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The shooting approach in analyzing bang-bang extremals with simultaneous control switches*

by

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Abstract: The paper is devoted to stability investigation of optimal structure and switching points position for parametric bang-bang control problem with special focus on simultaneous switches of two control components. In contrast to problems where only simple switches occur, the switching points in general are no longer differentiable functions of input parameters. Conditions for Lipschitz stability are found which generalize known sufficient optimality conditions to nonsmooth situation. The analysis makes use of backward shooting representation of extremals, and of generalized implicit function theorems. The Lipschitz properties are illustrated for an example by constructing backward parameterized family of extremals and providing first-order switching points prediction.

Keywords: bang-bang control, Lipschitz stability in optimal control, nonsmooth optimization, shooting type methods.

1. Introduction

In parametric optimal control problems with bang-bang type solutions, an important stability issue is the parameter dependency of switching structure. For Mayer type problem with systems dynamics depending linearly on control and given initial state, the question is investigated by means of a backward shooting approach for characterizing extremals. The analysis requires certain assumptions which are closely related to sufficient optimality conditions (Agrachev, Stefani and Zezza, 2002; Osmolovskii, 1998; Maurer and Osmolovskii, 2005, 2007) like bang-bang regularity and strict bang-bang property (assumptions 1, 2a below) together with appropriate second-order coercivity type conditions (assumption 3).

In recent years, stability properties for switching points localization had been obtained e.g. from the so-called induced finite-dimensional problem (Kim and

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Maurer, 2003; Felgenhauer, 2005) using standard sensitivity results from nonlinear programming, or from a shooting type approach applied to the first-order system of conditions in Pontryagin's maximum principle (see Felgenhauer 2003, 2006). From a numerical point of view, the traditional shooting method became an efficient tool for solving control problems after multiple-shooting techniques (see Bulirsch and Stoer, 1980) had been developed and implemented. On the other hand, its close relation to Newton's method in solving the underlying systems of equations makes it attractive for theoretical purpose as well.

The following stability results have been presented previously: (i) differentiability of switching points w.r.t. parameters under conditions 1, 2a for linear state systems in Felgenhauer (2003), (ii) differentiability under restriction to problems with only simple switches (i.e. switch of only one control component at each time) under assumptions 1, 2a, 3 in Kim and Maurer (2003), (iii) differentiable behavior and local uniqueness of structure of extremals for semilinear systems with possibly multiple switches, see Felgenhauer (2006).

In this paper, Lipschitz continuity of the shooting mapping is obtained and used, in particular, for deriving new results on local Lipschitz stability of switching points w.r.t. parameters. In Section 2, the problem class and regularity assumptions are given. Section 3 repeats the facts on the induced finite-dimensional problem and provides generalization to multiple switches. In the case of a simultaneous switching of two control components, certain local stability of principal control structure was proved in Felgenhauer (2007) (see also Theorem 1, section 3). The main result is derived by using a nonsmooth version of the Implicit Function Theorem in Section 4, Theorem 3. An example designed for illustrating nondifferentiability is considered in Section 5 and Appendix 2. Some numerical experiments on backward parameterized extremals and local switching points prediction are included.

It should be noticed that the investigation is restricted to at most double switches. Generalization to simultaneous switches of more than two control components are not covered yet and should be the subject of future research (see also Poggiolini and Stefani, 2006, for related optimality issues).

Notation. Let R^n be the Euclidean vector space with norm $|\cdot|$, and scalar product written as $(a,b) = a^T b$. Superscript T is generally used for transposition of matrices, respectively vectors. For $r > 0$, define $B_r(x)$ as the closed ball with center x and radius r in R^n. The Lebesgue space of order p of vector-valued functions on $[0,1]$ is denoted by $L_p(0,1;R^k)$. $W_p^l(0,1;R^k)$ is the related Sobolev space, and norms are given as $\|\cdot\|_p$ and $\|\cdot\|_{l,p}$, $(1 \le p \le \infty,\ l \ge 1)$, respectively. The symbol ∇_x denotes (partial) gradients whereas ∂_x is used for (partial) generalized derivative in the sense of Clarke. In several places, Lie brackets $[g,f] = \nabla_x g\, f \,-\, \nabla_x f\, g$ occur. By $convM$, respectively $cl\, M$, the convex hull and closure of a set M are described. For characterizing discontinuities, jump terms are denoted $[v]^s = v(t_s+0) - v(t_s-0)$, where the index s will become clear from the context.

2. Multi-input bang-bang optimal control

Consider a parametric optimal control problem with terminal cost functional and system dynamics linear w.r.t. control:

$$
\begin{aligned}
&(\mathbf{P}_h) \qquad && \min\ J_h(x,u) \;=\; k(x(1),h) && \\
&s.t. && \dot{x}(t) \;=\; f(x(t),h) \;+\; g(x(t),h)\,u(t) \qquad && a.e.\ \text{in } [0,1], \qquad (1)\\
& && x(0) \;=\; a(h), && \qquad (2)\\
& && |\,u_i(t)| \;\le\; 1, \quad i=1,\dots,m, && a.e.\ \text{in } [0,1]\,. \qquad (3)
\end{aligned}
$$

The function $x\,:\,[0,1]\to R^n$ denotes the state of the system, and $u\,:\,[0,1]\to R^m$ the (generally vector-valued) control. The parameter $h\in R$ is assumed to be close to the reference value $h_0=0$. We further assume that, at $h=0$, a (possibly local) solution $(x^0,u^0)\in W^1_\infty\times L_\infty$ exists with the control u^0 being bang-bang, i.e. taking only extremal values ± 1 in each component. More precise assumptions are given below.

All data functions are supposed to be sufficiently smooth w.r.t. their input variables at least in a certain neighborhood $N_\rho=\{(t,x,u):\ \ t\in[0,1],\ |u|\le 1,\ i=1,\dots,m,\ |x-x^0(t)|\,<\,\rho\}$ of the reference trajectory.

For problem (P_h), Pontryagin's maximum principle holds in normal form. Denote by H the Hamilton function,

$$H(x,u,p,h) \;=\; f(x,h)^T p \;+\; \sum_{i=1}^{m} u_i\, g_i(x,h)^T p,$$

then the switching function for the control is $\sigma\;=\;\nabla_u H\;=\;g(x,h)^T p$. The first-order necessary optimality conditions may be given in a *backward shooting* formulation as follows:

$$
\begin{array}{lclclcl}
\dot{x}(t) &=& f(x(t),h) \,+\, g(x(t),h)u(t), && x(1) &=& z,\\
\dot{p}(t) &=& -A(x(t),u(t),h)^T p(t), && p(1) &=& \nabla_x k(z),\\
\sigma(t) &=& g(x(t),h)^T p(t), && u(t) &\in& -Sign\,\sigma(t),
\end{array} \qquad (4)
$$

$$T(z,h) \;=\; x(0,z,h) \;-\; a(h) \;=\; 0\,. \qquad (5)$$

(The symbol $Sign$ is used for the *generalized* signum function defined as set-valued at zero, i.e. $Sign(0)=[-1,+1]$, and the matrix function A stands for $\nabla_x(f+g\,u)$.)

Solutions $x=x(t,z,h),\ u=u(t,z,h)$ and $p=p(t,z,h)$ of the generalized differential-algebraic system (4) yield extremals for the control problem at h if the initial condition (5) is fulfilled. In general, for varying $z\approx x^0(1)$, the curves $(x(t,z,h),p(t,z,h))$ from (4) with $t\in[0,1]$ and fixed h provide a backward parameterized family of extremals (in the sense of Noble and Schättler, 2002; Ledzewicz, Nowakowski and Schättler, 2004) related to (P_h) but with free initial state value. In the language of Hamiltonian methods, they form the flow $\mathcal{H}_t$ of the maximized Hamiltonian, see Poggiolini and Stefani (2006), Agrachev and Sachkov (2004).

The control u^0 related to (P_0) for $h = 0$ is assumed to be of strict and regular bang-bang type:

Assumption 1 (bang-bang regularity)
The pair (x^0, u^0) is a solution such that u^0 is piecewise constant and has no singular arcs. For every j and $\sigma_j^0 = g_j(x^0, 0)^T p$, the set $\Sigma_j^0 = \{ t \in [0,1] : \sigma_j^0(t) = 0 \}$ is finite, and $0, 1 \notin \Sigma_j^0$.

It will not be excluded that more than one control component switches at a time (simultaneous or *multiple* switches). However, for the aim of the given paper, we will restrict ourselves to at most double switches. Notice that, in case of multiple switches, the switching functions may be nondifferentiable at points of control discontinuity, since

$$\dot\sigma_j^0(t) \;=\; p(t)^T \,[g_j, f]\,(t) \;+\; \sum_{k \neq j} u_k^0(t)\, p(t)^T \,[g_j, g_k]\,(t) \tag{6}$$

where $[\,\cdot, \cdot\,]$ stand for Lie brackets.

