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Finite element error analysis for state-constrained optimal control of the Stokes equations*

by

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Abstract: An optimal control problem for 2d and 3d Stokes equations is investigated with pointwise inequality constraints on the state and the control. The paper is concerned with the full discretization of the control problem allowing for different types of discretization of both the control and the state. For instance, piecewise linear and continuous approximations of the control are included in the present theory. Under certain assumptions on the L^∞-error of the finite element discretization of the state, error estimates for the control are derived which can be seen to be optimal since their order of convergence coincides with the one of the interpolation error. The assumptions of the L^∞-finite-element-error can be verified for different numerical settings. Finally the results of two numerical experiments are presented.

Keywords: linear-quadratic optimal control problems, Stokes equations, state constraints, numerical approximation, finite elements

1. Introduction

This paper is concerned with the finite element discretization for the following linear quadratic optimal control problem subject to the Stokes equations and

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additional constraints on the control and the state:

$$
\text{(P)} \quad \begin{cases} \text{minimize} & J(v,u) := \dfrac{1}{2}\displaystyle\int_\Omega |v-z|^2_{\mathbb{R}^d}\,dx + \dfrac{\alpha}{2}\int_\Omega |u|^2_{\mathbb{R}^d}\,dx \\ \text{subject to} & -\Delta\, v + \nabla p = u \quad \text{in } \Omega \\ & \nabla \cdot v = 0 \quad \text{in } \Omega \\ & v = 0 \quad \text{on } \Gamma := \partial\Omega \\ \text{and} & v \in K \subset L^\infty(\Omega')^d \\ & a \le u(x) \le b \quad \text{a.e. in } \Omega, \end{cases}
$$

where u denotes the control, v and p are velocity and pressure, respectively, and z is the given desired state. Furthermore, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded domain with boundary Γ and $\alpha > 0$ is a given number. Moreover, $a, b \in \mathbb{R}^d$ are given vectors, whereas K denotes a closed and convex subset of $L^\infty(\Omega')^d$, where Ω' is a fixed (not necessarily proper) subset of Ω. Possible examples for K are box constraints for v or restrictions on the Euclidian norm of v, i.e.,

$$
\begin{aligned}
K^{(1)} &:= \left\{ v \in L^\infty(\Omega')^d | v_a \le v(x) \le v_b \text{ a.e. in } \Omega' \right\} \\
K^{(2)} &:= \left\{ v \in L^\infty(\Omega')^d | |v(x)|^2_{\mathbb{R}^d} \le \varrho \text{ a.e. in } \Omega' \right\}
\end{aligned}
$$

with given bounds $v_a, v_b \in \mathbb{R}^d$, and $\varrho > 0$. In view of the no-slip conditions on the boundary, it might be reasonable to require the state constraints only in the interior of Ω. The theory presented is applicable for both cases, i.e. $\Omega' \neq \Omega$ and $\Omega' = \Omega$. We point out that the subsequent analysis can be extended to the case of more general control constraints, i.e. $u \in K_u$ with a nonempty convex subset $K_u \in L^\infty(\Omega)^d$. For a better readability of the paper, we concentrate on box constraints for the control, while general constraints on the state are considered.

It is well known that, if certain constraint qualifications are satisfied, then the generalized Karush-Kuhn-Tucker theory allows to derive first-order necessary conditions that include the existence of Lagrange multipliers associated to the state constraints in $(L^\infty(\Omega')^d)^*$, i.e., the dual of $L^\infty(\Omega')^d$ with respect to the inner product of $L^2(\Omega')^d$ (see Zowe and Kurcyusz, 1979, or Casas, 1993). This lack of regularity of the multipliers complicates the numerical analysis of state-constrained optimal control problems. Nevertheless, in the recent past, some progress has been achieved concerning the finite element error analysis of state-constrained elliptic problems. We exemplarily mention Casas (2002), where a semilinear elliptic control problem with finitely many state constraints is considered, and Casas and Mateos (2002), where convergence of a finite element discretization for state-constrained semilinear elliptic problems is proved in a general setting. Moreover, we refer to Deckelnick and Hinze (2007 a,b), where a variational discretization of state-constrained elliptic problems is considered, and to Deckelnick, Günther and Hinze (2007) for problems with pointwise constraints on the gradient of the state variable. Furthermore, Deckelnick and Hinze

(2008) also investigated piecewise constant approximations of the control in the presence of pointwise state constraints and obtained an order of convergence of $h|\log h|$ in the two dimensional case and $h^{1/2}$ in case of three dimensions with respect to the L^2-norm on the control. Afore, slightly worse results for the same setting are proven in Meyer (2008) by employing a completely different analysis.

In this paper, we show that the analysis of Meyer (2008) can be transferred to the Stokes equations and more general discretizations for the control, e.g. *piecewise linear and continuous* ansatz functions. In particular, the use of more general ansatz functions, as e.g. piecewise linear ones, requires significant modification of the theory presented in Meyer (2008), which is performed by using a particular quasi-interpolant introduced in Carstensen (1999). Moreover, to deal with different discretization techniques for the Stokes equations, we have to allow for discrete states which may not be feasible for the continuous problem. This constitutes another significant difference to the existing theory. The presented analysis covers results for different settings such as for instance the following: Let $\Omega \in \mathbb{R}^2$ be a convex polygon and Ω' be strictly contained in Ω and suppose that the Stokes equations are discretized with the Taylor-Hood element, while we use piecewise linear ansatz functions for the control. Then there holds for every $\varepsilon > 0$

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^2} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq C\, h^{1-\varepsilon},$$

where $(\bar{u}, \bar{v}, \bar{p})$ is the solution of (P), while $(\bar{u}_h, \bar{v}_h, \bar{p}_h)$ denotes the solution of its discrete counterpart.

To the authors' best knowledge, this is the first note that deals with the discretization error for the optimal control of the Stokes equations in the presence of pointwise state constraints. There are several papers considering finite element discretizations of the unconstrained optimal control of the Stokes and Navier-Stokes equations (see for instance Bochev and Gunzburger, 2004; Deckelnick Hinze, 2004; Gunzburger, Hou and Svobodny, 1991a,b), as well as contributions for the purely control-constrained case, Rösch and Vexler (2006). However, the analysis in case of pointwise state constraints differs significantly from these settings since, among other things, optimal L^∞-error estimates for the finite element discretization of the Stokes equations are required.

The paper is organized as follows: after stating the main assumptions and known results for the continuous problem (P) in the following section, we introduce a general framework for a discretization of (P) in Section 3, which covers different concrete discrete schemes. Thereafter, in Section 4 we discuss some special interpolation results to be used in Section 5, where a priori error analysis for the problem under consideration is presented. Finally, Section 6 is devoted to concrete discretization schemes and their practical realization, whereas the numerical examples are presented in Section 7.

2. Notation and assumptions

In all what follows, $|z|_{\mathbb{R}^d} = \left(\sum_{i=1}^d z_i^2\right)^{1/2}$ denotes the Euclidian norm and inequalities of the form $z \leq w$ with $w, z \in \mathbb{R}^d$, are understood componentwise. Moreover the natural inner product of $L^2(\Omega)^d$ is abbreviated by $(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\Omega)^d}$. Furthermore, we introduce the Hilbert spaces

$$V := H_0^1(\Omega)^d, \quad L := \left\{p \in L^2(\Omega) | \int_\Omega p(x)\,dx = 0\right\}.$$

Throughout this article, let σ be a real number satisfying $1 < \sigma < d/(d-1)$. Then we define the conjugate exponent by $\sigma' = \sigma/(\sigma-1)$. In addition, W_σ denotes the Sobolev space $W^{1,\sigma}(\Omega)^d$, whereas we set $V_\infty := L^\infty(\Omega')^d$. The dual spaces associated to W_σ and V_∞ with respect to the inner product of $L^2(\Omega)^d$ and $L^2(\Omega')^d$, respectively, are denoted by W_σ^* and V_∞^*.

Assumption 2.1 *On the quantities in* (P)*, we impose the following conditions:*
- Ω *is an open, bounded, simply connected Lipschitz domain* $\Omega \subset \mathbb{R}^d$, $d = 2, 3$*, while* Ω' *denotes an open subset of* Ω.
- K *is a closed and convex subset of* $L^\infty(\Omega')^d$
- $a, b \in \mathbb{R}^d$ *with* $a \leq b$
- $z \in L^2(\Omega)^d$.

Let us introduce the variational formulation of the Stokes equations by

$$(\nabla v, \nabla\varphi) - (p, \nabla\cdot\varphi) + (\nabla\cdot v, \psi) = (u, \varphi) \quad \forall\,(\varphi,\psi) \in V \times L. \tag{1}$$

It is well known that, for a given right-hand side $u \in L^2(\Omega)^d$, there exists a unique solution to (1) and the associated solution operator, denoted by $G : u \mapsto (v,p)$, is continuous from $L^2(\Omega)^d$ to $V \times L$. Moreover, we introduce the control-to-state operator $S\colon L^2(\Omega)^d \to V$ which maps the control variable u to the velocity component of the solution $G\,u$, i.e., $S\colon u \mapsto v$. Sometimes S and G are considered in different spaces (e.g. $L^\infty(\Omega')^d$), for simplicity also denoted by S and G, respectively. Based on the control-to-state operator, we define the reduced control problem by:

$$(\mathrm{P}) \quad \begin{cases} \displaystyle\min_{u \in L^2(\Omega)^d} f(u) := J(S\,u, u) \\ \qquad \text{s.t.} \quad S\,u \in K \\ \qquad\qquad a \leq u(x) \leq b \quad \text{a.e. in } \Omega. \end{cases}$$

For the solution operator of the Stokes system, the following mapping properties are known. We refer to Brown and Shen (1995, Theorem 2.9) and the references therein.

Proposition 2.1 *Under the hypothesis on* Ω *in Assumption 2.1, there is a positive number* $2d/(d+2) \leq \bar\sigma < d/(d-1)$ *such that* S *continuously maps*

$W_\sigma^* = (W^{1,\sigma}(\Omega)^d)^*$ to $W^{1,\sigma'}(\Omega)^d$ for all $\sigma \in [\bar\sigma, d/(d-1)[$. Hence, due to $\bar\sigma' > d$, Sobolev embedding theorems give

$$S : L^2(\Omega)^d \hookrightarrow W_\sigma^* \to W^{1,\sigma'}(\Omega)^d \hookrightarrow V_\infty \quad \forall\, \sigma \in [\bar\sigma, d/(d-1)[. \tag{2}$$

For the rest of the paper, let σ be a fixed, but arbitrary number in $[\bar\sigma, d/(d-1)[$. As already mentioned in the introduction, a certain constraint qualification is needed to derive the existence of Lagrange multipliers by means of the generalized Karush-Kuhn-Tucker theory. Here, we rely on

Assumption 2.2 (Slater condition) *There is a $\hat u \in L^\infty(\Omega)^d \cap W_\sigma$, satisfying*

$$S\,\hat u \in \operatorname{int} K$$
$$a \le \hat u(x) \le b \text{ a.e. in } \Omega.$$

In order to state the necessary optimality conditions for the solution of (P) we introduce the set of admissible controls, which incorporates both the control and the state constraints:

$$U_{\text{ad}} := \big\{u \in L^\infty(\Omega)^d | a \le u(x) \le b \text{ a.e. in } \Omega,\ S\,u \in K\big\}.$$

Theorem 2.1 *Under Assumption 2.2 there exists a unique solution of (P), denoted by $\bar u$. This solution provides some additional regularity, namely $\bar u \in W_\sigma$, and satisfies the following variational inequality*

$$(S\,\bar u - z\,,\, S\,u - S\,\bar u) + \alpha\,(\bar u\,,\, u - \bar u) \ge 0 \quad \forall\, u \in U_{\text{ad}} \tag{3}$$

where U_{ad} is defined as above.

