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On the regularization error of state constrained Neumann control problems*

by

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Abstract: A linear elliptic optimal control problem with pointwise state constraints in the interior of the domain is considered. Furthermore, the control is given on the boundary with associated constraints. An artificial distributed control is introduced in the cost functional, in the state equation and in the state constraints. Since there are no control constraints for the artificial control, efficient numerical methods can be easily established. Based on a possible violation of the pure pointwise state constraints, an error estimate for the regularization error is derived. The theoretical results are illustrated by numerical tests.

Keywords: optimal control, elliptic equation, state constraints, boundary control, regularization, virtual control.

1. Problem formulation

We are interested in an optimal control problem with constrained boundary control and pointwise state constraints:

$$
(\mathrm{P})\left\{\begin{array}{lll}
\text{Minimize} & J(y,u) := \dfrac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u\|^2_{L^2(\Gamma)} & \\
\text{subject to} & -\triangle y + y = 0 & \text{in } \Omega \\
 & \partial_n y = u & \text{on } \Gamma \\
 & u_a \le u(x) \le u_b & \text{a.e. on } \Gamma \\
 & y(x) \ge y_c(x) & \text{a.e. in } \Omega',
\end{array}\right.
$$

where $\Omega \subset \mathbb{R}^2$ is bounded domain with $C^{1,1}$ boundary and $\Omega' \subset\subset \Omega$ is an inner subdomain with $\operatorname{dist}\{\Omega',\Gamma\} > 0$. Furthermore, $y_d \in L^2(\Omega)$ and $y_c \in C^{0,\alpha}(\Omega')$, $0<\alpha<1$ are given functions and $u_a \le u_b$, $\nu > 0$ are real numbers. We use the abbreviation a.e. for "almost everywhere".

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Instead of problem (P), we will investigate a family of regularized optimal control problems:

$$
(\mathrm{P}_\varepsilon)\left\{\begin{array}{lll}
\text{minimize} & J_\varepsilon(y,u,v) := \dfrac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u\|^2_{L^2(\Gamma)} + \dfrac{\psi(\varepsilon)}{2}\|v\|^2_{L^2(\Omega)} & \\
\text{subject to} & -\triangle y + y = \phi(\varepsilon)v & \text{in } \Omega \\
 & \partial_n y = u & \text{on } \Gamma \\
 & u_a \le u(x) \le u_b & \text{a.e. on } \Gamma \\
 & y(x) \ge y_c(x) - \xi(\varepsilon)v(x) & \text{a.e. in } \Omega',
\end{array}\right.
$$

with a regularization parameter $\varepsilon > 0$ and Ω, ν, y_d, y_c, u_a and u_b as defined above. The functions ψ, ϕ and ξ are positive real valued functions depending on the regularization parameter ε respectively.

At a first glance, regularization of problem (P) seems not to be necessary since the problem is well-posed. Nevertheless, there are several arguments that a regularization is reasonable. We comment on Casas (1993), where one can see that the Lagrange multipliers associated to the state constraints are in general only Borel measures. Of course, this causes low regularity of the dual variables and of the optimal control. Due to the structure of the optimality, the optimal control $\bar{u}$, the optimal state $\bar{y}$ and the adjoint state $\bar{p}$ at the boundary are unique. However, if the state constraints are active on an open set, then the Lagrange multiplier and the adjoint state are not unique. Therefore, different regularization concepts have been developed in the last years. In a recent paper of Tröltzsch and Yousept (2006) a source representation was introduced. Moreover, an alternative penalty concept was used by Hintermüller and Kunisch (2006). Furthermore, we mention that the regularization techniques allow for an efficient numerical solution of the control problem.

The concept of virtual control was introduced in Krumbiegel and Rösch (2006). However, we will not assume additional control constraints concerning the virtual control. In Krumbiegel and Rösch (2006) this requirement was essential to derive error estimates, since the L^∞-bound of the virtual control occurs in all estimates. The main advantage of this approach without additional control constraints for the virtual control is that it decreases the numerical difficulties essentially. If constraints are given, it may happen that the different constraints are active simultaneously. This leads to non-uniqueness of the adjoint variables. Consequently, singular matrices may occur in different numerical schemes. The separation of active sets is given by construction in our new approach. Hence, numerical methods for solving optimal control problems are directly applicable, for instance the interior point method (see, e.g., Meyer, Prüfert and Tröltzsch, 2005; Schiela and Weiser, 2004); and the Primal-dual active set strategy (see, e.g., Bergounioux, Ito and Kunisch, 1999; Hintermüller, Ito and Kunisch, 2003; Kunisch and Rösch, 2002).

Let us mention that for the special choice $\phi(\varepsilon) \equiv 0$ the proposed concept is very close to the approach discussed by Hintermüller and Kunisch (2006).

The virtual control v plays there the part of the penalization function in the augmented objective.

This paper is organized as follows. In Section 2 optimality conditions and regularity results for the optimal solutions are established. The feasibility of solutions is discussed in Section 3. Furthermore, regularization estimates are derived. In Section 4 the application of the Primal-dual active set strategy and numerical tests illustrating the theory are presented.

2. Optimality conditions and regularity

In this section we will establish first order optimality conditions for both problems (P) and (P_ε). Furthermore, we will recall regularity results of the solutions. Considering the state equation of problem (P_ε), we split this equation in two separate ones. First we consider the boundary value problem:

$$\begin{aligned} -\triangle y_1 + y_1 &= 0 \quad && \text{in } \Omega \\ \partial_n y_1 &= u && \text{on } \Gamma. \end{aligned} \tag{1}$$

Equation (1) admits for all $u \in L^2(\Gamma)$ a unique solution $y_1 \in H^1(\Omega) \cap C(\bar{\Omega})$ (see, for instance, Tröltzsch, 2005). If we consider the state y_1 as a function in $L^2(\Omega)$, we can introduce the control-to-state operator $S_1 : L^2(\Gamma) \to L^2(\Omega)$, that assigns u to y_1. The second part of the state equation is given by:

$$\begin{aligned} -\triangle y_2 + y_2 &= \phi(\varepsilon)v \quad && \text{in } \Omega \\ \partial_n y_2 &= 0 && \text{on } \Gamma. \end{aligned} \tag{2}$$

Referring also to Tröltzsch (2005), (2) admits for every $v \in L^2(\Omega)$ a unique solution $y_2 \in H^1(\Omega) \cap C(\bar{\Omega})$. Hence, we introduce the linear and continuous solution operator $S_2 : L^2(\Omega) \to L^2(\Omega)$, $y_2 = S_2\phi(\varepsilon)v$. Therefore, the weak solutions of the state equations of problem (P) and (P_ε) are given by:

$$y = S_1u + S_2(\phi(\varepsilon)0) = S_1u \quad \text{for (P)}; \qquad y_\varepsilon = S_1u_\varepsilon + S_2(\phi(\varepsilon)v_\varepsilon) \quad \text{for } (\mathrm{P}_\varepsilon), \tag{3}$$

where we set $v \equiv 0$ for the problem (P).

Furthermore, we define the admissible sets for both problems

$$U_{ad} = \{u \in L^2(\Gamma) |\, u_a \le u \le u_b \text{ a.e. on } \Gamma;\, S_1u \ge y_c \text{ a.e. in } \Omega'\} \qquad \text{for (P)}$$

and

$$\begin{aligned} V_{ad}^{\varepsilon} = \{&(u,v) \in L^2(\Gamma) \times L^2(\Omega) |\, u_a \le u \le u_b \text{ a.e. on } \Gamma; \\ &S_1u + S_2\phi(\varepsilon)v \ge y_c - \xi(\varepsilon)v \text{ a.e. in } \Omega'\} \qquad \text{for } (\mathrm{P}_\varepsilon), \end{aligned}$$

respectively. Note that the state constraints and the mixed control-state constraints are included in the admissible sets respectively. The admissible sets are convex and closed. In the next section we guarantee the existence of feasible points such that the admissible sets are nonempty. For this purpose, we assume the existence of an inner point according to the pure state constraints. Since the objective functional is strictly convex and radially unbounded in both problems and the admissible sets are nonempty, the existence and uniqueness of optimal solutions is obtained by standard methods for both problems (P) and (P_ε).

In order to formulate the optimality conditions, we introduce the adjoint state for both problems. Note that all inequality constraints are handled by admissible sets. Consequently, there occurs no Lagrange multiplier in the adjoint problem. Furthermore, the adjoint problem is equal for both problems (P) and (P_ε). By standard methods one obtains

$$\begin{aligned} -\triangle p + p &= y - y_d && \text{in } \Omega \\ \partial_n p &= 0 && \text{on } \Gamma. \end{aligned} \tag{4}$$

The optimality conditions are formulated in the following lemma.