Assumption 2a (strict bang-bang property)
For every $j = 1, \ldots, m$, for all $t_s \in \Sigma_j^0$: $\quad \dot\sigma_j^0(t_s + 0) \cdot \dot\sigma_j^0(t_s - 0) \;>\; 0.$

Assumption 2b (double switch restriction)
All $t_s \in \bigcup \Sigma_k^0$ related to $h = 0$ are switching points of at most two control components.

3. Switching points variation

An important tool for investigating optimality and stability properties of bang-bang solutions for the problems (P_h) consists in formulating and analyzing the *induced* finite-dimensional program with switching points positions as main unknown variables, see Agrachev, Stefani and Zezza (2002), Kim and Maurer (2003), Felgenhauer (2005), Poggiolini and Stefani (2006). The principal control structure, i.e. the number and type of control switches for each component u_i, are assumed to be fixed. For certain local solution examinations, this property is ensured by the following result, Felgenhauer (2007):

Theorem 1 *Suppose Assumptions 1, 2a and 2b hold true. Further, let $(x^h, u^h, p^h) \in W^1_\infty \times L_\infty \times W^1_\infty$ with $z = z^h = x^h(1)$ be a solution of (4) satisfying the initial condition (5) and the estimate*

$$\|x^h - x^0\|_\infty + \; \|u^h - u^0\|_1 < \epsilon\,. \tag{7}$$

If $\delta = |h| + \epsilon$ is sufficiently small then the following relations hold for (x^h, u^h, p^h) together with $\sigma^h = g(x^h, h)^T p^h$:

(i) $u^h(t) = -sign\,\sigma^h(t)$ *almost everywhere on* $[0,1]$, *and* u^h *has the same switching structure and number of switching points as* u^0,
(ii) *if* $\sigma_j^h(t_s^h) = 0$, *then* $\dot\sigma_j^h(t_s^h+0)\cdot\dot\sigma_j^h(t_s^h-0) > 0 \quad (j=1,\dots,m)$.

In particular, one can assume $u(1) = u^0(1)$ for extremals (x,u,p) which are close to (x^0,u^0,p^0) in the sense of (7). The switching points will be componentwise enumerated, i.e. $\Sigma_j = \{\,\tau_{js}:\ 1\le s\le l(j),\ 0=\tau_{j0}<\tau_{jk}<\tau_{j,k+1}<\tau_{j,l(j)+1} = 1$ for $k<l(j)\}$, $j=1,\dots,m$, and assembled to a vector $\Sigma = (\tau_{js}) \in R^L$ with $L=\sum_{j=1}^m l(j)$. Notice that all vectors satisfying the above monotone ordering conditions form an open subset D_Σ in R^L.

Let a vector $\Sigma\in D_\Sigma$ be given, sufficiently close to Σ^0, related to the reference solution of (P_0). We construct an admissible pair $x=x(t,\Sigma,h)$, $u=u(t,\Sigma)$ for (P_h) by solving

$$u_j(t,\Sigma) \;\equiv\; (-1)^{l(j)-s}u_j^0(1) \qquad \text{for} \quad t\in(\tau_{js},\tau_{j,s+1}), \tag{8}$$

$$\dot x(t) \;=\; f(x(t),h) + g(x(t),h)\,u(t,\Sigma), \qquad x(0) = a(h)\,, \tag{9}$$

and define the parametric auxiliary problem

$$\mathbf{(OP}_h\mathbf{)} \qquad \min\ \phi_h(\Sigma) \;=\; k\,(x(1,\Sigma,h),h) \qquad \text{w.r.t. } \Sigma\in D_\Sigma.$$

Then, at $h=0$, the vector $\Sigma=\Sigma^0$ is a local minimizer of ϕ_0 (see Kim and Maurer, 2003, and Felgenhauer, 2007) with

$$Z(\Sigma^0,0) \;=\; \nabla_\Sigma\phi_0(\Sigma^0) \;=\; -\Big(\big[\,u_j^0\big]^s\,\sigma_j^0(t_{js})\Big)_{j,s} \;=\; 0\,. \tag{10}$$

In general, relation $Z(\Sigma,h) = \nabla_\Sigma\phi_h(\Sigma) = 0$ is the stationary point map associated to (OP_h).

If all switching points in Σ are *simple* or if, for each double switch $\tau_\alpha=\tau_\beta$ with $\alpha=(j,s)$, $\beta=(i,r)$ and $i\neq j$, the vector fields $g_i(x(t,\Sigma,h),h)$ and $g_j(x(t,\Sigma,h),h)$ are *commuting* at $t=\tau_\alpha$, then ϕ_h is twice continuously differentiable in a neighborhood of $\Sigma^0\in D_\Sigma$, see Kim and Maurer (2003), Felgenhauer (2007), or Poggiolini and Stefani (2006). In case of double, respectively multiple switches of several control components, in general only $\phi_h\in C^{1,1}$ holds. However, one can find generalized second-order derivatives e.g. in the sense of Clarke (see Clarke, 1983; Klatte and Kummer, 2002). Suppose that there are k double switches in Σ^0, then in D_Σ one can find 2^k disjoint subsets D^ν such that, for each of the double switching pairs $t_\alpha=t_\beta$ from u^0, either $\tau_\alpha<\tau_\beta$ or $\tau_\alpha>\tau_\beta$ holds in D^ν. Since the switching points for $\Sigma\in D^\nu$ are well separated, $\phi_h\in C^2(D^\nu)$ follows for each $\nu=1,\dots,2^k$. Further, for vectors $\Sigma'\in\bigcap\{\,cl\,D^\nu\!:\ \nu=1,\dots,2^k\}$ one can find the limits

$$\nabla^\nu\,(\nabla\phi_h)|_{\Sigma'} \;=\; \lim_{\Sigma\in D^\nu,\Sigma\to\Sigma'}\nabla^2_\Sigma\phi_h(\Sigma)\,.$$

By a representation theorem for Clarke's generalized derivative in case of piecewise C^2 functions (Klatte and Kummer, 2002; Scholtes, 1994), we obtain

$$\partial_\Sigma(\nabla_\Sigma\phi_h(\Sigma')) = conv\{\ \nabla^\nu\,(\nabla\phi_h)|_{\Sigma'}\ ,\ \nu=1,\dots,2^k\}. \tag{11}$$

Thus, the matrices $\nabla^\nu(\nabla\phi_h)$ as particularly chosen limiting Hessians are spanning matrices of $\partial(\nabla\phi_h)$. Formulas for their respective elements including the case of double switches have been derived in Felgenhauer (2007).

In the following, it will be assumed that the solution Σ^0 of (OP_h) with $h=0$ satisfies a strong generalized second-order optimality condition in the sense of Clarke (see Clarke, 1983):

Assumption 3 *$\exists\, c>0$: $v^T Q\, v \;\geq\; c\,|v|^2$ for all $v \in R^L$ and each matrix $Q \in \partial_\Sigma(\nabla_\Sigma\phi_0)(\Sigma^0)$.*

It should be noticed that Assumption 3 holds if and only if each of the spanning Hessians $\nabla^\nu(\nabla\phi_0)$ is positive definite at Σ^0 (see Kojima, 1980, for nonlinear programming application). The given assumption could be also formulated by means of more general (and possibly nonconvex) subdifferentials, see Mordukhovich (2006).

The generalized coercivity condition on $\partial(\nabla\phi_h)$ ensures, in particular, full rank property of the matrices so that, by the generalized Implicit Function Theorem, the local Lipschitz invertibility of the stationary point map follows (for the proof see Felgenhauer, 2007):

Theorem 2 *Let (P_0) have a bang-bang solution (x^0,u^0) with switching points Σ^0 such that Assumptions 1, 2a,b and 3 are fulfilled. Then, a neighborhood U of $h_0=0$ exists such that the following statements hold:*

(i) *In R^L there exists a neighborhood S of Σ^0 such that $\forall\, h\in U$ equation*

$$Z(\Sigma,h) \;=\; \nabla_\Sigma\phi_h(\Sigma) \;=\; 0 \tag{12}$$

has an unique solution $\Sigma=\Sigma(h)\in S$. As a function of h, $\Sigma=\Sigma(h)$ is Lipschitz continuous on U.

(ii) *$\forall\, h\in U$: $\quad \Sigma \;=\; \Sigma(h) \;\in\; D_\Sigma$.*
In particular, t_s^h may be (an at most) double switch belonging to $\Sigma_j(h)\cap\Sigma_i(h)$ only if there is a neighboring $t_s^0\in\Sigma_j^0\cap\Sigma_i^0$, too.

