Control and Cybernetics

vol. **36** (2007) No. 3

On spaces with mixed modulars and some spaces of temperate distributions

by

Julian Musielak

Adam Mickiewicz University, Faculty of Mathematics and Computer Science
Umultowska 87, 61-614 Poznań, Poland

Abstract: Extending the Benedek-Panzone spaces of functions integrable with mixed powers in $\mathbb{R}^2$, the spaces are investigated of functions integrable with mixed Orlicz functions, with application to Besov-type spaces and Triebel-Lisorkin spaces of temperate distributions.

Keywords: spaces with mixed norm, modular spaces, temperate distributions, Besov-type spaces, Triebel-Lisorkin spaces.

1.

We are going to extend the Benedek-Panzone spaces with mixed L^p-norm on Cartesian product $\Omega = \Omega_1 \times \Omega_2$ (Benedek, Panzone, 1961) to the case of mixed modulars and norms of Orlicz type over general measure spaces with σ-finite and complete measure spaces. This will be applied to spaces of Besov type, $B^s_{p,q}$, and Triebel-Lisorkin spaces $F^s_{p,q}, p, q > 0,\ s \in \mathbb{R}$.

2.

Let $(\Omega_i, \sum_i, \mu_i),\ i = 1, 2$, be two measure spaces with σ-finite and complete measures μ_1, μ_2 on σ-algebras $\sum_1, \sum_2$ of subsets of spaces Ω_1, Ω_2, respectively. Let $(\Omega, \sum, \mu)$ be the space $\Omega = \Omega_1 \times \Omega_2$ with product measure $\mu = \mu_1 \times \mu_2$ in the smallest σ-algebra $\sum$ of subsets of Ω containing all sets of the form $A_1 \times A_2$ with $A_1 \in \sum_1$, $A_2 \in \sum_2$. Let $L_0(\Omega)$ be the space of all $\sum$-measurable functions $f : \Omega \to \mathbb{R}$, finite μ-a.e. with zero element in $L_0(\Omega)$ being the function equal to zero μ-a.e. in Ω. Let M, N be two even, convex functions on $\mathbb{R}$ such that $M(0) = N(0) = 0$, $0 < M(u) < \infty$ and $0 < N(u) < \infty$ for $u \neq 0$, $M(u) \to \infty$ and $N(u) \to \infty$ as $u \to \infty$. We denote

$$I_{M,N}(f) = \int_{\Omega_1} M\left(\int_{\Omega_2} N(f(x,y))d\mu_2(y)\right) d\mu_1(x)$$

for $f \in L_0(\Omega)$. Let us remark that in case when Ω_1, Ω_2 are intervals in $\mathbb{R}$ and $\mu_1 = \mu_2$ is the Lebesgue measure, the functional $I_{M,N}$ was investigated in Musielak (1998), where also further bibliography may be found. We shall prove that

PROPOSITION 1 *$I_{M,N}$ is a convex modular in $L_0(\Omega)$, i.e.*
(a) $I_{M,N}(0) = 0$,
(b) $0 < I_{M,N}(f) \leq \infty$ *if* $f \neq 0$,
(c) $I_{M,N}(-f) = I_{M,N}(f)$,
(d) $I_{M,N}(\alpha f + \beta g) \leq \alpha I_{M,N}(f) + \beta I_{M,N}(g)$ *for* $\alpha, \beta \geq 0$, $\alpha + \beta = 1$
for arbitrary f, $g \in L_0(\Omega)$ *(for the definition of a convex modular, compare Musielak, 1983).*

Proof. Properties (c) and (d) are obvious. In order to prove (a) and (b), let us denote for a set $A \in \sum$ by χ_A its characteristic function and let us write

$$A_x = \{y \in \Omega_2 : f(x,y) \in A\};$$

we have then $A_x \in \sum_2$ for all $x \in \Omega_1$. For an arbitrary $f \in L_0(\Omega)$ with

$$A = \{(x,y) \in \Omega : f(x,y) \neq 0\}$$

we have

$$\begin{aligned} I_{M,N}(f) &= \int\limits_{\Omega_1} M\left(\int\limits_{\Omega_2} N(f(x,y))\chi_A(x,y)d\mu_2(y)\right) d\mu_1(x) \\ &= \int\limits_{\Omega_1} M\left(\int\limits_{\Omega_2} N(f(x,y))\chi_{A_x}(y)d\mu_2(y)\right) d\mu_1(x) \qquad (1) \\ &= \int\limits_{\Omega_1} M\left(\int\limits_{A_x} N(f(x,y))d\mu_2(y)\right) d\mu_1(x). \end{aligned}$$

