (4) applies. Having fixed p, let

$$u = \text{ arg min } (E, f(\cdot) - < p, \cdot >),$$

then standard calculations (see, e.g., Strang and Fix, 1973, Chapter 1) yield

$$u = \sum_{k=1}^{+\infty} \frac{(p, \varphi_k)\varphi_k}{\lambda_k}, u_n = \sum_{k=1}^{n} \frac{(p, \varphi_k)\varphi_k}{\lambda_k} \text{ in } F,$$

so that (norms of F)

$$\frac{\|u_n - u\|^2}{\|p\|^2} \leq \sum_{k=n+1}^{+\infty} \frac{1}{\lambda_k^2}. \tag{6}$$

Since for every n and $p \neq 0$

$$\frac{\|u_n\|}{\|p\|} \leq \frac{\|u_n - u\|}{\|p\|} + \frac{\|u\|}{\|p\|} \tag{7}$$

we get by (6), as $n \to +\infty$, $\limsup \text{cond } f_n \leq \text{cond } f$. Exchanging the roles of u_n and u in (7) we obtain $\liminf \text{cond } f_n \geq \text{cond } f$, which ends the proof. ■

Now we consider a class of finite element methods employed in the numerical solution of the Dirichlet problem for a quadratic integral functional in the calculus of variations. Let

$$f(w) = \int_{\Omega} [\sum_{i,k=1}^{d} a_{ik}(x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_k} + c(x)w^2] dx, w \in E = H_0^1(\Omega)$$

where $d = 2$ or $d = 3, \Omega$ is the interior of a polygon of R^2 or a polyhedron of R^3; of course Ω is open, bounded and convex with a Lipschitz boundary. We consider $F = L^2(\Omega)$ and a sequence of positive numbers $h_n \to 0$ as $n \to +\infty$. Let T_n be a corresponding regular triangulation of $\bar{\Omega}$ as defined in Quarteroni and Valli (1994), Chapter 3, so that h_n is an upper bound of the diameter of each element of T_n. See also Ciarlet (1978), Chapter 2. Let E_n be the linear space of those $v \in C^0(\bar{\Omega})$ such that $v = 0$ on $\partial\Omega$ and the restriction of v to each element of T_n is a polynomial of degree less or equal to 1. Thus, every E_n is the space (corresponding to T_n) of linear triangular or parallelepipedal finite elements. We assume that every $a_{ik} = a_{ki} \in C^1(\bar{\Omega}), i, k = 1, \ldots, d$, that $c \in L^{\infty}(\Omega), c(x) \geq 0$ a.e., and that there exist positive constants α, ω such that

$$\alpha|\lambda|^2 \leq \sum_{i,k=1}^{d} a_{ik}(x)\lambda_i\lambda_k \leq \omega|\lambda|^2$$

for a.e $x \in \Omega$, every $\lambda \in R^d$.

THEOREM 2 *Under the previous assumptions all problems* $(E, f), (E_n, f_n)$ *are well-conditioned, and*

$$\text{cond } f_n \to \text{cond } f \text{ as } n \to +\infty.$$

Proof. Well-conditioning follows by Proposition 3.1 of Zolezzi (2002). For any fixed $p \in L^2(\Omega)$ denote by

$$u_n = m(f_n, p), u = m(f, p)$$

the unique minimizers of $f_n(w) - (p, w)$ on E_n, and of $f(w) - (p, w)$ on $H_0^1(\Omega)$. We apply Theorem 6.2.1, p. 171, of Quarteroni and Valli (1994), so we get

$$\|u - u_n\|_{H_0^1} \leq \text{ (constant) } h_n\|u\|_{H^2}. \tag{8}$$

Indeed, the regularity theorem applies, see Remark 6.2.1, p. 173, of Quarteroni and Valli (1994) due to the smoothness of the coefficients a_{ik}. By Proposition 6.2.2 of Quarteroni and Valli (1994) we know that

$$\|u\|_{H^2} \leq \text{ (constant) } \|p\|_{L^2}$$

hence by (8)

$$\|u - u_n\|_{L^2} \leq \text{ (constant) } h_n \|p\|_{L^2}.$$

Since $h_n \to 0$, by (7) we obtain as $n \to +\infty$ $\limsup \text{cond } f_n \leq \text{cond } f$, and by exchanging the roles of u_n, u in (7) we obtain $\liminf \text{cond } f_n \geq \text{cond } f$, hence the conclusion. ■

Similar convergence theorems can be obtained by using different finite elements as described in Quarteroni and Valli (1994) and Ciarlet (1978).

REMARK 1 There is no contradiction between Theorem 2 and the well known ill-conditioning of the stiffness matrix, corresponding to finite element methods, as $h_n \to 0$. The reason is that the condition number of the stiffness matrix is obtained by using a family of norms on E_n, depending on h_n, which are not equivalent to the norm of $F = L^2(\Omega)$ as $h_n \to 0$: see Quarteroni and Valli (1994), Proposition 6.3.1.

