Control and Cybernetics

vol. **36** (2007) No. 4

Systems of variational inequalities related to economic equilibrium

by

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Abstract: In the paper a new approach to the Walrasian general equilibrium model of economy is presented. The classical market clearing condition is replaced by suitably formulated variational inequality. It states that the market clears for a commodity if its equilibrium price is positive; otherwise, there may be an excess supply of the commodity in equilibrium and then its price is zero. Such approach enables establishing new existence results without assumptions which were fundamental for the currently used methods:
(i) Dis-utility functions are not required to be strictly convex and they may attain their minima in the consumption sets (the local nonsatiation of preferences is not required).
(ii) The boundary of the positive orthant is allowed for the price vector in equilibrium. It allows for investigation of certain new problems, e.g. bankruptcy conditions.

Keywords: optimization problem, variational inequalities, duality, competitive equilibria.

1. Introduction

Let $x_j \in \mathbb{R}^n_+$ be minimizers of a finite collection of convex objectives $V_j : \mathbb{R}^n_+ \to \mathbb{R} \cup \{+\infty\}$, $j = 1, \dots, m$. The minimizers are assumed to fulfill unilateral constraints $\langle A_j \pi, x_j \rangle \leq \phi_j(\pi)$ determined by given nonnegative continuous functions $\phi_j(\cdot)$ depending on a vector $\pi \in \mathbb{R}^n_+$. The problem is to find $\pi \in \mathbb{R}^n_+$ and $(x_j) \in (\mathbb{R}^n_+)^m$ which are linked together by a subdifferential relation of the form $\sum_{j=1}^m A_j^T x_j \in \partial \Phi_+(\pi)$, Φ_+ being a convex function.

The main feature of the aforementioned problem is that the feasible set for the unknowns π, x_j, $j = 1,\ldots,m$, is nonconvex and, hence, the standard theory of variational inequalities (see Kinderlehrer and Stampacchia, 1980, Ekeland and Temam, 1976) cannot be applied directly to obtain solutions. The approach presented here does not involve the notion of Pareto optimum or its generalizations (see Pallaschke and Rolewicz, 1997; Luc, 1989; Lee et al., 1998; Hadjisavvas and Schaible, 1998 and the references therein), but, roughly speaking, is based on the analysis of objectives' parametrized constrained minima $\big(x_j(\pi)\big)$ as functions of π. Some ideas from Naniewicz and Panagiotopoulos (1995) concerning the treatment of nonmonotone inequality problems are applied.

The considered problem has been first studied in Naniewicz (2002) under the hypothesis that $\phi_j(\tau) \geq \delta_j$, $\forall\, \tau \in \mathbb{R}^n_+$, $\delta_j > 0$. Now we examine the case in which the functions ϕ_j, $j = 1,\ldots,m$ are positive homogeneous of degree 1 and $\Phi = \sum_{j=1}^{m} \phi_j$. Sufficient conditions for existence of the solutions for the problem will be formulated.

The motivation for this work comes from mathematical economics (see, e.g., von Neumann, 1945-6; Nash, 1950; Arrow and Intrilligator, 1982; Arrow and Debreu, 1954; Aliprantis, Brown, Burkinshaw, 1989; Nagurney, 1999; Nagurney and Siokos, 1997; Panek, 2000, and the references quoted there). Under the aforementioned assumptions on ϕ_j, $j = 1,\ldots,m$, and Φ the problem to be studied is related to the general equilibrium model of an economy in finite dimensional commodity space, in particular to the models of Arrow-Debreu and Arrow-Debreu-McKenzie. There is a large literature on the general equilibrium model as given by both finite and infinite dimensional commodity spaces. For this issue we refer the reader to von Neumann (1945-46), Nash (1950), Arrow and Debreu (1954), Arrow and Intrilligator (1982), Aliprantis, Brown, Burkinshaw (1989), Bulavsky (1994), Chichilnisky and Heal (1993, 1998), Chichilnisky (1993), Nagurney and Siokos (1997), Mas-Colell, Whinston, Green (1995), Mas-Colell (1986), Mas-Colell and Richard (1991), McKenzie (1959), Gale and Mas-Colell (1975), Negishi (1960), Scarf (1973), Eaves (1972), Hirsh and Smale (1979), Smale (1976), Aliprantis, Tourky, Yannelis (2001), Aliprantis, Monteriro, Tourky (2004), Aliprantis, Florenzano, Tourky (2005), and the references therein. The use of the homotopy methods for economic equilibria can be found in Eaves (1972), Hirsh and Smale (1979), Smale (1976) (see also Chichilnisky, 1993 and the references quoted there).

Recall that in the Arrow-Debreu-McKenzie model there are m consumers (indexed by $j \in J = \{1,\ldots,m\}$), n firms (indexed by $i \in I\colon = \{1,\ldots,n\}$), and s goods (indexed by $l \in L\colon = \{1,\ldots,s\}$). In such economy, society's initial endowments and technological possibilities (i.e., the firms) are owned by consumers. The initial endowment of j's consumer is given by $\omega_j \in \mathbb{R}^n_+$. In addition, we suppose that consumer j owns a share κ_{ji} of firm i, where $\sum_{j\in J} \kappa_{ji} = 1$. Denote by $Y_i \subset \mathbb{R}^n$ the production set associated with i's firm.

Recall that allocation $(x_1^\star,\ldots,x_m^\star,y_1^\star,\ldots,y_n^\star)$, $x_j^\star \in \mathbb{R}^n_+$, $j \in J$, $y_i^\star \in \mathbb{R}^n$, $i \in I$, and price vector $p^\star \in \mathbb{R}^n_+$ constitute a competitive (or Walrasian) equilibrium

if the following conditions are satisfied (Mas-Colell, Whinston, Green, 1995):
Profit maximization: For each firm $i \in I$, $y_i^\star$ solves

$$\max_{y_i \in Y_i} \langle p^\star, y_i \rangle; \tag{1}$$

Dis-utility minimization: For each consumer $j \in J$, $x_j^\star$ solves

$$\min\Big\{ V_j(x_j) \colon \langle p^\star, x_j \rangle \le \langle p^\star, \omega_j \rangle + \sum_{i \in I} \kappa_{ij} \langle p^\star, y_i^\star \rangle, \;\; x_j \in \mathbb{R}^n_+ \Big\}; \tag{2}$$

Market clearing:

$$\sum_{j \in J} x_j^\star = \sum_{j \in J} \omega_j + \sum_{i \in I} y_i^\star. \tag{3}$$

In the presented approach we introduce convex, nonnegative valued, positive homogeneous of degree 1 functions $\phi_j(p) \colon = \langle p, \omega_j \rangle + \sum_{i \in I} \kappa_{ij} \sup_{y_i \in Y_i} \langle p, y_i \rangle$ and

$$\Phi(p) \colon = \sum_{j \in J} \phi_j(p) = \langle p, \sum_{j \in J} \omega_j \rangle + \sum_{i \in I} \sup_{y_i \in Y_i} \langle p, y_i \rangle, \quad p \in \mathbb{R}^n_+,$$

and instead of (3) the variational inequality, called the balance condition, will be considered

$$\Big\langle p - p^\star, -\sum_{j \in J} x_j^\star \Big\rangle + \Phi(p) - \Phi(p^\star) \ge 0, \quad \forall\, p \in \mathbb{R}^n_+. \tag{4}$$

It states that the market clears for a commodity if its equilibrium price is positive. Otherwise, there may be an excess supply of the commodity in equilibrium and then its price is zero.

Finally, a more general problem can be stated:

Find $\{x_j^\star\}_{j \in J} \subset \mathbb{R}^{nm}_+$ and $p^\star \in \mathbb{R}^n_+$ such that
Dis-utility minimization: For each consumer $j \in J$, $x_j^\star$ solves

$$\min\Big\{ V_j(x_j) \colon \langle A_j p^\star, x_j \rangle \le \phi_j(p^\star), \;\; x_j \in \mathbb{R}^n_+ \Big\}; \tag{5}$$

Balance condition:

$$\Big\langle p - p^\star, -\sum_{j \in J} A_j^T x_j^\star \Big\rangle + \Phi(p) - \Phi(p^\star) \ge 0, \quad \forall\, p \in \mathbb{R}^n_+, \tag{6}$$

which will be examined from the point of view of the existence issues.

2. Statement of the problem and preliminaries

First, basic notations are presented. Denote by $\mathbb{R}^n$ the Euclidean vector space of all vectors $x = [x_1, \ldots, x_n]$, $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, equipped with the inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\langle \pi, x \rangle = \sum_{i=1}^n x_i p_i, \quad x = [x_1, \ldots, x_n], \, \pi = [p_1, \ldots, p_n] \in \mathbb{R}^n.$$

By $\mathbb{R}^{n \times n}$ we denote all $n \times n$ real valued matrices. Moreover, the following notations will be used:

$$\mathbb{R}_+ = \{\alpha \in \mathbb{R} \colon \alpha \geq 0\},$$
$$\mathbb{R}^n_+ = \{x = [x_1, \ldots, x_n] \in \mathbb{R}^n \colon x_i \geq 0, \ \forall\, i = 1, \ldots, n\},$$
$$\mathbb{R}^{n \times n}_+ = \{A = (A_{ik}) \in \mathbb{R}^{n \times n} \colon A_{ik} \geq 0, \ \forall\, i, k = 1, \ldots, n\},$$
$$\mathbb{R}^n_- = \{x = [x_1, \ldots, x_n] \in \mathbb{R}^n \colon x_i \leq 0, \ \forall\, i = 1, \ldots, n\},$$

Moreover, denote by ind_K the indicator function of a set K i.e.

$$\mathrm{ind}_K(y) = \begin{cases} 0 & \text{if } y \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

Throughout the paper it will be assumed that the functions

$$V_j : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \quad j = 1, \ldots, m, \tag{7}$$

are convex, proper and lower semicontinuous functions and

$$\overline{V}_j := V_j + \mathrm{ind}_{\mathbb{R}^n_+}.$$

Assume that the functions

$$\phi_j : \mathbb{R}^n_+ \to \mathbb{R} \ \text{ with } \ \phi_j(\tau) \geq 0, \quad \forall\, \tau \in \mathbb{R}^n_+, \quad j = 1, \ldots, m, \tag{8}$$

are continuous functions. Moreover, let the matrices $A_j \in \mathbb{R}^{n \times n}_+$ satisfy

$$\mathrm{Ker}\, A_j = \{0\}, \quad j = 1, \ldots, m, \tag{9}$$

where $\mathrm{Ker}\, A_j = \{\tau \in \mathbb{R}^n_+ \colon A_j \tau = 0\}$. Furthermore, let

$$\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \tag{10}$$

be a convex, proper, lower semicontinuous function.