Now, suppose that $f = 0$ in $L_0(\Omega)$, then $\int\limits_{\Omega_1} \mu_2(A_x)d\mu_1(x) = \mu(A) = 0$. Hence, $\mu_2(A_x) = 0$ μ_1-a.e. in Ω_1. Consequently,

$$\int\limits_{A_x} N(f(x,y))d\mu_2(y) = 0 \quad \mu_1\text{-a.e. in } \Omega,$$

and so, by (1), $I_{M,N}(f) = 0$, what proves (a). In order to prove (b) let us suppose that $f \neq 0$, i.e. $\mu(A) > 0$. Then there exists a $\delta > 0$ such that the set

$$B = \{(x,y) : |f(x,y)| > \delta\}$$

is of measure $\mu(B) > 0$. By (1), applied to B in place of A, we have

$$\begin{aligned} I_{M,N}(f) &\geq \int_{\Omega_1} M\left(\int_{B_x} N(\delta)d\mu_2(y)\right)d\mu_1(x) \\ &= \int_{\Omega_1} M\left(N(\delta)\int_{B_x} d\mu_2(y)\right)d\mu_1(x) \\ &= \int_{\Omega_1} M(N(\delta)\mu_2(B_x))d\mu_1(x). \end{aligned}$$

Suppose now that $I_{M,N}(f) = 0$. Then, by the above inequality, we have

$$\int_{\Omega_1} M(N(\delta)\mu_2(B_x))d\mu_1(x) = 0,$$

which implies $\mu_2(B_x) = 0$ μ_1-a.e., whence

$$\mu(B) = \int_{\Omega_1} \mu_2(B_x)d\mu_1(x) = 0,$$

a contradiction. Hence $f \neq 0$ in $L_0(\Omega)$ implies $I_{M,N}(f) > 0$, that is, (b). ■

The modular $I_{M,N}$ generates the modular space

$$L_{M,N}(\Omega) = \{f \in L_0(\Omega) : I_{M,N}(\lambda f) < \infty \quad \text{for some} \quad \lambda > 0\},$$

which is a normed vector space with norm

$$||f||_{M,N} = \inf\{u > 0 : I_{M,N}\left(\frac{f}{u}\right) \leq 1\}$$

(see Musielak, 1983, Definition 1.4 and Theorem 1.5). In this space, convergence $f_k \to f$ in norm is equivalent to the condition $I_{M,N}(\lambda(f_k - f)) \to 0$ as $k \to \infty$ for every $\lambda > 0$. If we replace here "for every $\lambda > 0$" by "for some $\lambda > 0$", we say that (f_k) is *modular convergent* or $I_{M,N}-$*convergent* to f, writing $f_k \to f[I_{M,N}]$ as $k \to \infty$ (Musielak, 1983, Theorem 1.6 and Definition 5.1).

Moreover, we say that *a sequence* (f_k) *of functions* $f_k \in L_{M,N}(\Omega)$ *is* $I_{M,N}$*-Cauchy*, if $I_{M,N}(\lambda(f_k - f_l)) \to 0$ as $k, l \to \infty$ for some $\lambda > 0$. A modular space is called *modular complete*, if every modular Cauchy sequence of its elements is modular convergent to some of its elements. Now, we are going to establish the relation between $I_{M,N}$-convergence and convergence in measure μ in Ω. First, we prove

Proposition 2 *Let $f_k \in L_{M,N}(\Omega)$ for $k = 1, 2, \ldots$. Suppose that there exists a set $\Omega_0 \subset \Omega_1$, $\Omega_0 \in \Sigma_1$ of measure $\mu_1(\Omega_0) < \infty$ such that $f_k(x,y) = 0$ for $x \in \Omega_1 \backslash \Omega_0$, $y \in \Omega_2$, $k = 1, 2, \ldots$. Then*

(a) *if $f_k \to f[I_{M,N}]$, then $f \in L_{M,N}(\Omega)$ and $f_k \to f$ in measure μ in Ω,*