(iii) *All matrices Q in $\partial_\Sigma(\nabla_\Sigma\phi_h)(\Sigma(h))$ are positive definite with lower eigenvalue bound $c'>0$ independent of $h\in U$. The vector $\Sigma(h)$ is thus a strict local minimizer of ϕ_h from (OP_h).*

4. Backward shooting and extended shooting approach

In Theorem 1, local structural stability of prospective solutions to the shooting system (4), (5) has been established. However, the arguments did not allow to decide whether such solutions exist for $h\neq 0$. In this section the local existence of shooting extremals is shown. For their construction, a so-called *extended* shooting system is used combining the original shooting approach with the idea of varying switching point positions.

Let there be given a vector pair $(\Sigma, z) \in R^L \times R^n$ near the reference value (Σ^0, z^0) where $z^0 = x^0(1)$. Set

$$c = c(z) = -sign \left\{ g(z)^T \nabla_x k(z) \right\}.$$

Under Assumption 1, c coincides with $u^0(1)$ if only z is taken sufficiently close to z^0. Let us choose a bang-bang control function $u = u(\cdot, \Sigma)$ related to $c = c(z)$ and Σ by

$$u_j(t, \Sigma) \equiv (-1)^{l(j)-s} c_j \qquad \text{for} \quad t \in (\tau_{js}, \tau_{j,s+1}). \tag{13}$$

Then, for sufficiently small h, the following system has a unique solution $x = x(\cdot, \Sigma, z, h),\ p = p(\cdot, \Sigma, z, h)$:

$$\begin{aligned} \dot{x}(t) &= f(x(t), h) + g(x(t), h)\, u(t, \Sigma), \qquad x(1) = z, & (14)\\ \dot{p}(t) &= -A(x(t), u(t, \Sigma), h)^T p(t), \qquad p(1) = \nabla_x k(z), & (15) \end{aligned}$$

and we accomplish the construction by defining $\sigma(\cdot, \Sigma, z, h) = g(x(\cdot), h)^T p(\cdot)$.

Lemma 1 *Let Assumptions 1, 2a and 2b hold at $h = 0$, and assume $|(\Sigma, z, h) - (\Sigma^0, z^0, 0)| < \delta$ with δ given in Theorem 1. Then, the solution components x, u and p of (13) – (15) satisfy Pontryagin's maximum principle if and only if the initial condition for x and the switching criteria for u are fulfilled, i.e.*

$$\begin{aligned} V(\Sigma, z, h) &= x(0, \Sigma, z, h) - a(h) = 0, & (16)\\ W(\Sigma, z, h) &= -\Gamma(\Sigma) \cdot \sigma(\cdot, \Sigma, z, h)|_\Sigma = 0, & (17)\\ \Gamma(\Sigma) &= diag_{(js)} \left\{ \left[u_j^0 \right]^s \right\}. \end{aligned}$$

Notice that all entries in Γ are ± 2 so that this matrix is always regular. Thus, in equation (17), for each $\alpha = (j, s)$ with $\tau_\alpha \in \Sigma_j$ it is required that

$$W_\alpha(\Sigma, z, h) = -\left[u_j^0 \right]^s \sigma_j(\tau_\alpha, \Sigma, z, h) = 0.$$

Proof. (The proof follows Felgenhauer, 2007, proof of Theorem 1.)
Conditions (14) – (16) represent state and adjoint equations together with boundary and transversality conditions related to (P_h). It will be shown that, locally, the maximum condition (or equivalently: $u(t) \in -Sign\, \sigma(t)$ *a.e.* on $[0, 1]$) follows from (13) together with (17):
For (Σ, z, h) close to $(\Sigma^0, z^0, 0)$, from (13) and (14) the estimate (7) follows:

$$\|x - x^0\|_\infty + \|u - u^0\|_1 < \epsilon.$$

By Gronwall's Lemma, we further get

$$\|x - x^0\|_{1,1} + \|p - p^0\|_{1,1} + \|\sigma - \sigma^0\|_{1,1} = \mathrm{O}(\delta)$$

for $x = x(\cdot, \Sigma, z, h)$, $\sigma = \sigma(\cdot, \Sigma, z, h) = p(\cdot, \Sigma, z, h)^T g(x(\cdot, \Sigma, z, h))$ etc.

If δ is sufficiently small, $r > 0$ exists such that the following properties hold:
(i) for $j = 1, \ldots, m$, $\forall t_s \in \Sigma_j^0$, outside the balls $B_r(t_s)$ the functions $u_j = u_j(\cdot, \Sigma)$ and u_j^0 are continuous, and $\sigma_j(t)$, $\sigma_j^0(t)$ are of the same sign.
(ii) for $j = 1, \ldots, m$, $\forall t_s \in \Sigma_j^0$, $\dot\sigma_j(t)$ with $t \in B_r(t_s)$ has the same sign as $\dot\sigma_j^0(t_s \pm 0)$.
Indeed, in analogy to (6), by (14), (15) we have

$$\dot\sigma_j(t) \;=\; p(t)^T\,[g_j, f]\,(t) \;+\; \sum_{k \neq j} u_k(t)\, p(t)^T\,[g_j, g_k]\,(t).$$

Let us start from $t = 1$ where $-sign\,\sigma(1) = u(1) = u^0(1)$, and find backwards the next switching point, $\bar t_s \in \Sigma_i^0$: If $\bar t_s$ is a simple switch of u_i^0 then, inside $B_r(\bar t_s)$, all u_k with $k \neq i$ are continuous and equal to u_k^0, so that $|\dot\sigma_i(t) - \dot\sigma_i^0(t)| = \mathrm{O}(\delta)$. If $\bar t_s$ is a double switch, e.g. $\bar t_s \in \Sigma_j^0 \cap \Sigma_i^0$, then

$$\begin{aligned}\dot\sigma_i(t) \;\;\in\;\; & p(t)^T\,[g_i, f]\,(t) \;+\; \sum_{k \notin \{i,j\}} u_k(t)\, p(t)^T\,[g_i, g_k]\,(t) \\ & + conv\{u_j^0(\bar t_s + 0), u_j^0(\bar t_s - 0)\}\, p(t)^T\,[g_i, g_j]\,(t),\end{aligned}$$

so that

$$dist\,\big\{\,\dot\sigma_i(t),\, conv\{\dot\sigma_i^0(\bar t_s + 0), \dot\sigma_i^0(\bar t_s - 0)\}\big\} \;=\; \mathrm{O}(\delta + r).$$

Thus, for sufficiently small δ and r, the time derivatives are of the same sign, and σ_i is strictly monotone on $B_r(\bar t_s)$. Therefore, σ_i has a regular zero in $B_r(\bar t_s)$, and for $t = \bar t_s - r$ the values of $u = u(\cdot, \Sigma)$ and $-sign\,\sigma(\cdot, \Sigma, z, h)$ again coincide. The process will be repeated backwards from switching point to switching point and we conclude that, for all i, $-sign\,\sigma_i(\cdot, \Sigma, z, h)$ and $u_i(\cdot, \Sigma)$ have the same number and type of discontinuities on $[0, 1]$. Thus, they are equal if and only if (17) is fulfilled. ■

The system (16), (17) with (x, u, p) and σ from (13) – (15) can be shortly written as one equation in R^{n+L},

$$F(\Sigma, z, h) \;=\; 0 \tag{18}$$

where $F = (V, W)^T$ as a mapping to R^{n+L} is defined on some neighborhood of $(\Sigma^0, z^0, 0) \in R^{L+n} \times R$. It will be shown that, under assumptions 1, 2a, 2b and 3, equation (18) locally determines implicit functions $\Sigma = \Sigma(h)$, $z = z(h)$ depending Lipschitz continuously on the parameter h.

In Felgenhauer (2006), function F and the extended shooting system have been considered for the case of semilinear state system in (P_h) where g is independent of x. In this situation, F turned out to be continuously differentiable even if multiple switches occurred, and the partial Jacobian

$$\nabla_{(\Sigma,z)} F \;=\; \begin{pmatrix} \nabla_\Sigma V & \nabla_z V \\ \nabla_\Sigma W & \nabla_z W \end{pmatrix} \tag{19}$$

was proved to be regular under strict bang-bang assumptions. In the more general situation, with state equation of type (1), the function F may be non-differentiable if multiple control switches occur. However, one can find at least generalized derivatives by using similar techniques as in preceding section: indeed, the restriction of F to sets with $\Sigma \in D^\nu$ (i.e. to those areas where the switching points are well separated) provides differentiable parts and one can find respective limits $\nabla^\nu F$ for approximating possible double switches.