Recall that if H is Hilbert space and $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is a convex function, the subdifferential $\partial\varphi \colon H \to 2^H$ is defined by

$$\partial\varphi(u) = \{w \in H \colon \varphi(v) - \varphi(u) \geq \langle w, v - u \rangle, \ \forall v \in H\},$$

provided that $\varphi(u) < +\infty$ and $\partial\varphi(u) = \emptyset$, otherwise.

We are now in a position to formulate the main problem of the paper.

Problem (P): Find $\pi \in \mathbb{R}^n_+$ and $x_j \in \mathbb{R}^n_+$, $j = 1, \dots, m$, such as to satisfy the conditions:

$$(PM) \qquad V_j(x_j) = \min\left\{V_j(x)\colon \left\langle A_j\pi, x\right\rangle \le \phi_j(\pi) \text{ and } x \in \mathbb{R}^n_+\right\},\ j = 1, \dots, m,$$

$$(PE) \qquad \Big\langle -\sum_{j=1}^{m} A_j^T x_j, \tau - \pi \Big\rangle + \Phi(\tau) - \Phi(\pi) \ge 0, \quad \forall\, \tau \in \mathbb{R}^n_+.$$

If we assume that for any $j = 1, \dots, m$ the matrix A_j is the identity matrix, the function $-V_j$ describes the preferences of j-trader's, the function $\phi_j(\pi) = \langle \pi, \omega_j \rangle$ is a profit of j-trader's and $\Phi = \sum_{j=1}^m \phi_j$, then the problem (P) with $\pi > 0$ (each component of π is positive) is equivalent to finding of an equilibrium in the Arrow-Debreu model of pure exchange. In general, (PE) states that the market clears for a commodity if its equilibrium price is positive. Otherwise, there may be an excess supply of the commodity in equilibrium and then its price will be zero.

Moreover, a solution of the problem (P) can be treated as the problem of a competitive equilibrium of Arrow-Debreu-McKenzie model, when the functions $\phi_j(\pi) = \langle \pi, \omega_j \rangle + \sum_{k=1}^s \theta_{jk} \sup_{y_k \in Y_k} \langle \pi, y_k \rangle$ describe a profit of j-trader's and $\Phi = \sum_{j=1}^m \phi_j$ (see Aliprantis, Brown, Burkinshaw, 1989; Panek, 2000).

The case of $\phi_j(\tau) \ge \delta_j\ \forall\, \tau \in \mathbb{R}^n_+$, $\delta_j > 0$ is the starting point to an examination of the case in which the functions ϕ_j, $j = 1, \dots, m$ are positive homogeneous of degree 1 and $\Phi = \sum_{j=1}^m \phi_j$. Now we remind the main results for such a case.

The sufficient condition for (PM) to have solutions reads as follows.

Theorem 1 *(Theorem 1, p. 149, Naniewicz, 2002) Assume that for any $j = 1, \dots, m$, the hypotheses below hold:*

(H_1) $0 \in \mathrm{cl}(\mathrm{Dom}\, \partial \overline{V}_j)$, $\quad (\mathbb{R}^n_- \setminus \{0\}) \cap B_{\mathbb{R}^n}(0, r_j) \subset \mathrm{Int}\, \mathrm{Dom}\, \overline{V}_j^\star$ *for some* $r_j > 0$*;*

(H_2) $\{x \in \mathbb{R}^n_+ : \{\langle x^\star, x\rangle : x^\star \in \partial \overline{V}_j(x)\} \cap \mathbb{R}_- \ne \emptyset\} \subset B_{\mathbb{R}^n}(0, M_j)$ *for some* $M_j > 0$*;*

(H_4) $\phi_j(\tau) \ge \delta_j \ \ \forall \tau \in \mathbb{R}^n_+$ *for some* $\delta_j > 0$.

Then for any $\pi \in \mathbb{R}^n_+ \setminus \{0\}$ *the optimization problem: Find* $x_j \in \mathbb{R}^n_+$ *such that*

$$V_j(x_j) = \min\{V_j(y)\colon \forall y \in \mathbb{R}^n_+ \text{ with } \langle A_j\pi, y\rangle \le \phi_j(\pi)\} \tag{11}$$

has at least one solution. Moreover, there exists $\alpha_j \in \Lambda_j(\pi)$*,* $\Lambda_j(\pi)$ *being the set of all solutions of variational inequality*

$$\langle A_j\pi, -\partial\overline{V}_j^\star(-\alpha_j A_j \pi)\rangle (t - \alpha_j) + \phi_j(\pi)(t - \alpha_j) \ge 0, \quad \forall\, t \ge 0, \tag{12}$$

with the property that

$$x_j \in \partial \overline{V}_j^\star\big(-\alpha_j A_j \pi\big). \tag{13}$$

Additionally, $\Lambda_j : \mathbb{R}^n_+ \setminus \{0\} \to 2^{\mathbb{R}_+}$ when extended to $\mathbb{R}^n_+$ by setting $\Lambda_j(0) := \{0\}$ has nonempty, closed, convex and bounded values and it is an upper semicontinuous mapping from $\mathbb{R}^n_+$ into $2^{\mathbb{R}_+}$.

Remark 1 From the proof of the Theorem 1 we get that

$$\partial \overline{V}^\star_j(0) \neq \emptyset, \quad j = 1, \ldots, m.$$

Now the problem (PE) can be considered. Taking into account (13) we introduce a multivalued mapping $\mathcal{R} : \mathbb{R}^n_+ \to 2^{\mathbb{R}^n}$ by setting

$$\mathcal{R}(\pi) := -\sum_{j=1}^{m} A_j^T \partial \overline{V}^\star_j(-\Lambda_j(\pi) A_j \pi), \quad \pi \in \mathbb{R}^n_+, \tag{14}$$

(where $y \in \mathcal{R}(\pi)$ if and only if there exist $\alpha_j \in \Lambda_j(\pi)$, $x_j \in \partial \overline{V}^\star_j(-\alpha_j A_j \pi)$ for any $j = 1, \ldots, m$ such that $y = -\sum_{j=1}^m A_j^T x_j$).
It is easily seen that (PE) can be equivalently formulated as follows: Find $\pi \in \mathbb{R}^n_+$ and $X \in \mathcal{R}(\pi)$ such that

$$\langle X, \tau - \pi \rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \tau \in \mathbb{R}^n_+. \tag{15}$$

As far as $\mathcal{R}$ is concerned we have the result.

Proposition 1 *(Proposition 4, p.150, Naniewicz, 2002) Under the hypotheses of Theorem 1, $\mathcal{R}$ given by (14) is a multivalued, upper semicontinuous mapping from $\mathbb{R}^n_+$ into $2^{\mathbb{R}^n}$ with nonempty, convex, closed and bounded values.*

This allows the formulation of the following result.

Theorem 2 *(Theorem 2, p.151, Naniewicz, 2002) Suppose that for any $j = 1, \ldots, m$ the hypotheses below hold:*
(H_1) $0 \in \mathrm{cl}(\mathrm{Dom}\, \partial \overline{V}_j)$, $\quad (\mathbb{R}^n_- \setminus \{0\}) \cap B_{\mathbb{R}^n}(0, r_j) \subset \mathrm{Int}\,\mathrm{Dom}\, \overline{V}^\star_j$ *for some* $r_j > 0$*;*
(H_2) $\{x \in \mathbb{R}^n_+ : \{\langle x^\star, x\rangle : x^\star \in \partial \overline{V}_j(x)\} \cap \mathbb{R}_- \neq \emptyset\} \subset B_{\mathbb{R}^n}(0, M_j)$ *for some* $M_j > 0$*;*
(H_4) $\phi_j(\tau) \geq \delta_j \ \forall \tau \in \mathbb{R}^n_+$ *for some* $\delta_j > 0$*;*
(H_5^0) $\{\tau \in \mathbb{R}^n_+ : \Phi(\tau) \leq \sum_{j=1}^m \phi_j(\tau) + \Phi(0)\} \subset B_{\mathbb{R}^n}(0, M)$ *for some* $M > 0$*;*
(H_6) $\partial \Phi_+(0) \neq \emptyset$, *where* $\Phi_+ := \Phi + \mathrm{ind}_{\mathbb{R}^n_+}$.
Then the problem: Find $\pi \in \mathbb{R}^n_+$ and $X \in \mathcal{R}(\pi)$ such as to satisfy the variational inequality

$$\langle X, \tau - \pi \rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \forall \tau \in \mathbb{R}^n_+, \tag{16}$$

has at least one solution.
Therefore, there exists $\pi \in \mathbb{R}^n_+$, $x_j \in \mathbb{R}^n_+$, $j = 1, \ldots, m$ such that

$$(PM) \quad V_j(x_j) = \min \{V_j(x) \colon \langle A_j \pi, x \rangle \leq \phi_j(\pi) \text{ and } x \in \mathbb{R}^n_+\}, \ j = 1, \ldots, m,$$

$$(PE) \qquad \Big\langle -\sum_{j=1}^{m} A_j^T x_j, \tau - \pi \Big\rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \forall\, \tau \in \mathbb{R}_+^n.$$

Equivalently, there exists $\big(\pi, (x_j), (\alpha_j)\big) \in \mathbb{R}_+^n \times (\mathbb{R}_+^n)^m \times (\mathbb{R}_+)^m$ *such that*

$$\left.\begin{array}{c} -\alpha_j A_j \pi \in \partial \overline{V}_j(x_j), \\ \big\langle A_j \pi, x_j \big\rangle - \phi_j(\pi) \in \partial \operatorname{ind}_{\geq 0}(\alpha_j) \\ \Phi(\tau) - \Phi(\pi) \geq \Big\langle \tau - \pi, \sum_{j=1}^m A_j^T x_j \Big\rangle, \quad \forall\, \tau \in \mathbb{R}_+^n \end{array}\right\}. \tag{17}$$

3. Approximation result

The multivalued mapping $\mathcal{R}$ defined in (14) is not upper semicontinuous in 0 and this mapping does not have bounded values for the functions ϕ_j positive homogeneous of degree 1. In order to get the solution of the problem (P) in this case we construct the appropriate approximation. We proceed in two steps. First, we shift the argument π in (17) to have boundary points of $\mathbb{R}_+^n$, $\partial\mathbb{R}_+^n := \mathbb{R}_+^n \setminus \operatorname{Int} \mathbb{R}_+^n$, out of the domain. This will allow for taking advantage of Theorem 2. Second, we extend the dimension and construct the family of extended dimensional problems parametrized by a small parameter $\varepsilon > 0$. The examination of the behaviour of the corresponding solutions when $\varepsilon \to 0$ enables establishing of new existence results for the problem under consideration.