Proof. The existence and uniqueness result is standard. To show the additional regularity of $\bar u$, we make use of the generalized Karush-Kuhn-Tucker theory (see Zowe and Kurcyusz, 1979). To this end, set $\bar v = S(\bar u)$. Under the Slater condition in Assumption 2.2, the generalized KKT theory guarantees the existence of a Lagrange multiplier $\bar\mu \in V_\infty^*$ such that $\bar u$ satisfies

$$\bar u = \Pi_{[a,b]}\big\{-\frac{1}{\alpha}\, S^*\big(E_2(\bar v - z) + E_\infty \bar\mu\big)\big\} \tag{4}$$

with the adjoint operator $S^* \colon (W^{1,\sigma'}(\Omega)^d)^* \to W_\sigma$ (see Proposition 2.1). Moreover, $E_2 \colon L^2(\Omega)^d \to (W^{1,\sigma'}(\Omega)^d)^*$ and $E_\infty \colon V_\infty^* \to (W^{1,\sigma'}(\Omega)^d)^*$ are the associated embedding operators. Furthermore, $\Pi_{[a,b]}$ denotes the component- and pointwise projection operator on the interval $[a,b]$. Since this projection operator maps W_σ to itself, we have $\bar u \in W_\sigma$. Finally, the derivation of the variational inequality follows standard arguments. ∎

Remark 2.1 *We point out that the convergence analysis, presented below, does not involve dual variables, i.e., the adjoint state or Lagrange multipliers. In this context, the existence of Lagrange multipliers is just required to guarantee the additional regularity of $\bar u$ which is needed for the derivation of interpolation error estimates (see Lemmata 4.3 and 4.4 below).*

3. Discretization

Now we turn to the discretization of (P). First, let us introduce a family of meshes $\{\mathcal{T}_h\}$ with mesh size $h > 0$. The mesh $\mathcal{T}_h$ consists of open cells T (triangles, tetrahedra, quadrilaterals, hexahedra) such that

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} \bar{T}$$

fulfilling usual assumptions on the finite element mesh, see, e.g., Brenner and Scott (1994). Notice that this implies that the cells lying on the boundary of Ω may be curved if Γ is smooth (see Section 6.1 for details). The mesh size is defined by

$$h := \max_{T \in \mathcal{T}_h} h_T \quad \text{with} \quad h_T := \operatorname{diam}(T).$$

With each $T \in \mathcal{T}_h$, we associate the diameter of the largest ball contained in T, denoted by R_T. We suppose the following regularity assumptions for $\{\mathcal{T}_h\}$:

Assumption 3.1 *There exist two positive constants ρ and R such that*

$$\frac{h_T}{R_T} \leq R, \quad \frac{h}{h_T} \leq \rho$$

hold for all cells $T \in \cup_{h>0} \mathcal{T}_h$.

To each mesh, we associate finite dimensional subspaces of V and L, denoted by V_h and L_h. The discrete counterpart of (1) is then given by

$$(\nabla v_h, \nabla \varphi_h) - (p_h, \nabla \cdot \varphi_h) + (\nabla \cdot v_h, \psi_h) = (u, \varphi_h) \quad \forall\, (\varphi_h, \psi_h) \in V_h \times L_h \tag{5}$$

with associated solution operator $G_h u = (v_h, p_h) \in V_h \times L_h$. Concrete choices for the pairs (V_h, L_h), allowing for existence of the solution operator G_h, will be discussed in Section 6. Analogously to the above, we define the discrete control-to-state operator S_h mapping given control u to the velocity component v_h of $G_h u$. In all what follows, we rely on the following conditions on S_h, that will be verified in Section 6 for different settings.

Assumption 3.2 *The following error estimates hold true*

$$\|S\, u - S_h\, u\|_{V_\infty} \leq c\, \delta(h)\, \|u\|_{L^\infty(\Omega)^d} \tag{6}$$

with some function $\delta\colon \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\delta(h) \to 0$ if $h \downarrow 0$, and a constant c independent of h and u.

Next, we turn to the discretization of the control. To this end, we define the associated ansatz functions.

Assumption 3.3 *For each mesh $\mathcal{T}_h$ there is a family of control ansatz functions consisting of n ansatz functions $\phi_i \in L^\infty(\Omega)$, $1 \le i \le n$. Here, the number n as well as the ansatz functions themselves are allowed to depend on the mesh and therefore on the mesh size. Moreover, for every $i \in \{1, ..., n\}$ there holds*

$$\operatorname*{ess\,sup}_{x\in\bar\Omega} \phi_i(x) = 1, \quad \phi_i(x) \ge 0 \text{ a.e. in } \Omega, \quad \sum_{i=1}^{n} \phi_i(x) = 1 \text{ a.e. in } \Omega. \qquad (7)$$

In addition, we assume that the patch $\omega_i := \operatorname{supp}\phi_i$ is a connected set of positive measure and contained in the union of M_i adjacent cells that share at least one common vertex. Further, there exists a constant $M \in \mathbb{N}$, independent of h, such that $M_i \le M$ for all $i \in \{1, ..., n\}$. Finally, each cell $T \in \mathcal{T}_h$ is contained in at most N supports ω_i, N independent of h.

Remark 3.1 *If Ω is a polygon ($d = 2$) or polyhedron ($d = 3$), the assumptions on the ansatz functions ϕ_i, $i = 1, ..., n$ are clearly fulfilled for different common finite elements such as:*

- piecewise constant *elements,*
- linear *finite elements in case of triangles and tetrahedrons, respectively,*
- bi-/trilinear *elements for quadrilaterals and hexahedrons, respectively.*

The assumption $\phi_i(x) \ge 0$ a.e. in Ω is not needed for the derivation of interpolation error estimates, but for the feasibility of interpolated controls (see Lemma 5.5).

The discrete control space is given by $U_h := \operatorname{span}\{\phi_i \,|\, 1 \le i \le n\}^d$. Now we are in the position to define the discrete counterpart to (P):

$$(\mathrm{P}_h) \quad \begin{cases} \min\limits_{u_h\in U_h} & f(u) := J(S_h\, u_h, u_h) \\ \text{s.t.} & S_h\, u_h \in K, \\ & a \le u_h(x) \le b \quad \text{a.e. in } \Omega. \end{cases}$$

Notice that (P_h) is not a fully discrete problem, since K and z are not discretized. The discretization of K and z is postponed to Section 6.3. One shows by standard arguments:

Theorem 3.1 *Assume that the feasible set for (P_h) is not empty, i.e., there exists a discrete control $u_h \in U_h$ with $a \le u_h(x) \le b$ a.e. in Ω and $S_h\, u_h \in K$. Then there exists unique solution of (P_h), denoted by $\bar u_h \in U_h$, which satisfies the following discrete variational inequality*

$$(S_h\, \bar u_h - z\, ,\, S_h\, u_h - S_h\, \bar u_h) + \alpha\, (\bar u_h\, ,\, u_h - \bar u_h) \ge 0 \quad \forall\, u_h \in U^h_{\mathrm{ad}} \qquad (8)$$

with

$$U^h_{\mathrm{ad}} := \big\{u_h \in U_h | a \le u_h(x) \le b \text{ a.e. in } \Omega,\ S_h\, u_h \in K\big\}.$$

4. Interpolation estimates

In this section we discuss some interpolation estimates for functions in W_σ. For the error analysis in the next section we need an interpolation operator which provides interpolation estimates of optimal order among other things in negative Sobolev norms (see Lemma 4.4) and additionally has the following property:

$$a \le u(x) \le b \text{ a.e. in } \Omega \quad \Rightarrow \quad a \le (\Pi_h u)(x) \le b \text{ a.e. in } \Omega. \tag{9}$$

To this end we consider the quasi-interpolation operator introduced in Carstensen (1999). For an arbitrary $u \in L^1(\Omega)$, the construction is as follows:

$$\Pi_h u = \sum_i \pi_i(u)\phi_i, \tag{10}$$

where $\pi_i(u) \in \mathbb{R}$ is defined by

$$\pi_i(u) = \frac{\int_{\omega_i} u\phi_i \, dx}{\int_{\omega_i} \phi_i \, dx}. \tag{11}$$

Analogously, the quasi-interpolation operator for vectorial quantities is defined componentwise and for simplicity also is denoted by Π_h. The property (9) is obviously fulfilled due to the above construction and Assumption 7.

In the following we discuss error estimates for $u - \Pi_h u$ in different norms on the computational domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. To keep the discussion concise, we argue for a single component for the rest of this section. The results for vectorial quantities immediately follow from norm equivalence in $\mathbb{R}^d$.

LEMMA 4.1 *For each $i \in \{1, ..., n\}$, there is a constant c_i which may depend on diam ω_i such that*

$$\|u - \pi_i(u)\|_{L^2(\omega_i)} \le c_i \, \|\nabla u\|_{L^s(\omega_i)} \quad \forall\, u \in W^{1,s}(\omega_i)$$

for all $\frac{2d}{d+2} \le s < \infty$.

REMARK 4.1 *The condition $s \ge \frac{2d}{d+2}$ is required for the embedding $W^{1,s}(\omega_i) \hookrightarrow L^2(\omega_i)$. There obviously holds:*

$$\frac{2d}{d+2} < \frac{d}{d-1} \quad \textit{for } d = 2, 3.$$

Proof. Let $i \in \{1, ..., n\}$ be arbitrary. For the proof we use an indirect argument. If the proposed assertion is false, there exists a sequence $\{u_k\} \subset W^{1,s}(\omega_i)$ with

$$\|u_k - \pi_i(u_k)\|_{L^2(\omega_i)} = 1 \quad \text{and} \quad \|\nabla u_k\|_{L^s(\omega_i)} \le \frac{1}{k} \quad \forall\, k \in \mathbb{N}.$$

Now, let us consider $v_k = u_k - \pi_i(u_k)$. Since $\pi_i(u_k) \in \mathbb{R}$ by construction, we have $\nabla \pi_i(u_k) = 0$ and therefore

$$\|v_k\|_{L^2(\omega_i)} = 1 \quad \text{and} \quad \|\nabla v_k\|_{L^s(\omega_i)} \le \frac{1}{k} \quad \forall\, k \in \mathbb{N}. \tag{12}$$

Therefore, thanks to $s \le 2$, $\{v_k\}$ is bounded in $W^{1,s}(\omega_i)$ and there exists a subsequence denoted again by $\{v_k\}$ with

$$v_k \rightharpoonup v \quad \text{in } W^{1,s}(\omega_i)$$

and therefore

$$v_k \to v \quad \text{in } L^s(\omega_i).$$

Due to (12), ∇v_k is a Cauchy sequence in $L^s(\omega_i)$ and therefore

$$v_k \to v \quad \text{in } W^{1,s}(\omega_i).$$

Hence, $\nabla v = 0$ and $v = const$. Moreover there holds by the definition of π_i:

$$\int_{\omega_i} v_k \phi_i \, dx = 0,$$

and therefore

$$\int_{\omega_i} v \phi_i \, dx = 0,$$

which implies $v = 0$. Due to the embedding $W^{1,s}(\omega_i) \hookrightarrow L^2(\omega_i)$, we have $v_k \to v$ in $L^2(\omega_i)$ and therefore $\|v\|_{L^2(\omega_i)} = 1$. This is a contradiction. ∎

LEMMA 4.2 *There is a constant c which is independent of h such that*

$$\|u - \pi_i(u)\|_{L^2(\omega_i)} \le c\, h^{d\left(\frac{1}{2}-\frac{1}{s}\right)+1} \|\nabla u\|_{L^s(\omega_i)} \quad \forall\, u \in W^{1,s}(\omega_i)$$

for all $i \in \{1, ..., n\}$ and all $\frac{2d}{d+2} \le s < \infty$.

