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On singular arcs in nonsmooth optimal control*

by

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Abstract: In this paper we consider general optimal control problems (OCP) which are characterized by a nonsmooth ordinary state differential equation. However, we allow only mild types of nonsmoothness. More precisely, we assume that the right-hand side of the state equation is piecewise smooth and that the switching points, which separate these pieces, are determined as points, where a state- and possibly control-dependent (smooth) switching function changes sign. For this kind of optimal control problems necessary optimality conditions are developed. Attention is paid to the situation when the switching function vanishes identically along a nontrivial subarc. Such subarcs, which we call singular state subarcs, are investigated with respect to necessary conditions and to junction conditions. In extension to earlier results of the authors, Oberle and Rosendhal (2006), in this paper nonsmooth OCPs are considered with respect to the order of the switching function. Especially, the case of a zero-order switching function is included and examples of order zero, one and two are treated.

Keywords: nonsmooth optimal control problems, necessary conditions, singular state subarcs, Zermelo's problem.

1. Introduction

The paper is concerned with general optimal control problems (OCP) which are characterized by a nonsmooth ordinary state differential equation. More precisely, we assume that the right-hand side of the state equation is piecewise smooth and that the switching points, which separate these pieces, are determined as those points where a state- and possibly control-dependent (smooth) switching function changes sign. Nonsmooth optimal control problems of this type rarely have been mentioned in the literature, see for example Baumann (2002), Chudej (1995), and Moyer (2002). Of course, they are special examples for the rather general theory of Clarke (1983). Such problems sometimes occur in applications.

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It should be remarked that a similar type of optimal control problems, namely multiprocess control problems or multi-stage control problems, have been treated in the literature, see Clark, Vinter (1989), and Augustin, Maurer (2000). However, for this kind of optimal control problems the switching times between different stages are determined in a different way. Either these points are fixed or they are determined by boundary conditions. In this paper, however, we are interested in the optimal junction conditions, which are caused by the use of switching functions.

In a recent paper, Oberle and Rosendahl (2006), the authors have considered nonsmooth optimal control problems of the above type and have given an application to economics. However, only switching functions of order one have been treated and necessary optimality conditions have been derived. Especially the case of a so-called singular-state subarc has been considered.

In the present paper, we extend these results including the case of an order-zero switching function and give necessary optimality conditions for regular and singular OCPs of this type. Further, we consider two classical examples. The first example describes the optimal control of an electric circuit which includes a diode and a capacitor. This problem has already been investigated in the book of Clarke (1983). It is a nonsmooth OCP with a switching function of order zero. We apply the necessary conditions and present regular and singular solutions to this problem. By a slight modification we obtain a nonsmooth OCP with an order-two switching function. For this problem we present regular solutions.

The second example is the classical Zermelo's navigation problem. Here, one has to determine optimal control functions for a time-minimal horizontal plane flight of an aircraft within a prescribed space-depending wind field. If we assume that the wind field contains certain lines of discontinuities (atmospheric fronts), we end up with a nonsmooth OCP with a switching function of order one. We apply the necessary conditions and present numerical solutions as well for the regular as for the singular case.

The paper is organized as follows: In the first part we consider a general nonsmooth OCP and derive corresponding necessary conditions in the form of a multipoint boundary value problem. In section two, we further assume that the switching function along the solution trajectory changes sign only at isolated points (regularity assumption). The necessary conditions, we derive, differ for control dependent switching functions (order zero), on the one hand, and for switching functions which only depend on the state (positive order), on the other hand. In section three, in addition, we admit singular state subarcs. Here, the necessary conditions can be derived only for order zero and order one problems. In the remaining three sections we investigate the examples, mentioned before.

2. Nonsmooth optimal control problems, regular case

We consider a general OCP with a piecewise defined state differential equation. The problem has the following form:

Problem (P) Determine a piecewise continuous control function $u\colon [a,b] \to \mathbb{R}^m$, such that the functional

$$I = g(x(b)) \tag{1}$$

is minimized, subject to the following constraints (state equations, boundary conditions, and control constraints)

$$x'(t) \;=\; f(x(t), u(t))\,, \quad t \in [a,b] \quad \text{a.e.}, \tag{2a}$$
$$r(x(a), x(b)) \;=\; 0\,, \tag{2b}$$
$$u(t) \;\in\; \mathcal{U} \;\subset\; \mathbb{R}^m. \tag{2c}$$

The control region $\mathcal{U}$ is assumed to be a compact and convex cuboid of the form $\mathcal{U} = \Pi_i[u_{i,\min}, u_{i,\max}]$. Further, we assume that the right-hand side of the state equation (2a) is of the special form

$$f(x,u) = \begin{cases} f_1(x,u), & \text{if} \quad S(x,u) < 0, \\ f_s(x,u), & \text{if} \quad S(x,u) = 0, \\ f_2(x,u), & \text{if} \quad S(x,u) > 0, \end{cases} \tag{3}$$

where the functions $S : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ $(k = 1, 2, s)$, and $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$, $\ell \in \{0, \ldots, 2n\}$, are sufficiently smooth.

S is called the *switching function* of Problem (P). Note that in many cases the dynamics f_s – the index s stands for *singular* – along the singular surface $S = 0$ will be given either by $f_s := f_1$ or by $f_s := f_2$.

Our aim is to derive necessary conditions for Problem (P). To this end, let (x^0, u^0) denote a solution of the problem with a piecewise continuous optimal control function u^0. Piecewise continuity is understood in the sense that there exists a finite partition $a < t_1 < \ldots < t_q < b$ such that u^0 is continuous in each open subinterval and at a and b as well, and that all one-sided limits $u^0(t_j^\pm)$, $j = 1, \ldots, q$, exist.

We assume that the problem is *regular* with respect to the minimum principle, that is: For suitable λ, $x \in \mathbb{R}^n$ the *Hamiltonians*

$$\mathcal{H}_j(x,u,\lambda) \;:=\; \lambda^{\mathrm{T}} f_j(x,u), \quad j = 1, 2, s \tag{4}$$

possess a unique minimum u_j^0 with respect to the control $u \in \mathcal{U}$.

Finally, for this section, we assume that the following regularity assumption holds.

Regularity Condition (R) There exists a finite grid $a =: t_0 < t_1 < \ldots < t_q < t_{q+1} := b$ such that the optimal switching function $S[t] := S(x^0(t), u^0(t))$ is either positive or negative in each open subinterval $]t_{j-1}, t_j[$, $j = 1, \ldots, q+1$.

In the following, we distinguish two cases. On the one hand, if the switching function is independent of the control u, the switching function along the solution, $S[\cdot] := S(x^0(\cdot))$, is a continuous function, so that t_j is an isolated root of $S[\cdot]$. We indicate this case by $p > 0$.

On the other hand, if the switching function depends explicitly on the control, $S[\cdot] := S(x^0(\cdot), u^0(\cdot))$ may have discontinuities at the t_j. In this case, we say that the switching function is of order zero, $p = 0$.