Step 1. In order to take advantage of Theorem 2 we choose $\pi_0 \in \mathbb{R}_+^n \setminus \{0\}$ in the way described below and for any $\varepsilon > 0$ consider the problem with shifted argument, namely

$$\left.\begin{array}{c} -\alpha_j A_j(\pi + \varepsilon\pi_0) \in \partial \overline{V}_j(x_j), \\ \big\langle A_j(\pi + \varepsilon\pi_0), x_j \big\rangle - \phi_j(\pi + \varepsilon\pi_0) \in \partial \operatorname{ind}_{\geq 0}(\alpha_j) \\ \Phi(\tau) - \Phi(\pi) \geq \big\langle \tau - \pi, \sum_{j=1}^m A_j^T x_j \big\rangle, \quad \forall\, \tau \in \mathbb{R}_+^n. \end{array}\right\} \tag{18}$$

This problem is a modification of (17) by shifting $\pi \mapsto \pi + \varepsilon\pi_0$, $\pi \in \mathbb{R}_+^n$, and replacing $\Phi(\cdot)$ by $\Phi(\cdot - \varepsilon\pi_0)$. An element π_0 should be chosen in such a way that for each $j = 1, \ldots, m$, and any $\varepsilon > 0$ we can find $\delta_{j\varepsilon} > 0$ to fulfill the estimates

$$\phi_j(\tau + \varepsilon\pi_0) \geq \delta_{j\varepsilon}, \quad \forall \tau \in \mathbb{R}_+^n. \tag{19}$$

Then, (H_4) holds and the application of Theorem 2 is allowed. By reformulating (H_5^0) we are led to the following result.

THEOREM 3 *Let us assume that there exists* $\pi_0 \in \mathbb{R}_+^n \setminus \{0\}$ *such that* (19) *is satisfied for any* $\varepsilon > 0$. *Suppose that for each* $j = 1, \ldots, m$ *the hypotheses below hold:*

(H_1) $0 \in \operatorname{cl}(\operatorname{Dom} \partial \overline{V}_j)$, $\quad (\mathbb{R}_-^n \setminus \{0\}) \cap B_{\mathbb{R}^n}(0, r_j) \subset \operatorname{Int} \operatorname{Dom} \overline{V}_j^{\star}$ *for some* $r_j > 0$*;*

(H_2) $\{x \in \mathbb{R}^n_+ : \{\langle x^\star, x\rangle : x^\star \in \partial \overline{V}_j(x)\} \cap \mathbb{R}_- \neq \emptyset\} \subset B_{\mathbb{R}^n}(0, M_j)$ *for some* $M_j > 0$;

($\hat{H}^\varepsilon_5$) $\{\tau \in \mathbb{R}^n_+ : \Phi(\tau) \leq \sum_{j=1}^m \phi_j(\tau + \varepsilon\pi_0) + \Phi(0)\} \subset B_{\mathbb{R}^n}(0, M_\varepsilon)$ *for some* $M_\varepsilon > 0$;

(H_6) $\partial\Phi_+(0) \neq \emptyset$, *where* $\Phi_+ := \Phi + \mathrm{ind}_{\mathbb{R}^n_+}$.

Then there exists $(\pi^\varepsilon, (x^\varepsilon_j), (\alpha^\varepsilon_j)) \in \mathbb{R}^n_+ \times (\mathbb{R}^n_+)^m \times (\mathbb{R}_+)^m$ *fulfilling* (18).

Step 2. The idea is to introduce an additional dimension for the problem under consideration. For suitable modified data we shall formulate the approximation problem of (18)-type that makes possible to take advantage of Theorem 3.

Let us consider the data system $(W^\varepsilon_j, \widetilde{\phi}^\varepsilon_j, \widetilde{\Phi})$, where

(i) $W^\varepsilon_j : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$:

$$W^\varepsilon_j(x, z) := V_j(x) + \varphi^\varepsilon_j(z), \quad x \in \mathbb{R}^n,\ z \in \mathbb{R}_+,\ \varepsilon > 0, \tag{20}$$

with $\varphi^\varepsilon_j(z) := -\varepsilon s_j z^{1-\varepsilon} + \mathrm{ind}_{\leq s_j}(z)$, $s_j > 0$, $\sum_{j=1}^m s_j = s < 1$, $\overline{W}^\varepsilon_j := W^\varepsilon_j + \mathrm{ind}_{\mathbb{R}^n_+ \times \mathbb{R}_+}$;

(ii) $\widetilde{\phi}^\varepsilon_j : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}_+$:

$$\widetilde{\phi}^\varepsilon_j(\pi, q) := \phi^\varepsilon_j(\pi) + q s_j, \quad \pi \in \mathbb{R}^n_+,\ q \in \mathbb{R}_+, \tag{21}$$

with $\phi^\varepsilon_j(\tau) := \min\{\phi_j(\tau), \frac{1}{\varepsilon}\}$;

(iii) $\widetilde{\Phi} : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$:

$$\widetilde{\Phi}(\pi, q) = \Phi(\pi) + q, \quad \pi \in \mathbb{R}^n_+,\ q \in \mathbb{R}_+. \tag{22}$$

The problem is to find $(\pi^\varepsilon, q^\varepsilon) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, $(x^\varepsilon_j, z^\varepsilon_j) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ and $\alpha^\varepsilon_j \in \mathbb{R}_+$ such that

$$\left.\begin{aligned} &-\alpha^\varepsilon_j(A_j\pi^\varepsilon, q^\varepsilon + \varepsilon q_0) \in \partial\overline{W}^\varepsilon_j(x^\varepsilon_j, z^\varepsilon_j),\\ &\big\langle (A_j\pi^\varepsilon, q^\varepsilon + \varepsilon q_0), (x^\varepsilon_j, z^\varepsilon_j)\big\rangle - \widetilde{\phi}^\varepsilon_j(\pi^\varepsilon, q^\varepsilon + \varepsilon q_0) \in \partial\,\mathrm{ind}_{\geq 0}(\alpha^\varepsilon_j)\\ &\widetilde{\Phi}(\tau, q) - \widetilde{\Phi}(\pi^\varepsilon, q^\varepsilon) \geq \big\langle \tau - \pi^\varepsilon, \textstyle\sum_{j=1}^m A^T_j x^\varepsilon_j\big\rangle + \big(\textstyle\sum_{j=1}^m z^\varepsilon_j\big)(q - q^\varepsilon),\\ &\hspace{6cm}\forall\,(\tau, q) \in \mathbb{R}^n_+ \times \mathbb{R}_+. \end{aligned}\right\} \tag{23}$$

In the foregoing system the "π"-argument has been shifted by an element $\pi_0 = (0, q_0) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \setminus \{(0,0)\}$ to deal with the admissible set of the form $\mathbb{R}^n_+ \times \mathbb{R}_+ + \varepsilon(0, q_0)$ on which one can more clearly control the boundedness of the corresponding α^ε_j's. Moreover, we have replaced ϕ_j by ϕ^ε_j for better handling the boundedness of the corresponding π^ε's.

Now we are ready to state the hypotheses under which the existence of solutions of(23) follows. From now on, for convenience we set $q_0 = s_j = \frac{s}{m}$.

First, let us notice that from

$$\widetilde{\phi}_j^\varepsilon(\tau, q+\varepsilon\tfrac{s}{m}) \geq \varepsilon\big(\tfrac{s}{m}\big)^2 > 0, \quad \forall\,(\tau,q)\in\mathbb{R}^n_+\times\mathbb{R}_+,$$

it follows that (19) holds with $\delta_{j\varepsilon}=\varepsilon\big(\frac{s}{m}\big)^2>0$.
Assume that

(H_2^ε) $\big\{x\in\mathbb{R}^n_+ : \{\langle x^\star, x\rangle : x^\star\in\partial\overline{V}_j(x)\}\cap(\mathbb{R}_-+\varepsilon)\neq\emptyset\big\}\subset B_{\mathbb{R}^n}(0,M_{j\varepsilon})$ for some $M_{j\varepsilon}>0$;

(H_5^ε) $\big\{\tau\in\mathbb{R}^n_+ : \Phi(\tau)\leq\sum_{j=1}^m\phi_j^\varepsilon(\tau)+\Phi(0)+\varepsilon\frac{s}{m}\big\}\subset B_{\mathbb{R}^n}(0,M_\varepsilon)$ for some $M_\varepsilon>0$.