Lemma 1 *Let $(\bar{u}, \bar{y})$ be the optimal solution of problem (P). The necessary and sufficient optimality condition is given by*

$$(\bar{p}|_\Gamma + \nu\bar{u}, u - \bar{u})_{L^2(\Gamma)} \geq 0 \qquad \forall u \in U_{ad}, \tag{5}$$

where the associated adjoint state $\bar{p}$ is the solution of (4).
Moreover, let $(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{y}_\varepsilon)$ be the optimal solution of problem (P_ε). The optimality condition is given by

$$(\bar{p}_\varepsilon|_\Gamma + \nu\bar{u}_\varepsilon, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} + (\phi(\varepsilon)\bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon, v - \bar{v}_\varepsilon)_{L^2(\Omega)} \geq 0 \quad \forall (u,v) \in V_{ad}^\varepsilon, \tag{6}$$

where the associated adjoint state $\bar{p}_\varepsilon$ is the solution of (4).

In the sequel we introduce the optimality conditions of problem (P_ε) for a fixed regularization parameter $\varepsilon > 0$ using the classical approach with a Lagrange multiplier associated with the mixed constraint. In the case of pointwise control-state-constraints the existence of regular Lagrange multipliers was proved in Rösch and Tröltzsch (2006). Further on, the control constraints are treated by an admissible set

$$U_{ad} := \{u \in L^2(\Gamma) : \quad u_a \leq u \leq u_b \text{ a.e. on } \Gamma\}.$$

By introducing a Lagrange multiplier μ corresponding to the state-control-constraint, we obtain the following optimality system:

$$\begin{aligned} -\triangle \bar{y}_\varepsilon + \bar{y}_\varepsilon &= \phi(\varepsilon)\bar{v}_\varepsilon & \qquad -\triangle \tilde{p}_\varepsilon + \tilde{p}_\varepsilon &= \bar{y}_\varepsilon - y_d - \mu \\ \partial_n \bar{y}_\varepsilon &= \bar{u}_\varepsilon & \partial_n \tilde{p}_\varepsilon &= 0 \end{aligned} \tag{7}$$

$$(\tilde{p}_\varepsilon|_\Gamma + \nu\bar{u}_\varepsilon, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} \geq 0, \quad \forall u \in U_{ad} \tag{8}$$

$$\phi(\varepsilon)\tilde{p}_\varepsilon + \psi(\varepsilon)v_\varepsilon - \xi(\varepsilon)\mu = 0 \quad \text{a.e. in } \Omega \tag{9}$$

$$(\mu, y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon)_{L^2(\Omega')} = 0, \quad \mu \geq 0, \quad \bar{y}_\varepsilon \geq y_c - \xi(\varepsilon)\bar{v}_\varepsilon \quad \text{a.e. in } \Omega'. \tag{10}$$

To keep the notation simple, we extend the Lagrange multiplier in (7) and (9) by zero on the whole domain Ω. Since the Lagrange multiplier acts in the right hand side of the adjoint equation, the optimal adjoint state is denoted different from the adjoint state in the optimality condition (6). Now, we consider the adjoint equation in (7). From Grisvard (1985) we obtain the following regularity result.

THEOREM 1 *Let Ω be a bounded domain with $C^{1,1}$ boundary. Then for every $\bar{y}_\varepsilon \in L^2(\Omega)$ and $\mu \in L^2(\Omega)$ the adjoint equation in (6) admits a unique solution $\tilde{p}_\varepsilon \in H^2(\Omega)$.*

This result ensures that the boundary values of the adjoint state are at least of $H^1(\Gamma)$-regularity. In order to derive the regularity of the optimal control, we replace the optimality condition (9) by the equivalent and pointwise projection formula

$$\bar{u}_\varepsilon = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\tilde{p}_\varepsilon|_\Gamma\right). \tag{11}$$

Due to the fact that the projection operator in (11) provides for H^1-regularity, the optimal control $\bar{u}_\varepsilon$ belongs to $H^1(\Gamma)$. For the state equation we will use another result from Grisvard (1985).

THEOREM 2 *Let Ω be a bounded domain with $C^{1,1}$ boundary. Then the state equation in problem (6) admits for every $\bar{v}_\varepsilon \in L^2(\Omega)$ and $\bar{u}_\varepsilon \in H^{1/2}(\Gamma)$ a unique solution $\bar{y}_\varepsilon \in H^2(\Omega)$.*

Later, we will be interested in a L^∞-estimate of the virtual control $\bar{v}_\varepsilon$. Therefore, we state the following lemma.

LEMMA 2 *Let $(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{y}_\varepsilon)$ satisfy the optimality system (7)-(10) with associated adjoint state $\tilde{p}_\varepsilon$ and Lagrange multiplier μ. Then the following projection formula*

$$\xi(\varepsilon)\mu = \max\left\{0, \frac{\psi(\varepsilon)}{\xi(\varepsilon)}(y_c - \bar{y}_\varepsilon) + \phi(\varepsilon)\tilde{p}_\varepsilon\right\} \tag{12}$$

is valid on Ω'. Moreover, $\bar{v}_\varepsilon$ and μ belong to $L^\infty(\Omega)$.

Proof. We begin by splitting the domain Ω' into two disjoint subsets $\Omega' = \Omega_1 \cup \Omega_2$, where we define

$$\Omega_1 := \{x \in \Omega' : \mu(x) > 0\}$$
$$\Omega_2 := \{x \in \Omega' : \mu(x) = 0\}.$$

First, we consider Ω_1. The complementary slackness condition (10) yields

$$\bar{v}_\varepsilon = \frac{1}{\xi(\varepsilon)}(y_c - \bar{y}_\varepsilon).$$

Thus, we obtain by (9)

$$\xi(\varepsilon)\mu = \frac{\psi(\varepsilon)}{\xi(\varepsilon)}(y_c - \bar{y}_\varepsilon) + \phi(\varepsilon)\tilde{p}_\varepsilon$$

and the projection formula (12) is valid.

Considering Ω_2, we have $\mu = 0$. Therefore, equations (9) and (10) imply

$$0 = \phi(\varepsilon)\tilde{p}_\varepsilon + \psi(\varepsilon)v_\varepsilon \geq \frac{\psi(\varepsilon)}{\xi(\varepsilon)}(y_c - \bar{y}_\varepsilon) + \phi(\varepsilon)\tilde{p}_\varepsilon.$$

Hence, the projection formula is also satisfied.

Due to Theorems 1 and 2, we have H^2-regularity for $\bar{y}_\varepsilon$ and $\tilde{p}_\varepsilon$ respectively. Hence, the function inside the max-function belong to $L^\infty(\Omega)$ since the space $H^2(\Omega)$ is embedded in $L^\infty(\Omega)$. Consequently, the Lagrange multiplier is from $L^\infty(\Omega)$ and with the help of (9) we obtain $\bar{v}_\varepsilon \in L^\infty(\Omega)$. ■

Later, we want to observe the regularization error of the optimal solution $\bar{u}_\varepsilon$ concerning the optimal solution of the original problem. Therefore, we need uniform boundedness of the state $y_1 = S_1\bar{u}_\varepsilon$ in the space $C^{0,\alpha}(\Omega)$ for all ε. Due to the original problem, we have to deal with less regularity in the control. However, we can consider $u \in L^\infty(\Omega)$ in (1) since the control is bounded by the constraints. The following result is obtained from Mateos (2000).

THEOREM 3 *Let Ω be a bounded domain with $C^{1,1}$ boundary. Then for every $\bar{u}_\varepsilon \in L^p(\Gamma)$, $p \geq 1$ the partial differential equation* (1) *admits a unique solution $y_1 \in W^{1,p}(\Omega)$.*

Further, we obtain from the Sobolev embedding theorem, that $W^{1,p}(\Omega)$, $p > 2$ is embedded in the space $C^{0,\alpha}(\Omega)$ for $\alpha \leq 1 - 2/p$. Hence, we obtain with Theorem 3 the following a-priori bound for $y_1 = S_1\bar{u}_\varepsilon$ in $C^{0,\alpha}(\Omega)$:

$$\|y_1\|_{C^{0,\alpha}(\Omega)} \leq C\|y_1\|_{W^{1,p}(\Omega)} \leq C\|\bar{u}_\varepsilon\|_{L^p(\Gamma)} \leq C\max\{|u_a|, |u_b|\}. \tag{13}$$

3. New regularization error estimate

In this section we will derive an error estimate between the solution of the original problem (P) and the regularized problem (P_ε). Furthermore, we will point out the relationships between the different parameter functions.