Now, we can summarize the necessary conditions for the OCP (P). Here, on each subinterval $[t_j, t_{j+1}]$, we denote $\mathcal{H}(x, u, \lambda) := \mathcal{H}_j(x, u, \lambda)$ where $j \in \{1, 2\}$ is chosen according to the sign of S in the corresponding (open) subinterval. The following theorem is a generalization of our previous results in the related paper Oberle and Rosendahl (2006).

THEOREM 1 *With the assumptions above the following necessary conditions hold. There exist an adjoint variable $\lambda : [a, b] \to \mathbb{R}^n$, which is a piecewise C^1–function, and Lagrange multipliers $\nu_0 \in \{0, 1\}$, $\nu \in \mathbb{R}^\ell$, $\kappa \in \mathbb{R}^q$, such that (x^0, u^0) satisfies $(t \in [a, b])$*

$$\lambda'(t) = -\mathcal{H}_x(x^0(t), u^0(t), \lambda(t)), \quad a.e., \tag{5a}$$

$$u^0(t) = \mathit{argmin}\{\mathcal{H}(x^0(t), u, \lambda(t)) : \ u \in U\}, \tag{5b}$$

$$\lambda(a) = -\frac{\partial}{\partial x^0(a)} [\nu^T r(x^0(a), x^0(b))], \tag{5c}$$

$$\lambda(b) = \frac{\partial}{\partial x^0(b)} [\nu_0\, g(x^0(b)) + \nu^T r(x^0(a), x^0(b))], \tag{5d}$$

$$\lambda(t_j^+) = \begin{cases} \lambda(t_j^-), & \text{if } p = 0, \quad j = 1, \ldots, q, \\ \lambda(t_j^-) + \kappa_j \nabla_x S(x^0(t_j)), & \text{if } p > 0, \end{cases} \tag{5e}$$

$$\mathcal{H}[t_j^+] = \mathcal{H}[t_j^-], \quad j = 1, \ldots, q. \tag{5f}$$

Proof. We assume that there is just *one* point $t_1 \in]a, b[$, where the switching function $S[\cdot]$ changes sign. Moreover, we assume that the following *switching structure* holds

$$S[t] \begin{cases} < 0, & \text{if } \ a \le t < t_1 \\ > 0, & \text{if } \ t_1 < t \le b. \end{cases} \tag{6}$$

We compare the optimal solution (x^0, u^0) with those admissible solutions (x, u) of (P) which have the same switching structure. Each candidate of this type can be associated with its separated parts $(\tau \in [0, 1])$

$$\begin{array}{llll} x_1(\tau) := x(a + \tau(t_1 - a)), & x_2(\tau) := x(t_1 + \tau(b - t_1)), \\ u_1(\tau) := u(a + \tau(t_1 - a)), & u_2(\tau) := u(t_1 + \tau(b - t_1)). \end{array} \tag{7}$$

Now, $(x_1, x_2, t_1, u_1, u_2)$ is an admissible and $(x_1^0, x_2^0, t_1^0, u_1^0, u_2^0)$ an optimal solution of the following auxilliary optimal control problem.

PROBLEM (P') Determine a piecewise continuous control function $u = (u_1, u_2) : [0, 1] \to \mathbb{R}^{2\,m}$, such that the functional

$$I = g(x_2(1)) \tag{8}$$

is minimized, subject to the constraints $(\tau \in [0,1])$

$$x_1'(\tau) = (t_1 - a)\, f_1(x_1(\tau), u_1(\tau)), \qquad \text{a.e.,} \tag{9a}$$

$$x_2'(\tau) = (b - t_1)\, f_2(x_2(\tau), u_2(\tau)), \qquad \text{a.e.,} \tag{9b}$$

$$t_1'(\tau) = 0, \tag{9c}$$

$$r(x_1(0), x_2(1)) = 0, \tag{9d}$$

$$x_2(0) - x_1(1) = 0, \tag{9e}$$

$$S(x_1(1)) = 0, \quad \text{only if} \quad p > 0, \tag{9f}$$

$$u_1(\tau),\ u_2(\tau) \in \mathcal{U} \subset \mathbb{R}^m. \tag{9g}$$

Problem (P') is a classical optimal control problem with a smooth right-hand side, and $(x_1^0, x_2^0, t_1^0, u_1^0, u_2^0)$ is a solution of this problem. Therefore, we can apply the well–known necessary conditions of optimal control theory: There exist continuous and piecewise continuously differentiable adjoint variables $\lambda_j : [0,1] \to \mathbb{R}^n$, $j = 1,2$, and Lagrange-multipliers $\nu_0 \in \{0,1\}$, $\nu \in \mathbb{R}^\ell$, $\nu_1 \in \mathbb{R}^n$, and $\kappa \in \mathbb{R}$, such that, with the Hamiltonian

$$\widetilde{\mathcal{H}} := (t_1 - a)\, \lambda_1^{\mathrm{T}} f_1(x_1, u_1) + (b - t_1)\, \lambda_2^{\mathrm{T}} f_2(x_2, u_2), \tag{10}$$

and the augmented performance index

$$\Phi := \nu_0\, g(x_2(1)) + \nu^{\mathrm{T}} r(x_1(0), x_2(1)) + \nu_1^{\mathrm{T}} (x_2(0) - x_1(1)) + \kappa\, S(x_1(1)), \tag{11}$$

$(\kappa = 0$, if $p = 0)$ the following conditions hold

$$\lambda_1' = -\widetilde{\mathcal{H}}_{x_1} = -(t_1 - a)\, \left(\lambda_1^{\mathrm{T}} f_1(x_1, u_1)\right)_{x_1}, \tag{12a}$$

$$\lambda_2' = -\widetilde{\mathcal{H}}_{x_2} = -(b - t_1)\, \left(\lambda_2^{\mathrm{T}} f_2(x_2, u_2)\right)_{x_2}, \tag{12b}$$

$$\lambda_3' = -\widetilde{\mathcal{H}}_{t_1} = -\lambda_1^{\mathrm{T}} f_1(x_1, u_1) + \lambda_2^{\mathrm{T}} f_2(x_2, u_2), \tag{12c}$$

$$u_k(\tau) = \operatorname{argmin}\{\lambda_k(\tau)^{\mathrm{T}} f_k(x_k(\tau), u) :\ u \in U\},\ k = 1,2, \tag{12d}$$

$$\lambda_1(0) = -\Phi_{x_1(0)} = -(\nu^{\mathrm{T}} r)_{x_1(0)}, \quad \lambda_1(1) = \Phi_{x_1(1)} = -\nu_1 + \kappa\, S_x(x_1(1)), \tag{12e}$$

$$\lambda_2(0) = -\Phi_{x_2(0)} = -\nu_1, \quad \lambda_2(1) = \Phi_{x_2(1)} = (\nu_0\, g + \nu^{\mathrm{T}} r)_{x_2(1)}, \tag{12f}$$

$$\lambda_3(0) = \lambda_3(1) = 0. \tag{12g}$$

Due to the autonomy of the state equations and due to the regularity assumptions above, both parts $\lambda_1^{\mathrm{T}} f_1$ and $\lambda_2^{\mathrm{T}} f_2$ of the Hamiltonian are constant on $[0,1]$. Thus, λ_3 is a linear function which vanishes due to the boundary conditions (12g). Together with the relation (12c) one obtains the continuity of the Hamiltonian (5f).