Proof. The proof uses the assertion from Lemma 4.1 on a reference patch $\hat{\omega}_i$ and a standard transformation argument. For convenience of the reader, we shortly sketch the arguments for a domain with polygonal ($d = 2$) or polyhedral ($d = 3$) boundary and the case of triangles and tetrahedra, respectively. Let ω_i be an arbitrary patch consisting of the cells $T_j^{(i)}$, $j = 1, ..., M_i$. By Assumption 3.3, $M = \max_i\{M_i\}$ is bounded independently of h. To each patch ω_i, we associate a reference patch $\hat{\omega}_i$ whose vertices lie on the surface of the unit ball in $\mathbb{R}^d$. Moreover, it consists of M_i congruent cells $\hat{T}_j^{(i)}$. Due to $M_i \le M$, the number of possible reference patches is finite and they can be constructed so that $|\hat{T}_j^{(i)}|$ is bounded from below and above by constants independent of h (see Bernardi, 1989, Section 4, for the construction of suitable reference patches). Now denote by F_i, $F_i\hat{x} = x$, the bi-Lipschitz transformation from $\hat{\omega}_i$ to ω_i, and set $F_j^{(i)} := F_i|_{\hat{T}_j^{(i)}}$, i.e. the affine-linear transformation from $\hat{T}_j^{(i)}$ to $T_j^{(i)}$. Analogously to (11), let $\hat{\pi}_i$ be defined by

$$\hat{\pi}_i(v) := \frac{\int_{\hat{\omega}_i} \hat{\phi}_i\, v\, d\hat{x}}{\int_{\hat{\omega}_i} \hat{\phi}_i\, d\hat{x}} = \frac{\int_{\hat{\omega}_i} (\phi_i \circ F_i)\, v\, d\hat{x}}{\int_{\hat{\omega}_i} \phi_i \circ F_i\, d\hat{x}},$$

where $\hat{\phi}_i$ denotes the ansatz function on $\hat{\omega}_i$. Then, due to $u \circ F_i \in W^{1,s}(\hat{\omega}_i)$, we obtain

$$\begin{aligned}
\|u - \pi_i(u)\|^2_{L^2(\omega_i)} &= \sum_{j=1}^{M_i} \frac{|T_j^{(i)}|}{|\hat{T}_j^{(i)}|} \int\limits_{\hat{T}_j^{(i)}} \Big(u(F_j^{(i)}\hat{x}) - \pi_i(u)\Big)^2 d\hat{x} \\
&\le c\, h^d \int\limits_{\hat{\omega}_i} \big(u \circ F_i - \hat{\pi}_i(u \circ F_i)\big)^2 d\hat{x} \le c\, h^d \Bigg(\int\limits_{\hat{\omega}_i} |\nabla_{\hat{x}}(u \circ F_i)|^s d\hat{x} \Bigg)^{\frac{2}{s}} \\
&\le c\, h^d \Bigg(\sum_{j=1}^{M_i} \frac{|\hat{T}_j^{(i)}|}{|T_j^{(i)}|} \int\limits_{T_j} |\nabla_x u|^s \Big|\frac{\partial x}{\partial \hat{x}}\Big|^s dx \Bigg)^{\frac{2}{s}} \le c\, h^{d\left(1-\frac{2}{s}\right)+2} \|\nabla u\|^2_{L^s(\omega_i)}
\end{aligned}$$

with a constant $c > 0$ independent of h. If quadrilaterals or hexahedra are used, one argues analogously using suitably defined reference patches. In case of smooth boundaries, where $F_j^{(i)}$ is not longer affine-linear, the result follows from similar transformation arguments known from the theory of interpolation on curved domains (see Bernardi, 1989, Lemma 2.3). ■

LEMMA 4.3 *There is a constant c which is independent of h such that*

$$\|u - \Pi_h u\|_{L^2(\Omega)} \le c\, h^{d\left(\frac{1}{2}-\frac{1}{s}\right)+1}\, \|\nabla u\|_{L^s(\Omega)} \quad \forall\, u \in W^{1,s}(\Omega),$$

with $\frac{2d}{d+2} \le s \le 2$.

Proof. Due to $\sum_{i=1}^n \phi_i \equiv 1$ and the definition of Π_h, we find for all $v \in L^2(\Omega)$

$$\begin{aligned}
(u - \Pi_h u, v) &= \left(u \sum_{i=1}^n \phi_i - \sum_{i=1}^n \pi_i(u)\phi_i, v \right) = \sum_{i=1}^n \int_{\omega_i} (u - \pi_i(u))\, \phi_i\, v\, dx, \\
&\le c\, h^{d\left(\frac{1}{2}-\frac{1}{s}\right)+1} \sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)}\, \|v\|_{L^2(\omega_i)} \\
&\le c\, h^{d\left(\frac{1}{2}-\frac{1}{s}\right)+1} \left(\sum_{i=1}^n \|\nabla u\|^s_{L^s(\omega_i)} \right)^{1/s} \left(\sum_{i=1}^n \|v\|^{s'}_{L^2(\omega_i)} \right)^{1/s'} .
\end{aligned}$$

Using the fact that $\frac{s'}{2} \geq 1$ since $s \leq 2$, we have

$$\sum_{i=1}^{n} \|v\|_{L^2(\omega_i)}^{s'} = \sum_{i=1}^{n} \left(\|v\|_{L^2(\omega_i)}^{2}\right)^{\frac{s'}{2}} \leq \left(\sum_{i=1}^{n} \|v\|_{L^2(\omega_i)}^{2}\right)^{\frac{s'}{2}}.$$

Hence,

$$|(u - \Pi_h u, v)| \leq c\, h^{d\left(\frac{1}{2}-\frac{1}{s}\right)+1} \|\nabla u\|_{L^s(\Omega)} \|v\|_{L^2(\Omega)}.$$

Notice that Assumption 3.3 implies $\sum_{i=1}^{n} \|\nabla w\|_{L^q(\omega_i)}^{q} \leq c \|\nabla w\|_{L^q(\Omega)}^{q}$ for every $w \in W^{1,q}(\Omega)$ and every $1 \leq q < \infty$. Setting $v = u - \Pi_h u$, we complete the proof. ∎

LEMMA 4.4 *There exists a constant c, independent of h, such that*

$$\|u - \Pi_h u\|_{W^{1,s}(\Omega)^*} \leq c\, h^{2d\left(\frac{1}{2}-\frac{1}{s}\right)+2} \|u\|_{W^{1,s}(\Omega)} \quad \forall\, u \in W^{1,s}(\Omega)$$

with $\frac{2d}{d+2} \leq s \leq 2$.

Proof. Similarly to the beginning of the proof of the previous lemma, one has for all $v \in W^{1,s}(\Omega)$:

$$(u - \Pi_h u, v) = \left(u \sum_{i=1}^{n} \phi_i - \sum_{i=1}^{n} \pi_i(u)\phi_i, v\right) = \sum_{i=1}^{n} \int_{\omega_i} (u - \pi_i(u))\, \phi_i\, v\, dx.$$

The definition of π_i implies

$$\int_{\omega_i} (u - \pi_i(u))\, \phi_i\, dx = 0,$$

and therefore we continue with

$$\begin{aligned}
(u - \Pi_h u, v) &= \sum_{i=1}^{n} \int_{\omega_i} (u - \pi_i(u))\, \phi_i\, (v - \pi_i(v))\, dx \\
&\leq c\, h^{2d\left(\frac{1}{2}-\frac{1}{s}\right)+2} \sum_{i=1}^{n} \|\nabla u\|_{L^s(\omega_i)} \|\nabla v\|_{L^s(\omega_i)} \\
&\leq c\, h^{2d\left(\frac{1}{2}-\frac{1}{s}\right)+2} \left(\sum_{i=1}^{n} \|\nabla u\|_{L^s(\omega_i)}^{s}\right)^{\frac{1}{s}} \left(\sum_{i=1}^{n} \|\nabla v\|_{L^s(\omega_i)}^{s'}\right)^{\frac{1}{s'}}.
\end{aligned}$$

Using the fact $\frac{s'}{s} \geq 1$ since $s \leq 2$, we obtain

$$\sum_{i=1}^{n} \|\nabla v\|_{L^s(\omega_i)}^{s'} = \sum_{i=1}^{n} \left(\|\nabla v\|_{L^s(\omega_i)}^{s}\right)^{\frac{s'}{s}} \leq \left(\sum_{i=1}^{n} \|\nabla v\|_{L^s(\omega_i)}^{s}\right)^{\frac{s'}{s}}$$

so that

$$|(u - \Pi_h u, v)| \leq c\, h^{2d\left(\frac{1}{2}-\frac{1}{s}\right)+2} \left(\sum_{i=1}^{n} \|\nabla u\|_{L^s(\omega_i)}^s\right)^{\frac{1}{s}} \left(\sum_{i=1}^{n} \|\nabla v\|_{L^s(\omega_i)}^s\right)^{\frac{1}{s}}$$
$$\leq c\, h^{2d\left(\frac{1}{2}-\frac{1}{s}\right)+2} \|u\|_{W^{1,s}(\Omega)} \|v\|_{W^{1,s}(\Omega)}.$$

This completes the proof. ■

LEMMA 4.5 *For every $u \in L^\infty(\Omega)$, there holds*

$$\|\Pi_h\, u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}.$$

Proof. In view of (11), we obtain

$$|\pi_i(u)| \leq \|u\|_{L^\infty(\Omega)} \quad \forall\, i \in \{1, ..., n\}.$$

Together with (7), this implies

$$\left|\sum_{i=1}^{n} \pi_i(u)\, \phi_i(x)\right| \leq \max_i\{|\pi_i(u)|\} \sum_{i=1}^{n} \phi_i(x) \leq \|u\|_{L^\infty(\Omega)} \quad \forall\, x \in \Omega,$$

which gives the assertion. ■

5. Convergence analysis

With the above results at hand, in particular Lemmata 4.3 and 4.4, one can extend the theory from Meyer (2008) to problem (P). The analysis of Meyer (2008) is mainly based on the existence of functions $u_d \in U_h$ and $u_c \in U$ which are feasible for one of the problems (P) or (P$_h$), but in some sense close to the solution of the other problem. In Meyer (2008), the proofs are presented for the case of box constraints on the state. With the help of the support functional, the arguments can easily be adapted to the more general state constraint in (P). For convenience of the reader, this is demonstrated in the following section. We characterize the convex set K by means of the support functional: since the interior of K is not empty by Assumption 2.2, the supporting hyperplane theorem implies

$$\operatorname{int} K = \bigcap_{\mu \in V_\infty^*,\, \mu \neq 0} \{v \in V_\infty \mid \langle \mu\,,\, v\rangle_{V_\infty^*, V_\infty} < s(\mu)\}, \tag{13}$$

where $s : V_\infty^* \to \mathbb{R}$ denotes the support functional, i.e. $s(\mu) = \sup_{v \in K} \langle \mu\,,\, v\rangle_{V_\infty^*, V_\infty}$ (see, e.g., Luenberger, 1969). Based on Assumption 2.2, we find the following

LEMMA 5.1 *There is a constant $\tau > 0$ such that*

$$\langle \mu\,,\, S\,\hat{u}\rangle_{V_\infty^*, V_\infty} \leq s(\mu) - \tau \quad \textit{for all } \mu \in V_\infty^* \textit{ with } \|\mu\|_{V_\infty^*} = 1 \tag{14}$$

holds true.

Proof. First, since $S\,\hat{u} \in \operatorname{int} K$ by Assumption 2.2, there exists an $r > 0$ such that $\{v \in V_\infty \mid \|v - S\,\hat{u}\|_{V_\infty} \le r\} \subset K$. Hence, for all $w \in V_\infty$ with $\|w\|_{V_\infty} = 1$, we have $S\,\hat{u} \pm r\,w \in K$. Now let μ with $\|\mu\|_{V_\infty^*} = 1$ be arbitrary. Since

$$\|\mu\|_{V_\infty^*} = \sup_{\|w\|_{V_\infty}=1} |\langle \mu\,,\, w\rangle_{V_\infty^*,V_\infty}| = 1,$$

there is a $\hat{w}$ with $\|\hat{w}\|_{V_\infty} = 1$ such that $|\langle \mu\,,\, \hat{w}\rangle| \ge 1/2$. For the rest of the proof assume that $\langle \mu\,,\, \hat{w}\rangle \ge 1/2$ and define $\hat{v} := S\,\hat{u} + r\,\hat{w}$. If $-\langle \mu\,,\, \hat{w}\rangle \ge 1/2$, the proof can be carried out analogously with $\hat{v} := S\,\hat{u} - r\,\hat{w}$. Clearly, by construction, $\hat{v} \in K$ such that

$$s(\mu) \ge \langle \mu\,,\, \hat{v}\rangle_{V_\infty^*,V_\infty} = \langle \mu\,,\, S\,\hat{u}\rangle_{V_\infty^*,V_\infty} + r\langle \mu\,,\, \hat{w}\rangle_{V_\infty^*,V_\infty} \ge \langle \mu\,,\, S\,\hat{u}\rangle_{V_\infty^*,V_\infty} + \frac{r}{2}.$$

Hence, setting $\tau = r/2$ finally gives the assertion. ∎

Next recall that σ is a fixed, but arbitrary number in $[\bar{\sigma}, d/(d-1)[$ and $W_\sigma = W^{1,\sigma}(\Omega)^d$.

DEFINITION 5.1 *Given* $\sigma \in [\bar{\sigma}, d/(d-1)[$ *and* $h > 0$, *we set*

$$\eta(\sigma, h) := h^{2d(\frac{1}{2}-\frac{1}{\sigma})+2}$$
$$\beta(\sigma, h) := \max\{\eta(\sigma, h), \delta(h)\},$$

where $\delta(h)$ *is defined as in Assumption 3.2. Moreover, we define*

$$u_c := \bar{u}_h + \gamma_c \delta(h)(\hat{u} - \bar{u}_h)$$
$$u_d := \Pi_h\,\bar{u} + \gamma_d\,\beta(\sigma, h)(\Pi_h\,\hat{u} - \Pi_h\,\bar{u}),$$

with constants $\gamma_c, \gamma_d > 0$ *defined in the subsequent.*

LEMMA 5.2 *There exists a constant* γ_c *independent of* h *such that the function* u_c *is feasible for* (P) *for all* $h < h_1$, *where* $h_1 > 0$ *is chosen sufficiently small so that* $\gamma_c\,\delta(h) \le 1$.