In analogy to (19), Clarke's subdifferential ∂F is obtained after replacing $\nabla_\Sigma W$ by the generalized partial derivative term $\partial_\Sigma W$. In case of double switches, $\partial_\Sigma W$ is determined due to the representation theorem in Scholtes (1994) as

$$\begin{aligned} \partial_\Sigma(W(\Sigma', z, h)) &= conv\{ \nabla^\nu_\Sigma W(\Sigma, z, h)|_{\Sigma=\Sigma'}\ ,\ \nu = 1, \ldots, 2^k\}, \\ \nabla^\nu_\Sigma W(\Sigma, z, h)|_{\Sigma=\Sigma'} &= \lim_{\Sigma \in D^\nu, \Sigma \to \Sigma'} \nabla_\Sigma W(\Sigma, z, h). \end{aligned} \tag{20}$$

For the notation we refer to (11). The matrix elements will be described in detail by formulas (34) – (36) in the Appendix.

LEMMA 2 *Under Assumptions 1, 2a,b and 3, the matrix* $\nabla_z V(\Sigma^0, z^0, 0)$ *together with all matrices from* $\partial_\Sigma W(\Sigma^0, z^0, 0)$ *is regular. In particular, all* $M \in \partial_\Sigma W(\Sigma^0, z^0, 0)$ *have positive determinants.*

The proof of the lemma is left to the Appendix.

After this preliminary analysis of the the principal structure of ∂F, local solvability and Lipschitz stability of the extended shooting system is obtained:

LEMMA 3 *Let Assumptions 1 – 3 hold for* P_h *at* $h_0 = 0$ *for the solution* (x^0, u^0), *switching set* Σ^0, *and the related adjoint function* p. *Then, near* $(\Sigma^0, z^0, 0)$ *with* $z^0 = x^0(1)$, *equation* $F(\Sigma, z, h) = 0$ *defines locally unique functions* $\Sigma = \Sigma(h)$ *and* $z = z(h)$ *depending Lipschitz continuously on the parameter* h.

Proof. Consider $F = (V, W)^T$ and its generalized partial Jacobian,

$$\partial_{(\Sigma,z)} F(\Sigma, z, h) = \begin{pmatrix} \nabla_\Sigma V(\Sigma, z, h) & \nabla_z V(\Sigma, z, h) \\ \partial_\Sigma W(\Sigma, z, h) & \nabla_z W(\Sigma, z, h) \end{pmatrix}$$

near $(\Sigma^0, z^0, 0)$. The underlying extended shooting approach can be interpreted as one way for obtaining a solution to the stationary point map $Z(\Sigma, h) = 0$ related to the finite-dimensional program (OP_h) in section 3, see (10). To this aim we follow a primal-dual construction where the mapping $(\Sigma, h) \to (x, u)$ given by equations (8), (9) is completed by adjoint information, i.e. $\tilde{p} = \tilde{p}(t, \Sigma, h)$, $\tilde{\sigma} = \tilde{\sigma}(t, \Sigma, h)$ solving

$$\begin{aligned} \dot{\tilde{p}}(t) &= -(\nabla_x f(x(t, \Sigma, h), h) + \nabla_x g(x(t, \Sigma, h), h) u(t, \Sigma))^T \tilde{p}(t)\,, \\ \tilde{p}(1) &= \nabla_x k(x(1, \Sigma, h), h)\,, \\ \tilde{\sigma}(t) &= g(x(t), h)^T \tilde{p}(t). \end{aligned}$$

Obviously, $\tilde{p}$ can be equivalently obtained by finding $\tilde{z} = \tilde{z}(\Sigma, h)$ from (16),

$$V(\Sigma, z, h) \;=\; 0\,,$$

and inserting $z = \tilde{z}$ into (15). The partial Jacobian of V w.r.t. z can be expressed as $\nabla_z V = \Phi^T(1)$ where $\Phi = \Phi(\cdot, \Sigma, z, h)$ and $\Psi = \Psi(\cdot, \Sigma, z, h)$ are the fundamental matrix solutions of the linearized state resp. adjoint equations,

$$\dot{\Psi} \;=\; A\,\Psi, \qquad \dot{\Phi} \;=\; -A^T\Phi, \qquad \Psi(0) \;=\; \Phi^T(0) \;=\; I. \tag{21}$$

Since the matrices $\Phi(t)$, $\Psi(t)$ are regular for all $t \in [0,1]$ and (Σ, z, h) near $(\Sigma^0, z^0, 0)$, the function $\tilde{z}$ turns out to be differentiable at $(\Sigma^0, 0)$.
Further, we find

$$Z_\alpha(\Sigma, h) \;=\; W_\alpha(\Sigma, \tilde{z}(\Sigma, h), h) \;=\; -\left[u_j^0\right]^s \tilde{\sigma}_j(\tau_\alpha) \tag{22}$$

so that the generalized partial Jacobian $\partial_\Sigma Z \;=\; \partial_\Sigma \left(\nabla_\Sigma \phi_h(\Sigma)\right)$ satisfies

$$\begin{aligned} \partial_\Sigma Z \;&=\; \partial_\Sigma W \,+\, \nabla_z W \cdot \nabla_\Sigma \tilde{z} \\ &=\; \partial_\Sigma W \,-\, \nabla_z W \, \left(\nabla_z V\right)^{-1} \nabla_\Sigma V \end{aligned} \tag{23}$$

Under Assumption 3, all matrices in the right hand side are thus regular at $(\Sigma^0, z^0, 0)$. On the other hand, each of them represents the Schur complement of the regular matrix bloc $\nabla_z V$ in the related matrix from $\partial_{(\Sigma,z)} F(\Sigma, z, h)$ so that all elements in Clarke's generalized Jacobian of the mapping F turn out to be regular near the reference point. The partial invertibility of equation (18) and Lipschitz continuity of the solution functions follow by Generalized Implicit Function Theorem from Clarke (1983). ∎

For sufficiently small h, the solution $(z(h), \Sigma(h))$ of (18) in their z-components contains a solution of the backward shooting process (4), (5):

According to Lemma 2, equation $W(\Sigma, z, h) = 0$ from (17) has the locally unique solution $\tilde{\Sigma} = \tilde{\Sigma}(z, h)$ which as a function of (z, h) is Lipschitz continuous. By Lemma 1, locally the related control $u(\cdot, \tilde{\Sigma})$ satisfies the maximum condition. Inserting now $\tilde{\Sigma}$ into $V(\Sigma, z, h) = 0$ we end up with

$$T(z, h) \;=\; V(\tilde{\Sigma}(z, h), z, h) \;=\; 0,$$

i.e. the backward-shooting relation. Obviously, $z = z(h)$ from (17), (16) is a solution, and the corresponding $\Sigma = \Sigma(h)$ coincides with $\tilde{\Sigma}(z(h), h)$. By Theorem 1 and Lemma 3, local uniqueness (in restriction to sets where (7) is fulfilled) and Lipschitz stability of $z = z(h)$ follow.
Summarizing the obtained results leads to

THEOREM 3 *Let the Assumptions 1 – 3 hold together with the parameter restrictions from Theorem 1. Then the backward shooting procedure (4), (5) near* $z^0 = x^0(1)$ *and* $h = 0$ *has a solution* $z = z(h)$ *which is uniquely determined in the neighborhood (7) of the reference state-control pair. The function* $z = z(h)$

as well as the related switching times vector $\Sigma = \Sigma(h)$ *are Lipschitz continuous near* $h = 0$*. Further, the solution satisfies regularity and coercivity conditions* (ii), (iii) *from Theorem 2, respectively.*

Corollary 1 *Let the Assumptions of Theorem 3 hold true. If, in addition, the mapping* W *from (17) is differentiable w.r.t.* (Σ, z) *at* $(\Sigma^0, z^0, 0)$ *then* $z = z(h)$ *is differentiable at* $h = 0$*. The matrix*

$$\nabla_z T \;=\; \nabla_z V \;-\; \nabla_\Sigma V \; (\nabla_\Sigma W)^{-1} \, \nabla_z W \tag{24}$$

is a regular matrix then, and

$$\left.\frac{dz}{dh}\right|_{h=0} \;=\; -\,(\nabla_z T)^{-1}\,\frac{\partial T}{\partial h} \qquad \text{with} \quad \frac{\partial T}{\partial h} = \frac{\partial V}{\partial h} \;-\; \nabla_\Sigma V \; (\nabla_\Sigma W)^{-1}\,\frac{\partial W}{\partial h}.$$

Notice that differentiability of W in particular holds in case of simple switches, or for semilinear state system (see Felgenhauer, 2006). The regularity of (24) follows from Lemma 2 and the regularity of ∇F (see proof of Lemma 3), again by Schur complement argumentation.

In general, i.e. the possibly nondifferentiable case, it is an open question whether one can find, e.g., directional or generalized derivatives for $(\Sigma, z) = (\Sigma(h), z(h))$ from the backward, respectively extended shooting approach near $h = 0$. In special cases, when one can predict structure and prove that $\Sigma(\lambda h)$ belongs to a fixed subset D^ν for $\lambda \in (0, \bar\lambda)$ and certain $\bar\lambda > 0$, one-sided derivatives are available and can be calculated in analogy to dz/dh from the above Corollary after replacing ∇W by $\nabla^\nu W$.