If one recombines the adjoints

$$\lambda(t) := \begin{cases} \lambda_1\left(\dfrac{t-a}{t_1-a}\right), & t \in [a, t_1[, \\ \lambda_2\left(\dfrac{t-t_1}{b-t_1}\right), & t \in [t_1, b], \end{cases} \tag{13}$$

one obtains the adjoint equation (5a) from Eqs. (12a-b), the minimum principle (5b) from Eq. (12d), and the natural boundary conditions and the continuity and jump conditions (5c-e) from Eqs. (12e-f). ∎

It should be remarked that the results of Theorem 1 easily can be extended to nonautonomous OCPs with nonsmooth state equations and to problems with free final-time t_b. This holds too, if the performance index contains an additional integral term, i.e.

$$I \;=\; g(t_b, x(t_b)) \;+\; \int_a^{t_b} f_0(t, x(t), u(t))dt. \tag{14}$$

These extensions can be treated by standard transformation techniques, which transform the problems into the form of Problem (P). The result is that for the extended problems one has to redefine the Hamiltonian by

$$\mathcal{H}(t, x, u, \lambda, \nu_0) \;:=\; \nu_0\, f_0(t, x, u) \;+\; \lambda^{\mathrm{T}} f(t, x, u). \tag{15}$$

3. Nonsmooth optimal control problems, singular case

In this section we continue the investigation of the general optimal control problem (P). However, we drop the regularity condition (R). We assume that a solution (x^0, u^0) of (P) contains a finite number of nontrivial subarcs, where the switching function vanishes identically. More precisely:

Singularity Condition (S) We assume that there exists a finite grid $a =: t_0 < t_1 < \ldots < t_q < t_{q+1} := b$ such that in each open subinterval $]t_{j-1}, t_j[$, $j = 1, \ldots, q+1$, the optimal switching function $S[t] = S(x^0(t), u^0(t))$ is either totally positive, totally negative, or vanishes identically. The later subarcs are called *singular state subarcs*, see Bell, Jacobson (1975) and Bryson, Ho (1969), for the analogous situation of singular control subarcs.

Thus, the grid points t_j are either isolated points, where the switching function $S[\cdot]$ changes sign, or they are entry or exit points of a singular state subarc.

By J_{reg} we denote the set of indices of grid points t_j where the switching function changes sign, by J_{entry} those of the entry points, and by J_{exit} those of the exit points of the singular state subarcs.

We give a more precise definition of the *order* of a singular state subarc, in analogy to the order of state variable inequality constraint. To this end, we use the following recursive definition

$$S^{(0)}(x, u) := S(x, u), \quad S^{(k)}(x, u) := S_x^{(k-1)}(x, u)^{\mathrm{T}} f_s(x, u), \quad k = 1, 2, \ldots \tag{16}$$

We say that, for the solution (x^0, u^0), the switching function S is of order $p \geq 0$, if the first total time derivatives $S^{(k)}$, $k = 0, \ldots, p-1$, are independent of the control u, and further, if $S^{(p)}$ satisfies the following regularity condition (constraint qualification)

$$\frac{\partial}{\partial u} S^{(p)}(x^0(t), u^0(t)) \neq 0, \quad \forall\, t \in [t_j, t_{j+1}], \quad j \in J_{entry}. \tag{17}$$

Order Condition (O) We assume that the switching function is either of order zero, $p = 0$, or of order one, $p = 1$, with respect to the fixed solution (x^0, u^0) of problem (P), i.e.

$$\begin{array}{lll} \text{for} & p = 0: & S_u(x^0(t), u^0(t)) \neq 0, \\ \text{for} & p = 1: & S = S(x),\ S_u^{(1)}(x^0(t), u^0(t)) \neq 0 \end{array} \tag{18}$$

may hold along each singular state subarc.

Now, we introduce the extended Hamiltonian (here also denoted by $\mathcal{H}$)

$$\mathcal{H}(x, u, \lambda, \mu) := \mathcal{H}_k(x, u, \lambda, \mu) := \lambda^{\mathrm{T}} f_k(x, u) + \mu\, S^{(p)}(x, u), \tag{19}$$

where $k \in \{1, 2, s\}$ is chosen according to the sign of S in the corresponding subinterval, and μ denotes a Lagrange multiplier. We set $\mu := 0$ for $k = 1, 2$. Again, we assume regularity with respect to the minimum principle.

In the following, we summarize the necessary conditions for Problem (P).

Theorem 2 *With the assumptions above the following necessary conditions hold. There exist an adjoint variable $\lambda : [a, b] \to \mathbb{R}^n$, which is a piecewise C^1–function, and Lagrange multipliers $\nu_0 \in \{0, 1\}$, $\nu \in \mathbb{R}^\ell$, $\kappa_j \in \mathbb{R}$ ($j \in J_{reg} \cup J_{entry}$), and a piecewise continuous Lagrange multiplier $\mu : [a, b] \to \mathbb{R}$, such that (x^0, u^0) satisfies the conditions ($t \in [a, b]$)*

$$\lambda'(t) = -\mathcal{H}_x(x^0(t), u^0(t), \lambda(t), \mu(t)), \quad a.e. \tag{20a}$$

$$u^0(t) = \mathit{argmin}\{\mathcal{H}(x^0(t), u, \lambda(t), \mu(t)) :\ u \in U\}, \tag{20b}$$

$$\mu(t)\, S(x^0(t), u^0(t)) = 0, \tag{20c}$$

$$\lambda(a) = -\frac{\partial}{\partial x^0(a)} [\nu^T r(x^0(a), x^0(b))], \tag{20d}$$

$$\lambda(b) = \frac{\partial}{\partial x^0(b)} [\nu_0\, g(x^0(b)) + \nu^T r(x^0(a), x^0(b))], \tag{20e}$$

$$\lambda(t_j^+) = \begin{cases} \lambda(t_j^-) + \kappa_j\, \nabla_x S(x^0(t_j)), & \textit{for } p = 1,\ j \in J_{reg} \cup J_{entry}, \\ \lambda(t_j^-), & \textit{for } p = 0, \end{cases} \tag{20f}$$

$$\mathcal{H}[t_j^+] = \mathcal{H}[t_j^-], \quad j = 1, \ldots, q. \tag{20g}$$

Proof. For simplicity, we assume that the switching function $S[\cdot]$ along the optimal trajectory has just *one* singular state subarc $[t_1, t_2] \subset]a, b[$, and that the following *switching structure* holds

$$S[t] \quad \begin{cases} < 0, & \text{if} \quad a \le t < t_1, \\ = 0, & \text{if} \quad t_1 \le t \le t_2, \\ > 0, & \text{if} \quad t_2 < t \le b. \end{cases} \tag{21}$$