(b) *if (f_k) is $I_{M,N}$-Cauchy, then (f_k) is Cauchy in measure μ in Ω.*

Proof. First, we prove (a). From the inequality

$$I_{M,N}\left(\frac{1}{2}\lambda f\right) \leq \frac{1}{2} I_{M,N}(\lambda(f_k - f)) + \frac{1}{2} I_{M,N}(\lambda f_k)$$

for $\lambda > 0$, $k = 1, 2, \ldots$ we conclude that $f \in L_{M,N}(\Omega)$. Let us denote

$$A_k(\varepsilon) = \{(x,y) \in \Omega : |f_k(x,y) - f(x,y)| \geq \varepsilon\}$$

for any $\varepsilon > 0$. Let us fix a $\lambda > 0$ such that $I_{M,N}(\lambda(f_k - f)) \to 0$ as $k \to \infty$. Applying the notation in the proof of Proposition 1 we obtain, by Jensen's inequality for convex functions

$$\begin{aligned}
&\mu_1(\Omega_0) M\left(\frac{1}{\mu_1(\Omega_0)} \int\limits_{\Omega_0} N(\lambda,\varepsilon)\mu_2[(A_k(\varepsilon))_x] d\mu_1(x)\right) \\
&\quad\leq \int\limits_{\Omega_0} M(N(\lambda,\varepsilon)\mu_2[(A_k(\varepsilon))_x]) d\mu_1(x)) \\
&\quad\leq \int\limits_{\Omega_0} M\left(N(\lambda,\varepsilon) \int\limits_{\Omega_2} \chi_{A_k(\varepsilon)}(x,y) d\mu_2(y)\right) d\mu_1(x) \\
&\quad\leq \int\limits_{\Omega_0} M\left[\int\limits_{\Omega_2} N(\lambda(f_k(x,y) - f(x,y)))\chi_{A_k(\varepsilon)}(x,y) d\mu_2(y)\right] d\mu_1(x) \\
&\quad\leq I_{M,N}(\lambda(f_k - f)).
\end{aligned}$$

Denoting by M_{-1} the inverse of M in $\mathbb{R}_+$, we thus obtain

$$\begin{aligned}
\mu[A_k(\varepsilon) \cap \Omega_0] &= \int\limits_{\Omega_0)} \mu_2[(A_k(\varepsilon))_x] d\mu_1(x) \\
&\leq \frac{\mu_1(\Omega_0)}{N(\lambda\varepsilon)} M_{-1}\left(\frac{1}{\mu_1(\Omega_0)} I_{M,N}(\lambda(f_k - f))\right).
\end{aligned}$$

By assumption, we have $A_k(\varepsilon) \subset \Omega_0$, and so we have also

$$\mu(A_k(\varepsilon)) \leq \frac{\mu_1(\Omega_0)}{N(\lambda\epsilon)} M_{-1}\left(\frac{1}{\mu_1(\Omega_0)} I_{M,N}(\lambda(f_k - f))\right). \tag{2}$$

for $k = 1, 2, \ldots.$ Since $I_{M,N}(\lambda(f_k - f)) \to 0$ as $k \to \infty$, we have $\mu(A_k(\varepsilon)) \to 0$ as $k \to \infty$. This means that $f_k \to f$ in measure μ in Ω.

The proof of (b) follows also from the inequality (2) with f replaced by f_l, and $k, l \to \infty$. ■

THEOREM 1 *Let the measure μ_1 be σ-finite in such a manner that $\Omega_1 = \bigcup_{i=1}^{\infty} \Omega_i^*$, where the sets $\Omega_i^* \in \Sigma_1$ are pairwise disjoint and such that for some constants $K_1, K_2 > 0$ there holds $K_1 \leq \mu_1(\Omega_i^*) \leq K_2$ for $i = 1, 2, \ldots.$ Let $f_k \in L_{M,N}(\Omega)$ for $k = 1, 2, \ldots.$ Then the properties (a) and (b) from Propositon 2 hold.*

Proof. First, we prove (a). Taking notation from the proof of Proposition 2 and applying inequality (2) to Ω_i^* in place of Ω_0, we obtain for every $\varepsilon > 0$

$$\begin{aligned}\mu(A_k(\varepsilon)) &\leq \frac{\mu_1(\Omega_i^*)}{N(\lambda\varepsilon)} M_{-1}\left(\frac{1}{\mu_1(\Omega_i^*)} I_{M,N}(\lambda(f_k - f))\right) \\ &\leq \frac{K_2}{N(\lambda\varepsilon)} M_{-1}\left(\frac{1}{K_1} I_{M,N}(\lambda(f_k - f))\right) \to 0\end{aligned}$$

as $k \to \infty$. Thus, $f_k \to f$ in measure μ in Ω. The proof of (b) follows in the same manner as in Proposition 2. ■