3. Abstract finite element method

We present an abstract extension of Theorem 2 to a class of uniformly convex functionals, as follows. Let $f : E \to [0, +\infty), f(0) = 0$ be a given Gâteaux differentiable functional which is absolutely continuous on the segments of E, i.e. $t \to f(tx + (1-t)y)$ is absolutely continuous on $[0, 1]$ for every x and y. Suppose that there exist continuous functions

$$\alpha, \beta : [0, +\infty) \to [0, +\infty)$$

such that

$$\alpha(\|x - y\|_E) \leq < \nabla f(x) - \nabla f(y), x - y > \leq \beta(\|x - y\|_E) \tag{9}$$

for all $x, y \in E$. Moreover, we assume that the functions $\bar{\alpha}, \bar{\beta}$ given by

$$\bar{\alpha}(t) = \int_0^1 \frac{\alpha(tx)}{x} dx, \ \bar{\beta}(t) = \int_0^1 \frac{\beta(tx)}{x} dx$$

are well-defined on the whole interval $[0, +\infty)$, continuous, strictly increasing, such that

$$\bar{\alpha}(0) = \bar{\beta}(0) = 0, \frac{\bar{\alpha}(t)}{t} \to +\infty \text{ as } t \to +\infty. \tag{10}$$

Then, as well known (e.g. see Melkes, 1970, Lemma 2), for every $p \in E^*$ there exists a unique global minimizer $m(f, p)$ of $f(\cdot) - < p, \cdot >$ on E. Let E_n be a sequence of nonempty closed convex subsets of E. Suppose that there exists a sequence $\varepsilon_n \to 0$ such that for every sufficiently small $p \in E^*$ we have

$$\text{dist } (m(f, p), E_n) \leq \varepsilon_n \varphi(\|p\|_F) \tag{11}$$

for some $\varphi : [0, +\infty) \to R$, which is continuous and increasing; the distance from E_n is taken with respect to the norm of E. Furthermore we assume that, for some $\delta > 0$,

$$\sup\{\frac{\gamma[q\varphi(x)]}{x} : 0 < x \le \delta\} \to 0 \text{ as } q \to 0+ \tag{12}$$

where $\gamma = \bar{\alpha}^{-1}(\bar{\beta})$. We denote again by f_n the restriction of f to E_n, and consider the condition numbers as in (3) and (4).

THEOREM 3 *If (9), (10), (11) and (12) are fulfilled, then*

$$\text{cond } f_n \to \text{cond } f \ \text{ as } \ n \to +\infty.$$

Proof. Write

$$u_n = \text{ arg min } (E_n, f(\cdot) - < p, \cdot >), u = \text{ arg min } (E, f(\cdot) - < p, \cdot >)$$

having fixed $p \in E^*$ (sufficiently small). From Theorem 1 of Melkes (1970), applied to $f(\cdot) - < p, \cdot >$ it follows that

$$\|u - u_n\|_E \le \gamma(\|u - x\|_E), \gamma = \bar{\alpha}^{-1}(\bar{\beta}),$$

for every n and $x \in E_n$. Since γ is continuous and strictly increasing, by taking the infimum with respect to x we get

$$\|u - u_n\|_E \le \gamma[\text{ dist } (u, E_n)].$$

Then by (11) and (7)

$$\frac{\|u_n\|}{\|p\|} \le \frac{C}{\|p\|}\gamma[\varepsilon_n \varphi(\|p\|)] + \frac{\|u\|}{\|p\|}, \tag{13}$$

where C is a suitable constant and the norm is that of F. By (12), given $\varepsilon > 0$, we have

$$\frac{\gamma[q\varphi(x)]}{x} \le \varepsilon$$

if $0 < x \le \delta$ and q is sufficiently small. Then, by (13), if n is sufficiently large (independently of p)

$$\frac{\|u_n\|}{\|p\|} \le \varepsilon + \frac{\|u\|}{\|p\|}$$

whence

$$\limsup_{p \to 0} \frac{\|u_n\|}{\|p\|} \le \varepsilon + \limsup_{p \to 0} \frac{\|u\|}{\|p\|}$$

for all sufficiently large n. It follows that $\limsup_{n \to +\infty} \text{cond } f_n \le \text{cond } f$. By exchanging the roles of u_n and u in (13), we get $\liminf \text{cond } f_n \ge \text{cond } f$, which completes the proof in cases of both $\text{cond } f < +\infty$ and $\text{cond } f = +\infty$. ∎

Remark 2 Absolute continuity of f on segments, as required in Theorem 3, is needed in order to apply the Lagrange formula in the proof of Theorem 1 of Melkes (1970). Such a condition is automatically fulfilled provided f is continuous (see Phelps, 1989, Proposition 2.8).