Again, we compare the optimal solution (x^0, u^0) with those admissible solutions (x, u) of the problem which have the same switching structure. Each candidate is associated with its separated parts $(\tau \in [0,1])$

$$\begin{array}{rclrcl} x_1(\tau) & := & x(a+\tau(t_1-a)), & u_1(\tau) & := & u(a+\tau(t_1-a)), \\ x_s(\tau) & := & x(t_1+\tau(t_2-t_1)), & u_s(\tau) & := & u(t_1+\tau(t_2-t_1)), \\ x_2(\tau) & := & x(t_2+\tau(b-t_2)), & u_2(\tau) & := & u(t_2+\tau(b-t_2)). \end{array} \tag{22}$$

Now, $(x_1, x_s, x_2, t_1, t_2, u_1, u_s, u_2)$ performs an admissible and $(x_1^0, x_s^0, x_2^0, t_1^0, t_2^0, u_1^0, u_s^0, u_2^0)$ an optimal solution of the following auxilliary optimal control problem.

Problem (P"). Determine a piecewise continuous control function $u = (u_1, u_s, u_2) : [0,1] \to \mathbb{R}^{3m}$, such that the functional

$$I = g(x_2(1)) \tag{23}$$

is minimized, subject to the constraints $(\tau \in [0,1])$

$$x_1'(\tau) = (t_1 - a)\, f_1(x_1(\tau), u_1(\tau)), \quad \text{a.e.,} \tag{24a}$$
$$x_s'(\tau) = (t_2 - t_1)\, f_s(x_s(\tau), u_s(\tau)), \quad \text{a.e.,} \tag{24b}$$
$$x_2'(\tau) = (b - t_2)\, f_2(x_2(\tau), u_2(\tau)), \quad \text{a.e.,} \tag{24c}$$
$$t_k'(\tau) = 0, \quad k = 1, 2, \tag{24d}$$
$$r(x_1(0), x_2(1)) = 0\,, \tag{24e}$$
$$x_s(0) - x_1(1) = x_2(0) - x_s(1) = 0, \tag{24f}$$
$$S(x_s(\tau), u_s(\tau)) = 0\,, \tag{24g}$$
$$u_1(\tau),\ u_s(\tau),\ u_2(\tau) \in \mathcal{U}. \tag{24h}$$

Problem (P") again is a classical OCP with a smooth right-hand side. However, it contains, depending on the order p, a (regular) control equality constraint, or a pure state equality constraint of first order, respectively. We can apply the classical necessary conditions of optimal control theory, see Hestenes (1966). If the constraint qualification (18) is satisfied, there exist a continuous Lagrange multiplier $\widetilde{\mu}$, and continuously differentiable adjoint variables λ_k, $k = 1, s, 2, 3, 4$, such that with the Hamiltonian

$$\begin{array}{rcl} \widetilde{\mathcal{H}} & := & (t_1 - a)\, \lambda_1^{\mathrm{T}} f_1(x_1, u_1) \ + \ (t_2 - t_1)\, \lambda_s^{\mathrm{T}} f_s(x_s, u_s) \\ & + & (b - t_2)\, \lambda_2^{\mathrm{T}} f_2(x_2, u_2) \ + \ \widetilde{\mu}\, (t_2 - t_1)\, S^{(p)}(x_s, u_s), \end{array} \tag{25}$$

and the augmented performance index (with $\kappa = 0$ for $p = 0$)

$$\begin{array}{rcl} \Phi & := & \nu_0 g(x_2(1)) - \kappa S(x_s(0), u_s(0)) + \nu^{\mathrm{T}} r(x_1(0), x_2(1)) \\ & + & \nu_1^{\mathrm{T}} (x_s(0) - x_1(1)) + \nu_2^{\mathrm{T}} (x_2(0) - x_s(1)), \end{array} \tag{26}$$

the following conditions hold $(\tau \in [0,1])$

$$
\begin{aligned}
\lambda_1' &= -\widetilde{\mathcal{H}}_{x_1} = -(t_1-a)\,\left(\lambda_1^{\mathrm{T}} f_1\right)_{x_1}, && \text{(27a)}\\
\lambda_s' &= -\widetilde{\mathcal{H}}_{x_s} = -(t_2-t_1)\big[(\lambda_s^{\mathrm{T}} f_s)_{x_s} + \widetilde{\mu}(\tau) S^{(p)}_{x_s}(x_s,u_s)\big], && \text{(27b)}\\
\lambda_2' &= -\widetilde{\mathcal{H}}_{x_2} = -(b-t_2)\,(\lambda_2^{\mathrm{T}} f_2)_{x_2}, && \text{(27c)}\\
\lambda_3' &= -\widetilde{\mathcal{H}}_{t_1} = -\lambda_1^{\mathrm{T}} f_1 + \lambda_s^{\mathrm{T}} f_s + \widetilde{\mu}(\tau)\, S^{(p)}(x_s,u_s), && \text{(27d)}\\
\lambda_4' &= -\widetilde{\mathcal{H}}_{t_2} = -\lambda_s^{\mathrm{T}} f_s + \lambda_2^{\mathrm{T}} f_2 - \widetilde{\mu}(\tau)\, S^{(p)}(x_s,u_s), && \text{(27e)}\\
u_j(\tau) &= \operatorname{argmin}\{\lambda_j(\tau)^{\mathrm{T}} f_j(x_j(\tau),u) :\ u \in \mathcal{U}\},\ \ j=1,\,2, && \text{(27f)}\\
u_s(\tau) &= \operatorname{argmin}\{\lambda_s(\tau)^{\mathrm{T}} f_s(x_s(\tau),u) + \widetilde{\mu}(\tau)\, S^{(p)}(x_s(\tau),u) :\ u \in \mathcal{U}\}, && \text{(27g)}\\
\lambda_1(0) &= -\Phi_{x_1(0)} = -(\nu^{\mathrm{T}} r)_{x_1(0)}, \quad \lambda_1(1) = \Phi_{x_1(1)} = -\nu_1, && \text{(27h)}\\
\lambda_s(0) &= -\Phi_{x_s(0)} = -\nu_1 + \kappa\, S_{x_s(0)}, \quad \lambda_s(1) = \Phi_{x_s(1)} = -\nu_2, && \text{(27i)}\\
\lambda_2(0) &= -\Phi_{x_2(0)} = -\nu_2, \quad \lambda_2(1) = \Phi_{x_2(1)} = (\ell_0\, g + \nu^{\mathrm{T}} r)_{x_2(1)}, && \text{(27j)}\\
\lambda_3(0) &= \lambda_3(1) = \lambda_4(0) = \lambda_4(1) = 0. && \text{(27k)}
\end{aligned}
$$

Due to the autonomy of the optimal control problem, all three parts $\lambda_1^{\mathrm{T}} f_1$, $\lambda_s^{\mathrm{T}} f_s$, and $\lambda_2^{\mathrm{T}} f_2$ of the Hamiltonian are constant. Due to Eq. (24g) we get $S^{(p)}(x_s(\tau),u_s(\tau)) = 0$, $\tau \in [0,1]$. Because of Eqs. (27d), (27e), and (27k), the adjoints λ_3 and λ_4 vanish and we obtain the global continuity of the augmented Hamiltonian (19).