3.1. Auxiliary results and feasibility

Now, we will construct feasible solutions for each problem based on the solutions of the other problem. However, we have to assume the existence of an inner point concerning the state constraint:

ASSUMPTION 1 There exists a function $\hat{u} \in L^2(\Gamma)$ with $u_a \leq \hat{u}(x) \leq u_b$ a.e. on Γ and $\hat{y}(x) \geq y_c + \tau$ a.e. in Ω' with $\tau > 0$, where $\hat{y}$ is the weak solution of the state equation in problem (P) for $\hat{u}$.

The following lemma ensures the feasibility of the solution $(\bar{u}, \bar{y})$ of the problem (P) for the regularized problem.

LEMMA 3 *For every $\varepsilon > 0$ the control $(\bar{u}, 0)$ is feasible for (P_ε).*

Proof. Since $\bar{u}$ is feasible for (P),

$$\xi(\varepsilon)0 + \bar{y} = \bar{y} \geq y_c \quad \text{a.e. in } \Omega'$$

holds true for all $\varepsilon > 0$. Therefore, $(\bar{u}, 0)$ also fulfills the constraints of (P_ε). ■

Next, we will derive an auxiliary result, which is needed later.

LEMMA 4 *Let $E \subset \mathbb{R}^d$ be open and bounded and the function f in $C^{0,\alpha}(E)$ for some $0 < \alpha \leq 1$ with $\|f\|_{C^{0,\alpha}(E)} \leq \sigma$. Then there exists a constant $C > 0$, such that the estimate*

$$\|f\|_{L^\infty(E)} \leq C\sigma^{\frac{d}{2\alpha+d}} \|f\|_{L^2(E)}^{\frac{2\alpha}{2\alpha+d}} \tag{14}$$

is satisfied.

Proof. Let $\bar{x} \in E$ be the point, where we obtain

$$M := |f(\bar{x})| = \max_{x \in E} \{|f(x)|\}.$$

Moreover, let $\mathcal{U}_\delta(\bar{x})$ be a ball with center $\bar{x}$ and sufficiently small radius δ such that

$$|f(x) - f(\bar{x})| \leq \frac{M}{2}, \qquad \forall x \in \mathcal{U}_\delta(\bar{x})$$

is satisfied. The definition of Hölder-continuous functions and $\|f\|_{C^{0,\alpha}(E)} \leq \sigma$ yields

$$|f(x) - f(\bar{x})| \leq \sigma\delta^\alpha, \qquad \forall x \in \mathcal{U}_\delta(\bar{x}).$$

By choosing δ as follows

$$\delta := \left(\sigma^{-1}\frac{M}{2}\right)^{1/\alpha},$$

we ensure the validity of the first inequality. Hence, we guarantee

$$|f(x)| \geq \frac{M}{2}, \qquad \forall x \in \mathcal{U}_\delta(\bar{x}).$$

Now, we will estimate the L^2-norm from below.

$$\|f\|^2_{L^2(E)} = \int\limits_E f^2 dx \geq \int\limits_{\mathcal{U}_\delta(\bar{x})} f^2 dx \geq \left(\frac{M}{2}\right)^2 \int\limits_{\mathcal{U}_\delta(\bar{x})} dx$$
$$= \left(\frac{M}{2}\right)^2 C\delta^d = C\sigma^{-d/\alpha} M^{2+d/\alpha} = C\sigma^{-d/\alpha} \|f\|^{\frac{2\alpha+d}{\alpha}}_{L^\infty(E)}.$$

This estimate implies the assertion. ∎

In the next step, we construct a feasible solution for the original problem (P). Therefore, we consider the violation of the optimal regularized control $\bar{u}_\varepsilon$ of problem (P_ε) with respect to the pure state constraints of problem (P). We define the violation function by

$$d[\bar{u}_\varepsilon, (P)] := (y_c - S_1\bar{u}_\varepsilon)_+ = \max\{0, y_c - S_1\bar{u}_\varepsilon\}. \tag{15}$$

Furthermore, the $L^\infty(\Omega')$-norm of this function is called maximal violation of $\bar{u}_\varepsilon$ to problem (P).

LEMMA 5 *The maximal violation* $\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')}$ *of* $\bar{u}_\varepsilon$ *to (P) can be estimated by*

$$\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} \leq C(\xi(\varepsilon) + \phi(\varepsilon))^{\frac{\alpha}{\alpha+1}} \|\bar{v}_\varepsilon\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega)}, \tag{16}$$

where C is a positive constant independent of ε.

Proof. Since $y_1 = S_1\bar{u}_\varepsilon$, $y_c \in C^{0,\alpha}(\Omega')$ and the max-function is continuous, the function $d[S_1\bar{u}_\varepsilon, (P)]$ belongs also to the space $C^{0,\alpha}(\Omega')$. Furthermore, we obtain

$$\|d[\bar{u}_\varepsilon, (P)]\|_{C^{0,\alpha}(\Omega')} = \|(y_c - S_1\bar{u}_\varepsilon)_+\|_{C^{0,\alpha}(\Omega')} \leq \|y_c\|_{C^{0,\alpha}(\Omega')} + \|S_1\bar{u}_\varepsilon\|_{C^{0,\alpha}(\Omega')}.$$

Using (13), we obtain an upper bound k independent of ε for $d[\bar{u}_\varepsilon, (P)]$. By Lemma 4 with $E = \Omega' \subset \mathbb{R}^2$ and some further estimates we obtain

$$\begin{aligned}
\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} &\leq C\|d[\bar{u}_\varepsilon, (P)]\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega')} \\
&\leq C\|(y_c - S_1\bar{u}_\varepsilon)_+\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega)} \\
&= C\|(y_c - S_1\bar{u}_\varepsilon - S_2\phi(\varepsilon)\bar{v}_\varepsilon + S_2\phi(\varepsilon)\bar{v}_\varepsilon)_+\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega)} \\
&\leq C\|(y_c - \bar{y}_\varepsilon)_+ + (S_2\phi(\varepsilon)\bar{v}_\varepsilon)_+\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega)} \\
&\leq C\left(\|(\xi(\varepsilon)\bar{v}_\varepsilon)_+\|_{L^2(\Omega)} + \|S_2\phi(\varepsilon)\bar{v}_\varepsilon\|_{L^2(\Omega)}\right)^{\frac{\alpha}{\alpha+1}} \\
&\leq C(\xi(\varepsilon) + \phi(\varepsilon))^{\frac{\alpha}{\alpha+1}} \|\bar{v}_\varepsilon\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega)}.
\end{aligned}$$

In the next Lemma, we will construct a feasible solution u_δ for the problem (P) depending on the inner point $\hat{u}$ and the optimal regularized control $\bar{u}_\varepsilon$.

LEMMA 6 *Let the Assumption 1 be satisfied. Then for every $\varepsilon > 0$ there exists a $\delta_\varepsilon \in (0,1)$, such that $u_\delta := (1-\delta)\bar{u}_\varepsilon + \delta\hat{u}$ is feasible for (P) for all $\delta \in [\delta_\varepsilon, 1]$.*

Proof. Since the control constraints of the problems (P) and (P_ε) are equal, the convex combination

$$u_\delta := (1-\delta)\bar{u}_\varepsilon + \delta\hat{u} \tag{17}$$

fulfills the constraints

$$u_a \leq \hat{u} \leq u_b, \qquad \text{a.e. in } \Gamma.$$

Furthermore, we know for the corresponding state

$$y_\delta = S_1 u_\delta + S_2\phi(\varepsilon)v_\delta.$$

Hence, we have to set $v_\delta \equiv 0$, such that the pair (u_δ, y_δ) becomes feasible for problem (P). We continue by

$$\begin{aligned} y_\delta = S_1 u_\delta &= (1-\delta)S_1\bar{u}_\varepsilon + \delta S_1\hat{u} \\ y_\delta - y_c &= (1-\delta)(S_1\bar{u}_\varepsilon - y_c) + \delta(\hat{y} - y_c) \\ &\geq -(1-\delta)d[\bar{u}_\varepsilon, (P)] + \delta\tau \\ &\geq -(1-\delta)\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} + \delta\tau. \end{aligned}$$

One can easily see that $\delta\tau - (1-\delta)\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} \geq 0$ implies the feasibility of u_δ for problem (P). Hence, we set

$$\delta_\varepsilon := \frac{\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')}}{\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} + \tau} \in (0,1) \qquad \forall \varepsilon > 0, \tag{18}$$

and u_δ is feasible for all $\delta \in [\delta_\varepsilon, 1]$. ∎

3.2. Error estimates

In this section we will derive the main result. The next result provides a preliminary error estimate of the optimal solutions of problem (P) with respect to the optimal regularized one of problem (P_ε).