Proof. First we show $S\,u_c \in K$. To this end, let $\mu \in V_\infty^*$, $\mu \ne 0$, be arbitrary and define

$$\tilde{\mu} := \frac{1}{\|\mu\|_{V_\infty^*}}\,\mu.$$

such that $\|\tilde{\mu}\|_{V_\infty^*} = 1$. Then, by Lemma 5.1, one obtains

$$\begin{aligned}
&\langle \tilde{\mu}\,,\, S\,u_c\rangle_{V_\infty^*,V_\infty} = (1 - \gamma_c\delta(h))\langle \tilde{\mu}\,,\, S\,\bar{u}_h\rangle_{V_\infty^*,V_\infty} + \gamma_c\delta(h)\langle \tilde{\mu}\,,\, S\hat{u}\rangle_{V_\infty^*,V_\infty} \\
&\le (1 - \gamma_c\delta(h))\big[\langle \tilde{\mu}\,,\, S_h\,\bar{u}_h\rangle_{V_\infty^*,V_\infty} + \langle \tilde{\mu}\,,\, (S - S_h)\bar{u}_h\rangle_{V_\infty^*,V_\infty}\big] + \gamma_c\delta(h)\big(s(\tilde{\mu}) - \tau\big) \\
&\le s(\tilde{\mu}) - \gamma_c\delta(h)\tau + (1 - \gamma_c\delta(h))\|\tilde{\mu}\|_{V_\infty^*}\,\|(S - S_h)\bar{u}_h\|_{V_\infty} \\
&\le s(\tilde{\mu}) - \delta(h)\big(\gamma_c\tau - c(1 - \gamma_c\delta(h))\|\bar{u}_h\|_{L^\infty(\Omega)^d}\big) \qquad (15)
\end{aligned}$$

where we used Assumption 3.2, (14), and the feasibility of $\bar{u}_h$ for (P_h) which implies $\langle \tilde{\mu}\,,\, S_h\, \bar{u}_h\rangle \leq s(\tilde{\mu})$. In view of the control constraints in (P), we obtain for the second addend in the last inequality

$$\gamma_c \tau - c(1 - \gamma_c \delta(h))\|\bar{u}_h\|_{L^\infty(\Omega)^d} \geq \gamma_c \tau - c\, \max\{|a|, |b|\}$$

such that $\langle \tilde{\mu}\,,\, S\, u_c\rangle < s(\tilde{\mu})$ is fulfilled if we choose $\gamma_c > c\, \max\{|a|, |b|\}/\tau$. Hence, u_c satisfies

$$\langle \mu\,,\, S\, u_c\rangle_{V_\infty^*, V_\infty} = \|\mu\|_{V_\infty^*}\, \langle \tilde{\mu}\,,\, S\, u_c\rangle_{V_\infty^*, V_\infty} < \|\mu\|_{V_\infty^*}\, s(\tilde{\mu}) = s(\mu),$$

since the support functional is clearly sublinear. As μ was chosen arbitrary, (13) implies $S\, u_c \in K$ if $\gamma_c > c\, \max\{|a|, |b|\}/\tau$. Furthermore, if we choose h_1 small enough such that $\gamma_c \delta(h) \leq 1$, then u_c is a convex linear combination of two functions in $\{u \in L^\infty(\Omega)^d \,|\, a \leq u(x) \leq b \text{ a.e. in } \Omega\}$ and therefore also satisfies the control constraints in (P). Consequently, the assertion holds true. ∎

To prove a similar result for the other direction, i.e., the feasibility of u_d for (P_h), we need some auxiliary results which are presented in the subsequent.

LEMMA 5.3 *Suppose $u \in W_\sigma$ is given. Then*

$$\|S(u - \Pi_h\, u)\|_{V_\infty} \leq c\, \eta(\sigma, h)\, \|u\|_{W_\sigma}$$

holds true with a constant $c > 0$ independent of h.

Proof. The mapping properties of S in Proposition 2.1 imply

$$\|S(u - \Pi_h\, u)\|_{V_\infty} \leq c\, \|S\|_{\mathcal{L}(W_\sigma^*, W^{1,\sigma'}(\Omega)^d)}\, \|u - \Pi_h\, u\|_{W_\sigma^*} \leq c\, \eta(\sigma, h)\, \|u\|_{W_\sigma},$$

where we used Lemma 4.4 and the definition of η. ∎

LEMMA 5.4 *Let $\tilde{\mu} \in V_\infty^*$ with $\|\tilde{\mu}\|_{V_\infty^*} = 1$ be arbitrary. Then, for every $u \in W_\sigma \cap L^\infty(\Omega)^d$,*

$$\langle \tilde{\mu}\,,\, S_h\, \Pi_h\, u\rangle_{V_\infty^*, V_\infty} \leq \langle \tilde{\mu}\,,\, S\, u\rangle_{V_\infty^*, V_\infty} + c\, \beta(\sigma, h)\, \big(\|u\|_{W_\sigma} + \|u\|_{L^\infty(\Omega)^d}\big)$$

is satisfied with a constant $c > 0$ independent of h and u.

Proof. In view of $\|\tilde{\mu}\|_{V_\infty^*} = 1$, we find

$$\begin{aligned}
&\langle \tilde{\mu}\,,\, S_h\, \Pi_h\, u\rangle_{V_\infty^*, V_\infty} \\
&\qquad = \langle \tilde{\mu}\,,\, S\, u\rangle_{V_\infty^*, V_\infty} + \langle \tilde{\mu}\,,\, S(\Pi_h\, u - u)\rangle_{V_\infty^*, V_\infty} + \langle \tilde{\mu}\,,\, (S_h - S)\Pi_h\, u\rangle_{V_\infty^*, V_\infty} \\
&\qquad \leq \langle \tilde{\mu}\,,\, S\, u\rangle_{V_\infty^*, V_\infty} + \|\tilde{\mu}\|_{V_\infty^*}\, \|S(\Pi_h\, u - u)\|_{V_\infty} + \|\tilde{\mu}\|_{V_\infty^*}\, \|(S_h - S)\Pi_h\, u\|_{V_\infty} \\
&\qquad \leq \langle \tilde{\mu}\,,\, S\, u\rangle_{V_\infty^*, V_\infty} + c\, \big(\eta(\sigma, h)\, \|u\|_{W_\sigma} + \delta(h)\, \|u\|_{L^\infty(\Omega)^d}\big)
\end{aligned}$$

where we used Lemma 5.3, Assumption 3.2, and Lemma 4.5. With the definition of β (see Definition 5.1), the assertion is verified. ∎

LEMMA 5.5 *There exists a constant γ_d depending on $\bar{u}$ and σ, but not on h, such that u_d is feasible for* (P_h) *if $h < h_2$, where $h_2 > 0$ is chosen so that $\gamma_d\, \beta(\sigma, h) \leq 1$.*

Proof. Let $\mu \in V_\infty^*$ again be arbitrary and define $\tilde{\mu} = \mu/\|\mu\|_{V_\infty^*}$ as in the proof of Lemma 5.2. Similarly to (15), we estimate

$$
\begin{aligned}
&\langle \tilde{\mu}\,,\, S_h\, u_d\rangle_{V_\infty^*, V_\infty} \\
&= (1 - \gamma_d\beta(\sigma, h))\langle \tilde{\mu}\,,\, S_h\, \Pi_h\, \bar{u}\rangle_{V_\infty^*, V_\infty} + \gamma_d\beta(\sigma, h)\,\langle \tilde{\mu}\,,\, S_h\, \Pi_h\, \hat{u}\rangle_{V_\infty^*, V_\infty} \\
&\leq\, (1 - \gamma_d\beta(\sigma, h))\Big[\langle \tilde{\mu}\,,\, S\,\bar{u}\rangle_{V_\infty^*, V_\infty} + c\,\beta(\sigma, h)\,\big(\|\bar{u}\|_{W_\sigma} + \|\bar{u}\|_{L^\infty(\Omega)^d}\big)\Big] \\
&\qquad + \gamma_d\beta(\sigma, h)\Big[\langle \tilde{\mu}\,,\, S\,\hat{u}\rangle_{V_\infty^*, V_\infty} + c\,\beta(\sigma, h)\,\big(\|\hat{u}\|_{W_\sigma} + \|\hat{u}\|_{L^\infty(\Omega)^d}\big)\Big] \\
&\leq\, s(\tilde{\mu}) - \beta(\sigma, h)\Big[\gamma_d\tau - c\,\big(\underbrace{\|\bar{u}\|_{W_\sigma} + \|\bar{u}\|_{L^\infty(\Omega)^d} + \|\hat{u}\|_{W_\sigma} + \|\hat{u}\|_{L^\infty(\Omega)^d}}_{:=\, c_u}\big)\Big].
\end{aligned}
\tag{16}
$$

Hence, if we choose $\gamma_d > c\, c_u/\tau$, then one obtains $\langle \tilde{\mu}\,,\, S_h\, u_d\rangle < s(\tilde{\mu})$ which gives, in turn, $S_h\, u_d \in K$ by the same arguments as in the proof of Lemma 5.2. Notice that γ_d is independent of h, but depends on $\|\bar{u}\|_{W_\sigma}$ and therefore on $\bar{u}$ and σ. Moreover, we have that

$$a \leq (\Pi_h u)(x) \leq b \quad \text{a.e. in } \Omega,$$

see (9). Hence, the same arguments as in the proof of Lemma 5.2 give

$$a \leq u_d(x) \leq b \quad \text{a.e. in } \Omega,$$

for all $h < h_2$ if h_2 is sufficiently small, namely fulfills $\gamma_d\beta(\sigma, h) \leq 1$. Since $u_d \in U_h$ by construction, we therefore end up with $u_d \in U_{\mathrm{ad}}^h$. ∎

Now we are in the position to prove our main result which reads as follows:

THEOREM 5.1 *Let $\bar{u}$ and $\bar{u}_h$ denote the optimal solutions of* (P) *and* (P_h)*, respectively. Then, under Assumptions 2.1–2.2 and 3.1–3.3, the following estimate holds true for all $h < \min\{h_1, h_2\}$*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^d} + \|S\,\bar{u} - S_h\,\bar{u}_h\|_{L^2(\Omega)^d} \leq C\,\sqrt{\max\{\eta(\sigma, h), \delta(h)\}}$$

with a constant $C > 0$ which depends on $\bar{u}$ and σ, but not on h.

Proof. Based on a technique introduced in Falk (1973), it is shown in Meyer (2008) that the variational inequalities (3) and (8) imply for all $u \in U_{\mathrm{ad}}$ and all $u_h \in U_{\mathrm{ad}}^h$

$$
\begin{aligned}
&\frac{\alpha}{2}\,\|\bar u - \bar u_h\|^2_{L^2(\Omega)^d} + \frac{1}{2}\,\|S\,\bar u - S_h\,\bar u_h\|^2_{L^2(\Omega)^d}\\
&\leq c\Big[\|u_h - \bar u\|^2_{L^2(\Omega)^d} + \big(\|\bar u\|_{W_\sigma} + \|S\,\bar u - z\|_{L^2(\Omega)^d}\big)\big(\|u - \bar u_h\|_{W_\sigma^*} + \|u_h - \bar u\|_{W_\sigma^*}\big)\\
&\qquad + \|u_h - \bar u\|^2_{W_\sigma^*} + \|(S - S_h)u_h\|^2_{L^2(\Omega)^d}\\
&\qquad + \|S\,\bar u - z\|_{L^2(\Omega)^d}\big(\|(S - S_h)\bar u_h\|_{L^2(\Omega)^d} + \|(S - S_h)u_h\|_{L^2(\Omega)^d}\big)\Big].
\end{aligned}
\tag{17}
$$

Here, the constant c depends on α, but not on $\bar u$, $\bar u_h$, u, and u_h. Thanks to Lemmata 5.2 and 5.5, we are allowed to insert $u = u_c$ and $u_h = u_d$ in (17). Then, by means of Lemmata 4.3 and 4.4 and the definition of β, we obtain

$$
\begin{aligned}
\|u_d - \bar u\|_{L^2(\Omega)^d} &\leq \|\Pi_h \bar u - \bar u\|_{L^2(\Omega)^d} + \gamma_d\,\beta(\sigma, h)\,\|\Pi_h \hat u - \Pi_h \bar u\|_{L^2(\Omega)^d}\\
&\leq c\,\big(\|\bar u\|_{W_\sigma} + \|\hat u\|_{W_\sigma}\big)\,\max\{\sqrt{\eta(\sigma, h)}, \beta(\sigma, h)\}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\|u_d - \bar u\|_{W_\sigma^*} &\leq \|\Pi_h \bar u - \bar u\|_{W_\sigma^*} + \gamma_d\,\beta(\sigma, h)\,\|\Pi_h \hat u - \Pi_h \bar u\|_{W_\sigma^*}\\
&\leq c\,\big(\|\bar u\|_{W_\sigma} + \|\hat u\|_{W_\sigma}\big)\,\beta(\sigma, h).
\end{aligned}
\tag{19}
$$

