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Continuous characters and joint topological spectrum

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Abstract: It is well known that there is a one-to-one correspondence between the characters of a finitely generated commutative Banach algebra and the joint spectrum of its generators. In this paper we show that this fact is also true for an arbitrary semitopological algebra and its continuous characters, provided we replace the concept of a joint spectrum by concept of a topological joint spectrum. In particular, we show that a finitely generated semitopological algebra has a continuous character if and only if the topological joint spectrum of its generators is non-void.

Keywords: semitopological algebra, character, (joint) topological spectrum, generators.

1. Introduction

All algebras in this paper are commutative unital real or complex, with the unity denoted by e.

A *semitopological algebra* is a (real or complex) algebra, which is a topological vector space and the multiplication $(x, y) \mapsto xy$ is separately continuous, i.e. the operator $x \mapsto xy$ is continuous for each fixed y in the algebra in question. If A is an algebra over a field $\mathbf{K}$ ($\mathbf{K} = \mathbf{C}$ or $\mathbf{R}$), then the spectrum of an element x in A, $\sigma_{\mathbf{K}}(x)$ (written also $\sigma(x)$ in the case when it does not matter which field K is taken into account), is defined by

$$\sigma(x) = \{\lambda \in \mathbf{K} : x - \lambda e \notin G(A)\},$$

where $G(A)$ denotes the set (group) of invertible elements in A.

Similarly, the joint spectrum $\sigma(x_1, \ldots, x_n)$ of an n-tuple $x_1, \ldots, x_n \in A$ is defined as

$$\sigma(x_1, \ldots, x_n) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{K}^n : id(x_1 - \lambda_1 e, \ldots, x_n - \lambda_n e) \neq A\},$$

where $id(u_1, \ldots, u_k)$ denotes the smallest ideal of A containing the elements $u_1, \ldots, u_k$, i.e. the set

$$\{\sum_1^k u_i y_i : y_i \in A\}.$$

Such a definition (with $\mathbf{K} = \mathbf{C}$) is equivalent to a generally accepted definition (given by means of Gelfand transforms) for commutative Banach algebras.

The concepts of *topological spectrum* $\sigma^{(t)}(x)$ and *topological joint spectrum* $\sigma^{(t)}(x_1, \ldots, x_n)$ (the latter is introduced in this paper), are obtained from the above concepts by replacing there the group $G(A)$ by the semigroup $G^{(t)}(A)$ of all topologically invertible elements, or replacing the ideal $id(u_1, \ldots, u_k)$ by its closure, which is also a (proper or not) ideal. Recall that an element x of a semitopological algebra A is said to be topologically invertible if there is a net (z_α) of elements of A such that

$$\lim_\alpha z_\alpha x = e,$$

or, equivalently, when the ideal xA is non-dense. Clearly

$$(\lambda_1, \ldots, \lambda_n) \in \sigma^{(t)}(x_1, \ldots, x_n)$$

if and only if the ideal $id(x_1 - \lambda_1 e, \ldots, x_n - \lambda_1 e)$ is not dense in A.

We have, obviously, $G(A) \subset G^{(t)}(A)$, so that $\sigma^{(t)}(x) \subset \sigma(x)$ and both sets can be different (in Abel and Żelazko, 2006, in remark after Proposition 3, an example is given of a topological algebra A and an element x in A with void $\sigma^{(t)}(x)$ but non-void $\sigma(x)$). Similarly, we have

$$\sigma^{(t)}(x_1, \ldots, x_n) \subset \sigma(x_1, \ldots, x_n). \tag{1}$$

A character of a semitopological algebra A is a non-zero linear functional which satisfies $f(xy) = f(x)f(y)$ for all $x, y \in A$. Clearly, we have $f(e) = 1$ for such a functional. Similarly as for Banach algebras we denote by $\mathfrak{M}(A)$ the set of all (scalar valued) continuous characters of a semitopological algebra A.

A semitopological algebra A is generated by elements x_i if it coincides with its smallest closed subalgebra containing these elements. It is called *finitely generated* if it is generated by a finite set of its elements.