If one recombines the adjoints

$$
\lambda(t) := \begin{cases} \lambda_1\big(\dfrac{t-a}{t_1-a}\big), & t \in [a,t_1[,\\[2mm] \lambda_s\big(\dfrac{t-t_1}{t_2-t_1}\big), & t \in [t_1,t_2],\\[2mm] \lambda_2\big(\dfrac{t-t_2}{b-t_2}\big), & t \in\,]t_2,b], \end{cases} \tag{28}
$$

and the state and control variables accordingly, one obtains all the necessary conditions of the theorem. ■

Again, we mention that the results of Theorem 2 can be easily extended to nonautonomous nonsmooth OCPs, to problems with free final-time, and to optimal control problems with performance index of Bolza type, as well.

4. A nonsmooth OCP of order zero

The following example is taken from the well-known book of Clarke (1983). It describes the control of an electronic circuit, which includes a diode and a condenser. The diode is treated as a resistor with two values of resistance depending on the direction of the current.

If $u := U$ denotes the initializing voltage (control), and $x := U_C$ denotes the voltage at the condenser (state), one obtains the following nonsmooth OCP.

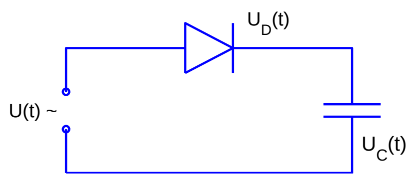

Figure 1. Electric circuit with a diode and a capacitor

Problem (D1). Minimize the functional

$$I(u) \;=\; \frac{1}{2}\int_0^2 u(t)^2\,dt \tag{29}$$

with respect to the state equation

$$x'(t) \;=\; \begin{cases} \alpha\,(u-x), & \text{if} \quad S = x-u \le 0, \\ \beta\,(u-x), & \text{if} \quad S = x-u > 0, \end{cases} \tag{30}$$

and the boundary conditions $x(0) = 4$, $x(2) = 3$.

In the smooth case, we choose $\alpha = \beta = 2$, the (unique) solution easily can be found applying the classical optimal control theory, see Fig. 2.

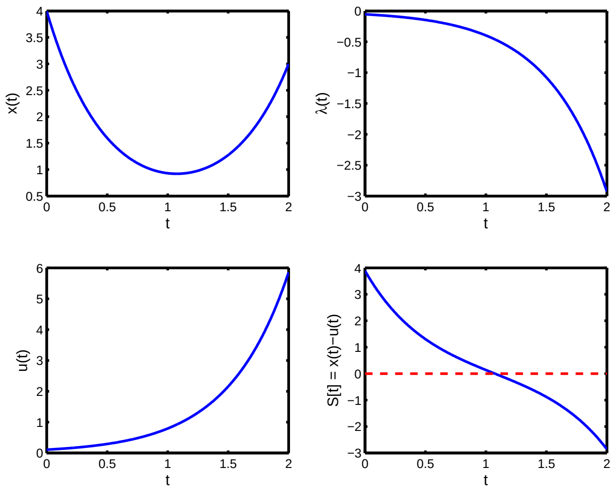

Figure 2. Problem (D1): Smooth case, $\alpha = \beta = 2$.

For the nonsmooth case, $\alpha \neq \beta$, we assume that there is just one point $t_1 \in]0, 2[$ where the switching function changes sign. Further, due to the results for the smooth case, we assume the solution structure

$$S[t] \begin{cases} > 0, & \text{if} \quad 0 \le t < t_1, \\ < 0, & \text{if} \quad t_1 < t \le 2. \end{cases} \tag{31}$$

According to Theorem 1 we obtain the following necessary conditions:

(i) $t \in [0, t_1^-]$: $\quad \mathcal{H} = \mathcal{H}_2 = \frac{1}{2}u^2 + \beta\lambda(u - x),$
$\lambda' = \beta\lambda, \quad u = -\beta\lambda.$

(ii) $t \in [t_1^+, 2]$: $\quad \mathcal{H} = \mathcal{H}_1 = \frac{1}{2}u^2 + \alpha\lambda(u - x),$
$\lambda' = \alpha\lambda, \quad u = -\alpha\lambda.$

The continuity condition (5f) yields

$$\mathcal{H}[t_1^+] - \mathcal{H}[t_1^-] = (\beta - \alpha)\lambda(t_1)\left[\frac{\alpha+\beta}{2}\lambda(t_1) + x(t_1)\right] = 0.$$

Thus, we obtain the following *three-point boundary value problem.*

$$\begin{aligned} x' &= \begin{cases} -\beta(\beta\lambda + x) & : \ t \in [0, t_1^-], \\ -\alpha(\alpha\lambda + x) & : \ t \in [t_1^+, 2], \end{cases} \\ \lambda' &= \begin{cases} \beta\lambda & : \ t \in [0, t_1^-], \\ \alpha\lambda & : \ t \in [t_1^+, 2], \end{cases} \\ x(0) &= 4, \qquad x(2) = 3, \qquad \frac{\alpha+\beta}{2}\lambda(t_1) + x(t_1) = 0. \end{aligned} \tag{32}$$

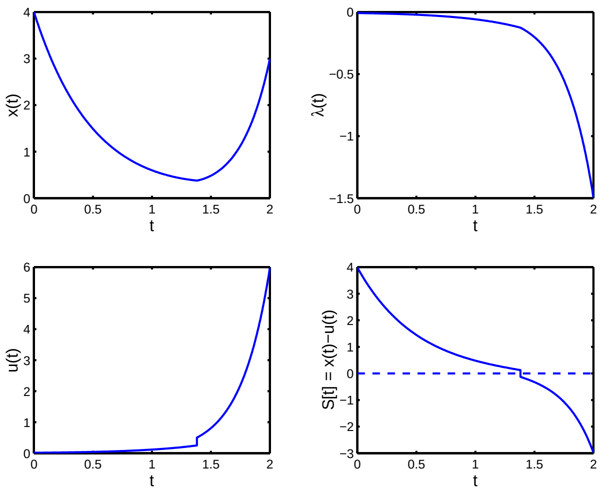

Figure 3. Problem (D1): Nonsmooth and regular case, $\alpha = 4$, $\beta = 2$.