In case of $u = u_c$, we have

$$
\|u_c - \bar u_h\|_{W_\sigma^*} \leq c\,\gamma_c\,\delta(h)\,\|\hat u - \bar u_h\|_{W_\sigma^*}. \tag{20}
$$

For the remaining expressions in (17), (6) implies

$$
\begin{aligned}
\|(S_h - S)u_d\|_{L^2(\Omega)^d} &\leq c\,\delta(h)\,\|\Pi_h \bar u + \gamma_d\,\delta(h)(\Pi_h \hat u - \Pi_h \bar u)\|_{L^\infty(\Omega)^d}\\
&\leq c\,\big(\|\bar u\|_{L^\infty(\Omega)^d} + \|\hat u\|_{L^\infty(\Omega)^d}\big)\,\delta(h)
\end{aligned}
\tag{21}
$$

$$
\|(S_h - S)\bar u_h\|_{L^2(\Omega)^d} \leq c\,\delta(h)\,\|\bar u_h\|_{L^\infty(\Omega)^d}, \tag{22}
$$

where we used Lemma 4.5 for the estimation of the right hand side in (21). Notice that $\bar u$ and $\hat u$ are bounded in W_σ and $L^\infty(\Omega)^d$ due to Assumption 2.2 and Theorem 2.1, whereas $\bar u_h$ is uniformly bounded in $L^\infty(\Omega)^d$ due to the control constraints. Inserting (18)–(22) in (17) finally implies

$$
\begin{aligned}
&\frac{\alpha}{2}\,\|\bar u - \bar u_h\|^2_{L^2(\Omega)^d} + \frac{1}{2}\,\|S\,\bar u - S_h\,\bar u_h\|^2_{L^2(\Omega)^d}\\
&\leq c\Big[\max\{\eta(\sigma, h), \beta(\sigma, h)^2\} + \big(\|\bar u\|_{W_\sigma} + \|S\,\bar u - z\|_{L^2(\Omega)^d}\big)\,\big(\beta(\sigma, h) + \delta(h)\big)\\
&\qquad + \beta(\sigma, h)^2 + \delta(h)^2 + \|S\,\bar u - z\|_{L^2(\Omega)^d}\,\delta(h)\Big]\\
&\leq C^2\,\max\{\eta(\sigma, h), \delta(h)\}
\end{aligned}
$$

thanks to the definition of β. For the last estimate we tacitly assumed that $\beta(\sigma, h) \leq 1$, which is clearly fulfilled if the mesh size is small enough. An inspection of the proof yields that C depends on $\bar u$ and σ, but not on h. ∎

Corollary 5.1 *Suppose that, in addition to the assumptions of Theorem 5.1,*

$$\|(G_h - G)u\|_{H^1(\Omega)^d \times L^2(\Omega)} \leq c\,\vartheta(h)\,\|u\|_{L^\infty(\Omega)^d}$$

is fulfilled with $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$, $\vartheta(h) \to 0$ *as* $h \downarrow 0$. *Then,* $(\bar v, \bar p) = G\,\bar u$ *and* $(\bar v_h, \bar p_h) = G_h\,\bar u_h$ *satisfy*

$$\|\bar v - \bar v_h\|_{H^1(\Omega)^d} + \|\bar p - \bar p_h\|_{L^2(\Omega)} \leq C\,\max\{\vartheta(h), \sqrt{\delta(h)}, \sqrt{\eta(\sigma,h)}\}$$

with a constant C *independent of* h.

Proof. The proof is almost standard. The mapping properties of G imply

$$\begin{aligned}
&\|G\bar u - G_h \bar u_h\|_{H^1(\Omega)^d \times L^2(\Omega)} \\
&\quad\leq \|G(\bar u - \bar u_h)\|_{H^1(\Omega)^d \times L^2(\Omega)} + \|(G - G_h)\bar u_h\|_{H^1(\Omega)^d \times L^2(\Omega)} \\
&\quad\leq \|G\|_{\mathcal{L}(L^2(\Omega)^d, H^1(\Omega)^d \times L^2(\Omega))} \|\bar u - \bar u_h\|_{L^2(\Omega)^d} + c\,\vartheta(h)\,\|\bar u_h\|_{L^\infty(\Omega)^d},
\end{aligned}$$

so that Theorem 5.1 yields the assertion. ∎

6. Concrete numerical settings

In the subsequent, several control problems and discretization techniques are discussed that are covered by the above theory. The critical point is to verify (6) for a concrete discretization such that $\delta(h)$, i.e., the L^∞-error of the finite element approximation, is not worse than $\eta(\sigma,h)$, i.e., the interpolation error. To keep the discussion concise, we restrict ourselves to discretization schemes that fulfill the discrete inf-sup condition so that there is no need for stabilization. We rely on the following assumptions:

Assumption 6.1 *The spaces* $V_h \subset V$ *and* $L_h \subset L$ *satisfy the following conditions*

- *There is a number* $k \in \mathbb{N}$, $k \geq 1$, *such that*

$$V_h \subset C(\bar\Omega)^d,\ \mathcal{P}_k(T)^d \subseteq V_h|_T, \quad \mathcal{P}_{k-1}(T) \subseteq L_h|_T \quad \forall\, T \in \mathcal{T}_h. \tag{23}$$

  Consequently, there exist interpolation operators i_h^v *and* i_h^p *that fulfill standard* approximation properties. *In particular, if* $t \in \{0,1\}$ *and* $q,r,s \in [1,\infty]$ *are given such that* $W^{2,r}(\Omega) \hookrightarrow W^{t,q}(\Omega)$ *and* $W^{1,s}(\Omega) \hookrightarrow L^q(\Omega)$, *then there holds:*

$$\|\nabla^t(v - i_h^v\,v)\|_{L^q(T)} \leq c\,h^{2-t+d(1/q-1/r)}\,\|\nabla^2 v\|_{L^r(T)} \quad \forall\, v \in W^{2,r}(T) \tag{24}$$

$$\|p - i_h^p\,p\|_{L^q(T)} \leq c\,h^{1+d(1/q-1/s)}\,\|\nabla p\|_{L^s(\omega_T)} \quad \forall\, p \in W^{1,s}(\omega_T) \tag{25}$$

  for all $T \in \mathcal{T}_h$. *Here,* ω_T *denotes the union of patches associated to the ansatz functions that are non-zero on* T, *and* $c > 0$ *does not depend on* h.

- Inverse property*: For all $v_h \in V_h$,

$$\|v_h\|_{L^\infty(T)^d} \leq c\, h^{-\frac{d}{2}}\, \|v_h\|_{L^2(T)^d} \quad \forall T \in \mathcal{T}_h \tag{26}$$

is valid with a constant $c > 0$ independent of h.

- Discrete inf-sup condition*: There is a real number $\gamma > 0$, independent of h, such that*

$$\sup_{\phi_h \in V_h} \frac{(p_h, \nabla \cdot \phi_h)}{\|\nabla \phi_h\|} \geq \gamma\, \|p_h\| \quad \forall\, p_h \in L_h.$$

The conditions in Assumption 6.1 are fulfilled by many standard finite elements, in particular by all examples mentioned in the following. *Beside Assumption 6.1, we suppose Assumptions 2.1–2.2, 3.1, and 3.3 to be satisfied in all what follows.* We again point out that the assumptions on the control discretization are fulfilled by constant and linear (bilinear) ansatz functions (see Remark 7). Furthermore, we assume the mesh size to be small enough to ensure $h < \min\{h_1, h_2\}$ throughout the following (see Theorem 5.1). *The aim of the subsequent sections is to verify Assumption 3.2.*

6.1. Smooth domains with $\Omega' = \Omega$

In this section we consider the following setting: The boundary Γ is of class C^2 and the subdomain Ω', where the state constraints are imposed, coincides with the domain Ω. Before we start the discussion, let us point out that we assume a triangulation that exactly fits the boundary which is fairly artificial in case of a smooth boundary. Moreover, we tacitly supposed that the integrals in (5) are exactly evaluated which is clearly hard to implement if Ω is not polygonally bounded. Therefore, a realistic discretization would cause other types of errors, which are neglected here, since this would go beyond the scope of this paper. Notice that these problems do clearly not arise if Ω has a polygonal boundary as in the case of the subsequent sections. We apply the result of Chen (2006), which requires some additional assumptions on the discretization, in particular a local L^2-error estimate of the Ritz-projection, see Chen (2006, Section 2) for details. The additional conditions are verified by Arnold and Liu (1995) for different types of finite elements such as

- all stable discretizations formed with *Lagrange elements* such as for instance the *Taylor-Hood element* (i.e. $\mathcal{P}_2/\mathcal{P}_1$-element)
- the *Mini element*, i.e., the unstable $\mathcal{P}_1/\mathcal{P}_1$-element enriched with bubble functions.

Using a technique developed in Schatz (1998), Chen proved the following result:

THEOREM 6.1 *Assume that the solution of* (1) *satisfies* $(v, p) \in W^{1,\infty}(\Omega)^d \times L^\infty(\Omega)$*. There is a constant $c > 0$, independent of h, v, and p, such that the* *solution of* (5)*, denoted by* $(v_h, p_h) \in V_h \times L_h$*, satisfies*

$$\|v - v_h\|_{L^\infty(\Omega)^d} \le c\, h\, |\log(h)|^m \big(\inf_{w \in V_h} \|v - w\|_{W^{1,\infty}(\Omega)^d} + \inf_{q \in L_h} \|q - p\|_{L^\infty(\Omega)} \big),$$

where $m = 0$ *if* $k > 1$ *and* $m = 1$ *if* $k = 1$.

If Ω is of class C^2, then $G\colon L^p(\Omega)^d \to W^{2,p}(\Omega)^d \times W^{1,p}(\Omega)$ for all $1 < p < \infty$ (see Temam, 1977, Proposition 2.3). Therefore, together with (24) and (25), Chen's result yields

Corollary 6.1 *For every* $\varepsilon > 0$*, there is a constant* $c_\varepsilon > 0$*, independent of* h *and* u*, so that*

$$\|v - v_h\|_{L^\infty(\Omega)^d} \le c_\varepsilon\, h^{2-\varepsilon}\, \|u\|_{L^\infty(\Omega)^d}.$$

Theorem 6.2 *For every* $\varepsilon > 0$*, there holds*

$$\|\bar u - \bar u_h\|_{L^2(\Omega)^d} + \|\bar v - \bar v_h\|_{H^1(\Omega)^d} + \|\bar p - \bar p_h\|_{L^2(\Omega)} \le C\, h^{2-\frac{d}{2}-\varepsilon} \tag{27}$$

with a constant $C > 0$ *which depends on* ε*, but not on* h.

Proof. Let $\varepsilon > 0$ be given. In view of Corollary 6.1, Assumption 3.2 is fulfilled with a constant c depending on ε and $\delta(h) = h^{2-2\varepsilon}$. Moreover, by choosing $\sigma = \max\left\{\bar\sigma, \frac{d}{d-1+\varepsilon}\right\}$, we obtain $\eta(\sigma, h) \le h^{4-d-2\varepsilon}$ (see Definition 5.1). Thus, Theorem 5.1 and Corollary 5.1 together with standard finite element results give the assertion. ∎

Remark 6.1 *Notice that* C *depends on* ε *firstly because of the constant* c_ε *from Corollary 6.1 and secondly due to the coupling of* σ *and* ε.

Remark 6.2 *As above, let* $\sigma = \sigma(\varepsilon) = \max\left\{\bar\sigma, \frac{d}{d-1+\varepsilon}\right\}$ *with a fixed, but arbitrary* $\varepsilon > 0$*. Then, Lemma 4.3 implies*

$$\|u - \Pi_h u\|_{L^2(\Omega)^d} \le c\, h^{2-\frac{d}{2}-\varepsilon}\, \|u\|_{W_{\sigma(\varepsilon)}} \quad \forall\, u \in W_{\sigma(\varepsilon)} \tag{28}$$

and therefore, the order in (27) *coincides with the one of the interpolation error.*

6.2. Convex domains with polygonal or polyhedral boundary

First, we consider the case $\Omega' = \Omega$. In case of polygons and polyhedrons, respectively, the following regularity result is known. For the proof, we refer to Dauge (1989) and Kellog and Osborn (1976).

Theorem 6.3 *Let* Ω *be a convex domain with polygonal* ($d = 2$) *or polyhedral* ($d = 3$) *boundary. Then, for all* $u \in L^2(\Omega)^d$*, the unique solution* $(v, p) \in V \times L$ *of* (1) *belongs to* $H^2(\Omega)^d \times H^1(\Omega)$.