THEOREM 2 *If the measure μ_1 satisfies the assumption from Theorem 1, then the space $L_{M,N}(\Omega)$ is $I_{M,N}$-complete and is a Banach space in the norm $\|\ \|_{M,N}$.*

Proof. Let (f_k) be an $I_{M,N}$-Cauchy sequence of elements of $L_{M,N}(\Omega)$. By Theorem 1, it is a Cauchy sequence in measure. Hence, it is convergent in measure μ to a function f. Consequently, (f_k) contains a subsequence (f_{k_i}) such that $f_{k_i}(x, y) \to f(x, y)$ μ-a.e. in Ω. Hence

$$N(\lambda f_{k_i}(x, y)) \to N(\lambda f(x, y)) \quad \text{as} \quad i \to \infty$$

for every $\lambda > 0$, μ-a.e. in Ω. By the Fatou lemma applied to the variables y and x, successively, we obtain

$$\begin{aligned}I_{M,N}(\lambda f) &= \int_{\Omega_1} M\left(\int_{\Omega_2} N(\lambda f(x, y)) d\mu_2(y))\right) d\mu_1(x) \\ &\leq \underline{\lim}_{i\to\infty} \int_{\Omega_1} M\left(\int_{\Omega_2} N(\lambda f_{k_i}(x, y)) d\mu_2(y))\right) d\mu_1(x) \\ &= \underline{\lim}_{i\to\infty} I_{M,N}(\lambda f_{k_i}) < \infty\end{aligned}$$

for sufficiently small $\lambda > \lambda_0$. Hence, $f \in L_{M,N}(\Omega)$. In an analogous manner we may show that

$$I_{M,N}(\lambda(f_{k_i} - f)) \leq \underline{\lim}_{j\to\infty} I_{M,N}(\lambda(f_{k_i} - f_{k_j})).$$

Since (f_{k_i}) is also $I_{M,N}$-Cauchy, we have

$$\lim_{i,j\to\infty} I_{M,N}(\lambda(f_{k_i} - f_{k_j})) = 0$$

for sufficiently small $\lambda > 0$, say for $\lambda \le \lambda_0$. Thus,

$$I_{M,N}(\lambda(f_{k_i} - f)) \to 0 \quad \text{as} \quad i \to \infty$$

for $\lambda < \lambda_0$. But we have for every k

$$I_{M,N}\left(\frac{1}{2}\lambda(f_k - f)\right) \le \frac{1}{2} I_{M,N}(\lambda(f_k - f_{k_i})) + \frac{1}{2} I_{M,N}(\lambda(f_{k_i} - f)). \tag{3}$$

The second term of the right-hand side of this inequality tends to 0 as $i \to \infty$ for $0 < \lambda \le \lambda_0$. So, taking arbitrary $\varepsilon > 0$ we may find an index i_0 such that

$$I_{M,N}(\lambda(f_{k_i} - f)) < \varepsilon \quad \text{for} \quad i \ge i_0$$

and any $0 < \lambda \le \lambda_0$. Since (f_k) is $I_{M,N}$-Cauchy, taking k and k_i sufficiently large we get

$$I_{M,N}(\lambda(f_k - f_{k_i})) < \varepsilon \quad \text{for} \quad 0 < \lambda \le \lambda_0.$$

Thus, from the inequality (3) it follows that

$$I_{M,N}\left(\frac{1}{2}\lambda(f_k - f)\right) < \varepsilon.$$

This means that $f_k \to f[I_{M,N}]$. Thus, $L_{M,N}(\Omega)$ is $I_{M,N}$-complete. Now, if we replace above the phrase "for $\lambda > 0$ sufficiently small" by "for all $\lambda > 0$", we obtain completeness of $L_{M,N}(\Omega)$ with respect to the norm $\|\,\|_{M,N}$. ■