In Fig. 3 the numerical solution for the parameters $\alpha = 4$ and $\beta = 2$ is shown. The result is obtained via the multiple shooting code BNDSCO, see Oberle, Grimm (1989), and Stoer, Bulirsch (1996). One observes that the preassumed sign distribution of the switching function is satisfied. Further, the optimal control and the optimal switching function are discontinuous at the switching point.

For parameters $\alpha < \beta$ the solution of the boundary value problem (32) does not satisfy the preassumed sign distribution of the switching function, see Fig. 4.

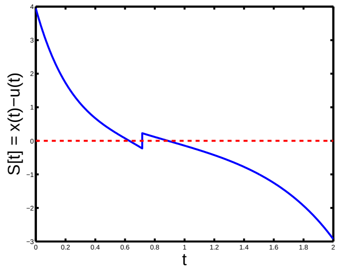

Figure 4. Problem (D1): Nonadmissible solution, $\alpha = 2$, $\beta = 4$.

So, for this choice of parameters we have to consider the singular case, i.e. the switching function vanishes along a nontrivial subarc. If we assume that there is exactly one singular state subarc,

$$S[t] \begin{cases} > 0, & \text{if} \quad 0 \le t < t_1, \\ = 0, & \text{if} \quad t_1 \le t \le t_2, \\ < 0, & \text{if} \quad t_2 < t \le 2, \end{cases} \tag{33}$$

we obtain the following necessary conditions due to Theorem 2.

$$\begin{array}{lll} \text{(i)} & t \in [0, t_1]: & \mathcal{H} = \mathcal{H}_2 = \frac{1}{2} u^2 + \beta \lambda (u - x), \\ & & \lambda' = \beta \lambda, \quad u = -\beta \lambda. \\ \text{(ii)} & t \in [t_1, t_2]: & \mathcal{H} = \mathcal{H}_s = \frac{1}{2} u^2 + \alpha \lambda (u - x) + \mu (x - u), \\ & & \lambda' = \alpha\lambda - \mu, \quad u = -\alpha \lambda + \mu = x. \\ \text{(iii)} & t \in [t_2, 2]: & \mathcal{H} = \mathcal{H}_1 = \frac{1}{2} u^2 + \alpha \lambda (u - x), \\ & & \lambda' = \alpha \lambda, \quad u = -\alpha \lambda. \end{array}$$

The continuity of the Hamiltonian, say at t_1, yields with

$$\begin{aligned} \mathcal{H}[t_1^-] = \mathcal{H}_2[t_1^-] &= \frac{1}{2} \beta^2 \lambda(t_1)^2 + \beta \lambda(t_1) (-\beta \lambda(t_1) - x(t_1)) \\ &= -\frac{1}{2} \beta \lambda(t_1) (\beta \lambda(t_1) + 2 x(t_1)) \\ \mathcal{H}[t_1^+] = \mathcal{H}_s[t_1^+] &= \frac{1}{2} x(t_1)^2 \end{aligned}$$

the interior boundary condition $x(t_1) + \beta \lambda(t_1) = 0$. The analogous condition holds at the second switching point t_2.

Altogether we obtain the following *multipoint boundary value problem.*

$$
\begin{aligned}
x' &= \begin{cases} -\beta\,(\beta\,\lambda + x) & : \quad t \in [0, t_1], \\ 0 & : \quad t \in [t_1, t_2], \\ -\alpha\,(\alpha\,\lambda + x) & : \quad t \in [t_2, 2], \end{cases} \\
\lambda' &= \begin{cases} \beta\,\lambda & : \quad t \in [0, t_1], \\ -x & : \quad t \in [t_1, t_2], \\ \alpha\,\lambda & : \quad t \in [t_2, 2], \end{cases} \\
x(0) &= 4, \qquad x(2) = 3, \\
x(t_1) &+ \beta\,\lambda(t_1) = 0, \quad x(t_2) + \alpha\,\lambda(t_2) = 0.
\end{aligned} \tag{34}
$$

For the parameters $\alpha = 2$, $\beta = 4$ the numerical solution is shown in Fig. 5. One observes a singular state subarc with the switching points $t_1 \doteq 0.632117$, $t_2 \doteq 0.882117$.

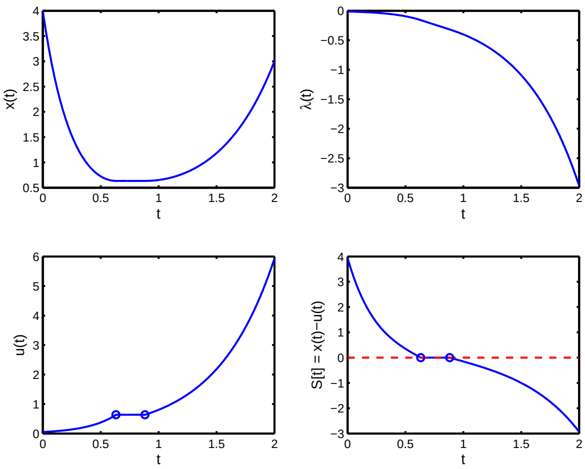

Figure 5. Problem (D1): Nonsmooth and singular case, $\alpha = 2$, $\beta = 4$.

In difference to the regular case, one observes that for the singular-state subarc the control and adjoint variable are continuous functions. This is a consequence of the necessary conditions when treating linear-quadratic OCPs, see Rosendahl (2008).

5. A modification of Clarke's example

In the following section we consider an OCP for a modified electric circuit, which contains a diode, a capacitor and a coil.

The relations between the initializing voltage U, the current I, and the voltages at the electric elements are given by

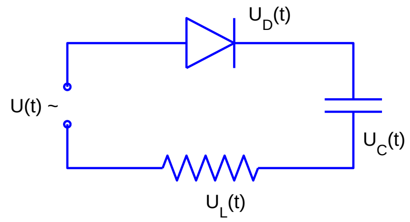

Figure 6. Electric circuit with a diode, a capacitor, and a coil

$$
\begin{aligned}
U(t) &= U_D(t) + U_C(t) + U_L(t), && \text{(35a)}\\
I(t) &= \begin{cases} U_D(t)/R_1, & \text{if } U_D \ge 0,\\ U_D(t)/R_2, & \text{if } U_D < 0,\end{cases} && \text{(35b)}\\
I(t) &= C\,\dot U_C(t), && \text{(35c)}\\
\dot I(t) &= U_L(t)/L. && \text{(35d)}
\end{aligned}
$$

By differentiation of Kirchhoff's law (35a) and using the abbreviations $u := \dot U$, $x_1 := I$, $x_2 := \dot I$, $\alpha := R_1/L$, $\beta := R_2/L$, and $\gamma := 1/(L\,C)$, we obtain the following OCP.