Based on this result and standard finite element error estimates, one proves for an arbitrary $u \in L^2(\Omega)$

$$\|v - v_h\|_{L^\infty(\Omega)^d} \le c\, h^{2-\frac{d}{2}}\, \|u\|_{L^2(\Omega)^d},$$

where $v = S\,u$ and $v_h = S_h\,u$ and $c > 0$ only depends on Ω (see, for instance, Rösch and Vexler, 2006, Lemma 3.2). Therefore, by setting $\delta(h) = h^{2-d/2}$ and $\sigma = \max\{\bar{\sigma}, 4/3\}$ (notice that $4/3 < d/(d-1)$ for $d = 2, 3$) such that $\eta(\delta, h) \le h^{2-d/2}$, Theorem 5.1 and Corollary 5.1 imply

Theorem 6.4 *Suppose that Ω is a convex domain with polygonal ($d = 2$) or polyhedral ($d = 3$) boundary. Then, we have*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^d} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^d} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le C\, h^{1-\frac{d}{4}}$$

with a constant $C > 0$ independent of h.

Notice that the order of convergence now differs from the one of the interpolation error. The situation changes if we restrict ourselves to two-dimensional domains with polygonal boundary and a maximum angle less or equal $\pi/2$. To see this, let us define the weighted L^2-norm as follows:

$$\|q\|_{\varsigma^\nu}^2 := \int_\Omega |q(x)|^2\, \varsigma(x)^\nu\, dx, \quad q \in L^2(\Omega)^d, \tag{29}$$

where $\varsigma : \bar{\Omega} \to \mathbb{R}_+$ is defined by

$$\varsigma(x) := \sqrt{|x - x_0|^2 + \theta^2}, \tag{30}$$

with given $x_0 \in \Omega$ and $\theta > h > 0$.

Theorem 6.5 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon whose maximum aperture angle is less or equal $\pi/2$. Moreover, suppose that (V_h, L_h) satisfies the* discrete weighted inf-sup condition, *i.e., for every $\theta > h$ and every point $x_0 \in \Omega$, there holds*

$$\sup_{\phi_h \in V_h} \frac{(p_h, \nabla\cdot\phi_h)}{\|\nabla\phi_h\|_{\varsigma^2}} \ge c\,|\log\theta|^{-1/2}\|p_h\|_{\varsigma^{-2}} \quad \forall\, p_h \in L_h, \tag{31}$$

with a constant $c > 0$ independent of h, θ, and x_0. Then, for every $\varepsilon > 0$, the discrete solution satisfies

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^2} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le C\, h^{1-\varepsilon}$$

with a constant $C > 0$ which depends on ε, but not on h.

Proof. According to a result from Kozlov, Maz'ya and Rossmann (2001, Section 5.8.1), for all $q \in [1, \infty[$, the solution $v \in (V \times L)$ of (1) belongs to $W^{2,q}(\Omega)^2 \times W^{1,q}(\Omega)$, provided that $u \in L^q(\Omega)^2$, and there holds

$$\|v\|_{W^{2,q}(\Omega)^2} + \|p\|_{W^{1,q}(\Omega)} \leq c\, \|u\|_{L^q(\Omega)^2}. \tag{32}$$

Moreover, Duran and Nochetto (1990) proved that, for all discretizations fulfilling Assumption 6.1 and (31), there exists a constant $c > 0$ independent of h such that

$$\|v - v_h\|_{L^\infty(\Omega)^2} \leq c\, h\, |\log(h)|^3 \big(\inf_{w \in V_h} \|v - w\|_{W^{1,\infty}(\Omega)^2} + \inf_{q \in L_h} \|q - p\|_{L^\infty(\Omega)} \big).$$

Hence, together with (32), (24) and (25) give the existence of a constant $c_\varepsilon > 0$, depending on ε, but not on h, such that for every $\varepsilon > 0$

$$\|v - v_h\|_{L^\infty(\Omega)^2} \leq c_\varepsilon\, h^{2-\varepsilon}\, \|u\|_{L^\infty(\Omega)^2}.$$

Then an argument, analogous to the proof of Theorem 6.2, finally implies the assertion. ∎

REMARK 6.3 *The discrete weighted inf-sup condition* (31) *is satisfied by various common stable finite elements, as proven in Duran and Nochetto (1990). We only mention*

- *the Taylor-Hood element on triangles or quadrilaterals (i.e., $\mathcal{P}_2/\mathcal{P}_1$- and $\mathcal{Q}_2/\mathcal{Q}_1$-elements, respectively)*
- *the Mini element*
- *the Crouzeix-Raviart element of different order $k \geq 2$, i.e., the $\mathcal{P}_k/\mathcal{P}_{k-1}$-element enriched with bubble functions.*

If the state constraints are only imposed in a compact subset of Ω, the results of Duran and Nochetto (1990) allow to get same the order of convergence as in the interpolation error (28), even if the maximum angle is larger than $\pi/2$. Notice that, in the presence of no-slip boundary conditions, it appears natural to consider the state constraints only in the interior of Ω, as illustrated in the introduction.

THEOREM 6.6 *Assume that Ω is a convex polygon and let $\Omega' \subset \Omega$ be given. Furthermore, we assume that, for every h, a union of cells of $\mathcal{T}_h$, denoted by Ω'', exists that contains Ω' and fulfills* $\operatorname{dist}(\overline{\Omega'}, \overline{\Omega \setminus \Omega''}) =: d > 0$ *and* $\operatorname{dist}(\overline{\Omega''}, \Gamma) =: \delta > 0$ *with d and δ independent of h. Furthermore, suppose that (V_h, L_h) satisfies the* discrete weighted inf-sup condition (31). *Then, for every $\varepsilon > 0$, there is a constant $C > 0$ depending on ε, but not on h, such that*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^2} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq C\, h^{1-\varepsilon}.$$

Proof. The proof is similar to the proof of Duran and Nochetto (1990, Theorem 4.1). In view of Theorem 6.3 and embedding theorems for $d = 2$, we have $\nabla v \in L^q(\Omega)$ for all $q < \infty$. Thus, Theorem 4.1 in Galdi (1994) yields for every $q \in [1, \infty[$ that $(v, p) \in W^{2,q}_{loc}(\Omega)^2 \times W^{1,q}_{loc}(\Omega)$ if $u \in L^q_{loc}(\Omega)^2$, which is clearly fulfilled due to the control constraints. Thus we obtain $(v, p) \in W^{2,q}(\Omega'')^2 \times W^{1,q}(\Omega'')$ for all $q < \infty$. Based on (31), it is shown in Duran and Nochetto (1990) that

$$\|v-v_h\|^2_{\varsigma^{-4}} \leq c\,\frac{h^2}{\theta^2}\,|\log\theta|^3\left(\|\nabla(v-i^v_h v)\|^2_{\varsigma^{-2}}+\|v-i^v_h v\|^2_{\varsigma^{-4}}+\|p-i^p_h p\|^2_{\varsigma^{-2}}\right) \quad (33)$$

holds for all $\theta > h > 0$ provided that Ω is a convex polygon. Here, ς and the associated norms are defined as in (30) and (29). Recall that $V_\infty = L^\infty(\Omega')^2$. We start by estimating

$$\|v - v_h\|_{V_\infty} \leq \|v - i^v_h v\|_{V_\infty} + \|v_h - i^v_h v\|_{V_\infty}.$$

Since $|v_h - i^v_h v| \in C(\bar{\Omega}')$, there is an $x_0 \in \bar{T}_0 \subseteq \overline{\Omega'}$ such that $\|v_h - i^v_h v\|_{V_\infty} = |v_h(x_0) - i^v_h v(x_0)|$. In all what follows, we use this x_0 in the definition of ς in (30). The inverse estimate (26) implies

$$\begin{aligned}|v_h(x_0) - i^v_h v(x_0)| &\leq \|v_h - i^v_h v\|_{L^\infty(T_0)^2}\\ &\leq c\,h^{-1}\,\|v_h - i^v_h v\|_{L^2(T_0)^2} \leq c\,\frac{\theta^2}{h}\,\|v_h - i^v_h v\|_{\varsigma^{-4}},\end{aligned}$$

where the last estimate follows from the definition of $\|\cdot\|_{\varsigma^{-4}}$ because of $\theta > h$. Now, one can apply (33) and continue with

$$\begin{aligned}\|v - v_h\|_{V_\infty} \leq\ & \|v - i^v_h v\|_{V_\infty} + c\,\theta|\log\theta|^{\frac{3}{2}}\left(\|\nabla(v - i^v_h v)\|_{\varsigma^{-2}} + \|p - i^p_h p\|_{\varsigma^{-2}}\right)\\ &+ c\Big(\frac{\theta^2}{h} + \theta|\log\theta|^{3/2}\Big)\|v - i^v_h v\|_{\varsigma^{-4}}.\end{aligned}$$

For an arbitrary $w \in L^\infty(\Omega)$ and $\nu \geq 0$, we obtain

$$\begin{aligned}\|w\|_{\varsigma^{-(2+\nu)}} &\leq \|w\,\varsigma^{-(1+\nu/2)}\|_{L^2(\Omega'')} + \|w\,\varsigma^{-(1+\nu/2)}\|_{L^2(\Omega\setminus\Omega'')}\\ &\leq \|w\|_{L^\infty(\Omega'')}\int_{\Omega''}\varsigma^{-(2+\nu)}\,dx^{\frac{1}{2}} + c\,\|w\|_{L^2(\Omega)},\end{aligned}$$

where we used the norm equivalence of $\|\cdot\|_{\varsigma^{-(2+\nu)}}$ and $\|\cdot\|_{L^2}$ on $\Omega\setminus\Omega''$ which holds due to $\operatorname{dist}(x_0, \overline{\Omega\setminus\Omega''}) \geq d > 0$. Together with

$$\int_{\Omega''}\varsigma^{-(2+\nu)}\,dx \leq \begin{cases} c\,\theta^{-\nu}, & \nu > 0\\ c\,|\log\theta|, & \nu = 0\end{cases}$$

(see Duran and Nochetto, 1990), it follows with $\nu = 0$ and $\nu = 2$, respectively, that

$$\begin{aligned}\|v - v_h\|_{V_\infty} \le\ & \|v - i_h^v v\|_{L^\infty(\Omega'')^2} \\ & + c\,\theta|\log\theta|^2\Big(\|\nabla(v - i_h^v v)\|_{L^\infty(\Omega'')^2} + \|\nabla(v - i_h^v v)\|_{L^2(\Omega)^2} \\ & \qquad + \|p - i_h^p p\|_{L^\infty(\Omega'')} + \|p - i_h^p p\|_{L^2(\Omega)}\Big) \\ & + c\Big(\frac{\theta}{h} + |\log\theta|^{3/2}\Big)\Big(\|v - i_h^v v\|_{L^\infty(\Omega'')^2} + \|v - i_h^v v\|_{L^2(\Omega)^2}\Big).\end{aligned}$$

Now we choose $\theta = h|\log h|$ such that $\theta > h$ for sufficiently small h. Because of the regularity of (v, p) stated at the beginning of the proof, applying (24) and (25) then yields the existence of a constant $c_\varepsilon > 0$, depending on ε, such that

$$\|v - v_h\|_{V_\infty} \le c_\varepsilon\, h^{2-\varepsilon}\, \|u\|_{L^\infty(\Omega)^2} \quad \forall\, \varepsilon > 0.$$

Here, we tacitly assumed that $h \le 1/e = 0.3679$ to ensure $|\log(h|\log h|)| \le |\log h|$. Notice, moreover, that the assumption $\mathrm{dist}(\overline{\Omega''}, \Gamma) =: \delta > 0$ implies $\mathrm{dist}(\omega_T, \Gamma) > 0$ for all $T \in \Omega''$ if h is sufficiently small. Hence, the above regularity result implies

$$p \in W^{1,q}\left(\bigcup_{T\subset\Omega''} \omega_T\right) \qquad \forall\, q < \infty$$

such that (25) applies to $\|p - i_h^p p\|_{L^\infty(\Omega'')}$. For the rest of the proof, we argue as in the proof of Theorem 6.2, which gives the assertion. ∎

6.3. Discretization of the data

Up to now, problem (P_h) is no finite dimensional optimization problem since we have not discretized the problem data, i.e., the desired state z and the set K. To this end, let us introduce the space of linear (bilinear) finite elements $V_h^{(1)} \subset V_h$ and the standard nodewise linear interpolant $i_h^{(1)}\colon C(\bar\Omega)^d \to V_h^{(1)}$. In addition, we introduce a discretization of K, denoted by $K_h \subset V_\infty$. The corresponding completely discrete problem for

$$u_h = \sum_{i=1}^{n} u_i \phi_i,$$

for simplicity also denoted by (P_h), is then given with

$$(\mathrm{P}_h)\quad \begin{cases} \min & J_h(v_h, u_h) := \dfrac{1}{2}\,\|v_h - i_h^{(1)} z\|^2_{L^2(\Omega)^d} + \dfrac{\alpha}{2}\,\|u_h\|^2_{L^2(\Omega)^d} \\ \text{s.t.} & v_h = S_h\, u_h \\ \text{and} & i_h^{(1)} v_h \in K_h \\ & u_h \in U_h,\ a \le u_i \le b \quad \forall\, i \in \{1, ..., n\}. \end{cases}$$

REMARK 6.4 *Notice that it depends on the concrete structure of K and its discretization whether* (P_h) *is straight forward to implement as a finite dimensional optimization problem or not. In the cases, discussed in this paper, the linear (bilinear) interpolation operator $i_h^{(1)}$ allows a nodewise evaluation of the state constraints and hence an easy implementation of* (P_h) *if K is discretized properly (see Remark 6.6 below).*

To shorten the description, we assume in all what follows that Assumption 3.2 is fulfilled with $\delta(h) = c\, h^{2-\varepsilon}$ with a fixed but arbitrary $\varepsilon > 0$ (see Sections 6.1 and 6.2). Moreover, for the sake of simplicity, we suppose that Ω' is a union of cells. If these assumptions are not fulfilled, the subsequent analysis can easily be modified.