The following result is a modification of Theorem 3 (not requiring assumption (12)). Let again $f : E \to [0, +\infty)$, $f(0) = 0$, be Gâteaux differentiable such that

$$< \nabla f(x) - \nabla f(y), x - y > \geq \|x - y\|_E \, \theta(\|x - y\|_E) \tag{14}$$

where $\theta = \theta(t), t \geq 0$, is strictly increasing, $\theta(0) = 0$, $\theta(t) \geq C_1 \, t$ for every t, some $C_1 > 0$. Assume that:

(a) for every $r > 0$ and every x, y in E with $\|x\|_E \leq r$, $\|y\|_E \leq r$ we have

$$\|\nabla f(x) - \nabla f(y)\| \leq \text{ (constant) } r \, \|x - y\|_E; \tag{15}$$

(b) there exists a sequence $\varepsilon_n \to 0$ such that

$$\text{dist } (m(f, p), E_n) \leq \varepsilon_n \tag{16}$$

for every n and sufficiently small $p \in F$; the distance is considered with respect to the norm of E.

Theorem 4 *If (14), (15) and (16) hold then*

$$\text{cond } f_n \to \text{cond } f \ \text{ as } \ n \to +\infty.$$

Proof. We follow the proof of Theorem 5.3.4 in p. 323 of Ciarlet (1978) obtaining for every n

$$\theta(\|u - u_n\|_E) \leq \text{ (constant) } \theta^{-1}(\|p\|_{E^*})\|u - x\|_E$$

for every $x \in E_n$, where u, u_n are as in the proof of Theorem 3. Then

$$\|u - u_n\|_E \leq C_1 \theta(\|u - u_n\|_E) \leq \text{ (constant) } \|p\|_F \text{ dist } (u, E_n)$$

and, by (16),

$$\|u - u_n\|_F \leq \alpha_n \|p\|_F$$

where $\alpha_n \to 0$ as $n \to +\infty$ (α_n independent of p) provided p is sufficiently small. Then the conclusion follows as in Theorem 3. ■

Remark 3 Suppose that $E = H_0^1(\Omega), F = L^2(\Omega)$ and E_n are finite dimensional subspaces corresponding to a finite element method. Then, conditions (11) and (16) are fulfilled provided the minimizers $m(f, p) \in H^2(\Omega)$ with $\|m(f, p)\| \leq$ constant if p is sufficiently small, and the finite elements allow the inequality

$$\|m(f, p) - m(f_n, p)\|_{H^1(\Omega)} \leq \text{ (constant) } h_n \|m(f, p)\|_{H^2(\Omega)}, \tag{17}$$

h_n being the supremum of the diameters of the considered finite elements. See Ciarlet (1973) for examples where (17) is fulfilled.

4. Extended Ritz method

We fix again a function $f : E \to [0, +\infty)$, $f(0) = 0$, within a different functional analytic setting than before. Namely we consider a single Hilbert space E, the embedding of (E, f) defined by

$$(p, x) \to f(x) - < p, x >, x \in E, p \in E^*$$

and the corresponding new condition number

$$\bar{c}(f) = \limsup_{p \to 0} \frac{\|m(f, p)\|_E}{\|p\|_{E^*}},$$

where $p \to 0$ in E^*. This setting is appropriate whenever a single space suffices in order to describe the features of the optimization problem we consider.

Let G be a fixed nonempty bounded subset of E. A modification of the Ritz method for approximately minimizing f, called in Zoppoli, Sanguineti and Parisini (2002) the *extended Ritz method,* works as follows. Let, for $n = 1, 2, 3, \ldots$

$$E_n = \text{ sp }_n G = \{\sum_{i=1}^{n} \alpha_i g_i : g_1, \ldots, g_n \in G, \alpha_1, \ldots, \alpha_n \in R\}$$

be the set of all linear combinations of at most n elements from G. We consider the restriction f_n of f to $sp_n G$ and we assume that for every sufficiently small $p \in E^*$ there exist unique minimizers

$$u_n = m(f_n, p) = \text{ arg min } (E_n, f(\cdot) - < p, \cdot >);$$
$$u = m(f, p) = \text{ arg min } (E, f(\cdot) - < p, \cdot >).$$