The hypothesis (H_5^ε) together with the assumption that $s<1$ in (i), allows easily to ensure that

$$\big\{(\tau,q)\in\mathbb{R}^n_+\times\mathbb{R}_+ : \widetilde{\Phi}(\tau,q)\leq\textstyle\sum_{j=1}^m\widetilde{\phi_j^\varepsilon}(\tau,q+\varepsilon\frac{s}{m})+\widetilde{\Phi}(0,0)\big\}\text{ is bounded.} \quad (24)$$

Thus $(\hat{H}_5^\varepsilon)$ holds. Moreover, from

$$\big\langle\partial\overline{W}_j^\varepsilon(x,z),(x,z)\big\rangle\leq 0, \quad (x,z)\in\mathbb{R}^n_+\times\mathbb{R}_+,$$

it easily follows that

$$\big\langle\partial\overline{V}_j(x),x\big\rangle\leq\varepsilon, \quad\text{and}\quad z\in(0,\tfrac{s}{m}].$$

Consequently, (H_2^ε) yields $|x|\leq M_{j\varepsilon}$ which implies (H_2). Finally, we have checked that for the data system $(W_j^\varepsilon,\widetilde{\phi}_j^\varepsilon,\widetilde{\Phi})$ all the requirements of Theorem 3 are fulfilled. Therefore one allows to conclude the existence of a system $\big((\pi^\varepsilon,q^\varepsilon),(x_j^\varepsilon,z_j^\varepsilon),(\alpha_j^\varepsilon)\big)\in(\mathbb{R}^n_+\times\mathbb{R}_+)\times\big(\mathbb{R}^n_+\times\mathbb{R}_+\big)^m\times(\mathbb{R}_+)^m$ such that (23) is fulfilled. It can be written equivalently as

$$\left.\begin{array}{c}
-\alpha_j^\varepsilon(A_j\pi^\varepsilon,q^\varepsilon+\varepsilon\frac{s}{m})\in\big(\partial\overline{V}_j(x_j^\varepsilon),\partial\overline{\varphi}_j^\varepsilon(z_j^\varepsilon)\big)\\
\big\langle A_j\pi^\varepsilon,x_j^\varepsilon\big\rangle+(q^\varepsilon+\varepsilon\frac{s}{m})z_j^\varepsilon-\phi_j^\varepsilon(\pi^\varepsilon)-(q^\varepsilon+\varepsilon\frac{s}{m})\frac{s}{m}\in\partial\,\mathrm{ind}_{\geq 0}(\alpha_j^\varepsilon)\\
\Phi(\tau)-\Phi(\pi^\varepsilon)\geq\big\langle\tau-\pi^\varepsilon,\sum_{j=1}^m A_j^T x_j^\varepsilon\big\rangle, \quad\forall\,\tau\in\mathbb{R}^n_+,\\
q-q^\varepsilon\geq\big(\sum_{j=1}^m z_j^\varepsilon\big)(q-q^\varepsilon), \quad\forall\,q\in\mathbb{R}_+.
\end{array}\right\} \quad (25)$$

We summarize the obtained result as follows.

Proposition 2 *Suppose that for any $\varepsilon>0$ the hypotheses (H_1), (H_2^ε), (H_5^ε) and (H_6) are assumed to hold. Then there exists a system $\big(\pi^\varepsilon,(x_j^\varepsilon),(\alpha_j^\varepsilon)\big)\in\mathbb{R}^n_+\times(\mathbb{R}^n_+)^m\times(\mathbb{R}_+)^m$ such that*

$$-\alpha_j^\varepsilon A_j\pi^\varepsilon\in\partial\overline{V}_j(x_j^\varepsilon), \quad (26)$$

$$z_j^\varepsilon=(1-\varepsilon)^{\frac{1}{\varepsilon}}\big(\tfrac{1}{\alpha_j^\varepsilon+r_j^\varepsilon}\big)^{\frac{1}{\varepsilon}}, \quad r_j^\varepsilon\geq 0, \quad \tfrac{s}{m}\geq z_j^\varepsilon, \quad (\tfrac{s}{m}-z_j^\varepsilon)r_j^\varepsilon=0 \quad (27)$$

$$\big\langle A_j\pi^\varepsilon,x_j^\varepsilon\big\rangle=\phi_j^\varepsilon(\pi^\varepsilon)+\varepsilon\tfrac{s}{m}(\tfrac{s}{m}-z_j^\varepsilon), \quad (\textit{if } \alpha_j^\varepsilon>0), \quad (28)$$

$$\big\langle A_j\pi^\varepsilon,x_j^\varepsilon\big\rangle\leq\phi_j^\varepsilon(\pi^\varepsilon), \quad (\textit{if } \alpha_j^\varepsilon=0), \quad (29)$$

$$\Phi(\tau)-\Phi(\pi^\varepsilon)\geq\big\langle\tau-\pi^\varepsilon,\textstyle\sum_{j=1}^m A_j^T x_j^\varepsilon\big\rangle, \quad\forall\,\tau\in\mathbb{R}^n_+. \quad (30)$$

Proof. Let us notice that $0 < z_j^\varepsilon \leq \frac{s}{m}$, $j = 1, \ldots, m$, and $\sum_{j=1}^m z_j^\varepsilon \leq \sum_{j=1}^m \frac{s}{m} = s < 1$. In view of $(25)_4$ this means that $q^\varepsilon = 0$. Therefore we can express (25) as (26)-(30). ■

REMARK 2 From the proof of the Proposition 2 we get that

$$\partial \overline{V}_j^\star(0) \neq \emptyset, \quad j = 1, \ldots, m.$$

4. Positive homogeneity of degree one

We shall investigate the problem (P) under the hypotheses

(H_4^3) $\phi_j(t\tau) = t\phi_j(\tau)$ $\forall \tau \in \mathbb{R}_+^n$, $\forall t > 0$;
(H_6^1) $\Phi = \sum_{j=1}^m \phi_j$ is convex and $\Phi(\tau) \geq \gamma |\tau|$ $\forall \tau \in \mathbb{R}_+^n$, $\gamma > 0$,

involved, for instance, in the models of Arrow-Debreu or Arrow-Debreu-McKenzie.

Before the formulation of the next Theorem, recall that for a convex set K and $x \in K$, $\partial \operatorname{ind}_K(x)$ is called the normal cone to K at x and is denoted by $N_K(x) := \partial \operatorname{ind}_K(x)$ (see Aubin, 1993). Moreover, recall that for a convex function $f : X \to \mathbb{R} \cup \{+\infty\}$, X being a Banach space, the asymptotic generalized gradient of f at x, denoted $\partial^\infty f(x)$, which is defined

$$\partial^\infty f(x) := \left\{x^\star \in X^\star : (x^\star, 0) \in N_{\operatorname{epi} f}(x, f(x))\right\},$$

where $\operatorname{epi} f$ denotes the epigraph of f (see Clarke, 1983; Rockafellar and Wets, 1998).

THEOREM 4 *Suppose that for any $j = 1, \ldots, m$ the following hypotheses hold:*
(H_1) $0 \in \operatorname{cl}(\operatorname{Dom} \partial \overline{V}_j)$, $(\mathbb{R}_-^n \setminus \{0\}) \cap B_{\mathbb{R}^n}(0, r_j) \subset \operatorname{Int} \operatorname{Dom} \overline{V}_j^\star$ *for some* $r_j > 0$;
(H_2^1) $\left\{x \in \mathbb{R}_+^n : \{\langle x^\star, x\rangle : x^\star \in \partial \overline{V}_j(x)\} \cap (\mathbb{R}_- + \varepsilon_0) \neq \emptyset\right\} \subset B_{\mathbb{R}^n}(0, M_j)$, *for some* $M_j > 0$, $\varepsilon_0 > 0$;
(H_4^3) $\phi_j(t\tau) = t\phi_j(\tau)$ $\forall \tau \in \mathbb{R}_+^n$, $\forall t > 0$;
(H_6^1) $\Phi = \sum_{j=1}^m \phi_j$ *is convex and* $\Phi(\tau) \geq \gamma |\tau|$ $\forall \tau \in \mathbb{R}_+^n$, *for some* $\gamma > 0$;
(H_0) $\partial \overline{V}_j^\star(0)$ *is compact;*
(H_9) $\langle A_j \tau, x_j\rangle > \phi_j(\tau)$ *for any* $\tau \in \mathbb{R}_+^n \setminus \{0\}$ *and* $x_j \in \partial \overline{V}_j^\star(0)$.
Moreover, for any $j = 1, \ldots, m$ assume that one of the conditions holds:
(H_8^0) $\operatorname{Dom} \overline{V}_j$ *is closed*
or
(H_8^1) *if* $x_j^k \to x_j$, $p^k \to p$, $|p^k| = 1$, *as* $k \to +\infty$, $\alpha_j^k > 0$ *such that* $-\alpha_j^k A_j p^k \in \partial \overline{V}_j(x_j^k)$ *and* $\phi_j(p) = 0$, *then* $\liminf_{k\to\infty} \overline{V}_j^\star(-\alpha_j^k A_j p^k) > -\infty$.

Then there exists a system $\big(\pi,(x_j),(\alpha_j)\big)$ *with* $\pi\in\mathbb{R}^n_+\setminus\{0\}$, $x_j\in\mathbb{R}^n_+$ *and* $\alpha_j\in\mathbb{R}_+\cup\{+\infty\}$ *for* $j=1,\ldots,m$, *such that*

$$\left.\begin{array}{c}-\alpha_jA_j\pi\in\partial\overline{V}_j(x_j),\ \big\langle A_j\pi,x_j\big\rangle-\phi_j(\pi)\in\partial\,\mathrm{ind}_{\geq 0}(\alpha_j),\ \text{if }\alpha_j\in\mathbb{R}_+,\\ -A_j\pi\in\partial^\infty\overline{V}_j(x_j),\ \big\langle A_j\pi,x_j\big\rangle=\phi_j(\pi)=0,\ \text{if }\alpha_j=+\infty,\\ \Phi(\tau)-\Phi(\pi)\geq\big\langle\tau-\pi,\sum_{j=1}^mA_j^Tx_j\big\rangle,\quad\forall\,\tau\in\mathbb{R}^n_+,\end{array}\right\}\ (\widetilde{P})$$

Proof. The proof will be divided into two steps.

Step 1. We assume that

$$\min\{\phi_j(\tau):\tau\in\mathbb{R}^n_+,|\tau|=1\}=\gamma_j,\quad\gamma_j>0,\quad j=1,\ldots,m.\tag{31}$$

Let us begin with checking the validity of the hypotheses of Proposition 2. Notice that by (H_4^3) and (H_6^1) the hypothesis (H_5^ε) holds. The remaining ones are satisfied immediately. Therefore one can assume the existence of $\big(\pi^\varepsilon,(x_j^\varepsilon),(\alpha_j^\varepsilon)\big)\in\mathbb{R}^n_+\times(\mathbb{R}^n_+)^m\times(\mathbb{R}_+)^m$, $0<\varepsilon\leq\varepsilon_0$, fulfilling (26)-(30).