Problem (D2). Minimize the functional

$$I(u) = \frac{1}{2}\int_0^2 u(t)^2\,dt \tag{36}$$

with respect to the state equation

$$
\begin{aligned}
x_1'(t) &= x_2, && \text{(37a)}\\
x_2'(t) &= \begin{cases} u - \alpha\,x_2 - \gamma\,x_1, & \text{if } S := x_1 \ge 0,\\ u - \beta\,x_2 - \gamma\,x_1, & \text{if } S := x_1 < 0,\end{cases} && \text{(37b)}
\end{aligned}
$$

and the boundary conditions

$$x_1(0) = 1,\ x_2(0) = -4,\ x_1(2) = x_2(2) = 0. \tag{38}$$

One observes that the switching function of this nonsmooth OCP $S := x_1$ is of the order $p = 2$. For this situation, only the regular case is tractable with our theory above. If we use this regularity assumption (R) and apply Theorem 1 for one switching point, we get the following three-point boundary value problem:

$$
\begin{aligned}
x_1' &= x_2, && && \text{(39a)}\\
x_2' &= u - \delta\,x_2 - \gamma\,x_1, && u = -\lambda_2, && \text{(39b)}\\
\lambda_1' &= \gamma\,\lambda_2, && \delta := \begin{cases} \alpha, & \text{if } t \in [0, t_1],\\ \beta, & \text{if } t \in\,]t_1, 2],\end{cases} && \text{(39c)}\\
\lambda_2' &= -\lambda_1 + \delta\,\lambda_2, && && \text{(39d)}\\
\lambda_1(t_1^+) &= \lambda_1(t_1^-) + (\beta - \alpha)\,\lambda_2(t_1), && \lambda_2(t_1^+) = \lambda_2(t_1^-), && \text{(39e)}
\end{aligned}
$$

$$
\begin{aligned}
x_1(t_1) &= 0, && (39f)\\
x_1(0) = 1, \quad x_2(0) = -4, \quad x_1(2) &= x_2(2) = 0. && (39g)
\end{aligned}
$$

This boundary value problem can be solved numerically. For both cases $\alpha < \beta$ and $\alpha > \beta$ we obtain admissible solutions, which satisfy the regularity assumption.

In Fig. 7 the solution of the boundary-value problem (39) for the parameters $\alpha = 2$, $\beta = 3$ is shown. Fig. 8 gives the solution for $\alpha = 3$, $\beta = 2$.

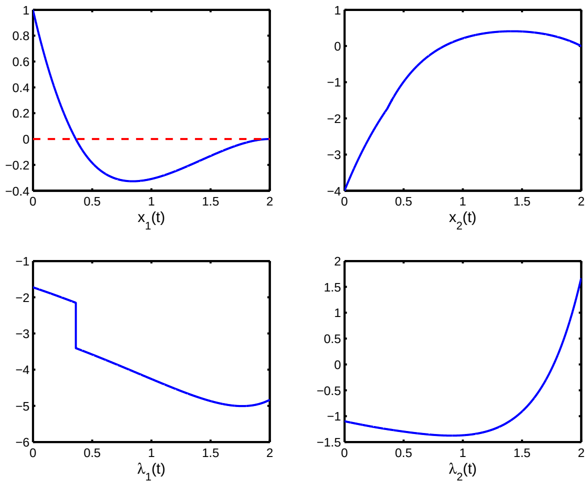

Figure 7. Problem (D2): Nonsmooth and regular case, $\alpha = 2$, $\beta = 3$, $\gamma = 1$.

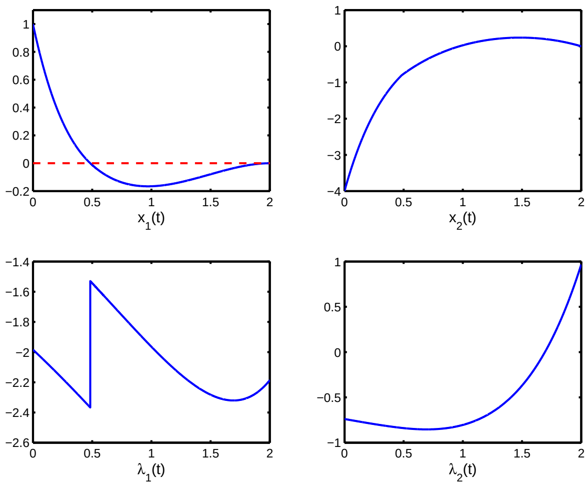

Figure 8. Problem (D2): Nonsmooth and regular case, $\alpha = 3$, $\beta = 2$, $\gamma = 1$.

6. The nonsmooth Zermelo's problem

In this section we consider a modification of the classical problem of Zermelo, see Arrow (1949), Zermelo (1930, 1931). In the literature of optimal control the problem is well known as the ship navigation problem. In its original notation, however, the problem is given as follows. One has to determine the heading control for the horizontal plane flight of an aircraft within a prescribed space-depending horizontal wind field such that the transfer time from a given initial- to a given endpoint is minimized.

In mathematical notation the problem can be formulated as an optimal control problem.

PROBLEM (Z) Determine the transfer time t_f and a piecewise continuous control function $\Theta(t)$, $0 \le t \le t_f$, such that

$$I(\Theta, t_f) \;:=\; t_f \tag{40}$$

is minimized, subject to following state equations and boundary conditions:

$$\begin{aligned} x'(t) &= v_0 \cos(\Theta(t)) + u(x(t), y(t)), && (41a)\\ y'(t) &= v_0 \sin(\Theta(t)) + v(x(t), y(t)), && (41b)\\ x(0) &= x_0, \quad x(t_f) = x_f, && (41c)\\ y(0) &= y_0, \quad y(t_f) = y_f. && (41d) \end{aligned}$$

Here, v_0 is the (constant) magnitude of the aircraft velocity relative to the wind field, Θ is the heading angle (control function), (u, v) is the velocity of the wind field relative to the ground. For simplicity, we assume that (u, v) depends only on the state (x, y), the position of the aircraft.

Further modifications of this problem, including, for example, wind fields which vary in space and time, or a three-dimensional modelling, are more or less straightforward.