The example in the next section illustrates such special case for the situation of two control components allowing for one control switch each, with a double switch occuring for appropriate parameter choice.

5. Double-switch example

Let us consider the following control problem with initial state value depending on a parameter h, and fixed terminal time $T > 0$:

$$\begin{array}{llcll} (\mathbf{P}_h(\epsilon)) & \min\; J(x,u) & = & 0.5\,|\,x(T)\,|^2 & \\ s.t. & \dot x_1 & = & x_2 + \epsilon\,(\epsilon\, x_2 + x_1 + 1)\,u_1, & \\ & \dot x_2 & = & u_2 & a.e. \text{ in } [0,T], \\ & x(0) & = & a(h), & \\ & |\,u_i(t)| & \le & 1, \quad i = 1,2, & a.e. \text{ in } [0,T]\,. \end{array} \tag{25}$$

By ϵ we denote an auxiliary constant from $(0, 1)$. It will be assumed that, for all h from a certain neighborhood U of $h_0 = 0$, the time parameter T is smaller than the optimal termination time for the system with initial position in $a(h)$.

The given problem is two-dimensional in both the state and control variables, and bilinear in the sense that the coefficient vector functions g_i related to u_i,

$i = 1, 2$, are affine-linear functions of the state variables. It is worth noticing that, in this case, the generalized coercivity condition follows directly from the strict and regular bang-bang behavior of the optimal control. The Hamiltonian related to $(\mathrm{P}_h(\epsilon))$ has the form:

$$H_\epsilon = p_1 x_2 + \epsilon p_1 \, (\epsilon x_2 + x_1 + 1) \, u_1 + p_2 u_2,$$

so that, by Pontryagin's maximum principle, we obtain first-order optimality conditions

$$\begin{array}{rclrcl} \dot{p}_1 & = & -\epsilon p_1 u_1, & p_1(T) & = & z_1 = x_1(T), \\ \dot{p}_2 & = & -p_1(1+\epsilon^2 u_1), & p_2(T) & = & z_2 = x_2(T), \\ \sigma_1 & = & \epsilon p_1(\epsilon x_2 + x_1 + 1), & u_1 & \in & -Sign\,\sigma_1, \\ \sigma_2 & = & p_2, & u_2 & \in & -Sign\,\sigma_2. \end{array} \tag{26}$$

If $x_1(T) \neq 0$ then each switching function σ_j is nonsmooth at t_s^i, $i \neq j$, (i.e. where σ_i vanishes) since

$$p^T \, [g_1, g_2] = \epsilon^2 p_1 \neq 0.$$

For appropriately chosen parameters T, ϵ and $a^0 = a(0)$ one can show that the corresponding reference extremal (x^0, u^0) has bang-bang control components switching simultaneously at a point $t_s \in (0, T)$. In order to find such trajectory, in a first step we construct certain backward parameterized family of extremals $(x(\cdot, z, \epsilon), p(\cdot, z, \epsilon))$ related to $(\mathrm{P}_h(\epsilon))$ where the terminal state variables $x(T) = z$ are taken as parameters, and the initial state is considered free, see Noble and Schättler (2002). The parameters are supposed to satisfy condition (39), Appendix 2, guaranteeing strict monotonicity of both switching functions σ_1 and σ_2 for z from some set $Z = Z(\epsilon, T) \subset R^2$. Next, for (ϵ, T) being fixed, we find values $z = z^d$ for which u_1 and u_2 change their signs at the same time, $t_s^1 = t_s^2$: to this aim, one has to solve the system $W(\Sigma, z) = 0$, or equivalently ask for $(t, z) \in (0, T) \times Z$ such that

$$\sigma_1(t, z) = 0, \qquad \sigma_2(t, z) = 0.$$

In Appendix 2, the solution is described by $t_s = T - \delta(r), z^d = z^d(r)$ parameterized via $r = -z_2/z_1$, where $\delta = \delta(r)$ and $z^d = z^d(r)$ are differentiable w.r.t. r on some interval I_r (see also (40) and (43)).

Now we are able to choose the reference data for $(\mathrm{P}_0(\epsilon))$ corresponding to a control with double switch behavior: to this aim, set $\epsilon = 0.5, T = 2$ and $r^0 = 1.5$. The corresponding switching point and terminal state are $t_s^0 = 1.06$ and $z^0 = z^d(r^0) \approx (-0.4839, 0.7258)$. From the backward solution method, the initial state vector $a^0 = a(0) \approx (-3.6547, 0.6059)$ is obtained. By the choice of z with (39) and (42), Assumptions 1 and 2a, 2b are satisfied. Formulas (23) and (37), Appendix 1, allow further for calculating the matrices $\nabla^{1,2}\,(\nabla\phi_0(\Sigma_0))$ spanning the generalized Hessian (11) of ϕ from the induced mathematical program (OP_0). For the test parameters corresponding to z^0 resp. a^0, we get

$$\nabla^1_\Sigma (\nabla_\Sigma \phi_0)\big|_{\Sigma^0} \approx \begin{pmatrix} 1.677 & 0.000 \\ 0.000 & 14.935 \end{pmatrix}, \quad \nabla^2_\Sigma (\nabla_\Sigma \phi_0)\big|_{\Sigma^0} \approx \begin{pmatrix} 0.903 & 0.774 \\ 0.774 & 14.161 \end{pmatrix}$$

which are both positive definite. Consequently, Assumption 3 holds for $h = 0$, $\Sigma^0_1 = \Sigma^0_2 = \{t^0_s\}$. By Theorem 3, the Lipschitz invertibility of the mapping $T = T(z, h)$ w.r.t. z follows, and $z = z(h)$, $\Sigma = \Sigma(h)$ related to $a = a(h)$ are locally uniquely determined near a^0.

For the given example one can further observe that, near a^0, the initial values corresponding to simultaneous switch of both control components form a (differentiable) curve C_x in R^2. For $r = 1.1 : 0.1 : 1.7$, initial values $a^d \in C_x$ and terminal parameters $z^d \in C$ corresponding to double-switch situation have been found from formula (43) and backward shooting. The results are given in Table 1.

Table 1. Double-switch data for different r

r	1.1	1.2	1.3	1.4	1.5	1.6	1.7
t_s	1.2707	1.2159	1.1626	1.1106	1.0600	1.0106	0.9624
a^d_1	-3.8434	-3.8313	-3.7901	-3.7290	-3.6547	-3.5726	-3.4861
a^d_2	0.3116	0.4025	0.4801	0.5471	0.6059	0.6584	0.7061
z^d_1	-0.7755	-0.6953	-0.6194	-0.5488	-0.4839	-0.4247	-0.3711
z^d_2	0.8530	0.8344	0.8052	0.7683	0.7258	0.6796	0.6309

The curve C_x divides a certain small neighborhood of a^0 into parts X^1, X^2 such that the extremals corresponding to $a(h) \in X^\nu$ have switching structure $\Sigma(h) \in D^\nu$, $\nu = 1, 2$, resp. For simplicity, the switching points are denoted by τ^i_s, and we declare

$$D^1 = \{(\tau^1_s, \tau^2_s) : 0 < \tau^1_s < \tau^2_s < T\}, \quad D^2 = \{(\tau^1_s, \tau^2_s) : 0 < \tau^2_s < \tau^1_s < T\}.$$

The partition is constructed from the corresponding subsets Z^1, Z^2 of a related neighborhood of the reference terminal state z^0 and the curve $C = cl(Z^1) \cap cl(Z^2)$, see (44), Appendix 2, for details. The related switching vectors localization in one of the sets D^ν is described in Table 2.

Table 2. Order of switches depending on z

$z_1 =$	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7
$r = 1.1$	D^1	D^1	D^1	D^1	D^1	D^1
1.3	D^1	D^1	D^1	D^1	D^1	D^2
1.5	D^1	D^1	D^1	$\approx \partial D^2$	D^2	D^2
1.7	D^1	$\approx \partial D^1$	$\approx \partial D^2$	D^2	D^2	D^2
1.8	D^1	$\approx \partial D^1$	D^2	D^2	D^2	D^2

In situation where the two switching points nearly coincide we use symbol ∂D^ν to reflect approaching of respective boundaries.

For general bang-bang control problems with simultaneous switching of two controls at a time, assumptions 1 – 3 alone do not guarantee yet the existence of a manifold dividing neighborhoods of a^0 (respectively of z^0) into subsets where the order of switchings is invariant. Conditions for such kind of non-degeneracy can be found e.g. in Poggiolini and Stefani (2006) where they are discussed in the context of second-order sufficient optimality conditions.