2. The results

It is well known (see e.g. Larsen, 1973, Theorem 4.5.1) that if A is a finitely generated Banach algebra, with generators $x_1, \ldots, x_n$, then its maximal ideal space $\mathfrak{M}(A)$ can be identified with $\sigma(x_1, \ldots, x_n)$ both as a set and as a topological (compact) space ($\mathfrak{M}(A)$ is equipped with the Gelfand topology). Here we extend this result onto much larger class of semitopological algebras, replacing joint spectrum by joint topological spectrum (note that for a Banach algebra both spectra coincide). However, we identify the points of the joint spectrum of generators with elements of $\mathfrak{M}(A)$ only set-wise, becase the Gelfand topology of $\mathfrak{M}(A)$ can be different from the topology of $\sigma^{(t)}(x_1, \ldots, x_n)$ inherited form the space $\mathbf{K}^n$. This is a generalization of the Proposition 2 in Abel and Żelazko (2006) obtained for singly generated semitopological algebras. In the paper of Abel and Żelazko (2006) it is also shown (example after Proposition 2) that $\sigma^{(t)}(x)$ can be topologically different from $\mathfrak{M}(A)$, for an algebra A singly generated by x.

Our main result reads as follows.

Theorem 1 *Let A be a commutative unital (real or complex) finitely generated semitopological algebra. Then there is a one-to-one correspondence between functionals in $\mathfrak{M}(A)$ and points in the topological joint spectrum of a set $x_1, \ldots, x_n$ of generators of A.*

This theorem is a consequence of the following two propositions (only in the first of them we use the assumption that our algebra is finitely generated).

Proposition 1 *Let A be as above, and let $\{x_1, \ldots, x_n\}$ be a set of its generators. Then for every $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \sigma^{(t)}(x_1, \ldots x_n)$ there is a continuous character $f_{\tilde{\lambda}}$ on A such that*

$$f_{\tilde{\lambda}}(x_i) = \lambda_i, \; i = 1, 2, \ldots, n \tag{2}$$

(clearly, $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$ implies $f_{\tilde{\lambda}_1} \neq f_{\tilde{\lambda}_2}$).

Proof. Denote by P_n the set of all polynomials in n variables $t_1, \ldots, t_n$ with coefficients in the field $\mathbf{K}$. Since A is generated by $x_1, \ldots, x_n$, the set

$$A_0 = \{p(x_1, \ldots, x_n) : p \in P_n\}$$

is a dense subalgebra of A. We shall make use of the following (well known and easily proved) formula true for an arbitrary polynomial p in P_n:

$$p(t_1, \ldots, t_n) - p(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} (t_i - \lambda_i) q_i(t_1, \ldots, t_n), \tag{3}$$

where $\lambda_1, \ldots, \lambda_n$ is a fixed n-tuple of scalars in $\mathbf{K}$ and q_i are suitably chosen (not uniquely determined) polynomials in P_n. Substituting in (3) the elements x_i instead of variables t_i and taking as λ_i the coordinates of a $\tilde{\lambda}$ in $\sigma^{(t)}(x_1, \ldots, x_n)$, we obtain

$$p(x_1, \ldots, x_n) - p(\lambda_1, \ldots, \lambda_n)e = \sum_{i=1}^{n} (x_i - \lambda_i e) q_i(x_1, \ldots, x_n) \tag{4}$$

and the right hand side of (4) is in the ideal I_0 of A_0, generated by elements $x_i - \lambda_i e, 1 \leq i \leq n$. The ideal I_0 of A_0 is not dense in A_0, otherwise it would be also dense in A, so that the elements $x_i - \lambda_i e$ would generate a dense ideal in A. But this is in contradiction with the relation $\tilde{\lambda} \in \sigma^{(t)}(x_1, \ldots, x_n)$.

We define now a linear functional $f_{\tilde{\lambda}}$ on A_0 setting

$$f_{\tilde{\lambda}}(p(x_1, \ldots, x_n)) = p(\lambda_1, \ldots, \lambda_n). \tag{5}$$