From (H_9) it follows that $\pi^\varepsilon\neq 0$. Moreover, by the hypothesis (H_9) combined with (29) and the fact that $\phi_j^\varepsilon(\pi^\varepsilon)\leq\phi_j(\pi^\varepsilon)$ it follows that $\alpha_j^\varepsilon>0$ for each $j=1,\ldots,m$. Thus from (28) we get

$$\big\langle A_j\pi^\varepsilon,x_j^\varepsilon\big\rangle=\phi_j^\varepsilon(\pi^\varepsilon)+\varepsilon\tfrac{s}{m}(\tfrac{s}{m}-z_j^\varepsilon),\quad j=1,\ldots,m.\tag{32}$$

Hence

$$\textstyle\sum_{j=1}^m\big\langle A_j\pi^\varepsilon,x_j^\varepsilon\big\rangle=\sum_{j=1}^m\phi_j^\varepsilon(\pi^\varepsilon)+\varepsilon\frac{s}{m}\big(s-\sum_{j=1}^mz_j^\varepsilon\big),\quad s-\sum_{j=1}^mz_j^\varepsilon\geq 0.\tag{33}$$

By means of (30) it follows that

$$\textstyle\Phi(\pi^\varepsilon)=\big\langle\pi^\varepsilon,\sum_{j=1}^mA_j^Tx_j^\varepsilon\big\rangle,\quad\sum_{j=1}^mA_j^Tx_j^\varepsilon\in W,\tag{34}$$

because Φ is positively homogeneous of degree 1. Therefore

$$\textstyle\sum_{j=1}^m\phi_j(\pi^\varepsilon)=\Phi(\pi^\varepsilon)=\sum_{j=1}^m\big\langle A_j\pi^\varepsilon,x_j^\varepsilon\big\rangle=\sum_{j=1}^m\phi_j^\varepsilon(\pi^\varepsilon)+\varepsilon\frac{s}{m}\big(s-\sum_{j=1}^mz_j^\varepsilon\big).\tag{35}$$

Now let us suppose that $\limsup_{\varepsilon\to 0}|\pi^\varepsilon|<+\infty$. Then, for sufficiently small $\varepsilon>0$, $\phi_j^\varepsilon(\pi^\varepsilon)=\phi_j(\pi^\varepsilon)$, $j=1,\ldots,m$ which, by (35), yields

$$\textstyle s-\sum_{j=1}^mz_j^\varepsilon=0.$$

But such an equality can happen only when $\frac{s}{m}=z_j^\varepsilon$ for each $j=1,\ldots,m$. This means that $\big(\pi^\varepsilon,(x_j^\varepsilon),(\alpha_j^\varepsilon)\big)$ is a solution of $(\widetilde{P})$ whenever $\varepsilon>0$ is small enough.

Now, suppose that $\limsup_{\varepsilon\to 0}|\pi^\varepsilon|=+\infty$. From (35) we get

$$\lim_{\varepsilon\to 0}\big(\phi_j(\pi^\varepsilon)-\phi_j^\varepsilon(\pi^\varepsilon)\big)=0,\ j=1,\ldots,m.\tag{36}$$

First, we show that $\{\alpha_j^\varepsilon |\pi^\varepsilon|\}_{\varepsilon\le\varepsilon_0}$ is bounded. Suppose, on the contrary, that $\alpha_j^\varepsilon |\pi^\varepsilon| \to +\infty$, as $\varepsilon \to 0$ (by choosing a subsequence, if necessary). Taking into account $-\alpha_j^\varepsilon A_j \pi^\varepsilon \in \partial \overline{V}_j(x_j^\varepsilon)$ we get

$$\left\langle -\alpha_j^\varepsilon A_j \pi^\varepsilon, x_j^\varepsilon \right\rangle = V_j(x_j^\varepsilon) + \overline{V}_j^\star(-\alpha_j^\varepsilon A_j \pi^\varepsilon).$$

From $\partial \overline{V}_j^\star(0) \neq \emptyset$ we obtain that there exists $c_j \in \mathbb{R}$ that for any $y \in \operatorname{Dom} \overline{V}_j$, $V_j(y) \geq -c_j$. Using (32) and the definition of the Fenchel conjugate function we get the following estimate

$$\alpha_j^\varepsilon \phi_j(\pi^\varepsilon) + \alpha_j^\varepsilon(\phi_j^\varepsilon(\pi^\varepsilon) - \phi_j(\pi^\varepsilon)) \leq c_j + V_j(y) + \langle \alpha_j^\varepsilon A_j \pi^\varepsilon, y\rangle, \quad \forall y \in \operatorname{Dom} \overline{V}_j. \tag{37}$$

Dividing (37) by ${\alpha_j}^\varepsilon |\pi^\varepsilon|$, letting $\varepsilon \to 0$ and using the fact that $\min\{\phi_j(\tau) : \tau \in \mathbb{R}_+^n,\ |\tau| = 1\} = \gamma_j$, $\gamma_j > 0$ we get

$$0 < \gamma_j \leq |A_j|\,|y|\,, \quad \forall y \in \operatorname{Dom} \overline{V}_j,$$

which contradicts the assumption $0 \in \operatorname{cl}(\operatorname{Dom} \partial \overline{V}_j)$.
Therefore in this case it can be assumed that $p^\varepsilon := \frac{\pi^\varepsilon}{|\pi^\varepsilon|} \to p$, $x_j^\varepsilon \to x_j$, $\alpha_j^\varepsilon |\pi^\varepsilon| \to \widetilde{\alpha}_j$, as $\varepsilon \to 0$, for some $p \in \mathbb{R}_+^n$, $|p| = 1$, $x_j \in \mathbb{R}_+^n$, $\widetilde{\alpha}_j \in \mathbb{R}_+$ (by passing to subsequence, if necessary). Using positive homogeneity of degree one the functions ϕ_j, $j = 1, \ldots, m$, Φ, the conditions (26), (28), (30) obtain the following form

$$\left.\begin{array}{c} -\alpha_j^\varepsilon |\pi^\varepsilon|\, A_j p^\varepsilon \in \partial \overline{V}_j(x_j^\varepsilon), \\ \left\langle A_j p^\varepsilon, x_j^\varepsilon \right\rangle - \phi_j(p^\varepsilon) = \frac{1}{|\pi^\varepsilon|}\left(\phi_j^\varepsilon(\pi^\varepsilon) - \phi_j(\pi^\varepsilon)\right) + \frac{\varepsilon}{|\pi^\varepsilon|}\frac{s}{m}\left(\frac{s}{m} - z_j^\varepsilon\right) \\ \Phi(\tau) - \Phi(p^\varepsilon) \geq \left\langle \tau - p^\varepsilon, \sum_{j=1}^m A_j^T x_j^\varepsilon \right\rangle, \quad \forall\, \tau \in \mathbb{R}_+^n, \end{array}\right\}$$

(substituting $\frac{\tau}{|\pi^\varepsilon|}$ into τ). By letting $\varepsilon \to 0$ we obtain

$$\left.\begin{array}{c} -\widetilde{\alpha}_j A_j p \in \partial \overline{V}_j(x_j), \\ \left\langle A_j p, x_j \right\rangle = \phi_j(p) \\ \Phi(\tau) - \Phi(p) \geq \left\langle \tau - p, \sum_{j=1}^m A_j^T x_j \right\rangle, \quad \forall\, \tau \in \mathbb{R}_+^n. \end{array}\right\}$$

This means that $\left(p, (x_j), (\widetilde{\alpha}_j)\right)$ is a solution of $(\widetilde{P})$.

Step 2. Now we assume that for some $j_0 \in 1, \ldots, m$

$$\min\{\phi_{j_0}(\tau) : \tau \in \mathbb{R}_+^n, |\tau| = 1\} = 0.$$

We apply the results from Step 1 to the system $\left(V_j(\cdot), \phi_j(\cdot) + \varepsilon\,|\cdot|\,, \Phi(\cdot) + m\varepsilon\,|\cdot|\right)$ for sufficiently small $\varepsilon > 0$. First, we notice the conditions (H_0), (H_9) imply that there exists $\widetilde{\varepsilon} \leq \varepsilon_0$ such that for any $j = 1, \ldots, m$ we get

$$\left\langle A_j \tau, x_j \right\rangle > \phi_j(\tau) + \widetilde{\varepsilon}\,|\tau|\,, \quad \forall \tau \in \mathbb{R}_+^n \setminus \{0\},\ \forall x_j \in \partial \overline{V}_j^\star(0). \tag{38}$$

Therefore, we obtain that for any $\varepsilon \le \widetilde{\varepsilon}$ there exist $\pi^\varepsilon \in \mathbb{R}^n_+$, $x^\varepsilon_j \in \mathbb{R}^n_+$, $\alpha^\varepsilon_j \in \mathbb{R}_+$, $j = 1, \dots, m$ such that

$$\left.\begin{aligned} -\alpha^\varepsilon_j A_j \pi^\varepsilon &\in \partial \overline{V}_j(x^\varepsilon_j), \\ \langle A_j \pi^\varepsilon, x^\varepsilon_j \rangle - \phi_j(\pi^\varepsilon) - \varepsilon\,|\pi^\varepsilon| &\in \partial\,\mathrm{ind}_{\ge 0}(\alpha^\varepsilon_j), \\ \Phi(\tau) - \Phi(\pi^\varepsilon) + \varepsilon m(|\tau| - |\pi^\varepsilon|) &\ge \langle \tau - \pi^\varepsilon, \textstyle\sum_{j=1}^m A_j^T x^\varepsilon_j \rangle, \quad \forall\, \tau \in \mathbb{R}^n_+. \end{aligned}\right\} \tag{39}$$

From (38), (39) it follows that $\pi^\varepsilon \ne 0$, $\left|x^\varepsilon_j\right| \le M_j$, $j = 1, \dots, m$. As in Step 1 we get the estimate

$$\alpha^\varepsilon_j \phi_j(\pi^\varepsilon) \le c_j + V_j(y) + \langle \alpha^\varepsilon_j A_j \pi^\varepsilon, y \rangle, \quad \forall y \in \mathrm{Dom}\,\overline{V}_j. \tag{40}$$

Let $p^\varepsilon = \frac{\pi^\varepsilon}{|\pi^\varepsilon|}$, $\varepsilon > 0$. Therefore, there exist $p \in \mathbb{R}^n_+$, $|p| = 1$, $x_j \in \mathbb{R}^n_+$, $j = 1, \dots, m$ such that $p^\varepsilon \to p$, $x^\varepsilon_j \to x_j$, $j = 1, \dots, m$ as $\varepsilon \to 0$ (by passing to a subsequence, if necessary).