Finally, in the particular example, one can try to find first-order approximations for the switching points under parameter perturbation. Although Theorem 3 in general does not provide yet tools for sensitivity calculation, one can utilize Lemma 2 for $W(\Sigma, z) = 0$, respectively,

$$\sigma_1(t_s^1, \Sigma, z) = \sigma_2(t_s^2, \Sigma, z) = 0$$

to analyze the influence of changes of the terminal parameter z.
Obviously, the following property holds:
If Δz is a small perturbation such that, for all $\rho \in (0,1)$, the switching vectors Σ_ρ corresponding to $z_\rho = z^0 + \rho \Delta z$ all belong to the same set D^i, $i \in \{1,2\}$, then

$$\nabla_\Sigma^i W \cdot \Delta\Sigma \,+\, \nabla_z W \,\Delta z \,=\, o(\Delta z).$$

In this situation, one can find the directional derivative $\partial^i(\Sigma(z^0), \Delta z)$ of Σ and

$$\Sigma_{lin} \,=\, \Sigma^0 - \left(\nabla_\Sigma^i W\right)^{-1} \nabla_z W \,\Delta z \,\in\, D^i$$

as the first-order switching point prediction.
If $z^0 \in Z^i$ then, under small perturbation, Σ will remain in D^i and the choice for Σ_{lin} is well-determined. The more interesting case is $z^0 \in C$ where the approximation requires predicting of switching order. In the example given one can utilize the special structure of $M \in \partial_\Sigma W$:
In case of two control components switching each exactly once on the time interval, the matrices $\nabla_\Sigma^i W$ are (2,2)-triangular and regular. By (37), with $e = (1,1)^T$ we get

$$\nabla_\Sigma^1 W\, e \,=\, \nabla_\Sigma^2 W\, e \,=:\, w \,\neq\, 0, \tag{27}$$

see Lemma 2. Moreover, the matrices have both positive determinants.
Denote $p = -\nabla_z W \Delta z$:

If the vector p is not parallel to w from (27) then, multiplying with (one of) $\left(\nabla_\Sigma^i W\right)^{-1}$, preserves the orientation between images: in case that the vector system (p, w) is positively oriented (i.e. they include an angle $\gamma \in (0, \pi)$) the same is true for the pre-images $d^i = \left(\nabla_\Sigma^i W\right)^{-1} p$ and e so that

$$\Sigma^0 + d^i \,=\, \Sigma^0 - \left(\nabla_\Sigma^i W\right)^{-1} \nabla_z W \Delta z \,\in\, D^1, \qquad i = 1, 2.$$

Consequently, $\Sigma_{lin} = \Sigma^0 + d^1$. Conversely, if the oriented angle between (p, w) is negative, the prediction directs into D^2 with $\Sigma_{lin} = \Sigma^0 + d^2$.

For the example of the data chosen above it could be further observed that the matrices $\nabla_z W$ are regular so that for almost all directions Δz the prediction is well-determined. The comparison to values found via backward solution of (13)–(15), (17) shows suitability of directional linearization. Moreover, the results confirm the possibility of non-differentiable behavior of the shooting mapping in case of simultaneous control switching.
For $r < 1.5$ we have chosen z-values from Z^1 leading to the case $t_s^1 < t_s^2 \, (= t_s^d)$, whereas for $r > 1.5$ the test values for z are taken from Z^2. Results are shown in Table 3.

Table 3. Switching points *vs.* first-order predictions

r	1.3	1.4	1.5	1.6	1.7
z_1	-0.4000	-0.4500	-0.4839	-0.5400	-0.6000
z_2	0.5200	0.6300	0.7258	0.8640	1.0200
t_s^1	1.0299	1.0520	1.0600	1.1369	1.2109
t_{lin}^1	1.0324	1.0523		1.1419	1.2326
t_s^2	1.1626	1.1106	1.0600	0.9069	0.7128
t_{lin}^2	1.1427	1.1065		0.9124	0.7383

Appendix 1: Partial derivatives of mapping F

Consider the function $F = (V, W)^T$ from (18) mapping some neighborhood of $(\Sigma^0, z^0, 0)$ into R^{n+L}. According to (16), (17) and the smoothness properties of σ, F is Lipschitz continuous and its generalized partial Jacobian,

$$\partial_{(\Sigma,z)} F \;=\; \begin{pmatrix} \nabla_\Sigma V & \nabla_z V \\ \partial_\Sigma W & \nabla_z W \end{pmatrix} \tag{28}$$

can be constructed as $conv_\nu \, \{\nabla^\nu F\}$ following the scheme from Scholtes (1994), see (11).

In a first step, explicit formulas for the derivatives of V and W are obtained. To this aim, we introduce partial derivative functions $\eta_z = \partial x/\partial z$, $\rho_z = \partial p/\partial z$, and $\eta_\alpha = \partial x/\partial t_\alpha$, $\rho_\alpha = \partial p/\partial t_\alpha$, respectively, which can be found from linearization of the state-adjoint system (14), (15). Differentiation of the canonical system w.r.t. z yields

$$\begin{aligned} \dot\eta_z &= A\,\eta_z, & \eta_z(1) &= I, \\ \dot\rho_z(t) &= -A^T \rho_z - C\,\eta_z, & \rho_z(1) &= \nabla_x^2 k(z) \end{aligned} \tag{29}$$

with $C = C(\cdot, \Sigma, z, h) = \nabla_x^2 (p^T f) = \nabla_x^2 H$ and $A = A(\cdot, \Sigma, z, h) = \nabla_x (f + g\,u)$ evaluated along $x = x(t, \Sigma, z, h)$, $p = p(t, \Sigma, z, h)$. The solutions are matrix functions which are continuously differentiable in time.
If the system is differentiated w.r.t. switching time parameter t_α, the following multi-point boundary value problem is obtained:

$$\begin{array}{rclll} \dot\eta_\alpha &=& A\,\eta_\alpha, & \eta_\alpha(1)=0, & [\,\eta_\alpha]^s = -[\,\nabla_p H]^\alpha\,, \\ \dot\rho_\alpha &=& -A^T\rho_\alpha - C\,\eta_\alpha, & \rho_\alpha(1)=0, & [\,\rho_\alpha]^s = [\,\nabla_x H]^\alpha\,. \end{array} \tag{30}$$

The solutions are piecewise continuous, and differentiable w.r.t. t on continuity intervals. For $\alpha = (j,s)$, the switching terms at t_α are given by

$$[\,\nabla_p H]^\alpha = g_j[t_\alpha]\,\left[\,u_j^0\right]^s, \qquad [\,\nabla_x H]^\alpha = \nabla_x g_j[t_\alpha]^T p(t_\alpha)\,\left[\,u_j^0\right]^s. \tag{31}$$

If t_α is a simple switch for u, then we have $[\,\nabla H]^\alpha = [\,\nabla H]^s$ – the full jump of ∇H at t_α. This is not true for case of multiple switches.

Using the above expressions, one can find the following partial derivatives of F (respectively V, W):

$$\begin{array}{rcll} \nabla_z V(\Sigma,z,h) &=& \Phi^T(1,\Sigma,z,h), & \\ \nabla_z W_\alpha(\Sigma,z,h) &=& \left[u_j^0\right]^s\,\left[\,p^T\nabla_x g_j\,\eta_z + g_j^T\rho_z\right]\big|_{\tau_\alpha}, & \alpha=(j,s)\in I_\Sigma, \quad (32) \\ \dfrac{\partial}{\partial t_\alpha}V(\Sigma,z,h) &=& \eta_\alpha(0) \;=\; \Phi[\tau_\alpha]^T\,[\nabla_p H]^\alpha & \end{array}$$

which are continuous functions of their arguments in a neighborhood of $(\Sigma^0, z^0, 0)$. In the representation, the fundamental matrix solutions $\Phi = \Phi(\cdot,\Sigma,z,h)$ from (21) are used as auxiliary functions.