THEOREM 4 *Let $(\bar{u}, \bar{y})$ and $(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{y}_\varepsilon)$ be the optimal solutions of (P) and (P_ε), respectively. Then, there exist positive constants C_1 and C_2 independent of ε, such that*

$$\nu\|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \leq \\ C_1(\xi(\varepsilon) + \phi(\varepsilon))^{\frac{\alpha}{\alpha+1}}\|\bar{v}_\varepsilon\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega)} + C_2\frac{(|\phi(\varepsilon)|)^2}{\psi(\varepsilon)}. \tag{19}$$

Proof. First, we introduce the weak formulations of the state equations in both problems (P) and (P_ε). The weak formulations are given by:

$$a(\bar{y}, z) = \int_\Gamma \bar{u} z ds, \quad \forall z \in H^1(\Omega) \tag{20}$$

and

$$a(\bar{y}_\varepsilon, z) = \int_\Gamma \bar{u}_\varepsilon z ds + \int_\Omega \phi(\varepsilon)\bar{v}_\varepsilon z dx, \quad \forall z \in H^1(\Omega), \tag{21}$$

respectively. The bilinear form $a(\cdot,\cdot)$ is defined by

$$a(y, z) := \int_\Omega \nabla y \cdot \nabla z dx + \int_\Omega yz dx.$$

Due to handling of all inequality constraints in admissible sets, the weak formulation for the adjoint equation is equal:

$$a(z, p) = (z, y - y_d)_{L^2(\Omega)} \qquad \forall z \in H^1(\Omega). \tag{22}$$

We consider the optimality conditions of both problems, where we use $(\bar{u}, 0)$ as a feasible test function in (6). According to Lemma 6, the control u_δ is useful as a test function in (5). Adding these two inequalities with the specific test functions, we obtain

$$(\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon, \bar{u} - \bar{u}_\varepsilon)_{L^2(\Gamma)} + (\phi(\varepsilon)\bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon, -\bar{v}_\varepsilon)_{L^2(\Omega)} + (\bar{p} + \nu\bar{u}, u_\delta - \bar{u})_{L^2(\Gamma)} \geq 0.$$

Next, we rewrite the previous inequality in a suitable form

$$\begin{aligned}(\bar{p} - \bar{p}_\varepsilon, \bar{u}_\varepsilon - \bar{u})_{L^2(\Gamma)} &+ \nu(\bar{u} - \bar{u}_\varepsilon, \bar{u}_\varepsilon - \bar{u})_{L^2(\Gamma)} \\ &+ (\bar{p} + \nu\bar{u}, u_\delta - \bar{u}_\varepsilon)_{L^2(\Gamma)} + (\phi(\varepsilon)\bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon, -\bar{v}_\varepsilon)_{L^2(\Omega)} \geq 0. \end{aligned} \tag{23}$$

Considering the first term in (23), we obtain with the help of (20) and (21)

$$(\bar{p} - \bar{p}_\varepsilon, \bar{u}_\varepsilon - \bar{u})_{L^2(\Gamma)} = a(\bar{y}_\varepsilon - \bar{y}, \bar{p} - \bar{p}_\varepsilon) - (\phi(\varepsilon)\bar{v}_\varepsilon, \bar{p} - \bar{p}_\varepsilon)_{L^2(\Omega)}.$$

Due to the weak formulation (22) of the adjoint equation of (P) and (P_ε), we get also

$$a(\bar{y}_\varepsilon - \bar{y}, \bar{p} - \bar{p}_\varepsilon) = (\bar{y}_\varepsilon - \bar{y}, \bar{y} - \bar{y}_\varepsilon)_{L^2(\Omega)} = -\|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)}.$$

In view of (23), we arrive at

$$\begin{aligned}\|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} + \nu\|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Gamma)} \leq (\bar{p} + \nu\bar{u}, u_\delta - \bar{u}_\varepsilon)_{L^2(\Gamma)} - \psi(\varepsilon)\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \\ - \phi(\varepsilon)(\bar{p}, \bar{v}_\varepsilon)_{L^2(\Omega)}.\end{aligned}$$

Inserting the definition (17) of u_δ and using the control constraints for $\hat{u}$ and $\bar{u}_\varepsilon$, we find

$$\begin{aligned}
&\|\bar{y}-\bar{y}_\varepsilon\|^2_{L^2(\Omega)}+\nu\|\bar{u}-\bar{u}_\varepsilon\|^2_{L^2(\Gamma)} \le \|\bar{p}+\nu\bar{u}\|_{L^2(\Gamma)}\|u_\delta \\
&\qquad -\bar{u}_\varepsilon\|_{L^2(\Gamma)}-\psi(\varepsilon)\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)}-\phi(\varepsilon)(\bar{p},\bar{v}_\varepsilon)_{L^2(\Omega)} \\
&= \|\bar{p}+\nu\bar{u}\|_{L^2(\Gamma)}\|\delta(\hat{u}-\bar{u}_\varepsilon)\|_{L^2(\Gamma)}-\psi(\varepsilon)\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)}-\phi(\varepsilon)(\bar{p},\bar{v}_\varepsilon)_{L^2(\Omega)} \\
&\le \|\bar{p}+\nu\bar{u}\|_{L^2(\Gamma)}\delta|u_b-u_a||\Gamma|^{1/2}-\psi(\varepsilon)\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)}+|\phi(\varepsilon)||(\bar{p},\bar{v}_\varepsilon)_{L^2(\Omega)}|.
\end{aligned}$$

Furthermore, the term $|(\bar{p},\bar{v}_\varepsilon)_{L^2(\Omega)}|$ can be estimated as follows:

$$|(\bar{p},\bar{v}_\varepsilon)_{L^2(\Omega)}| \le \frac{|\phi(\varepsilon)|}{2\psi(\varepsilon)}\|\bar{p}\|^2_{L^2(\Omega)}+\frac{\psi(\varepsilon)}{2|\phi(\varepsilon)|}\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)}.$$

According to Lemma 3 we choose the specific parameter

$$\delta_\varepsilon=\frac{\|d[\bar{u}_\varepsilon,(P)]\|_{L^\infty(\Omega')}}{\|d[\bar{u}_\varepsilon,(P)]\|_{L^\infty(\Omega')}+\tau}.$$

Without loss of generality, we note that there is a positive constant $C>0$ such that

$$\delta_\varepsilon \le C\|d[\bar{u}_\varepsilon,(P)]\|_{L^\infty(\Omega')}$$

is satisfied for all sufficiently small regularization parameters $\varepsilon>0$. Thus, with the help of (16) in Lemma 5 the estimate

$$\begin{aligned}
&\nu\|\bar{u}-\bar{u}_\varepsilon\|^2_{L^2(\Gamma)}+\|\bar{y}-\bar{y}_\varepsilon\|^2_{L^2(\Omega)}+\frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \le \\
&\qquad\qquad C_1(\xi(\varepsilon)+\phi(\varepsilon))^{\frac{\alpha}{\alpha+1}}\|\bar{v}_\varepsilon\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega)}+C_2\frac{(\phi(\varepsilon))^2}{\psi(\varepsilon)}
\end{aligned}$$

is obtained with

$$C_1=C|u_b-u_a||\Gamma|^{1/2}\|\bar{p}+\nu\bar{u}\|_{L^2(\Gamma)}, \quad C_2=\frac{1}{2}\|\bar{p}\|^2_{L^2(\Omega)}.$$

Let us mention that both constants can also be limited by expressions containing only the data of the problem. ■

In view of (19) $\frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}}$ has to become small for $\varepsilon\to 0$. Therefore, the following assumption is reasonable.

Assumption 2 For sufficiently small $\varepsilon>0$ we assume that

$$\frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}}<1. \tag{24}$$

Since $\bar{v}_\varepsilon$ is the optimal virtual control of problem (P$_\varepsilon$), one can easily see that $\|\bar{v}_\varepsilon\|_{L^2(\Omega)}$ is bounded by the objective functional as follows

$$\|\bar{v}_\varepsilon\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\psi(\varepsilon)}}, \quad C > 0.$$

However, we are able to improve this estimate with the help of the error estimate (19) in Theorem 4.

Corollary 1 *Let the assumptions of Theorem 4 and Assumption 2 be fulfilled. Then for sufficiently small $\varepsilon > 0$ the estimate*

$$\|\bar{v}_\varepsilon\|_{L^2(\Omega)} \leq C \frac{1}{\sqrt{\psi(\varepsilon)}} \left(\frac{\xi(\varepsilon) + \phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right)^{\frac{\alpha}{\alpha+2}} \tag{25}$$

is satisfied with some constant $C > 0$.