3. Examples

I. Let μ_1 be the Lebesgue measure m in the σ-algebra Σ_1 of all Lebesgue measurable subsets of the space $\Omega_1 = \mathbb{R}^n$. Moreover, let $\Omega_2 = \mathbb{N}$ be the set of natural numbers, Σ_2-the σ-algebra of all subsets of $\mathbb{N}$ and $\mu_2 = \nu$ - the counting measure, i.e. $\nu(\emptyset) = 0$, $\nu(A)$ - the number of elements of A if A is finite, $\nu(A) = \infty$ if A is infinite. Then, $L_0(\Omega)$ is the set of all sequences of functions, $f = (f_k)$, where $f_k : \mathbb{R}^n \to \mathbb{R}$ are Lebesgue measurable in $\mathbb{R}^n$, $k = 1, 2, ...$ In this case we have

$$I_{M,N}(f) = \int_{\mathbb{R}^n} M\left(\sum_{k=1}^{\infty} N(f_k(x))\right) dx \tag{4}$$

and

$$I_{N,M}(f) = \sum_{k=1}^{\infty} N\left(\int_{\mathbb{R}^n} M(f_k(x))\right) dx. \tag{5}$$

From Theorem 2 it follows that both spaces: $L_{M,N}(\mathbb{R}^n \times \mathbb{N})$ and $L_{N,M}(\mathbb{N}x\mathbb{R}^n)$ are Banach spaces with norms

$$||f||_{M,N} = ||(f_k)||_{M,N} = \inf\left\{u > 0 : I_{M,N}\left(\frac{f}{u}\right) \le 1\right\}$$

and

$$||f||_{N,M} = ||(f_k)||_{N,M} = \inf\left\{u > 0 : I_{N,M}\left(\frac{f}{u}\right) \le 1\right\},$$

respectively.

II. Let $\mathcal{S}'$ be the space of temperate distributions and $\mathcal{E}'$ - the space of distributions of compact support in $\mathbb{R}^n$. Let $\Phi_0 : \mathbb{R}^n \to \mathbb{R}$, $\Phi_0 \in C_0^\infty(\mathbb{R}^n)$ be such that the support supp $\Phi_0 \subset \{x \in \mathbb{R}^n, |x| \le \frac{3}{2}\}$, $\Phi_0(x) = 1$ for $|x| \le 1$, $0 \le \Phi_0(x) \le 1$ for all $x \in \mathbb{R}^n$. Let $\Phi_1(x) = \Phi_2(\frac{1}{2}x) - \Phi_0(x)$ and $\Phi_k(x) = \Phi_1(2^{-k+1}x)$ for $k \ge 2$, $x \in \mathbb{R}^n$. Let $\Phi = (\Phi_k)_{k=1}^\infty$ be fixed. We associate with every $f \in \mathcal{S}'$ the sequence (f_k), where

$$f_k = 2^{sk}\mathcal{F}^{-1}(\Phi_k \mathcal{F} f), s \in \mathbb{R}, \quad k = 1, 2, ...,$$

$\mathcal{F}$ being the Fourier transform and $\mathcal{F}^{-1}$-the inverse Fourier transform. Since $\Phi_k \in C_0^\infty(\mathbb{R}^n)$ and $\mathcal{F}f \in \mathcal{S}'$, so $\Phi_k\mathcal{F}f \in \mathcal{E}'$. Hence, $\mathcal{F}^{-1}(\Phi_k\mathcal{F}f)$ is a regular distribution (Hörmander, 1983, Theorem 7.1.14), generated by a function, denoted by the same symbol. Thus, the expressions in formulae (4) and (5) have a sense. Writing $\widetilde{f} = (f_k)$, we denote by $B^s_{M,N}$ the space of all $f \in \mathcal{S}'$ such that $\widetilde{f} \in L_{N,M}(\mathbb{R}^n)$, and by $F^s_{M,N}$ - the space of all $f \in \mathcal{S}'$ such that $\widetilde{f} \in L_{M,N}(\mathbb{R}^n)$. In case when $M(u) = |u|^p$, $N(u) = |u|^q$, $p, q > 1$, $s \in \mathbb{R}$, $B^s_{M,N}$ is the *Besov-type space* of temperate distributions and $F^s_{M,N}$ is the *Triebel-Lisorkin space* of temperate distributions (see Triebel, 1983).

References

Benedek, A. and Panzone, R. (1961) The spaces Lp with mixed norm. *Duke Math. Journal* **28**, 301-324.

Hörmander, L. (1983) *The Analysis of Linear Partial Differential Operators I. Grundlehren der Mathematischen Wissenschaften* **256**, Springer Verlag.

Musielak, J. (1983) *Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics* **1034**, Springer Verlag.

Musielak, J. (1998) On approximation of functions of two variables by integral means and their generalization. *Ann. Sem. Mat. Fiz. Univ. Modena*, Supplement at **XLVI**, 335-348.

Triebel, H. (1983) *Theory of Function Spaces. Monographs in Mathematics* **78**, Birkhäuser Verlag.