First we show that it is a well defined functional, i.e. the relation

$$p_1(x_1, \ldots, x_n) = p_2(x_1, \ldots, x_n) \tag{6}$$

implies $p_1(\lambda_1, \ldots, \lambda_n) = p_2(\lambda_1, \ldots, \lambda_n)$. So suppose, towards contradiction, that we have (6) but $p_1(\lambda_1, \ldots, \lambda_n) \neq p_2(\lambda_1, \ldots, \lambda_n)$. By substituting in (4) $p = p_j$, $j = 1, 2$, denoting the obtained polynomials q_i respectively by $q_i^{(1)}$ and $q_i^{(2)}$ and subtracting them, we obtain

$$e = C^{-1} \sum_{i=1}^{n} (x_i - \lambda_i e)(q_i^{(1)}(x_1, \ldots, x_n) - q_i^{(2)}(x_1, \ldots, x_n)), \tag{7}$$

where $C = p_2(\lambda_1, \ldots, \lambda_n) - p_1(\lambda_1, \ldots, \lambda_n) \neq 0$. Relation (7) shows that $(\lambda_1, \ldots, \lambda_n)$ does not belong to the joint spectrum $\sigma(x_1, \ldots, x_n)$, and so, by formula (1), it does not belong to $\sigma^{(t)}(x_1, \ldots, x_n)$, which gives the desired contradiction. Consequently, formula (5) defines on A_0 a linear functional which is clearly multiplicative. We shall show that it is continuous. If not, then its kernel $f_{\tilde{\lambda}}^{-1}(0)$ is dense in A_0. Since $f_{\tilde{\lambda}}(e) = 1$, the formulas (5) and (4) imply that this kernel coincides with the ideal I_0 defined after the formula (4) and shown to be not dense in A_0. Thus, the considered character $f_{\tilde{\lambda}}$ is continuous on A_0, which is dense in A, so that it extends by the continuity to a continuous character (denoted by the same symbol) on A. Clearly, the character $f_{\tilde{\lambda}}$ satisfies (2). The conclusion follows. ∎

PROPOSITION 2 *Let A be a semitopological algebra and let f be its continuous character. Then for any n-tuple $x_1, \ldots, x_n$ of elements of A the n-tuple $f(x_1), \ldots, f(x_n)$ is in the topological joint spectrum $\sigma^{(t)}(x_1, \ldots, x_n)$.*

The proof follows immediately from the fact that the ideal

$$id(x_1 - f(x_1)e, \ldots, x_n - f(x_n)e)$$

is contained in the kernel of f and so it cannot be dense in A.

COROLLARY 1 *A finitely generated semitopological unital algebra has a non-void set $\mathfrak{M}(A)$ of continuous characters if and only if the topologial joint spectrum of its set of generators is non-empty.*

REMARK 1 Theorem 1 implies that if $x_1, \ldots, x_n$ are generators of A, then

$$\sigma^{(t)}(x_1, \ldots, x_n) = \{(f(x_1), \ldots, f(x_n)) \in \mathbf{K}^n : f \in \mathfrak{M}(A)\}.$$

Such a formula, however, fails to be true if we replace generators by arbitrary elements in A. For instance, take the Cartesian product $A = L^\omega[0,1] \times C[0,1]$, where $L^\omega[0,1]$ is the Arens algebra (Arens, 1946, or Żelazko, 1971, Example 10.5), equal to the intersection of all $L^p[0,1]$ spaces with pointwise algebra operations and L_p-norms, $p \geq 1$, and $C[0,1]$ is the Banach (uniform) algebra of all continuous functions on the unit segment. It is known (Żelazko, 1971, 14.2) that L^ω has no characters at all. Each character on A is continuous and is of the form $f([x,y]) = y(t)$, for a fixed t satisfying $0 \leq t \leq 1$ (see Białynicki-Birula, Żelazko, 1957). Taking an element $\tilde{x} = [1,0]$ in A (a projection), we easily observe that $\sigma^{(t)}(\tilde{x}) = \{1,0\}$ while $\{f(\tilde{x}) : f \in \mathfrak{M}(A)\} = \{0\}$, so that the above equality does not hold true.

The following result implies the converse to the Proposition 2.

PROPOSITION 3 *Let A be a commutative unital (real or complex) finitely generated semitopological algebra. Let f be a not necessarily continuous linear functional on A such that for each k-tuple $y_1, \ldots, y_k$ of elements of A there is*

$$(f(y_1), \ldots, f(y_k)) \in \sigma^{(t)}(y_1, \ldots, y_k). \tag{8}$$

Then f is a continuous character.