See Kurkova and Sanguineti (2005), Zoppoli, Sanguineti and Parisini (2002) and the references therein for a detailed analysis of this extended Ritz method, motivations and applications to several optimization problems. Denote by M the Minkowski functional of the closed convex hull cl co $(G \cup -G)$, and consider

$$s = \sup\{\|x\| : x \in G\}.$$

We posit the following assumptions. The function f is Gâteaux differentiable and absolutely continuous on the segments of E. There exist real-valued functions α, β such that (9) and (10) hold. There exists a function $\varphi : [0, +\infty) \to [0, +\infty)$ such that

$$\|m(f, p)\| \leq \varphi(\|p\|) \tag{18}$$

for every sufficiently small $p \in E^*$, moreover, (12) holds. Finally, there exists a real number $z > 1/s$ such that

$$\{x \in E : \|x\| = 1\} \subset z \text{ cl co } (G \cup -G). \tag{19}$$

Lemma 1 *There exists a constant $k > 0$ such that*

$$\sqrt{s^2 M(u)^2 - \|u\|^2} \leq k\|u\|.$$

Proof. The conclusion is obvious if $u = 0$. If not, let $k = \sqrt{s^2 z^2 - 1}$. Then the inequality to be proved is equivalent to

$$\frac{M(u)^2}{\|u\|^2} \leq (1 + k^2)\frac{1}{s^2}$$

or, equivalently, to

$$M(\frac{u}{\|u\|}) \leq z$$

which is true because of (19). ■

Theorem 5 *If the previous assumptions are fulfilled, then*

$$\bar{c}(f_n) \to \bar{c}(f) \text{ as } n \to +\infty.$$

Proof. Let $\gamma = \bar{\alpha}^{-1}(\bar{\beta})$. Arguing as in the proof of Theorem 3 we have, for every n, that

$$\|u - u_n\| \leq \gamma[\text{ dist } (u, sp_n G)].$$

From Theorem 3.1 p. 467 of Kurkova and Sanguineti (2005) it follows that

$$\|u - u_n\| \leq \gamma(\sqrt{\frac{s^2 M(u)^2 - \|u\|^2}{n}})$$

for every n and sufficiently small p. Then by Lemma 1 and (18)

$$\|u - u_n\| \leq \gamma(\frac{k\|u\|}{\sqrt{n}}) \leq \gamma[\frac{k\varphi(\|p\|)}{\sqrt{n}}].$$

Then, from (12) we see that, given $\varepsilon > 0$,

$$\frac{\|u\|}{\|p\|} \leq \frac{1}{\|p\|}\gamma[\frac{k\varphi(\|p\|)}{\sqrt{n}}] + \frac{\|u_n\|}{\|p\|} \leq \varepsilon + \frac{\|u_n\|}{\|p\|}$$

for all sufficiently small $p \neq 0$ and for every sufficiently large n. Then

$$\lim \inf_{n \to +\infty} \bar{c}(f_n) \geq \bar{c}(f);$$

the conclusion obtains by exchanging the role of u and u_n. ■

REMARK 4 Assuming existence of global minimizers of f on sp_nG is a restrictive condition in general. A more natural pattern would use approximate solutions of (sp_nG, f). However, this would require a reasonable definition of a condition number of optimization problems based on approximate solutions only, which is by now an open problem.

Given two Banach spaces E, F and the real-valued function f as in Section 1, we compare now the condition numbers

$$\bar{c}(f) = \limsup_{p \to 0} \frac{\|m(f,p)\|_E}{\|p\|_{E^*}}$$

where $p \to 0$ in E^*, and

$$\text{cond } f = \limsup_{p \to 0} \frac{\|m(f,p)\|_F}{\|p\|_F}$$

where $p \to 0$ in F, we have employed in the previous sections.

PROPOSITION 1 *There exists a constant* $T > 0$ *(not depending of* f*) such that*

$$\text{cond } f \leq T \ \bar{c}(f).$$

Proof. By the continuous embedding $E \subset F$, for every $p \in F$ we have

$$\|m(f,p)\|_F \leq \text{ (constant) } \|m(f,p)\|_E.$$

Similarly

$$\|p\|_{E^*} \leq \text{ (constant) } \|p\|_F$$

hence

$$\frac{\|m(f,p)\|_F}{\|p\|_F} \leq \text{ (constant) } \frac{\|m(f,p)\|_E}{\|p\|_{E^*}},$$

whence

$$\limsup_{p \to 0} \frac{\|m(f,p)\|_F}{\|p\|_F} \leq \text{ (constant) } \limsup_{p \to 0} \frac{\|m(f,p)\|_E}{\|p\|_{E^*}}$$

where $p \to 0$ in F. Such constants depend only on the norms of the embedding maps. By the continuous embedding $F \subset E^*$,

$$\lim \sup_{p \to 0 \text{ in } F} \frac{\|m(f,p)\|_E}{\|p\|_{E^*}} \leq \lim \sup_{p \to 0 \text{ in } E^*} \frac{\|m(f,p)\|_E}{\|p\|_{E^*}},$$

ending the proof. ∎

Acknowledgements

We thank a referee whose suggestions improved the presentation. The work reported was partially supported by MURST and Università di Genova.

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