Proof. Considering the error estimate (19), we have the estimate:

$$\frac{\psi(\varepsilon)}{2} \|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \leq C_1 (\xi(\varepsilon) + \phi(\varepsilon))^{\frac{\alpha}{\alpha+1}} \|\bar{v}_\varepsilon\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega)} + C_2 \frac{(|\phi(\varepsilon)|)^2}{\psi(\varepsilon)}.$$

Moreover, this estimate implies

$$\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \leq \frac{C}{\psi(\varepsilon)} \max \left\{ (\xi(\varepsilon) + \phi(\varepsilon))^{\frac{\alpha}{\alpha+1}} \|\bar{v}_\varepsilon\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega)}, \frac{(|\phi(\varepsilon)|)^2}{\psi(\varepsilon)} \right\},$$

where $C = 2\max\{C_1, C_2\}$. Now, we consider the two cases where the maximum will be attained.

Case 1: First, we assume that the maximum in the right hand side of the previous inequality is attained by the second term. We note that the constant C is now generic. Hence, we obtain the following upper bound:

$$\|\bar{v}_\varepsilon\|_{L^2(\Omega)} \leq C \frac{\phi(\varepsilon)}{\psi(\varepsilon)}.$$

Case 2: Next, we consider the other case. We derive the following estimate:

$$\begin{aligned}
\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} &\leq \frac{C}{\psi(\varepsilon)} (\xi(\varepsilon) + \phi(\varepsilon))^{\frac{\alpha}{\alpha+1}} \|\bar{v}_\varepsilon\|^{\frac{\alpha}{\alpha+1}}_{L^2(\Omega)} \\
\|\bar{v}_\varepsilon\|^{\frac{\alpha+2}{\alpha+1}}_{L^2(\Omega)} &\leq \frac{C}{\psi(\varepsilon)} (\xi(\varepsilon) + \phi(\varepsilon))^{\frac{\alpha}{\alpha+1}} \\
\|\bar{v}_\varepsilon\|_{L^2(\Omega)} &\leq C \left((\psi(\varepsilon))^{-(\alpha+1)} (\xi(\varepsilon) + \phi(\varepsilon))^{\alpha} \right)^{\frac{1}{\alpha+2}}.
\end{aligned}$$

Concluding, we obtain the following upper bound

$$\begin{aligned}
\|\bar{v}_\varepsilon\|_{L^2(\Omega)} &\le C \max\left\{ \left((\psi(\varepsilon))^{-(\alpha+1)} (\xi(\varepsilon)+\phi(\varepsilon))^\alpha \right)^{\frac{1}{\alpha+2}}, \frac{\phi(\varepsilon)}{\psi(\varepsilon)} \right\} \\
&= \frac{C}{\sqrt{\psi(\varepsilon)}} \max\left\{ \sqrt{\psi(\varepsilon)} \left((\psi(\varepsilon))^{-(\alpha+1)} (\xi(\varepsilon)+\phi(\varepsilon))^\alpha \right)^{\frac{1}{\alpha+2}}, \frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right\} \\
&= \frac{C}{\sqrt{\psi(\varepsilon)}} \max\left\{ \left(\frac{(\xi(\varepsilon)+\phi(\varepsilon))^\alpha}{(\psi(\varepsilon))^{\alpha/2}} \right)^{\frac{1}{\alpha+2}}, \frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right\} \\
&= \frac{C}{\sqrt{\psi(\varepsilon)}} \max\left\{ \left(\frac{\xi(\varepsilon)+\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right)^{\frac{\alpha}{\alpha+2}}, \frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right\}
\end{aligned}$$

Due to (24) of Assumption 2 the maximum is attained by the first term inside the max-function, which is the assertion. ∎

Now we will provide the final error estimate of the optimal solutions of problem (P) concerning the optimal regularized one of problem (P$_\varepsilon$).

COROLLARY 2 *Let $(\bar{u}, \bar{y})$ and $(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{y}_\varepsilon)$ be the optimal solution of (P) and (P$_\varepsilon$), respectively. Then, there exists a positive constant C independent of ε, such that*

$$\nu \|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} \le C \left(\frac{(\xi(\varepsilon)+\phi(\varepsilon))^2}{\psi(\varepsilon)} \right)^{\frac{\alpha}{\alpha+2}}. \tag{26}$$

The error estimate (26) results directly from Theorem 4 and Corollary 1.

In practical applications one is interested in feasible controls, such that the infeasibility of the regularized control $\bar{u}_\varepsilon$ might be a problem. Therefore, we state the following remark, where we derive the same approximation properties for the *feasible* control as for the optimal regularized one $\bar{u}_\varepsilon$. For a proof we refer to Krumbiegel and Rösch (2006).

REMARK 1 Let u_δ be the control introduced in Lemma 6 with $\delta \sim \delta_\varepsilon$. Then there exists a constant $C > 0$ independent of ε such that

$$\nu \|\bar{u} - \bar{u}_\delta\|^2_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_\delta\|^2_{L^2(\Omega)} \le C \left(\frac{(\xi(\varepsilon)+\phi(\varepsilon))^2}{\psi(\varepsilon)} \right)^{\frac{\alpha}{\alpha+2}} \tag{27}$$

is satisfied.

One can easily see in the estimates (26) and (27) that an appropriate choice of the parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$ should satisfy the following conditions:

$$\lim_{\varepsilon\to 0} \frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} = 0, \qquad \lim_{\varepsilon\to 0} \frac{\xi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} = 0. \tag{28}$$

3.3. Results in the 3D-case

We mention that we can also derive a regularization error estimate for $\Omega \subset \mathbb{R}^3$, since the regularity result given in Theorem 3 is also valid in 3D. Moreover, by Sobolov embedding theorem $W^{1,p}(\Omega)$, $p > 3$ is embedded in the space of Hölder-continuous function $C^{0,\alpha}(\Omega)$ for $\alpha \le 1 - 3/p$.

Analogously to Lemma 5, the estimate of the maximal violation of the optimal regularized control $\bar{u}_\varepsilon$ to problem (P) is now given in the next corollary.

Corollary 3 *The maximal violation $\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')}$ of $\bar{u}_\varepsilon$ to (P) can be estimated by*

$$\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} \le C(\xi(\varepsilon) + \phi(\varepsilon))^{\frac{2\alpha}{2\alpha+3}} \|\bar{v}_\varepsilon\|_{L^2(\Omega)}^{\frac{2\alpha}{2\alpha+3}}, \tag{29}$$

where C is a positive constant independent of ε.

Further, the construction of a feasible control u_δ is equivalent to Lemma 6. Then the preliminary error estimate of the optimal solution of problem (P) according to the regularized one is provided in the next Corollary.

Corollary 4 *Let $(\bar{u}, \bar{y})$ and $(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{y}_\varepsilon)$ be the optimal solution of (P) and (P_ε), respectively. Then, there exist positive constants C_1 and C_2 independent of ε, such that*

$$\nu\|\bar{u} - \bar{u}_\varepsilon\|_{L^2(\Gamma)}^2 + \|\bar{y} - \bar{y}_\varepsilon\|_{L^2(\Omega)}^2 + \frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon\|_{L^2(\Omega)}^2 \le C_1(\xi(\varepsilon) + \phi(\varepsilon))^{\frac{2\alpha}{2\alpha+3}} \|\bar{v}_\varepsilon\|_{L^2(\Omega)}^{\frac{2\alpha}{2\alpha+3}} + C_2\frac{(|\phi(\varepsilon)|)^2}{\psi(\varepsilon)}. \tag{30}$$

With the help of Corollary 3 the proof can be done along the same lines as in Theorem 4. Deriving a similar estimate for $\|\bar{v}_\varepsilon\|_{L^2(\Omega)}$ as in Corollary 1, the final estimate for 3D-domains is given by:

Corollary 5 *Let $(\bar{u}, \bar{y})$ and $(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{y}_\varepsilon)$ be the optimal solution of (P) and (P_ε), respectively. Then, there exists a positive constant C independent of ε, such that*

$$\nu\|\bar{u} - \bar{u}_\varepsilon\|_{L^2(\Gamma)}^2 + \|\bar{y} - \bar{y}_\varepsilon\|_{L^2(\Omega)}^2 \le C\left(\frac{(\xi(\varepsilon) + \phi(\varepsilon))^{\frac{8\alpha+6}{2\alpha+3}}}{\psi(\varepsilon)}\right)^{\frac{\alpha}{3\alpha+3}}. \tag{31}$$

4. Numerical tests

The optimal control problem (P_ε) can be solved by several numerical methods, for instance an interior point method (see e.g. Meyer, Prüfert and Tröltzsch, 2005; Schiela and Weiser, 2004) or an active set strategy. For our purpose we want to apply a Primal-dual active set strategy, see e.g. Bergounioux, Ito and Kunisch (1999), Hintermüller, Ito and Kunisch (2003), Kunisch and Rösch

(2002). In order to realize this method, we have two possibilities. One can transform the problem (P_ε) to a completely control constrained problem by introducing new control

$$w_\varepsilon := \bar{y}_\varepsilon + \xi(\varepsilon)\bar{v}_\varepsilon.$$

Thus, one can apply the standard algorithm that is prescribed, for instance, in Bergounioux, Ito and Kunisch (1999). However, this transformation leads to a singular perturbed problem for $\varepsilon \downarrow 0$. Consequently, one has to deal with the specific difficulties of this problems.

