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Book review:

LIMIT CYCLES OF DIFFERENTIAL EQUATIONS

by

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In June 2006 Jaume Llibre and Armengol Gasull organized the "Advanced Course on Limit Cycles and Differential Equations" at the Centre de Recerca Matemàtica in Barcelona. There were three lecturers: two by the authors of this book and by Sergey Yakovenko. The book under review contains the notes of lectures of Christopher and Li. (Yakovenko, worked hard, then, on his book on analytic differential equations with Ilyashenko, which is now being published in AMS; so he did not prepare his notes for publication.)

The book is divided into two parts: 'Around the Center–Focus problem' by C. Christopher and 'Abelian Integrals and Applications to the Weak Hilbert's 16th Problem' by C. Li.

A singular point 0 of a real planar analytic vector field V is called the *center* if there exists a neighborhood U of 0 such that $U \setminus 0$ is filled with periodic integral curves of the field. The *problem of center* relies on finding conditions on the coefficients of the Taylor expansion of V at 0 which imply that 0 is a center. Here we must distinguish local and global problems.

The *local problem of center* (or the *center–focus problem*) was formulated by V. Arnold (1970) as follows: The space J^k of $k-$jets j^kV at 0 of germs $V : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is divided into three subspaces J^k_s, J^k_u, J^k_n. The jets from J^k_s (respectively from J^k_u) are such that any germ W with $j^kW = j^kV$ has 0 as asymptotically stable point (respectively as unstable point). The subspace J^k_n consists of 'neutral' jets. Arnold asked whether the sets $J^k_\#$ are semi-algebraic, or semi-analytic, and conjectured that the codimension of J^k_n grows to infinity with k. In the case $dV(0)$ is a rotation the problem is algebraically solvable (Lyapunov, 1966; Poincaré, 1951). In general, the sets $J^k_{s,u,n}$ can be not semi-algebraic (see Ilyashenko, 1972). Recently N. Medvedeva (2004) proved that Arnold's problem is analytically solvable.

In the *global center–focus problem* one considers the space $\mathcal{A}_n$ of polynomial vector fields V of degree $\leq n$ (with $V(0) = 0$) and tries to describe the subvariety $\mathcal{C}_n \subset \mathcal{A}_n$, called the *center variety*, consisting of fields with center at 0. Simultaneously, one asks about the maximal number of limit cycles which can bifurcate from a singular point of the center or the (weak) focus type for a vector field from $\mathcal{A}_n$.

The Christopher's part of the book concerns the global center–focus problem. He begins with the classical approach due to Poincaré and Lyapunov. Thus, he introduces the focus quantities, associated with a critical point of center or focus type, and estimates how many of them are independent, when considered as functions of the coefficients of the vector field of degree n. In particular, he presents examples where 12 focus quantities are independent in a family of cubic vector fields, and hence 11 small amplitude limit cycles can bifurcate from a center (see also Żołądek, 1995).

The main conjecture about the global center problem is that there exist only two general mechanisms for creating a polynomial vector field with center. One is the *Liouvillian integrability* and the other is the *algebraic reversibility.*

A particular case of the Liouvillian integrability is the Darboux integrability. It means that the vector field V has a first integral of the following *Darboux form*

$$e^{g/h} f_1^{a_1} \dots f_r^{a_r},$$

where $g(x,y)$, $h(x,y)$, $f_j(x,y)$ are polynomials and a_j are (complex) constants (see Darboux, 1878). Liouvillian first integrals are multivalued functions in $\mathbb{C}^2$ which can be expressed in quadratures: we start with the field rational functions and apply operations like adjoining a solution of an algebraic equation, adjoining an integral and adjoining an exponent of an integral. M. Singer (1992) proved that if V is Liouvillian integrable, then it has an integrating factor of the Darboux form. Christopher provides a proof of the Singer's theorem.

The simplest case of the algebraic reversibility is the *time reversibility* for differential systems of the form

$$\dot{x} = P(x^2, y), \qquad \dot{y} = xQ(x^2, y).$$

It is clear that this system is invariant with respect to the involution $(x,y,t) \to (-x,y,t)$. On the other hand, it can be obtained as a pull-back from the system $\dot{X} = P(X,Y)$, $\dot{Y} = Q(X,Y)$ via the fold map $(x,y) \to (x^2,y)$. If the second system has a trajectory tangent to the critical curve $X = 0$ from 'outside' then the preimage of this point is a center for the first system. The algebraic reversibility is a generalization of this construction when the fold map is defined by algebraic functions of x, y (see Żołądek, 1994).

Christopher analyses the *Liénard systems*

$$\dot{x} = y, \quad \dot{y} = -g(x) - yf(x)$$

and the *Cherkas' systems* (Cherkas, 1974)

$$\dot{x} = y, \quad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2$$

and confirms the above Liouvillian–reversibility conjecture in these cases.