Let $j \in \{1, \dots, m\}$. We consider two cases:

Case 1 $\phi_j(p) > 0$. Then, analogously as in Step 1, from (40) we get that $\{\alpha^\varepsilon_j\,|\pi^\varepsilon|\}_{\varepsilon \le \varepsilon_0}$ is bounded. Hence there exists $\widetilde{\alpha}_j \in \mathbb{R}$ (by passing to a subsequence if necessary) such that $\alpha^\varepsilon_j \to \widetilde{\alpha}_j$. From positive homogeneity of degree one of the function ϕ_j, letting $\varepsilon \to 0$ in $(39)_1$, $(39)_2$ we have

$$-\widetilde{\alpha}_j A_j p \in \partial \overline{V}_j(x_j), \quad \langle A_j p, x_j \rangle - \phi_j(p) \in \partial\,\mathrm{ind}_{\ge 0}(\widetilde{\alpha}_j).$$

Case 2. $\phi_j(p) = 0$. From $(39)_2$ we get

$$\langle A_j p, x_j \rangle = \phi_j(p), \quad \phi_j(p) = 0. \tag{41}$$

Moreover, if (H^0_8) holds, then it is easily seen that $x_j \in \mathrm{Dom}\,\overline{V}_j$. If (H^1_8) holds, then from the condition $-\alpha^\varepsilon_j\,|\pi^\varepsilon|\,A_j p^\varepsilon \in \partial \overline{V}_j(x^\varepsilon_j)$ we get

$$0 \ge V_j(x_j) + \liminf_{\varepsilon \to 0} \overline{V}^\star_j(-\alpha^\varepsilon_j\,|\pi^\varepsilon|\,A_j p^\varepsilon),$$

which implies $x_j \in \mathrm{Dom}\,\overline{V}_j$.
If $\liminf_{\varepsilon \to 0} \alpha_j\,|\pi^\varepsilon| = \widetilde{\alpha}_j \in \mathbb{R}_+$, then letting $\varepsilon \to 0$ in $(39)_1$ we get $-\widetilde{\alpha}_j A_j p \in \partial \overline{V}_j(x_j)$. It means that

$$-\widetilde{\alpha}_j A_j p \in \partial \overline{V}_j(x_j), \quad \langle A_j p, x_j \rangle - \phi_j(p) \in \partial\,\mathrm{ind}_{\ge 0}(\widetilde{\alpha}_j).$$

If $\liminf_{\varepsilon \to 0} \alpha^\varepsilon_j\,|\pi^\varepsilon| = +\infty$ then (41) means that $(-A_j p, 0) \in \partial\,\mathrm{ind}_{\mathrm{epi}\,\overline{V}_j}(x_j, V_j(x_j))$, which can be written equivalently as

$$-A_j p \in \partial^\infty \overline{V}_j(x_j).$$

From $(39)_3$ it follows that

$$\Phi(\tau) - \Phi(p^\varepsilon) + \varepsilon m(|\tau| - |p^\varepsilon|) \ge \langle \tau - p^\varepsilon, \textstyle\sum_{j=1}^m A_j^T x^\varepsilon_j \rangle, \quad \forall\, \tau \in \mathbb{R}^n_+,$$

(substituting $\frac{\tau}{|\pi^\varepsilon|}$ into τ). Thus by letting $\varepsilon \to 0$ we are led to

$$\Phi(\tau) - \Phi(p) \geq \left\langle \tau - p, \sum_{j=1}^{m} A_j^T x_j \right\rangle, \quad \forall\, \tau \in \mathbb{R}_+^n,$$

as desired. This completes the proof. ■

Remark 3 For the function Φ_+, which is convex, lower semicontinuous, positively homogeneous of degree 1, there exists a nonempty, convex, closed set $W \subset \mathbb{R}^n$ such that $\Phi_+(\tau) = \sup_{y \in W} \langle \tau, y \rangle$, $\tau \in \mathbb{R}^n$ (see Aubin, 1993). Therefore, problem $(\widetilde{P})$ from Theorem 4 can be formulated equivalently as to find $\big(\pi, (x_j), (\alpha_j)\big)$ such that

$$\left.\begin{array}{c} -\alpha_j A_j \pi \in \partial \overline{V}_j(x_j),\ \langle A_j\pi, x_j\rangle - \phi_j(\pi) \in \partial\, \mathrm{ind}_{\geq 0}(\alpha_j),\ \text{if } \alpha_j \in \mathbb{R}_+, \\ -A_j\pi \in \partial^\infty \overline{V}_j(x_j),\ \langle A_j\pi, x_j\rangle = \phi_j(\pi),\ \phi_j(\pi) = 0,\ \text{if } \alpha_j = +\infty, \\ \Phi(\pi) = \left\langle \pi, \sum_{j=1}^m A_j^T x_j \right\rangle, \quad \sum_{j=1}^m A_j^T x_j \in W. \end{array}\right\} \ (\widetilde{P}')$$

Remark 4 Note that from the proof of Theorem 1 it follows that $\partial \overline{V}_j^\star(0) \neq \emptyset$. In order to obtain the result similar to that of Theorem 4 for V_j's which may not attain global minimum on $\mathbb{R}_+^n$, it is sufficient to assume that the set $W \cap \mathbb{R}_+^n$ in Remark 3 is bounded. Then we can apply Theorem 4 for the functions $\widetilde{V}_j := V_j + \mathrm{ind}_{B_{\mathbb{R}^n}(0,K) \cap \mathbb{R}_+^n}$, $j = 1, \dots, m$, with suitably chosen constant $K > 0$. For such functions (H_1) reduces to the requirement that $0 \in \mathrm{cl}(\mathrm{Dom}\, \partial \overline{V}_j)$ and (H_2) is redundant (note that then $\mathrm{Dom}(\widetilde{V}_j^\star) = \mathbb{R}^n$). Therefore, for K sufficiently large and $W \cap \mathbb{R}_+^n$ bounded, any solution of the modified problem becomes a solution of the initial problem $(\widetilde{P})$.

Remark 5 In order to find a solution of the problem $(\widetilde{P})$ in the case when $\partial \overline{V}_j^\star(0)$ is not a compact set, it is sufficient to replace the assumption (H_9) by the following one
(H_9') there exists $\widetilde{\varepsilon} > 0$ for any $j = 1, \dots, m$, such that

$$\langle A_j \tau, x_j \rangle > \phi_j(\tau) + \widetilde{\varepsilon}\,|\tau|\,, \quad \forall \tau \in \mathbb{R}_+^n \setminus \{0\},\ x_j \in \partial \overline{V}_j^\star(0).$$

The main result reads as follows:

Theorem 5 *Suppose that for any $j = 1, \dots, m$ the following hypotheses hold:*
(H_1) $0 \in \mathrm{cl}(\mathrm{Dom}\, \partial \overline{V}_j)$, $(\mathbb{R}_-^n \setminus \{0\}) \cap B_{\mathbb{R}^n}(0, r_j) \subset \mathrm{Int}\, \mathrm{Dom}\, \overline{V}_j^\star$ *for some* $r_j > 0$;
(H_2^1) $\left\{x \in \mathbb{R}_+^n : \{\langle x^\star, x\rangle : x^\star \in \partial \overline{V}_j(x)\} \cap (\mathbb{R}_- + \varepsilon_0) \neq \emptyset\right\} \subset B_{\mathbb{R}^n}(0, M_j)$, *for some* $M_j > 0$, $\varepsilon_0 > 0$;
(H_4^3) $\phi_j(t\tau) = t\phi_j(\tau)$ $\forall \tau \in \mathbb{R}_+^n$, $\forall\, t > 0$;
(H_6^1) $\Phi = \sum_{j=1}^m \phi_j$ *is convex and* $\Phi(\tau) \geq \gamma\,|\tau|$ $\forall\, \tau \in \mathbb{R}_+^n$, $\gamma > 0$;
(H_7^1) $0 \notin \partial \overline{V}_j(0)$.

Moreover, for any $j = 1, \dots, m$, assume that one of the following conditions holds:

(H_8^0) $\mathrm{Dom}\,\overline{V}_j$ *is closed*

or

(H_8^1) *if* $x_j^k \to x_j$, $p^k \to p$, $|p^k| = 1$, *as* $k \to +\infty$, $\alpha_j^k > 0$ *such that* $-\alpha_j^k A_j p^k \in \partial \overline{V}_j(x_j^k)$ *and* $\phi_j(p) = 0$, *then* $\liminf_{k\to\infty} \overline{V}_j^\star(-\alpha_j^k A_j p^k) > -\infty$.

Then there exist $0 < r \le 1$ *and* $(\pi, (x_j), (\alpha_j))$, $\pi \in \mathbb{R}_+^n \setminus \{0\}$, $x_j \in \mathbb{R}_+^n$ *and* $\alpha_j \in \mathbb{R}_+ \cup \{+\infty\}$ *for* $j = 1, \dots, m$ *such that*

$$\left.\begin{aligned} -\alpha_j A_j \pi \in \partial \overline{V}_j(x_j),\ \langle A_j\pi, x_j\rangle - \phi_j(\pi) \in \partial\, \mathrm{ind}_{\ge 0}(\alpha_j),\ \text{if } \alpha_j \in \mathbb{R}_+,\\ -A_j\pi \in \partial^\infty \overline{V}_j(x_j),\ \langle A_j\pi, x_j\rangle = \phi_j(\pi) = 0,\ \text{if } \alpha_j = +\infty,\\ \Phi(\tau) - \Phi(\pi) \ge \langle \tau - \pi, \tfrac{1}{r}\textstyle\sum_{j=1}^m A_j^T x_j\rangle, \quad \forall\, \tau \in \mathbb{R}_+^n. \end{aligned}\right\}\ (PQ)$$

Proof. The proof will be divided into two steps.