In order to find generalized derivatives $\partial_\Sigma W$, we will first restrict W_α to such set of parameters Σ where $\Sigma_j \cap \Sigma_i = \emptyset$ for $i \neq j$. Differentiation of (17) gives

$$-\frac{1}{\left[u_j^0\right]^s}\frac{\partial}{\partial t_\beta}W_\alpha(\Sigma,z,h) = \delta_{\alpha\beta}\frac{d}{dt}\,(\sigma_j)|_{\tau_\alpha} + \left[p^T\nabla_x g_j\eta_\beta + g_j^T\rho_\beta\right]\big|_{\tau_\alpha} \tag{33}$$

(where the factor δ stands for the Kronecker symbol). By (30) we see that $\partial W_\alpha/\partial t_\beta$ vanishes for $\tau_\alpha > \tau_\beta$, and $\partial W_\alpha/\partial t_\alpha$ reduces to $\dot\sigma(\tau_\alpha)$ for each simple switching point $t = \tau_\alpha$: Indeed, the terms in squared brackets cancel out in the limit at $(\tau_\alpha - 0)$ by (31).
Similarly, we find the following limit values for a double switch $\tau_\alpha = \tau_\beta$ with $\alpha = (j,s)$, $\beta = (i,r)$ and $j \neq i$:

$$\begin{array}{rcll} \displaystyle\lim_{\tau_\beta-\tau_\alpha\to+0}\frac{\partial}{\partial t_\beta}W_\alpha(\Sigma,z,h) &=& -\left[\,u_j^0\right]^s\left(p^T\nabla_x g_j\,[\nabla_p H]^\beta - g_j^T\,[\nabla_x H]^\beta\right) & \\ &=& -\left[u_j^0\right]^s\left[u_i^0\right]^r\,\left(p^T\,[g_j,g_i]\right)\big|_{\tau_\alpha}, & (34) \\ \displaystyle\lim_{\tau_\beta-\tau_\alpha\to-0}\frac{\partial}{\partial t_\beta}W_\alpha(\Sigma,z,h) &=& 0, & (35) \end{array}$$

and analogously,

$$\frac{\partial}{\partial t_\alpha}W_\alpha(\Sigma,z,h) \to \begin{cases} -\left[u_j^0\right]^s\dot\sigma_j(\tau_\alpha-0) & \text{for} \quad \tau_\beta-\tau_\alpha\to+0 \\ -\left[u_j^0\right]^s\dot\sigma_j(\tau_\alpha+0) & \text{for} \quad \tau_\beta-\tau_\alpha\to-0 \end{cases}. \tag{36}$$

For simplicity assume that all entry points of Σ^0 are monotonically ordered in the whole (what can be always done by suitable index permutation). Then, by (34) – (36), important structural information on $\partial_\Sigma W$ is attainable: if all switching points are simple, $\partial_\Sigma W = \nabla_\Sigma W$ is a singleton with matrix elements given in (33). Obviously, the matrix is upper triangular.
In case of double switches, $\partial_\Sigma W$ is determined as

$$\begin{aligned} \partial_\Sigma(W(\Sigma', z, h)) &= conv\{ \nabla^\nu_\Sigma W(\Sigma, z, h)|_{\Sigma=\Sigma'}\ ,\ \nu = 1, \dots, 2^k\}, \\ \nabla^\nu_\Sigma W(\Sigma, z, h)|_{\Sigma=\Sigma'} &= \lim_{\Sigma \in D^\nu, \Sigma \to \Sigma'} \nabla_\Sigma W(\Sigma, z, h), \end{aligned}$$

(see (11) for notations), and elements are given by formulas (34) – (36). The matrices $\nabla^\nu_\Sigma W$ are now bloc-triangular. In particular, each simple switching point τ_γ corresponds to a nonzero diagonal element $\dot\sigma_j(\tau_\gamma)$ whereas for double switches $\tau_\alpha = \tau_\beta$ in the diagonal a $(2,2)$-bloc occurs, which is either upper or lower triangular. From (34) and (35) the following entries are obtained:

$$\nabla^\nu_{(\tau_\alpha,\tau_\beta)} W_{(\alpha,\beta)} = \begin{cases} \begin{pmatrix} \mu_j^- & \delta_{ij} \\ 0 & \mu_i^+ \end{pmatrix} & \text{if} \quad \tau_\beta > \tau_\alpha \quad \text{in} \quad D^\nu, \\ \begin{pmatrix} \mu_j^+ & 0 \\ \delta_{ji} & \mu_i^- \end{pmatrix} & \text{if} \quad \tau_\beta < \tau_\alpha \quad \text{in} \quad D^\nu, \end{cases} \tag{37}$$

with $\delta_{ij} = \left[u_i^0\right]^r \left[u_j^0\right]^s \left(p^T [g_i, g_j]\right)\big|_{\tau_\alpha}$, $\quad \mu_j^\pm = -\left[u_j^0\right]^s \dot\sigma_j(\tau_\alpha \pm 0)$ etc.
The derivative information above allows for proving the statements of Lemma 2:

Proof. Consider the (bloc-)triangular matrices M^ν related to D^ν by

$$M^\nu = \nabla^\nu_\Sigma W \in \partial_\Sigma W, \quad \nu = 1, \dots, 2^k.$$

Due to the maximum principle and Assumption 2a, their diagonal elements

$$m^\nu_{\alpha\alpha} = \mu_j^\pm = -\left[u_j^0\right]^s \dot\sigma_j(\tau_\alpha \pm 0)$$

are always positive. Thus, in case when only simple switches are present, the matrix $M = M^\nu$, $\nu = 1, \dots, 2^k$, has positive determinant and $\nabla_\Sigma W$ is regular.

Now assume that the vector Σ^0 contains double switch pairs, e.g. $\tau_\alpha = \tau_\beta$ for $\alpha = (j,s)$, $\beta = (i,r)$, and $i \neq j$. Again, the diagonal elements in M^ν are positive. Nondiagonal elements related to a double-switch bloc are found by (37) as

$$m^\nu_{\alpha\beta} \in \{0, \delta_{ij}\}, \qquad m^\nu_{\beta\alpha} \in \{0, -\delta_{ij}\} \tag{38}$$

where $\delta_{ij} = \mu_j^+ - \mu_j^- = \mu_i^- - \mu_i^+$, see (6).
It is easy to see that all matrices M^ν have positive determinants. If we consider arbitrary convex combinations of these bloc triangular matrices then, in diagonal blocs, we find convex combinations of two alternative forms which again have positive determinants: indeed, all diagonal elements are positive whereas

nondiagonal elements will have different signs, see (38).
Thus, all convex combinations of matrices from $\{M^\nu,\ \nu = 1,\dots,2^k\}$ assembling the set $\partial_\Sigma W$ are regular.

To complete the proof of the lemma, notice that the regularity of $\nabla_z V = \Phi^T(1)$ in (32) follows directly from properties of fundamental matrix solutions.■

Appendix 2: Parameter examples

Monotonicity of switching functions from (26). For $\epsilon \in (0,1)$ and appropriately chosen terminal parameters $z \in R^2$, the function $\sigma_1 = \epsilon\, p_1(\epsilon x_2 + x_1 + 1)$ as a function of time is monotone decreasing, and $\sigma_2 = p_2$ is monotone increasing:

Let z satisfy the conditions $z_1 \in (-1,0),\ z_2 > \epsilon$ together with

$$z_2 - T + 2\delta \,>\, \epsilon, \quad -z_2/z_1 \,<\, \frac{1+\epsilon^2}{\epsilon}\,\left(e^{\epsilon T} - 1\right) \tag{39}$$

where δ is given by

$$\frac{1+\epsilon^2}{\epsilon}\left(e^{\epsilon\delta} - 1\right) \;=\; -\frac{z_2}{z_1}. \tag{40}$$

Then, both σ_1 and σ_2 are strictly monotone functions of time.

Proof. Notice first that, due to the second part of condition (39), δ belongs to $(0,T)$.
Consider the time derivatives $\dot\sigma$: from the canonical system we get

$$\dot\sigma_1 \;=\; \epsilon\, p_1(\epsilon u_2 + x_2), \qquad \dot\sigma_2 \;=\; -\,p_1(1+\epsilon^2 u_1).$$

From (26) and $z_1 < 0$ it follows that $p_1 < 0$ all over $[0,T]$ and thus, $\sigma_2 = p_2$ is strictly monotone increasing. Since the function cannot change its sign more than once on R, e.g. at point t_s^2, from the state equation we find

$$x_2(t) \,\geq\, \begin{cases} \min\{z_2,\, z_2 + 2(T - t_s^2) - T\} & \text{if } \ \exists\, t_s^2 \in (0,T) \\ z_2 & \text{if } \ u_2 \equiv u_2(T) = -1 \end{cases}.$$

Finally, one can estimate t_s^2 using δ: indeed, from the adjoint equation and $z_1 < 0$ we see that, independently of switches of u_1, we always have $u_1 \leq 1$ and thus,

$$p_1(t) \,\geq\, z_1 e^{\epsilon(T-t)}, \qquad \dot p_2(t) \,\leq\, -z_1(1+\epsilon^2)\, e^{\epsilon(T-t)}. \tag{41}$$

Consequently,

$$p_2(t) \;=\; z_2 - \int_t^T \dot p_2(s)ds \,\geq\, z_2 + \, z_1\frac{1+\epsilon^2}{\epsilon}\left(e^{\epsilon(T-t)} - 1\right).$$

where equality holds at $t < T$ iff $u_1 \equiv +1$ on $[t, T]$. In particular, $p_2(T-\delta) \geq 0$ so that $t_s^2 \leq T - \delta < T$ follows. Using again (39) we conclude that $x_2(t) \geq \min\{z_2, z_2 + 2\delta - T\} > \epsilon$, or

$$\dot\sigma_1(t) < 0, \quad \dot\sigma_2(t) > 0 \qquad a.e. \text{ on } [0, T]. \tag{42}$$

∎

Remark 1 *An example of parameters satisfying assumptions (39) is given by* $\epsilon = 0.5, T = 2, z = (-0.4, 0.6)$ *with corresponding* $\delta = 2\ln 1.6 \approx 0.94$.