The second strategy is focused on directly solving the optimality system (7)-(10), where a Lagrange multiplier corresponding to the mixed control-state constraints was introduced. Therefore, we will use a Primal-dual active set strategy. Deriving this method, we need the pointwise formulation of the complementary slackness condition (10) that is given by:

$$\int_\Omega \mu(x)(y_c(x) - \bar{y}_\varepsilon(x) - \xi(\varepsilon)\bar{v}_\varepsilon(x))dx = 0.$$

Due to $\mu(x) \geq 0$ and $y_c(x) \leq \bar{y}_\varepsilon(x) + \xi(\varepsilon)\bar{v}_\varepsilon(x)$, this implies

$$\mu(x)(y_c(x) - \bar{y}_\varepsilon(x) - \xi(\varepsilon)\bar{v}_\varepsilon(x)) = 0, \quad \text{a.e. in } \Omega.$$

Given the optimal solution $(\bar{u}_\varepsilon, \bar{y}_\varepsilon, \tilde{p}_\varepsilon, \bar{v}_\varepsilon)$ for (P_ε), we will define the active and inactive sets. First we consider the control constraints acting at the boundary Γ. The active and inactive sets can be defined by

$$\begin{aligned} \mathcal{A}_-^\Gamma &:= \{x \in \Gamma \,|\, \bar{u}_\varepsilon(x) = u_a\} \\ \mathcal{A}_+^\Gamma &:= \{x \in \Gamma \,|\, \bar{u}_\varepsilon(x) = u_b\} \\ \mathcal{I}^\Gamma &:= \Gamma \setminus \{\mathcal{A}_-^\Gamma \cup \mathcal{A}_+^\Gamma\}. \end{aligned}$$

With the help of these sets, the variational inequality (8) in the optimality system can be replaced by the following explicit expression:

$$\bar{u}_\varepsilon(x) = \begin{cases} u_a \quad , & x \in \mathcal{A}_-^\Gamma \\ u_b \quad , & x \in \mathcal{A}_+^\Gamma \\ -\dfrac{\tilde{p}_\varepsilon(x)|_\Gamma}{\nu}, & x \in \mathcal{I}^\Gamma. \end{cases}$$

The active and inactive sets concerning the mixed control-state constraints are defined up to sets of measure zero as follows:

$$\begin{aligned} \mathcal{A}^\Omega &:= \{x \in \Omega' \,|\, \xi(\varepsilon)\bar{v}_\varepsilon(x) + \bar{y}_\varepsilon(x) - \mu(x) < y_c(x)\} \\ \mathcal{I}^\Omega &:= \Omega \setminus \mathcal{A}^\Omega. \end{aligned}$$

Hence, the inequalities in (10) can be replaced by associated equalities on the sets $\mathcal{A}^\Omega$ and $\mathcal{I}^\Omega$:

$$\begin{aligned} \xi(\varepsilon)\bar{v}_\varepsilon(x) + \bar{y}_\varepsilon(x) = y_c(x), \qquad & \text{a.e. on } \mathcal{A}^\Omega \\ \mu(x) = 0, \qquad & \text{a.e. on } \mathcal{I}^\Omega. \end{aligned}$$

Thus, the optimality system (7)-(10) can be transformed into

$$\left.\begin{array}{c} -\triangle\,\bar{y}_\varepsilon + \bar{y}_\varepsilon = \phi(\varepsilon)\bar{v}_\varepsilon \qquad -\triangle\,\tilde{p}_\varepsilon + \tilde{p}_\varepsilon = \bar{y}_\varepsilon - y_d - \mu \\ \partial_n \bar{y}_\varepsilon = \bar{u}_\varepsilon \qquad \partial_n \tilde{p}_\varepsilon = 0 \\ \bar{u}_\varepsilon(x) = \begin{cases} u_a \quad , & x \in \mathcal{A}_-^\Gamma \\ u_b \quad , & x \in \mathcal{A}_+^\Gamma \\ -\dfrac{\tilde{p}_\varepsilon(x)|_\Gamma}{\nu}, & x \in \mathcal{I}^\Gamma. \end{cases} \\ \phi(\varepsilon)\tilde{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\mu = 0 \quad \text{a.e. in } \Omega \\ \xi(\varepsilon)\bar{v}_\varepsilon(x) + \bar{y}_\varepsilon(x) = y_c(x), \quad \text{a.e. on } \mathcal{A}^\Omega \\ \mu(x) = 0, \quad \text{a.e. on } \mathcal{I}^\Omega. \end{array}\right\} \qquad (32)$$

The Primal-dual active set strategy proceeds as follows.

ALGORITHM:

1. Define initial sets $\mathcal{A}_-^{\Gamma,(0)}$, $\mathcal{A}_+^{\Gamma,(0)}$ and $\mathcal{A}^{\Omega,(0)}$. Set $\mathcal{I}^{\Gamma,(0)} = \Gamma \setminus \{\mathcal{A}_-^{\Gamma,(0)} \cup \mathcal{A}_+^{\Gamma,(0)}\}$, $\mathcal{I}^{\Omega,(0)} = \Omega \setminus \mathcal{A}^{\Omega,(0)}$ and $k = 0$.
2. Determine the solution $(u_\varepsilon^k, y_\varepsilon^k, p_\varepsilon^k, v_\varepsilon^k, \mu^k)$ of the optimality system (32) on the current active and inactive sets.
3. Determine the new active and inactive sets by

$$\begin{aligned} \mathcal{A}_-^{\Gamma,(k+1)} &= \left\{x \in \Gamma :\, u_\varepsilon^k(x) - {p_\varepsilon}^k(x) - \nu u_\varepsilon^k(x) < u_a\right\} \\ \mathcal{A}_+^{\Gamma,(k+1)} &= \left\{x \in \Gamma :\, u_\varepsilon^k(x) - {p_\varepsilon}^k(x) - \nu u_\varepsilon^k(x) > u_b\right\} \\ \mathcal{I}^{\Gamma,(k+1)} &= \Gamma \setminus \left\{\mathcal{A}_-^{\Gamma,(k+1)} \cup \mathcal{A}_+^{\Gamma,(k+1)}\right\} \\ \mathcal{A}^{\Omega,(k+1)} &= \{x \in \Omega' :\, \xi(\varepsilon)v_\varepsilon^k(x) + y_\varepsilon^k(x) - \mu^k(x) < y_c(x)\} \\ \mathcal{I}^{\Omega,(k+1)} &= \Omega \setminus \mathcal{A}^{\Omega,(k+1)}. \end{aligned}$$

4. If $\mathcal{A}_-^{\Gamma,(k+1)} = \mathcal{A}_-^{\Gamma,(k)}$, $\mathcal{A}_+^{\Gamma,(k+1)} = \mathcal{A}_+^{\Gamma,(k)}$ and $\mathcal{A}^{\Omega,(k+1)} = \mathcal{A}^{\Omega,(k)}$ then STOP, else:
Set $k := k + 1$ and goto 2.

The convergence of the algorithm and the justification of the termination condition in step 4 is discussed in several papers. We refer to Bergounioux, Ito and Kunisch (1999), Hintermüller, Ito and Kunisch (2003), Kunisch and Rösch (2002), and Rösch and Wachsmuth (submitted).