Step 1. We assume that

$$\min\{\phi_j(\tau) : \tau \in \mathbb{R}_+^n, |\tau| = 1\} = \gamma_j, \quad \gamma_j > 0, \quad j = 1, \dots, m. \tag{42}$$

Let $0 < \delta \le \varepsilon_0$. Let us make the following replacement in Proposition 2: $\Phi \mapsto \Phi^\delta$, $0 < \delta \le \varepsilon_0$ where

$$\Phi^\delta(\tau) := \big(\Phi(\tau)\big)^{1+\delta}, \quad \tau \in \mathbb{R}_+^n.$$

We have to check the validity of hypotheses stated in Proposition 2 in case of such replacement.

First it will be shown that (H_5^1) holds, i.e.

$$\big\{\tau \in \mathbb{R}_+^n : \big(\Phi(\tau)\big)^{1+\delta} \le \textstyle\sum_{j=1}^m \phi_j^\varepsilon(\tau) + \frac{\varepsilon}{m}\big\} \subset B_{\mathbb{R}^n}(0, \widetilde{M})$$

for some $\widetilde{M} > 0$ and any $0 < \varepsilon \le \delta$. Indeed, taking into account (H_6^1) and $\phi_j^\varepsilon(\tau) \le \phi_j(\tau)$, $j = 1, \dots, m$, we get the estimate for $\Phi(\tau) \ge 1$ and $\tau \in \big\{\tau \in \mathbb{R}_+^n : \big(\Phi(\tau)\big)^{1+\delta} \le \sum_{j=1}^m \phi_j^\varepsilon(\tau) + \frac{\varepsilon}{m}\big\}$

$$\Phi(\tau) \le (1 + \tfrac{\delta}{m})^{\frac{1}{\delta}} \le e^{\frac{1}{m}} \le e.$$

Having in mind that $\Phi(\tau) \ge \gamma\,|\tau|$, $\gamma > 0$, we finally arrive at

$$|\tau| \le \begin{cases} \frac{e}{\gamma} & \text{if } \Phi(\tau) \ge 1, \\ \frac{1}{\gamma} & \text{if } \Phi(\tau) \le 1. \end{cases} \tag{43}$$

Hence the assumption (H_5^1) follows with $\widetilde{M} = \frac{e}{\gamma}$. Further, it is not difficult to verify that $\partial \Phi_+^\varepsilon(0) \cap \mathbb{R}_+^n = \{0\}$, which yields (H_6) from Theorem 2.

Summing up, the hypotheses of Proposition 2 are fulfilled for the data system $\big((V_j),(\phi_j),\Phi^\delta\big)$. Thus, for any $0<\varepsilon\le\delta$ there exists a system $\big(\pi^\varepsilon,(x_j^\varepsilon),(\alpha_j^\varepsilon)\big)\in\mathbb{R}_+^n\times(\mathbb{R}_+^n)^m\times(\mathbb{R}_+)^m$ with the properties that

$$-\alpha_j^\varepsilon A_j\pi^\varepsilon\in\partial\overline{V}_j(x_j^\varepsilon), \tag{44}$$

$$z_j^\varepsilon=(1-\varepsilon)^{\frac{1}{\varepsilon}}\big(\tfrac{1}{\alpha_j^\varepsilon+r_j^\varepsilon}\big)^{\frac{1}{\varepsilon}},\quad r_j^\varepsilon\ge 0,\quad \tfrac{s}{m}\ge z_j^\varepsilon,\quad (\tfrac{s}{m}-z_j^\varepsilon)r_j^\varepsilon=0 \tag{45}$$

$$\big\langle A_j\pi^\varepsilon,x_j^\varepsilon\big\rangle=\phi_j^\varepsilon(\pi^\varepsilon)+\varepsilon\tfrac{s}{m}(\tfrac{s}{m}-z_j^\varepsilon),\quad (\text{if } \alpha_j^\varepsilon>0), \tag{46}$$

$$\big\langle A_j\pi^\varepsilon,x_j^\varepsilon\big\rangle\le\phi_j^\varepsilon(\pi^\varepsilon),\quad (\text{if } \alpha_j^\varepsilon=0), \tag{47}$$

$$\big(\Phi(\tau)\big)^{1+\delta}-\big(\Phi(\pi^\varepsilon)\big)^{1+\delta}\ge\big\langle\tau-\pi^\varepsilon,\textstyle\sum_{j=1}^m A_j^T x_j^\varepsilon\big\rangle,\quad\forall\,\tau\in\mathbb{R}_+^n. \tag{48}$$

From (H_7^1), (43), (44) we easily establish the estimate

$$|\pi^\varepsilon|\le\tfrac{e}{\gamma},\quad \liminf_{\varepsilon\to 0}|\pi^\varepsilon|>0,\quad \big|x_j^\varepsilon\big|\le M_j,\ j=1,\dots,m. \tag{49}$$

From (49) we obtain that for sufficiently small ε, $\phi_j^\varepsilon(\pi^\varepsilon)=\phi_j(\pi^\varepsilon)$, for $j=1,\dots,m$. Hence from (44), (46) we get the estimate for $\alpha_j^\varepsilon>0$

$$\alpha_j^\varepsilon\phi_j(\pi^\varepsilon)\le c_j+V_j(y)+\big\langle\alpha_j^\varepsilon A_j\pi^\varepsilon,y\big\rangle,\quad\forall y\in\operatorname{Dom}\overline{V}_j,$$

where the constant $c_j\in\mathbb{R}$ is such that $V_j(y)\ge -c_j$, for any $y\in\operatorname{Dom}\overline{V}_j$.

Analysis similar to that in the proof of Theorem 4 shows that $\{\alpha_j^\varepsilon|\pi^\varepsilon|\}_{\varepsilon\le\varepsilon_0}$ is bounded. Therefore, from (49) in this case it can be assumed that $\pi^\varepsilon\to\pi$, $x_j^\varepsilon\to x_j$ and $\alpha_j^\varepsilon\to\widetilde{\alpha}_j$ for some $\pi\in\mathbb{R}_+^n$, $\pi\ne 0$, $x_j\in\mathbb{R}_+^n$, $\widetilde{\alpha}_j\in\mathbb{R}_+$ (by passing to subsequence, if necessary) such that

$$\left.\begin{aligned}-\widetilde{\alpha}_jA_j\pi&\in\partial\overline{V}_j(x_j),\\ \big\langle A_j\pi,x_j\big\rangle-\phi_j(\pi)&\in\partial\,\mathrm{ind}_{\ge 0}(\widetilde{\alpha}_j),\\ (\Phi(\tau))^{1+\delta}-(\Phi(\pi))^{1+\delta}&\ge\big\langle\tau-\pi,\textstyle\sum_{j=1}^m A_j^Tx_j\big\rangle,\quad\forall\,\tau\in\mathbb{R}_+^n.\end{aligned}\right\} \tag{50}$$

Taking into account (50), we get for any $0<\varepsilon\le\varepsilon_0$ that there exist $\big(\pi^\varepsilon,(x_j^\varepsilon),(\alpha_j^\varepsilon)\big)$ such that

$$\left.\begin{aligned}-\alpha_j^\varepsilon A_j\pi^\varepsilon&\in\partial\overline{V}_j(x_j^\varepsilon),\\ \big\langle A_j\pi^\varepsilon,x_j^\varepsilon\big\rangle-\phi_j(\pi^\varepsilon)&\in\partial\,\mathrm{ind}_{\ge 0}(\alpha_j^\varepsilon),\\ (\Phi(\tau))^{1+\varepsilon}-(\Phi(\pi^\varepsilon))^{1+\varepsilon}&\ge\big\langle\tau-\pi^\varepsilon,\textstyle\sum_{j=1}^m A_j^Tx_j^\varepsilon\big\rangle,\quad\forall\,\tau\in\mathbb{R}_+^n.\end{aligned}\right\} \tag{51}$$

Moreover

$$0<|\pi^\varepsilon|\le\tfrac{e}{m},\quad \big|x_j^\varepsilon\big|\le M_j,\ j=1,\dots,m. \tag{52}$$

Analysis similar to that in the proof of Theorem 4 shows that $\{\alpha_j^\varepsilon|\pi^\varepsilon|\}_{\varepsilon\le\varepsilon_0}$, $j=1,\dots,m$ are bounded. Therefore, there exist $r\in[0,1]$, $p\in\mathbb{R}_+^n$, $|p|=1$,

$x_j \in \mathbb{R}^n_+$, $\widetilde{\alpha}_j \in \mathbb{R}$, $j = 1, \dots, m$ such that $p^\varepsilon \to p$ ($p^\varepsilon = \frac{\pi^\varepsilon}{|\pi^\varepsilon|}$), $|\pi^\varepsilon|^\varepsilon \to r$, $x_j^\varepsilon \to x_j$, $\alpha_j^\varepsilon |\pi^\varepsilon| \to \widetilde{\alpha}_j$. From positive homogeneity of degree one of the functions ϕ_j, letting $\varepsilon \to 0$ in $(51)_1$, $(51)_2$ we have

$$-\widetilde{\alpha}_j A_j p \in \partial \overline{V}_j(x_j), \quad \langle A_j p, x_j \rangle - \phi_j(p) \in \partial \operatorname{ind}_{\geq 0}(\widetilde{\alpha}_j).$$

Using positive homogeneity of degree one of the function ϕ_j the condition $(51)_3$ gets the equivalent form

$$|\pi^\varepsilon|^\varepsilon \left((\Phi(\tau))^{1+\varepsilon} - (\Phi(p^\varepsilon))^{1+\varepsilon}\right) \geq \langle \tau - p^\varepsilon, \textstyle\sum_{j=1}^m A_j^T x_j^\varepsilon \rangle, \quad \forall \tau \in \mathbb{R}^n_+$$