Backward Solution Method for first-order optimality system for $(\mathrm{P}_h(\epsilon))$. In the case that the backward shooting system (25), (26) has a solution such that the control function $u = u(\cdot, z)$ is bang-bang and satisfies Assumptions 1, 2a, the solution can be calculated by a prediction-correction algorithm:

step 1: For $z \in Z$ find $u(T) = -sign\,\sigma(T, z, \epsilon)$

step 2: Solve backwards the state-adjoint system with $u \equiv u(T)$,
$x(T) = p(T) = z$ on $[0, T]$. Set $k = 0$, $\theta_0 = T$.

step 3: Find $\sigma = \sigma(\cdot, z, \epsilon)$ from (26).
For $i = 1, 2$:
if σ_i changes sign on $(0, \theta_k)$ find $\hat t_s^i = \max\{t \in (0, \theta_k) : \sigma_i(t) = 0, \dot\sigma_i(t) \neq 0\}$.
Set $k = k + 1$ and $\theta_k = \max_i\{\hat t_s^i\}$.

step 4: Resolve state-adjoint equations on $[0, \theta_k]$ with $u \equiv -sign\,\sigma(\theta_k - 0)$.

step 5: Repeat steps 3–4 until structure of u is fixed.

Notice that, under monotonicity of $\sigma_{1,2}$, the solution is determined within at most two iteration steps. In our example, zeros of σ_2 in step 3 are found analytically whereas zeros of σ_1 are approximated by bisection.

The set C of double-switch parameters. As mentioned above, the estimates for p_1 and p_2 turn into equalities as far as $t_s^1 \leq t_s^2$ where the latter then will coincide with $(T-\delta)$. Consider now σ_1: the function will vanish at $t = T-\delta$ if and only if

$$D(\epsilon, z_1, z_2) := (\epsilon\, x_2 + x_1 + 1)|_{t=T-\delta} = 0.$$

The values for x_1, x_2 are found by backward solving the state system with $u \equiv u(T) = (+1, -1)^T$ what can be done analytically, e.g. with symbolic algebra tools. Inserting them into D yields

$$\begin{aligned} D &= \epsilon(z_2 + \delta) - \frac{1+\epsilon^2}{\epsilon^2}\left(e^{-\epsilon\delta} - 1 + \epsilon\delta\right) \\ &\quad + z_2\frac{1+\epsilon^2}{\epsilon}\left(e^{-\epsilon\delta} - 1\right) + (z_1 + 1)e^{-\epsilon\delta} \\ &=: a(\epsilon, r)\, z_1 + b(\epsilon, r), \qquad r = -z_2/z_1. \end{aligned} \tag{43}$$

The structure of the above expression allows (e.g. for $\epsilon = 0.5$ and r varying in the range $(0.7, 1.2)$) to find double-switch parameters $z_1^d = z_1^d(r)$ and $z_2^d =$

$z_2^d(r) = -r\, z_1^d$ by solving $D = 0$. The points (z_1^d, z_2^d) form the curve $C \subset Z$ which is smoothly parameterized by r. Asymptotic expansion for a shows that

$$a(\epsilon, r) = \mathrm{O}(1 + r^2) > 0 \qquad \text{for} \quad \epsilon \to 0,$$

and the positivity for (ϵ, r)-values considered above was confirmed numerically. Thus, for points lying to the right from curve C (i.e. (z_1, z_2^d) with $z_1 > z_1^d$) we get $D > 0$, and $D < 0$ for points to the left. In the first case, $t_s^1 < t_s^2 = T - \delta$ follows immediately. The second case tells us that the prediction for $\sigma_1(T - \delta)$ with $u \equiv u(T)$ is positive, i.e. $t_s^2 \le T - \delta < t_s^1$ must hold true. The result obtained is also confirmed by Tables 1–2, Section 5:

$$\begin{aligned} C &= \{(z_1^d, z_2^d) :\ D(\epsilon, z_1^d, z_2^d) = 0\}, \\ Z^1 &= \{(z_1, z_2) :\ D(\epsilon, z_1, z_2) > 0\}, \quad Z^2 = \{(z_1, z_2) :\ D(\epsilon, z_1, z_2) < 0\}. \end{aligned} \tag{44}$$

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References

Agrachev, A. and Sachkov, Yu. (2004) Control theory from the geometric viewpoint. *Encyclopaedia of Mathematical Sciences*, 87. *Control Theory and Optimization*, II. Springer, Berlin.

Agrachev, A., Stefani, G. and Zezza P.L. (2002) Strong optimality for a bang-bang trajectory. *SIAM J. Control Optim.* **41**, 991–1014.

Bulirsch, R. and Stoer, J. (1980) *Introduction to Numerical Analysis I, II.* Springer, New York.

Clarke, F.H. (1983) *Optimization and Nonsmooth Analysis.* Wiley Inc., New York.

Felgenhauer, U. (2003) On stability of bang-bang type controls. *SIAM J. Control Optim.*, **41**, 6, 1843–1867.

Felgenhauer, U. (2004) Optimality and sensitivity for semilinear bang-bang type optimal control problems. *Internat. J. Appl. Math. Computer Sc.*, **14**, 4, 447–454.

Felgenhauer, U. (2005) Optimality properties of controls with bang-bang components in problems with semilinear state equation. *Control & Cybernetics*, **34**, 3, 763–785.

Felgenhauer, U. (2006) Primal-dual stability approach for bang-bang optimal controls in semilinear systems. *Internat. J. Appl. Math. Computer Sc.*, to appear.

Felgenhauer, U. (2007) Lipschitz stability of broken extremals in bang-bang control problems. In: Lirkov et al., eds., *Proc. 6th Internat. Conf. Large-Scale Scientific Computing*, Sozopol 2007, *Lect. Notes Comp. Sci.* **4818**, Springer, 317–325.

Kim, J.R. and Maurer, H. (2003) Sensitivity analysis of optimal control problems with bang-bang controls. In: *Proc. 42nd IEEE-Conference on Decision and Control*, Maui (USA) 2003, **4**, 3281–3286.

Klatte, D. and Kummer, B. (2002) *Nonsmooth Equations in Optimization.* Dordrecht, Kluwer Acad. Publ.

Kojima, M. (1980) Strongly stable stationary solutions in nonlinear programs. In: *Analysis and Computing of Fixed Points.* Academic Press, 93–183.

Ledzewicz, U., Nowakowski, A. and Schättler, H. (2004) Stratifiable families of extremals and sufficient conditions for optimality in optimal control problems. *J. Optim. Theory Appl.* **122**, 2, 345–370.

Maurer, H. and Osmolovskii, N.P. (2005) Equivalence of second-order optimality conditions for bang-bang control problems, Part 1: Main results. *Control & Cybernetics*, **34**, 3, 2005, 927–950. Part 2: Proofs, variational derivatives, and representations. *Control & Cybernetics*, **36**, 1, 2007, 5–45.

Mordukhovich, B.S. (2006) *Variational Analysis and Generalized Differentiation I, II.* Springer, Berlin.

Noble, J. and Schättler, H. (2002) Sufficient conditions for relative minima of broken extremals in optimal control theory. *J. Math. Anal. Appl.* **269**, 98–128.

Osmolovskii, N.P. (2000) Second-order conditions for broken extremals. In: A. Ioffe et al., eds., *Calculus of Variations and Optimal Control.* Chapman & Hall/CRC Res. Notes Math. **411**, Boca Raton, FL, 198–216.

Poggiolini, L. and Stefani, G. (2004) State-local optimality of a bang-bang trajectory: a Hamiltonian approach. *Syst. Control Lett.* **53**, 3-4, 269–279.

Poggiolini, L. and Stefani, G. (2006) Sufficient optimality conditions for a bang-bang trajectory. In: *Proc. 45th IEEE-Conference on Decision and Control*, San Diego (USA), 2006, 6624–6629.

Schättler, H. (2006) Local fields of extremals for optimal control problems with state constraints of relative degree 1. *J. Dyn. Control Syst.* **12**, 4, 563–599.

Scholtes, S. (1994) Introduction to piecewise differentiable equations. *Preprint* **53**/1994, Inst. f. Statistik u. Mathemat. Wirtschaftstheorie, Universität Karlsruhe.