Another interesting problem, related with the center–focus problem, is the problem of center for the *Abel equation*

$$\frac{dy}{dx} = p(x)y^2 + q\,(x)\,y^3,$$

where p and q are polynomials of $x \in \mathbb{C}$. The authors of the problem, J.-P. Françoise and Y. Yomdin (Briskin, Françoise, Yomdin, 1999), ask for the conditions onto p and q that the solutions $y = \varphi(x, y_0)$, which satisfy the initial condition $\varphi(0, y_0) = y_0$, satisfy also $\varphi(1, y_0) = y_0$ for small y_0. There exists the *composition conjecture*, which claims that the polynomials $P(x) = \int_0^x p$ and $Q(x) = \int_0^x q$ take the form

$$P = \widetilde{P} \circ W, \quad Q = \widetilde{Q} \circ W$$

for a polynomial $W(x)$ such that $W(0) = W(1)$ (an analogue of the algebraic reversibility). Christopher presents his solution of a simplified version of this problem, when $q(x) = \varepsilon q_1(x)$ and ε is a small parameter. In the proof he uses monodromy properties of certain multivalued function of x (generating function for moments of certain measure).

A weakened version of the center–focus problem concerns systems of the form $\dot{x} = H_y + \varepsilon P$, $\dot{y} = -H_x + \varepsilon Q$, i.e. perturbations of *Hamiltonian systems* with center. Like in the lectures of C. Li the problem of limit cycles bifurcating from the center and the problem of finding the center conditions in this situation leads to consideration of the following *Abelian integrals* $I(c) = \int_{\gamma_c} Qdx - Pdy$ along ovals γ_c of the (real) algebraic curves $H(x, y) = c$. In particular, one asks for the conditions onto P, Q and H such that $I(c) \equiv 0$. The obvious condition is that the 1-form $\omega = Qdx - Pdy$ is relatively exact, i.e. $\omega = fdH + dR$. But there can exist situations with a kind of symmetry (or algebraic reversibility). Christopher presents his solution to this problem in the case of *hyperelliptic Hamiltonian*

$$H = y^2 + S(x).$$

In the proof he uses the monodromy theory and some results from the group theory.

One chapter of the Christopher's lecture is devoted to the *Lotka–Volterra systems*

$$\dot{x} = x(1 + ax + by), \qquad \dot{y} = y(-\lambda + cx + dy)$$

and the last chapter is devoted to discussion of other approaches to the center–focus problem. It is worth to note the experimental approach of H.C. Graf v. Bothmer (in press) who used the Weil–Deligne theorem, about the number of $\mathbb{Z}_p$−points of an algebraic variety, to detect some components of the center variety $\mathcal{C}_3$ (i.e. for cubic vector fields).

The Chengzhi Li's part of the book concerns the *weakened 16th Hilbert problem* (stated by Arnold, 1990) where one asks about the number of zeroes of the Abelian integrals

$$I(c) = \int_{\gamma_c} \omega, \qquad \omega = Qdx - Pdy,$$

associated with the perturbations

$$\dot{x} = H_y + \varepsilon P, \quad \dot{y} = -H_x + \varepsilon Q$$

(like in the weakened center–focus problem). It is well known that simple zeroes c_j of $I(c)$ correspond to generation of limit cycles (for $\varepsilon \neq 0$) from the ovals γ_{c_j} (see Ilyashenko, 1969).

Li begins his lectures with a review of results about progress in the problem of limit cycles for polynomial plane vector fields (the 16th Hilbert's problem). There are too many of them to be listed here.

Then he presents the general result of A. Khovanskii (1984) and A. Varchenko (1984) about existence of a bound $\leq C(m,n)$ for the number of zeroes of $I(c)$ depending only on the degrees m and n of ω and H, respectively. But the greatest part of his lecture is devoted to the case when the degree of H is small.

In particular, he recalls the solution of the weakened 16th Hilbert problem in the quadratic case by L. Gavrilov and I. Iliev (see Gavrilov, 2001). In the last chapter he presents a 'unified' proof of this result; probably this is the most technical part of his lecture.

Also the cases with elliptic Hamiltonian, i.e. $H = y^2 + S(x)$ with $\deg P = 3, 4$, are studied in detail. Here he demonstrates the main methods used in this theory. One such method uses the fact that the Abelian integrals satisfy differential equations of Fuchs type. There is a direct method of dividing the oval into segments, applied to Hamiltonians of the form $H = \Phi(x) + \Psi(y)$. There exists a method based on the argument principle (when c takes complex values), introduced by G. Petrov (1988). Finally, some people claim to use the averaging method (but in my opinion the Abelian integral itself is a kind of averaging).

There is some activity in studying the period function for periodic solutions of Hamiltonian systems, or (more generally) for systems with center. This problem also leads to some Abelian integrals (see Chicone, Jacobs, 1989). Li reviews numerous results obtained in this field.

In my opinion the book constitutes a very nice presentation of the main results in this very active area of mathematics. It should be especially useful for students, because they could find there a quick introduction to the main topics and an overview of results and methods.

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