(substituting $\frac{\tau}{|\pi^\varepsilon|}$ into τ). Thus by letting $\varepsilon \to 0$ we arrive at

$$r(\Phi(\tau) - \Phi(p)) \geq \langle \tau - p, \textstyle\sum_{j=1}^m A_j^T x_j \rangle, \quad \forall \tau \in \mathbb{R}^n_+.$$

From the hypotheses (H_7^1) we easily get that $r > 0$.
Therefore there exist $r \in (0, 1]$ and the system $(p, (x_j), (\widetilde{\alpha}_j))$, $|p| = 1$ such that

$$\left.\begin{array}{c} -\widetilde{\alpha}_j A_j \pi \in \partial \overline{V}_j(x_j), \\ \langle A_j \pi, x_j \rangle - \phi_j(\pi) \in \partial \operatorname{ind}_{\geq 0}(\widetilde{\alpha}_j), \\ \Phi(\tau) - \Phi(p) \geq \langle \tau - \pi, \frac{1}{r} \sum_{j=1}^m A_j^T x_j \rangle, \quad \forall\, \tau \in \mathbb{R}^n_+. \end{array}\right\}$$

Step 2. Now we assume that for some $j_0 \in \{1, \dots, m\}$

$$\min\{\phi_{j_0}(\tau) : \tau \in \mathbb{R}^n_+, |\tau| = 1\} = 0.$$

We apply the results from Step 1 to the system $\big(V_j(\cdot), \phi_j(\cdot) + \varepsilon\,|\cdot|\,, \Phi(\cdot) + m\varepsilon\,|\cdot|\big)$ for $\varepsilon > 0$. For any $0 < \varepsilon \leq 1$ there exist $r^\varepsilon \in (0, 1]$, $p^\varepsilon \in \mathbb{R}^n_+$, $|p^\varepsilon| = 1$, $x_j^\varepsilon \in \mathbb{R}^n_+$, $\alpha_j^\varepsilon \in \mathbb{R}_+$, $j = 1, \dots, m$ such that

$$\left.\begin{array}{c} -\alpha_j^\varepsilon A_j p^\varepsilon \in \partial \overline{V}_j(x_j^\varepsilon), \\ \langle A_j p, x_j^\varepsilon \rangle - \phi_j(p^\varepsilon) - \varepsilon \in \partial \operatorname{ind}_{\geq 0}(\alpha_j^\varepsilon), \\ r^\varepsilon\big(\Phi(\tau) - \Phi(p^\varepsilon) + \varepsilon m(|\tau| - 1)\big) \geq \langle \tau - p^\varepsilon, \sum_{j=1}^m A_j^T x_j^\varepsilon \rangle, \quad \forall\, \tau \in \mathbb{R}^n_+. \end{array}\right\} \quad (53)$$

From $(53)_1$ we get that $|x_j^\varepsilon| \leq M_j$. Moreover as in Step 1 we get the estimate

$$\alpha_j^\varepsilon \phi_j(p^\varepsilon) \leq c_j + V_j(y) + \langle \alpha_j^\varepsilon A_j p^\varepsilon, y \rangle, \quad \forall y \in \operatorname{Dom} \overline{V}_j. \quad (54)$$

From the fact that $|p^\varepsilon| = 1$ and $r^\varepsilon \in (0, 1]$ we get that there exist $p \in \mathbb{R}^n_+$, $|p| = 1$, $r \in [0, 1]$, $x_j \in \mathbb{R}^n_+$, $j = 1, \dots, m$ such that $r^\varepsilon \to r$, $p^\varepsilon \to p$, $x_j^\varepsilon \to x_j$, as $\varepsilon \to 0$ (by passing to a subsequence, if necessary).

Let $j \in \{1, \dots, m\}$. We consider two cases:

Case 1. $\phi_j(p) > 0$. Then, analogously as in Step 1, from (54) we get that $\{\alpha_j^\varepsilon\}_{\varepsilon\le 1}$ is bounded. Hence there exists $\widetilde{\alpha}_j \in \mathbb{R}$ (by passing to a subsequence if necessary) such that $\alpha_j^\varepsilon \to \widetilde{\alpha}_j$. Letting $\varepsilon \to 0$ in $(53)_1$, $(53)_2$ we obtain that

$$-\widetilde{\alpha}_j A_j p \in \partial \overline{V}_j(x_j), \quad \langle A_j p, x_j \rangle - \phi_j(p) \in \partial\,\mathrm{ind}_{\ge 0}(\widetilde{\alpha}_j). \tag{55}$$

Case 2. $\phi_j(p) = 0$. From $(53)_2$ we have that

$$\langle A_j p, x_j \rangle = \phi_j(p), \quad \phi_j(p) = 0.$$

Similarly to the proof of the Theorem 4 we get that $x_j \in \mathrm{Dom}\,\overline{V}_j$.
Moreover, if $\liminf_{\varepsilon\to 0} \alpha_j^\varepsilon |\pi^\varepsilon| = \widetilde{\alpha}_j \in \mathbb{R}_+$, then (55) holds.
If $\liminf_{\varepsilon\to 0} \alpha_j^\varepsilon |\pi^\varepsilon| = \widetilde{\alpha}_j = +\infty$, then the conditions $\langle A_j p, x_j \rangle = \phi_j(p)$, $\phi_j(p) = 0$ mean

$$-A_j p \in \partial^\infty \overline{V}_j(x_j).$$

Letting $\varepsilon \to 0$ in $(53)_3$ we get

$$r\Phi(\tau) - r\Phi(p) \ge \langle \tau - p, \textstyle\sum_{j=1}^m A_j^T x_j \rangle, \quad \forall\, \tau \in \mathbb{R}_+^n. \tag{56}$$

Moreover, from the assumption (H_6^1) we deduce that there exists $j' \in \{1,\dots,m\}$ such that $\phi_{j'}(p) > 0$. Note that then $x_{j'} \ne 0$. Suppose, on the contrary, that $x_{j'} = 0$. The condition $\langle A_{j'} p, x_{j'} \rangle - \phi_{j'}(p) \in \partial\,\mathrm{ind}_{\ge 0}(\widetilde{\alpha_{j'}})$ means $\widetilde{\alpha}_{j'} = 0$ and $0 \in \partial \overline{V}_{j'}(0)$, contrary to (H_7^1). Hence $\sum_{j=1}^m A_j^T x_j \ne 0$, which implies that that $0 < r \le 1$.

Summing up, there exist $0 < r \le 1$, $p \in \mathbb{R}_+^n$, $|p| = 1$, $x_j \in \mathbb{R}_+^n$, $\widetilde{\alpha_j} \in \mathbb{R}_+ \cup \{+\infty\}$, $j = 1,\dots,m$, such that

$$\left.\begin{array}{c} -\widetilde{\alpha}_j A_j p \in \partial \overline{V}_j(x_j),\ \langle A_j \pi, x_j \rangle - \phi_j(\pi) \in \partial\,\mathrm{ind}_{\ge 0}(\widetilde{\alpha}_j),\ \text{if } \widetilde{\alpha}_j \in \mathbb{R}_+, \\ -A_j p \in \partial^\infty \overline{V}_j(x_j),\ \langle A_j p, x_j \rangle = \phi_j(p) = 0,\ \text{if } \widetilde{\alpha}_j = +\infty, \\ \Phi(\tau) - \Phi(p) \ge \langle \tau - p, \frac{1}{r}\sum_{j=1}^m A_j^T x_j \rangle, \quad \forall\, \tau \in \mathbb{R}_+^n. \end{array}\right\}$$

This completes the proof. ■

COROLLARY 1 *Assume the hypotheses of Theorem 5. Then there exist* $\pi \in \mathbb{R}_+^n$, $\pi \ne 0$, $x_j \in \mathbb{R}_+^n$, $\alpha_j \in \mathbb{R}_+ \cup \{+\infty\}$, $j = 1,\dots,m$, *and* $0 < r \le 1$ *such that*

$$V_j(x_j) = \min\{V_j(y) \colon \langle A_j \pi, y \rangle \le \phi_j(\pi)\}, \ \text{if } \alpha_j \in \mathbb{R}_+, \tag{57}$$

$$-A_j \pi \in \partial^\infty \overline{V}_j(x_j),\ \langle A_j \pi, x_j \rangle = \phi_j(\pi) = 0,\ \text{if } \alpha_j = +\infty, \tag{58}$$

$$\Phi(\pi) = \textstyle\sum_{j=1}^m \phi_j(\pi), \quad \Phi(\pi) = \langle \pi, \frac{1}{r}\sum_{j=1}^m A_j^T x_j \rangle, \quad \frac{1}{r}\sum_{j=1}^m A_j^T x_j \in W. \tag{59}$$

If one the inequalities $\langle A_j \pi, x_j \rangle \le \phi_j(\pi)$ *happens to be strict, then* $r < 1$.

Remark 6 The results obtained lead to certain implications concerning the equilibrium price vector π. First, if $\phi_j(\pi) > 0$, i.e. the income of j's consumer-producer at prices π is positive, then he maximizes his utility over the set of all admissible commodity bundles $\{y \in \mathbb{R}^n_+ : \langle \pi, y \rangle \le \phi_j(\pi)\}$, as expected. But, if $\phi_j(\pi) = 0$, i.e. the income of j's consumer-producer at prices π vanishes, it is not possible to determine whether x_j maximizes his utility function $-V_j$ over the set $\{y \in \mathbb{R}^n_+ : \langle \pi, y \rangle = 0\}$ of all zero-valued commodity bundles (at prices π), what has quite reasonable explanation. Note that all of those zero-valued commodity bundles are equivalent in the sense of economy: for the customer they are equally worthless.

Moreover, if for at last one j the strict inequality $\langle \pi, x_j \rangle < \phi_j(\pi)$ occurs, i.e. the value of the optimal commodity bundle of j's consumer does not reach its budget line, then $r < 1$. Thus, $1 - r$ can be referred to as the measure of the difference between the value of the total endowment and aggregate demand at equilibrium prices π. This may happen when due to the preferences which are not bound to be strictly monotone, the Walras' law fails.

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