4.1. Example

Our aim is to illustrate the influence of the parameter functions on the regularization error. To this end we constructed optimal solutions of the original optimal control problem

$$
\text{(PT)} \left\{ \begin{array}{lll} \text{minimize} & J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} & \\ \text{subject to} & -\triangle y + y = f & \text{in } \Omega \\ & \partial_n y = u + g & \text{on } \Gamma \\ & u_a \le u(x) \le u_b & \text{a.e. on } \Gamma \\ & y(x) \ge y_c(x) & \text{a.e. in } \Omega' \end{array} \right. ,
$$

with $\Omega = \mathcal{B}(0,1) \subset \mathbb{R}^2$ denoting the unit circle. Furthermore, the functions y_d, $f \in L^2(\Omega)$, $y_c \in C^{0,\alpha}(\Omega)$, $\alpha < 1$ and $g \in L^2(\Gamma)$ are given. These functions and the associated optimal solutions are given in polar coordinates (r, φ). In our examples we consider boundary control constraints

$$
u_a \le u(x) \le u_b \qquad \text{a.e. on } \Gamma,
$$

where u_a, u_b are real numbers. Furthermore, the lower state constraints are given by

$$
y_c(x) \le y(x) \qquad \text{a.e. in } \Omega'.
$$

The Lagrange multipliers μ associated with pointwise state constraints are in general regular Borel measures, see Bergounioux and Kunisch (2002). In order to construct an analytical solution $(\bar{u}, \bar{y}, \bar{p})$, we have to satisfy the optimality system

$$
\left. \begin{array}{l} -\triangle \bar{y} + \bar{y} = f \quad \text{in } \Omega \qquad -\triangle \bar{p} + \bar{p} = \bar{y} - y_d - \mu \quad \text{in } \Omega \\ \partial_n \bar{y} = \bar{u} + g \quad \text{on } \Gamma \qquad \partial_n \bar{p} = 0 \quad \text{on } \Gamma \\ (\bar{p} + \nu\bar{u}, u - \bar{u})_{L^2(\Gamma)} \ge 0 \qquad \forall u \in U_{ad} \\ \int\limits_{\Omega} (y_c - y)\, d\mu = 0 \\ \mu \ge 0,\ y_c(x) \le y(x) \quad \text{a.e. in } \Omega. \end{array} \right\} \tag{33}
$$

In this example, we construct μ such that $d\mu = \mu(x)dx$ with a nonnegative function $\mu \in L^\infty(\Omega)$. Choosing $\bar{y}(r,\varphi) = r^4 - 2r^2 + 0.5r^2 \sin^2\varphi$ for the optimal state, the state equation in (33) implies

$$
f(r,\varphi) = r^4 - 18r^2 + 7 + 0.5r^2 \sin^2\varphi.
$$

In order to fulfill the boundary condition, we define first

$$
\tilde{u}(\varphi) := \partial_n \bar{y}(r,\varphi) = \sin^2\varphi.
$$

Moreover, the optimal control is given by the following pointwise projection

$$
\bar{u}(\varphi) = \Pi_{[u_a,u_b]}\left(\tilde{u}(\varphi)\right).
$$

With the help of $g = \tilde{u} - \bar{u}$, the boundary condition of the state equation is satisfied. Furthermore, the lower state constraint is given by

$$y_c(r,\varphi) = \begin{cases} \bar{y}(r,\varphi) & , \bar{y}(r,\varphi) > C \\ 2\bar{y}(r,\varphi) - C & , \bar{y}(r,\varphi) \leq C, \end{cases}$$

with $C = -0.1$. Further, with

$$\mu(r,\varphi) = \begin{cases} \bar{y}(r,\varphi) - C & , \bar{y}(r,\varphi) > C \\ 0 & , \bar{y}(r,\varphi) \leq C, \end{cases}$$

the complementary slackness condition in (33) is satisfied. Moreover, we define the adjoint state by

$$\bar{p}(r,\varphi) = -r^4 + 2r^2 + 1 + \nu(r^4 - 2r^2)\sin^2\varphi.$$

One can easily see that $\bar{p}$ fulfills the homogeneous Neumann boundary condition. Since $\bar{p}|_\Gamma = \bar{p}(r=1,\varphi) = -\nu\sin^2\varphi$, the optimality condition

$$\bar{u}(\varphi) = \Pi_{[u_a,u_b]}\left(-\frac{\bar{p}|_\Gamma}{\nu}\right) = \Pi_{[u_a,u_b]}\left(\tilde{u}(\varphi)\right).$$

in (33) is satisfied.

As described in Section 1, we regularize state constraints in problem (PT) by introducing a virtual control as follows:

$$(\mathrm{PT}_\varepsilon) \begin{cases} \text{minimize} \quad J_\varepsilon(y,u,v) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v\|^2_{L^2(\Omega)} & \\ \text{subject to} \quad -\triangle y + y = g + \phi(\varepsilon)v & \text{in } \Omega \\ \qquad \partial_n y = u + g & \text{on } \Gamma \\ \qquad u_a \leq u(x) \leq u_b & \text{a.e. on } \Gamma \\ \qquad y(x) \geq y_c(x) - \xi(\varepsilon)v(x) & \text{a.e. in } \Omega' \end{cases}$$

The regularized problem $(\mathrm{PT}_\varepsilon)$ was solved numerically by the Primal-dual active set strategy that we mentioned above. The method was implemented using Matlab and its PDE-toolbox for generating a uniform finite element mesh of triangles. All functions were discretized by piecewise linear functions. In the following, the numerical solutions of the regularized problem $(\mathrm{PT}_\varepsilon)$ are denoted by $(\cdot)_\varepsilon$ and the optimal control, optimal state, an the optimal adjoint state of the unregularized problem (PT) are $\bar{u}$, $\bar{y}$ and $\bar{p}$.

For all computations we set $\nu = 0.1$. Furthermore, the lower and upper bound according to the control constraints are chosen as $u_a = 0.05$ and $u_b = 0.9$. For the first numerical calculation we use the meshsize $h = 0.04$ and the regularization parameter $\varepsilon = 0.004$ with the following choice of parameter functions

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) = \sqrt{\varepsilon}, \quad \xi(\varepsilon) = \sqrt{\varepsilon}.$$

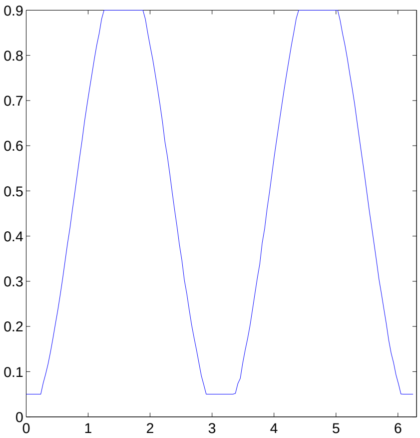

Figure 1. Control u_ε

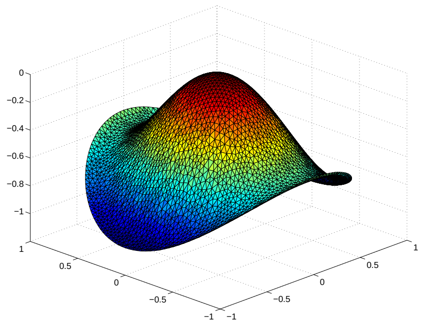

Figure 2. State y_ε

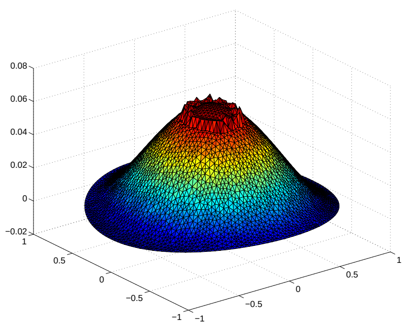

Figure 3. Virtual control v_ε

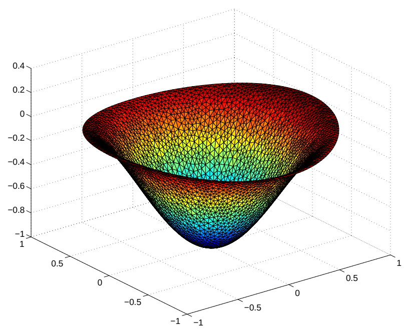

Figure 4. Adjoint state p_ε

Figs. 1-4 show the numerical solutions u_ε on the boundary for $\varphi \in [0, 2\pi)$, the state y_ε, the virtual control v_ε, and the adjoint state p_ε.

Next, we will investigate the behavior of the error between the regularized solutions and the] optimal solutions for $\varepsilon \to 0$. Moreover, we will consider two different settings of the parameter functions.