ASSUMPTION 6.2 *Beside Assumptions 2.1–2.2 and 3.1–3.3, assume that $z \in H^2(\Omega)^d$. Furthermore, let Assumption 3.2 hold with*

$$\delta(h) = c\, h^{2-\varepsilon} \tag{34}$$

with some fixed but arbitrary $\varepsilon > 0$. Moreover, let Ω' be a union of cells of $\mathcal{T}_h$ for every $h > 0$ and assume that $S\colon L^\infty(\Omega)^d \to W^{2,q}(\Omega')^d$ for all $q < \infty$. Moreover, suppose that K_h is convex with associated support functional $s_h : V_\infty^* \to \mathbb{R}$ that fulfills

$$|s(\mu) - s_h(\mu)| \leq c_s\, h^{2-\varepsilon}\, \|\mu\|_{V_\infty^*} \quad \forall\, \mu \in V_\infty^* \tag{35}$$

with a constant $c_s > 0$. To guarantee the existence of a solution to (P_h)*, we require the existence of a feasible point, i.e., there is a $\hat{u} \in U_h$ with $a \leq \hat{u}_i \leq b\ \forall\, i \in \{1, ..., n\}$ and $i_h^{(1)} S_h\, \hat{u}_h \in K_h$.*

REMARK 6.5 *Notice that the hypothesis on S and $\delta(h)$ agree with the theory presented in Sections 6.1 and 6.2 (see in particular, Corollary 6.1 and the proofs of Theorems 6.5 and 6.6).*

LEMMA 6.1 *Suppose that Assumption 6.2 holds. Let $u \in L^\infty(\Omega)^d$ be arbitrary and set as before $v_h = S_h\, u$. Then, for every $\varepsilon > 0$, there is a constant $c > 0$, independent of u and h, such that*

$$\|v_h - i_h^{(1)} v_h\|_{V_\infty} \leq c\, h^{2-\varepsilon}\, \|u\|_{L^\infty(\Omega)^d}.$$

Proof. The arguments are standard. For the convenience of the reader, we sketch the proof for a single component of v_h, for simplicity also denoted by v_h. Hence $V_\infty = L^\infty(\Omega')$. Let $\varepsilon > 0$ be arbitrary. We start by estimating

$$\|v_h - i_h^{(1)} v_h\|_{L^\infty(\Omega')} \leq \|i_h^{(1)}(v - v_h)\|_{L^\infty(\Omega')} + \|v - i_h^{(1)} v\|_{L^\infty(\Omega')} + \|v - v_h\|_{L^\infty(\Omega')}$$

with $v = S\,u$. Since Ω' is a union of cells by assumption, we find for the first addend

$$\begin{aligned}\|i_h^{(1)}(v - v_h)\|_{L^\infty(\Omega')} &= \max_{T\subset\Omega'} \|i_h^{(1)}(v - v_h)\|_{L^\infty(T)} = \max_{T\subset\Omega'} \max_{x_i\in\bar{T}} |v(x_i) - v_h(x_i)| \\ &\leq \max_{T\subset\Omega'} \|v - v_h\|_{L^\infty(T)} = \|v - v_h\|_{L^\infty(\Omega')},\end{aligned}$$

where we used that $i_h^{(1)}$ is the standard linear (bilinear) interpolation operator. Here x_i denotes a node of $\mathcal{T}_h$. Moreover, $i_h^{(1)}$ satisfies

$$\|v - i_h^{(1)}v\|_{L^\infty(\Omega')} \leq c\,h^{2-d/q}\|\nabla^2 v\|_{L^q(\Omega')} \quad \forall\, q < \infty$$

(see Brenner and Scott, 1994, or Bernardi, 1989). Thus, by choosing $q = d/\varepsilon < \infty$, the mapping properties of S together with Assumption 3.2 and (34), i.e.

$$\|v - v_h\|_{L^\infty(\Omega')} \leq c\,h^{2-\varepsilon}\|u\|_{L^\infty(\Omega)},$$

give the assertion. ∎

THEOREM 6.7 *Assume that Assumption 6.2 is fulfilled. Then, for every $\varepsilon > 0$, the unique solution of* (P_h) *satisfies*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^d} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^d} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)^d} \leq C\,h^{2-\frac{d}{2}-\varepsilon}$$

where the constant $C > 0$ depends on ε but not on h.

Proof. Since z is sufficiently smooth by assumption, we have $\|z - i_h^{(1)}z\|_{L^2(\Omega)^d} \leq c\,h^2\,\|z\|_{H^2(\Omega)^d}$ due to standard interpolation estimates. In view of this, the discretization of z can easily incorporated in the presented analysis. The underlying arguments are presented in detail in Meyer (2008, Section 7). In addition, due to Assumption 3.3, it is sufficient to require the control constraints only in the coefficients of u_h as done in (P_h). If K is discretized, then the proofs of Lemmata 5.2 and 5.5 have to be modified, more precisely (15) and (11), respectively. We exemplarily consider (11), the arguments in case of (15) are similar. Using (35) and Lemmata 5.1 and 6.1, we obtain for all $\tilde{\mu}$ with $\|\tilde{\mu}\|_{V_\infty^*} = 1$

$$\begin{aligned}\langle \tilde{\mu}\,,\, i_h^{(1)}\,S_h\,u_d\rangle_{V_\infty^*,V_\infty} &\leq \langle \tilde{\mu}\,,\, S_h\,u_d\rangle_{V_\infty^*,V_\infty} + \|S_h\,u_d - i_h^{(1)}\,S_h\,u_d\|_{V_\infty} \\ &\leq s(\tilde{\mu}) - c\,h^{2-\frac{d}{2}-\varepsilon}(\gamma_d\tau - c_u) + c\,h^{2-\varepsilon}\,\|u_d\|_{L^\infty(\Omega)^d} \\ &\leq s_h(\tilde{\mu}) - c\,h^{2-\frac{d}{2}-\varepsilon}(\gamma_d\tau - c_u - c_s),\end{aligned}$$

where c_u is defined as in (11). Hence, if we choose $\gamma_d > (c_u + c_s)/\tau$, then the same arguments as in the proof of Lemma 5.5 imply that u_d is feasible for (P_h). Again γ_d depends on $\bar{u}$ and σ, but not on h. Based on the feasibility of u_c and u_d, one can argue as in the proof of Theorem 5.1 to verify the assertion. ∎

Let us investigate two exemplary state constraints that are also used for the numerical tests in Section 7:

$$K^{(1)} := \{v \in V_\infty | v_a(x) \leq v(x) \leq v_b(x) \text{ a.e. in } \Omega'\}$$
$$K^{(2)} := \{v \in V_\infty | |v(x)|^2_{\mathbb{R}^d} \leq \varrho \text{ a.e. in } \Omega'\}.$$

First, we consider $K^{(1)}$, i.e., the cases of box constraints. Let us assume that Ω' coincides with a union of cells of $\mathcal{T}_h$ and denote the set of all nodes of $\mathcal{T}_h$ by $\mathcal{N}(\mathcal{T}_h)$. We consider the following finite dimensional optimization problem

$$(\mathrm{P}_h^{(1)}) \quad \begin{cases} \min\limits_{u_h \in U_h} & J_h(v_h, u_h) \\ \text{s.t.} & v_h = S_h\, u_h \\ \text{and} & v_{a,h}(x_i) \leq v_h(x_i) \leq v_{b,h}(x_i) \quad \forall\, x_i \in \mathcal{N}(\mathcal{T}_h) \cap \overline{\Omega'} \\ & a \leq u_i \leq b \quad \forall\, i \in \{1, ..., n\}, \end{cases}$$

with $v_{b,h} = i_h^{(1)}\, v_b$ and $v_{a,h}$ defined analogously.

Corollary 6.2 *Suppose that Ω is a convex polygon and let $\Omega' \subset \Omega$ be a union of cells of $\mathcal{T}_h$ for all $h > 0$. Assume in addition that Ω' fulfills the assumptions of Theorem 6.6. Furthermore, suppose that $z \in H^2(\Omega)^d$ and $v_a, v_b \in W^{2,\infty}(\Omega')^d$. Then the solution of $(\mathrm{P}_h^{(1)})$ satisfies for every $\varepsilon > 0$*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^2} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq C\, h^{1-\varepsilon},$$

where the constant $C > 0$ depends on ε, but not on h.

Proof. First observe that the state constraints in $(\mathrm{P}_h^{(1)})$ are equivalent to the ones in (P_h) if $K^{(1)}$ is discretized as indicated above, which is demonstrated in the following. We exemplarily consider the upper bound v_b. The case with lower constraint can be discussed analogously. Let φ_i, $i = 1, ..., m$, denote the ansatz functions associated to the linear (bilinear) interpolant $i_h^{(1)}$. Since they are non-negative and satisfy $\varphi_i(x_j) = \delta_{ij}$, the state constraints in $(\mathrm{P}_h^{(1)})$ are equivalent to

$$(i_h^{(1)}\, v_h)(x) \leq v_{b,h}(x) \quad \text{a.e. in } \Omega' \qquad \Leftrightarrow \qquad i_h^{(1)}\, v_h \in K_h^{(1)},$$

where $K_h^{(1)}$ is given by

$$K_h^{(1)} := \{v \in V_\infty \,|\, v(x) \leq v_{b,h}(x) \text{ a.e. in } \Omega'\}. \tag{36}$$

Thus, the nodewise state constraints in $(\mathrm{P}_h^{(1)})$ indeed agree with the state constraints in (P_h) if $K^{(1)}$ is discretized as done in (36).

To apply Theorem 6.7, we have to verify (35). Given an arbitrary $v \in K^{(1)}$, we define $\Pi_{K_h^{(1)}}(v)(x) := \min\{v(x), v_{b,h}(x)\}$, hence $\|v - \Pi_{K_h^{(1)}}(v)\|_{V_\infty} \le \|v_b - v_{b,h}\|_{V_\infty}$. Therefore we have for every $\mu \in V_\infty^*$

$$\langle \mu\,,\, v\rangle_{V_\infty^*,V_\infty} \le \langle \mu\,,\, \Pi_{K_h^{(1)}}(v)\rangle_{V_\infty^*,V_\infty} + \|\mu\|_{V_\infty^*}\,\|v_b - v_{b,h}\|_{V_\infty} \quad \forall\, v \in K^{(1)}.$$

Since $\Pi_{K_h^{(1)}}(v) \in K_h^{(1)}$, this gives

$$s(\mu) \le s_h(\mu) + \|\mu\|_{V_\infty^*}\,\|v_b - v_{b,h}\|_{V_\infty}. \tag{37}$$

An analogous argument with $\Pi_{K^{(1)}}(v)(x) := \min\{v(x), v_b(x)\}$, $v \in K_h^{(1)}$, implies

$$s_h(\mu) \le s(\mu) + \|\mu\|_{V_\infty^*}\,\|v_b - v_{b,h}\|_{V_\infty}. \tag{38}$$

Together with (37), this verifies (35) provided that v_b is sufficiently smooth, for instance $v_b \in W^{2,\infty}(\Omega')^d$. The remaining conditions in Assumption 6.2, in particular (34), are verified by the proof of Theorem 6.6 which gives the assertion. ∎

Remark 6.6 *We point out that the introduction of the standard linear (bilinear) interpolant $i_h^{(1)}$ in (P_h) allows to obtain the desired order of convergence even if the state constraints are only evaluated in the nodes of the triangulation which is easy to implement (see Remark 6.4). The situation changes if $K^{(1)}$ is for instance discretized using quadratic ansatz functions which complicates the implementation. Similar problems arise if the bound ϱ in $K^{(2)}$ is not constant and has to be discretized.*

Now, let us turn to $K^{(2)}$, i.e., constraints on the Euclidian norm of v. For this case we set $K_h^{(2)} = K^{(2)}$. The completely discrete problem is now given by

$$(\mathrm{P}_h^{(2)}) \qquad \begin{cases} \min\limits_{u_h \in U_h} & J_h(v_h, u_h) \\ \quad\text{s.t.} & v_h = S_h\, u_h \\ \quad\text{and} & |v_h(x_i)|^2_{\mathbb{R}^2} \le \varrho \quad \forall\, x_i \in \mathcal{N}(\mathcal{T}_h) \cap \overline{\Omega'} \\ & a \le u_i \le b \quad \forall\, i \in \{1, ..., n\}. \end{cases}$$

Corollary 6.3 *Suppose that Ω is a convex polygon and $\Omega' \subset \Omega$ fulfills the assumptions of Corollary 6.2. Furthermore, assume that $z \in H^2(\Omega)^d$. Then, the solution of $(\mathrm{P}_h^{(2)})$ satisfies for every $\varepsilon > 0$*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2} + \|\bar{v} - \bar{v}_h\|_{H^1(\Omega)^2} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le C\, h^{1-\varepsilon},$$

where the constant $C > 0$ depends on ε, but not on h.