Proof. Note that the relation (8) implies that $\sigma^{(t)}(y_1, \ldots, y_k)$ is non-void for any $y_1, \ldots, y_k \in A$ and so, by (1), $\sigma(y_1, \ldots, y_k)$ is non-void too. First we show that f is a character on A. Since $f(e) \in \sigma^{(t)}(e) \subset \sigma(e) = \{1\}$, we have $f(e) = 1$. Take now any y in A with $f(y) = 0$, we claim that for all z in A there is $f(yz) = 0$. If not, then for some z we have $f(yz) = \alpha \neq 0$ and so $f(yz - \alpha e) = f(yz) - \alpha = 0$. Therefore, by (8),

$$(f(y), f(yz - \alpha e)) = (0,0) \in \sigma^{(t)}(y, yz - \alpha e) \subset \sigma(y, yz - \alpha e).$$

Consequently, the ideal $I = id(y, yz - \alpha e)$ is proper. Since $y \in I$, we have $yz \in I$. But $yz - \alpha e$ is also in I and $\alpha \neq 0$. So $e \in I$ and I is not proper. The obtained contradiction proves our claim.

Let now y, z be arbitrary elements in A. We have $f(y - f(y)e) = 0$, and so, by the above claim, $f(yz - f(y)z) = 0$. The last equality implies $f(yz) = f(f(y)z) = f(y)f(z)$, thus f is a character. It remains to be shown that the character f is continuous. Let now $x_1, \ldots, x_n$ be generators of A and let choose arbitrarily y in A. Put $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = f(x_i), 1 \leq i \leq n$. By (8) and Theorem 1, there is a uniquely determined continuous character $f_{\tilde{\lambda}}$ on A satisfying (1). We shall be done if we show that $f = f_{\tilde{\lambda}}$. To this end observe that the elements $x_1, \ldots, x_n, y$ also generate A, and, by (8),

$$\tilde{\lambda}' = (\lambda_1, \ldots, \lambda_n, f(y)) \in \sigma^{(t)}(x_1, \ldots, x_n, y),$$

so that, by Theorem 1, there is a continuous character $f_{\tilde{\lambda}'}$ with $f_{\tilde{\lambda}'}(x_i) = \lambda_i, 1 \leq i \leq n$ and $f_{\tilde{\lambda}'}(y) = f(y)$. But the first n conditions already determine a continuous character on A, so that $f_{\tilde{\lambda}'} = f_{\tilde{\lambda}}$. Thus, $f(y) = f_{\tilde{\lambda}}(y)$, and since y was chosen arbitrarily, we have $f_{\tilde{\lambda}} = f$. The conclusion follows. ■

Remark 2 The assumption that A is finitely generated was essential in the proof of Proposition 3 and the proposition may fail to be true for an infinitely generated semitopological algebra. For instance, let Ω be the space of all at most countable ordinals provided with the order topology (the basis of neighbourhoods of a $\beta \in \Omega$ consists of segments $\alpha_1 < \beta \leq \alpha_2$, or $\alpha_1 \leq \beta \leq \alpha_2$ depending upon the fact whether β is a limit ordinal, or not). Consider the algebra $A = C(\Omega)$ provided with seminorms $||x||_\alpha = \max_{\beta \leq \alpha} |x(\beta)|, \alpha \in \Omega$. It is a complete multiplicatively convex algebra (see Michael, 1952, Example 3.7) and so a semitopological algebra. It is known that for each $x \in A$ there is an $\alpha_x \in \Omega$ with $x(\beta) = x(\alpha_x)$ for all $\beta \geq \alpha_x$. Clearly $F(x) = x(\alpha_x)$ is a discontinous character on A (see Michael, 1952, Proposition 12.2 and the following remark). On the other hand, for any finite number of elements $x_1, \ldots x_n \in A$ we have $F(x_i) = x_i(\alpha_0), 1 \leq i \leq n$, where $\alpha_0 = \max\{\alpha_{x_1}, \ldots, \alpha_{x_n}\}$, so that on these elements F agrees with the continuous character $f(x) = x(\alpha_0)$. Thus, by the Proposition 3, $(F(x_1), \ldots, F(x_n)) = (f(x_1), \ldots, f(x_n)) \in \sigma^{(t)}(x_1, \ldots, x_n)$, for each n-tuple $x_1, \ldots, x_n, n = 1, 2 \ldots$, while F is discontinuous.

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