First, we illustrate the dependence of the error on the parameter function $\xi(\varepsilon)$. Thus, we set

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) = \varepsilon^2, \quad \xi(\varepsilon) = \varepsilon^{i/4} \quad i = 1, 2. \tag{34}$$

In order to ensure a sufficiently small discretization error, we set the mesh size $\tilde{h} = 0.005$ for this calculation. The behavior of the error for this choice is shown in Fig. 5, where the left diagram illustrates the error $\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$

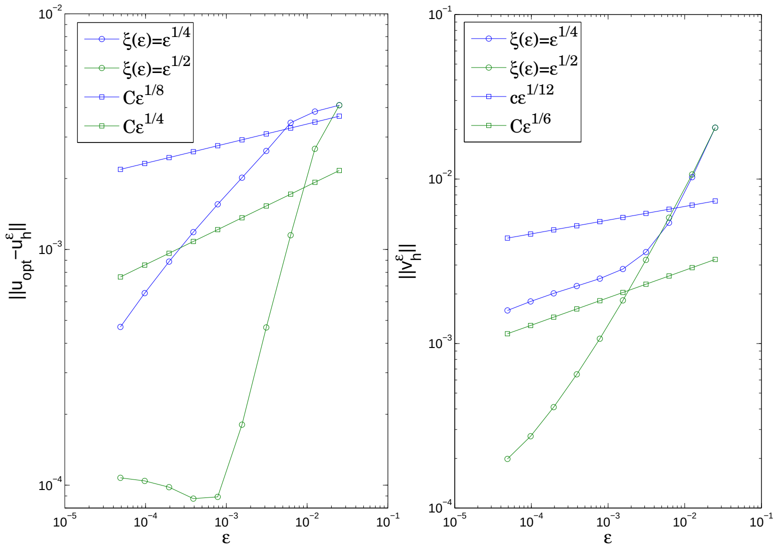

Figure 5. Error behavior for different settings in $\xi(\varepsilon)$

and the right diagram the $L^2(\Omega)$-norm of the virtual control v_ε, that tends to 0 according to Corollary 1.

The curves illustrate the validity of the error estimate given in Corollary 1 and Corollary 2. Furthermore, the descent rate of the error is increasing if the exponent of the regularization parameter in the choice of $\xi(\varepsilon)$ increases. Particularly in the circle-marked curve for $\xi(\varepsilon) = \varepsilon^{1/2}$ one can see that the discretization error dominates when the regularization parameter becomes smaller. Nevertheless, for all choices we obtain a better convergence rate than we expected by the theory.

For the case $i = 1$ in (34) we determined the experimental order of convergence with respect to ε. Therefore, we defined for positive error functionals $E(\varepsilon)$ with $\varepsilon > 0$ the value as follows: With two parameters $\varepsilon_1 \neq \varepsilon_2$ let

$$r_E := \frac{\ln E(\varepsilon_1) - \ln E(\varepsilon_2)}{\ln \varepsilon_1 - \ln \varepsilon_2}. \tag{35}$$

It follows from this definition that if $E(\varepsilon) = \mathcal{O}(\varepsilon^\beta)$ as $\varepsilon \downarrow 0$, then $r_E \approx \beta$. Moreover, the error functionals are given by

$$E_u(\varepsilon) = \|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}, \qquad E_v(\varepsilon) = \|v_\varepsilon\|_{L^2(\Omega)}. \tag{36}$$

Table 1 shows the values of the regularization errors according to the control and the values of the $L^2(\Omega)-$norm of the virtual control. Moreover, the experimental order of convergence with respect to ε is presented. According to Corollary 2 and the considered parameter functions, we expected for the error

Table 1. Errors and order of convergence for $i = 1$ in (34)

ε	$\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$	r_{E_u}	$\|v_\varepsilon\|_{L^2(\Omega)}$	r_{E_v}
$5e-2$	$5.0748e-3$	$-$	$4.1058e-2$	$-$
$2.5e-2$	$4.0931e-3$	0.31	$2.0531e-2$	0.99
$1.25e-2$	$3.8476e-3$	0.09	$1.0266e-2$	0.99
$6.25e-3$	$3.4497e-3$	0.16	$5.4201e-3$	0.92
$3.125e-3$	$2.6179e-3$	0.39	$3.5952e-3$	0.59
$1.5625e-3$	$2.0168e-3$	0.38	$2.8381e-3$	0.34
$7.8125e-4$	$1.5540e-3$	0.38	$2.4794e-3$	0.19
$3.9063e-4$	$1.1849e-3$	0.39	$2.2362e-3$	0.15
$1.9531e-4$	$8.8834e-4$	0.42	$2.0171e-3$	0.15
$9.7656e-5$	$6.5266e-4$	0.44	$1.8012e-3$	0.16
$4.8828e-5$	$4.6926e-4$	0.48	$1.5892e-3$	0.18

Table 2. Errors and order of convergence for $i = 1$ in (37)

ε	$\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$	r_{E_u}	$\|v_\varepsilon\|_{L^2(\Omega)}$	r_{E_v}
$1e-1$	$1.0503e-1$	$-$	$1.6255e-1$	$-$
$5e-2$	$5.5276e-2$	0.92	$8.1696e-2$	0.99
$2.5e-2$	$2.9725e-2$	0.89	$4.0955e-2$	1.00
$1.25e-2$	$1.6796e-2$	0.82	$2.0505e-2$	0.99
$6.25e-3$	$1.0293e-2$	0.70	$1.0259e-2$	0.99
$3.125e-3$	$7.0310e-3$	0.55	$5.1312e-3$	1.00
$1.5625e-3$	$5.4015e-3$	0.38	$2.5660e-3$	0.99
$7.8125e-4$	$3.5757e-3$	0.59	$1.2879e-3$	0.99
$3.9063e-4$	$1.7991e-3$	0.99	$6.4735e-4$	0.99
$1.9531e-4$	$8.7708e-4$	1.04	$3.2458e-4$	0.99
$9.7656e-5$	$4.1381e-4$	1.08	$1.6253e-4$	0.99

$\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$ a convergence rate of $\mathcal{O}(\varepsilon^{1/8})$. The experimental rates are much better than the theoretical ones. Particularly for smaller ε, the experimental convergence rate of the L^2-norm of the virtual control differs not so much from the expected one $\mathcal{O}(\varepsilon^{1/12})$.

Moreover, we observed the dependence on $\psi(\varepsilon)$ for the following settings

$$\phi(\varepsilon) \equiv 1, \quad \xi(\varepsilon) \equiv 1, \quad \psi(\varepsilon) = \varepsilon^{-(i+1)/2}, \quad i = 1, 2. \tag{37}$$

The results are shown in Fig. 6. Again, the behavior of the different curves illustrate the error estimates of Corollary 1 and Corollary 2. Furthermore, one can see different descent rates. Table 2 shows regularization error of the control and the $L^2(\Omega)$-norm of the virtual control v_ε for $i = 1$ in (37). Similarly to the first setting, the experimental convergence rates are better than the theoretical ones.

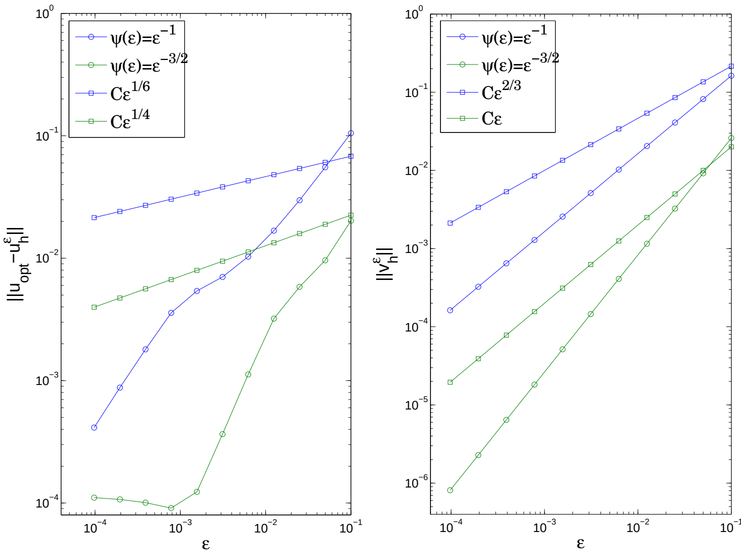

Figure 6. Error behavior for different settings in $\psi(\varepsilon)$

Let us summarize the numerical tests. We observed the convergence of the optimal control of problem (P_ε) for different choices of parameter functions. Moreover, behavior of the virtual control was considered and the tests confirmed the estimates of Corollary 1. However, for all settings of parameter functions the approximated rates were better than the expected ones. We note that the presented error estimates in Corollary 2 are the worst case scenarios. Moreover, in our numerical tests the regularization error is only one error. Of course, also a discretization error occurs.

Let us finish the paper with some comments on the computational effort. The computation time is essentially determined by the number of PDEs which have to be solved. We note that the new virtual control appears as a source term in the state equation. Thus, the effort for one iterate of solving the constrained linear quadratic optimization problem is hardly influenced by the virtual control. Furthermore, we computed the numerical results by the following strategy: We started with a moderate regularization parameter ε. Only a small number of iterations (< 10) of the Primal-dual active set strategy was needed to get the solution. The active and inactive sets of this solution were used for initialization to compute the results for the next smaller ε (a nested approach). The next solution was obtained after a few additional iterations.

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