Proof. Similar arguments as in the proof of Corollary 6.2, together with the convexity of $|\cdot|^2_{\mathbb{R}^2}$ imply that the state constraints in $(\mathrm{P}_h^{(2)})$ are equivalent to $|(i_h^{(1)} v_h)(x)|^2_{\mathbb{R}^2} \le \varrho$ a.e. in Ω'. Thus, Theorem 6.7 and the same arguments as in the proof of Theorem 6.6 give the assertion. ∎

7. Numerical experiments

In this section we perform numerical tests in order to verify the finite element error estimates obtained in the previous sections. The convex polygonal domain $\Omega = (0,1) \times (0,1)$ was discretized using a uniform triangular mesh. Boundary conditions of Dirichlet type were imposed on the boundary.

In the first example, the horizontal velocity on the upper boundary takes the value one, while the vertical component is zero. On the remaining boundary the condition is of no slip type. This problem is known in the literature as the "driven cavity flow".

In the second example we consider homogeneous Dirichlet boundary conditions and try to track the fluid to the vector field given by

$$z = \begin{pmatrix} sin(\pi x)^2 \cdot \sin(\pi y) \cdot \cos(\pi y) \\ -sin(\pi y)^2 \cdot \sin(\pi x) \cdot \cos(\pi x) \end{pmatrix}. \tag{39}$$

Let us point out that the latter test case is covered by the above theory, whereas the driven cavity example is strictly speaking not captured by the afore presented analysis due to a lack of regularity induced by the non-continuous inhomogeneity in the boundary conditions. Nevertheless, the driven cavity flow is investigated here as it can be seen as a benchmark for the Stokes system.

For the finite element discretization, we use Taylor-Hood elements with quadratic ansatz functions for the velocity and linear functions for the pressure. The controls were also discretized using piecewise linear polynomials, consistent with the conditions in Assumption 3.3. The discretized inequality constrained optimization problems are solved by applying a semi-smooth Newton method as stated in Hintermuller, Ito and Kunisch (2002). The inequality state constraints are added to the cost functional through a penalized Moreau-Yosida regularization term, see, e.g., De Los Reyes and Kunisch (2005).

For the solution of the discretized systems appearing in each semi-smooth Newton step a penalty method is applied (see Gunzburger, 2000, p. 125). This method considers, for $0 < \epsilon << 1$, the modified Stokes system

$$\begin{pmatrix} A & B^T \\ B & \epsilon I \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} M\vec{u} \\ 0 \end{pmatrix},$$

where A, B, and M are the matrices resulting from the finite element discretization of (1), I is the identity matrix, and $\vec{v}, \vec{p}$, and $\vec{u}$ are the vectors for the velocity, pressure, and control, respectively. A similar penalty scheme was used for the adjoint equations. For convergence results on this approach we refer to Gunzburger (2000).

The semi-smooth Newton algorithm stops if the L^2-residuum of the discretized control is lower than a given tolerance, typically set as 10^{-4}. The method is initialized setting the controls equal to 0 and solving successively the Stokes and the adjoint equations. With these values at hand, the active and inactive sets are determined for the first iteration.

The resulting linear systems in each semismooth Newton iteration were solved using MATLAB exact solver. All algorithms were implemented in MATLAB 7.4 and run on a 300 GHz machine with 24 GByte RAM and a precision of eps=2.2204e-16.

7.1. Example 1: box constraints

First, we consider pointwise box constraints on the state, i.e., constraints of the form $K^{(1)}$. To be more precise, the state constraint is given by $y_1 \geq -0.15$ in $\Omega_s = [0.1, 0.9] \times [0.1, 0.9]$. The target is to diminish the backward flow velocity and, as a consequence, the intensity of the vortex. The desired state is given by $z \equiv 0$. Thus, the example fits to the setting of Corollary 6.2. The Tikhonov regularization parameter is set to $\alpha = 0.1$, while we choose 10^5 as penalization parameter for the state constraints.

With a mesh size $h = \sqrt{2}/32$ the algorithm stops after 20 iterations. The horizontal and vertical components of the optimal control are depicted in Fig. 1, for $h = \sqrt{2}/64$. In Fig. 2 the optimal control vector field and the active set for the horizontal velocity component are depicted. From the graphics, the concentration of the irregular part of the horizontal control on the active set can be observed.

In Table 1 the convergence history is registered. The experimental error norms for different values of h are tabulated. We consider as optimal solution the one obtained numerically with a mesh step size $h = \sqrt{2}/160$, which will be denoted by u_h^* in all what follows. The quantity #it refers to the number of semi-smooth Newton iterations.

Table 1. Example 1, convergence history.

$\sqrt{2}/h$	5	10	20	40	80
#it	4	8	20	20	32
$\\|u_h - u_h^*\\|_{L^2}$	1.1601	0.7982	0.4804	0.2572	0.1098

To illustrate the convergence behavior, we define the quantity

$$EOC_2(u) := \frac{\log(\|u_h - u_h^*\|_{L^2}) - \log(\|u_{h_{\mathrm{ref}}} - u_h^*\|_{L^2})}{\log(h) - \log(h_{\mathrm{ref}})} \tag{40}$$

as the experimental order of convergence for the L^2-norm of u. Here, h_{ref} refers to the finest mesh size, hence in this case $h_{\mathrm{ref}} = \sqrt{2}/80$. The values for $EOC_2(u)$ are listed in Table 2. From this table, a rough coincidence between the theoretical and experimental convergence order can be inferred, since the experimental order of convergence order averages approximately $1 - \varepsilon$. This observation confirms the theoretical predictions of Corollary 6.2.

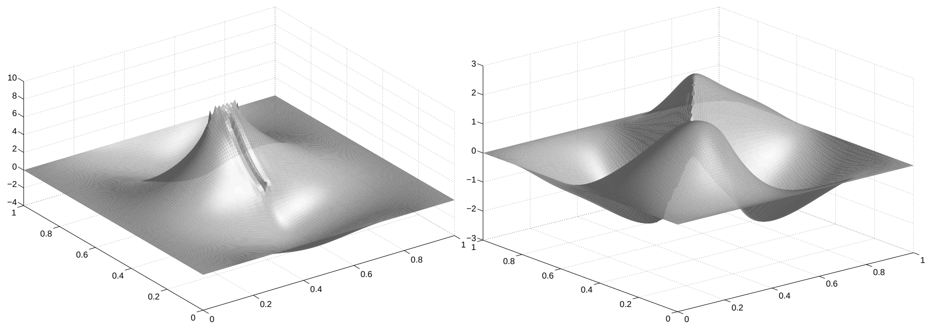

Figure 1. Example 1: horizontal and vertical components of the optimal control; $h = \sqrt{2}/64$.

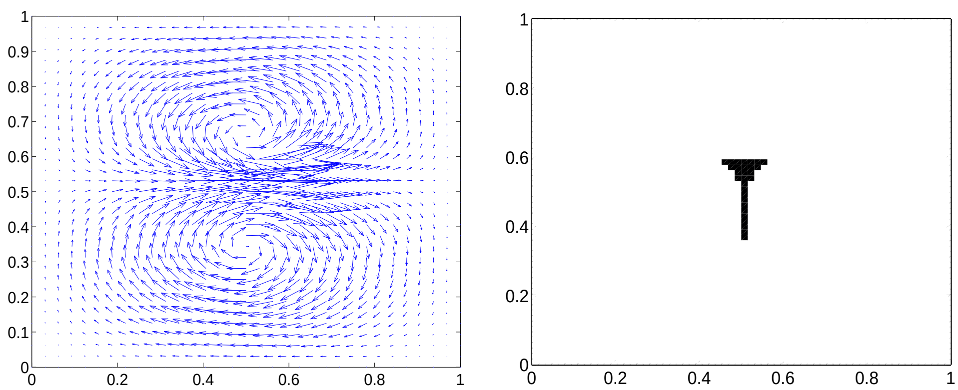

Figure 2. Example 1: control vector field (left) and active set for the horizontal component of the velocity (right); $h = \sqrt{2}/32$.

Table 2. Example 1, experimental order of convergence

$\sqrt{2}/h$	5	10	20	40	80
$EOC_2(u)$	0.85	0.95	1.06	1.23	–

7.2. Example 2: constraint on the Euclidian norm of the velocity vector

In this example we consider the state constraint $v_1^2(x) + v_2^2(x) \leq 10^{-4}$ in the center of the cavity. With this constraint, the norm of the velocity vector field is restricted pointwise in the subdomain $\Omega_s = [\frac{3}{8}, \frac{5}{8}]^2$. The desired state is given as in (39). As already mentioned before, this example is covered by the above analysis, to be more precise by Corollary 6.3. Thus, we expect a convergence rate of order $1 - \varepsilon$.

The resulting velocity vector field, with the Tikhonov parameter value $\alpha = 0.1$ and the Moreau-Yosida parameter value 10^5, is shown in Fig. 3, together with the optimal state without pointwise state constraints. The obstacle effect of the state constraint can be observed in the plot.

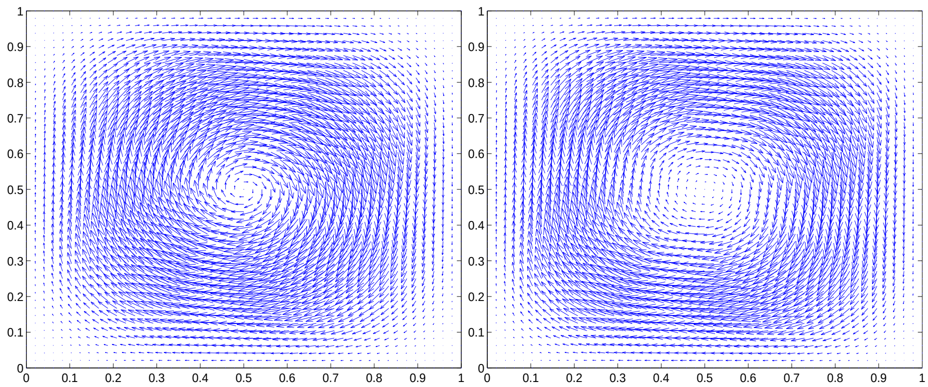

Figure 3. Example 2: optimal velocity vector field without state constraints (left) and with state constraint (right); $h = \sqrt{2}/24$.

The evolution of the finite element error and of the convergence rate as $h \to 0$ is registered in Table 3. The Tikhonov parameter is set to $\alpha = 0.2$, while we used 5×10^3 for the Moreau-Yosida penalization of the state constraints. Here, we consider as optimal solution u_h^* the one obtained numerically with a mesh step size $h = \sqrt{2}/240$. In average, an approximate order of $1 - \epsilon$ for the L^2-norms of control can be observed also in this example. Thus, the theoretical error estimate of Corollary 6.3 can be seen to be experimentally verified.

Table 3. Example 2, convergence history.

$\sqrt{2}/h$	5	15	30	60	120
#it	7	8	10	20	14
$\|u_h - u_h^*\|_{L^2}$	1.3108	0.5984	0.3637	0.2001	0.0863
$EOC_2(u)$	0.85	0.93	1.04	1.21	–

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