6.1. The smooth case

First, we summarize the necessary conditions for the smooth case, i.e. the wind field may be a smooth function of (x, y). The Hamiltonian is given by

$$\mathcal{H} \;=\; \lambda_1 (v_0 \cos(\Theta) + u) + \lambda_2 (v_0 \sin(\Theta) + v). \tag{42}$$

By the minimum principle we obtain the following optimal control law

$$\cos(\Theta) \;=\; -\frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \quad \sin(\Theta) \;=\; -\frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \tag{43}$$

and, thus, together with the adjoint equations, we obtain the following *two-point boundary value problem* with respect the independent variable $\tau \in [0, 1]$:

$$\begin{aligned} x' &= t_f (v_0 \cos(\Theta) + u(x, y)), && (44a)\\ y' &= t_f (v_0 \sin(\Theta) + v(x, y)), && (44b) \end{aligned}$$

$$\lambda_1' \;=\; t_f\,(-\lambda_1\,u_x(x,y) - \lambda_2\,v_x(x,y)), \tag{44c}$$
$$\lambda_2' \;=\; t_f\,(-\lambda_1\,u_y(x,y) - \lambda_2\,v_y(x,y)), \tag{44d}$$
$$t_f' \;=\; 0, \tag{44e}$$
$$x(0) \;=\; x_0, \quad x(1) \;=\; x_f, \tag{44f}$$
$$y(0) \;=\; y_0, \quad y(1) \;=\; y_f, \tag{44g}$$
$$\mathcal{H}[1] \;=\; [-v_0\sqrt{\lambda_1^2+\lambda_2^2} \;+\; \lambda_1\,u \;+\; \lambda_2\,v]|_{\tau=1} \;\;=\;\; -1. \tag{44h}$$

Following Bryson, Ho (1969), we choose the wind field (shear wind)

$$u(x,y) \;:=\; -\,v_s\,y, \qquad v(x,y) \;:=\; 0 \tag{45}$$

and the parameters

$$v_0 := 1, \quad v_s := 0.8, \quad x_0 := 3.66, \quad y_0 := -1.86, \quad x_f := 0, \quad y_f := 1. \tag{46}$$

Fig. 9 shows the optimal flight path. The aircraft heading is indicated at several points along the path. For the minimal flight time we obtain $t_f \doteq 4.9257352$.

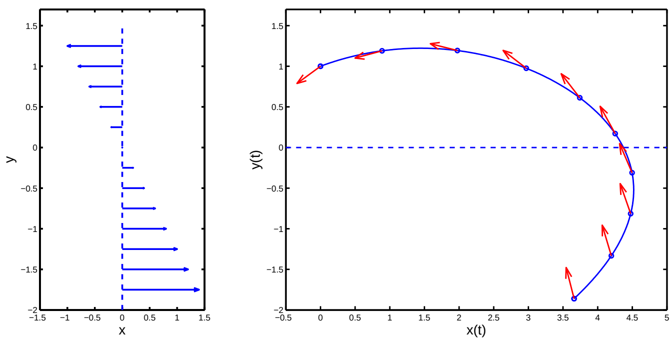

Figure 9. Problem (Z): Minimum time path for a smooth wind field; $v_0 = 1, v_s = 0.8$.

6.2. The nonsmooth case

Next, we consider the case of a nonsmooth wind field. With this ansatz an atmospheric front may be modeled. Again, we simplify the practical problem and choose the time-independent front line $y = 0$. Note, however, that the general theory allows to handle the case of time variant front lines too.

We choose the following wind field;

$$u(x,y) \;:=\; \begin{cases} -v_s\,y, & \text{if} \quad y \geq 0, \\ v_s, & \text{if} \quad y < 0, \end{cases} \qquad v(x,y) \;:=\; 0, \tag{47}$$

i.e., for $y < 0$, there is a constant head wind, whereas, for $y \geq 0$, there is a space-dependent rear wind. The switching function is given by $S(x, y, \Theta) := y$. Obviously, S is of the order $p = 1$. If we choose the data and boundary conditions as before, we may expect a regular solution with one switching point t_1, $0 < t_1 < t_f$.

For the necessary conditions we apply Theorem 1. Thus, a solution of the nonsmooth optimal control problem must satisfy the same boundary value problem (44) as before, however, augmented by the following jump and switching conditions

$$\lambda_1(t_1^+) = \lambda_1(t_1^-), \qquad \lambda_2(t_1^+) = \lambda_2(t_1^-) + \kappa_1 \tag{48a}$$

$$y(t_1) = 0, \qquad \mathcal{H}[t_1^+] = \mathcal{H}[t_1^-]. \tag{48b}$$

Note that, compared with the smooth case, the boundary value problem contains two additional unknowns, the switching time t_1 and the Lagrange multiplier κ_1. Both are determined by the switching conditions (48b).

The numerical solution of the resulting multipoint boundary value problem has been obtained by the multiple shooting code BNDSCO. In Fig. 10a the optimal flight path for the nonsmooth wind field is shown. The resulting minimal flight time is $t_f \doteq 4.9875063$.

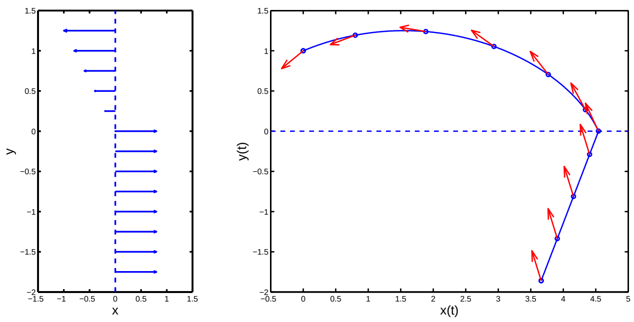

Figure 10a. Problem (Z): Minimum time path for the nonsmooth wind field (47).

In Fig. 10b the optimal state variables (x, y), the adjoint variable λ_2 corresponding to the state y, and the optimal control function on the scaled time interval $[0, 1]$ are given. One observes the discontinuity of the control and the adjoint variable λ_2 at the (nonscaled) switching point $t_1 \doteq 1.9912720$.

6.3. The singular case

If one substitutes the rear wind for $y \geq 0$ by a time variant head wind, the solution of this nonsmooth optimal control problem may contain a singular-

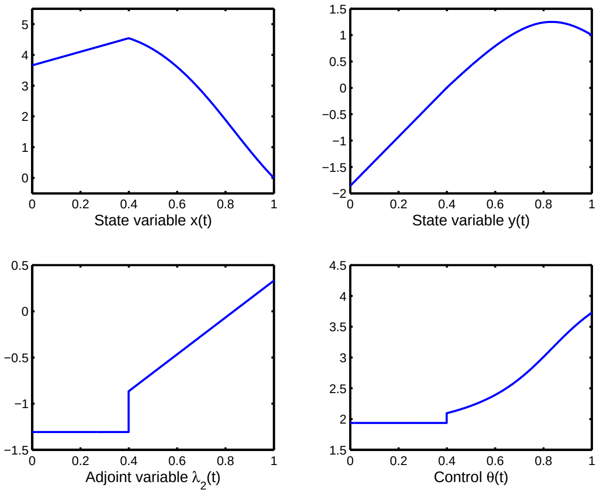

Figure 10b. Problem (Z): Corresponding optimal state, adjoint, and control functions.

state subarc. We choose the following wind field.

$$u(x,y) := \begin{cases} v_s\, y, & \text{if} \quad y \geq 0, \\ v_s, & \text{if} \quad y < 0, \end{cases} \qquad v(x,y) := 0, \tag{49}$$

The analysis of the singular subarc according to Theorem 2 yields the extended Hamiltonian

$$\mathcal{H} = \lambda_1 \left(v_0 \cos(\Theta) + u \right) + (\lambda_2 + \mu) \left(v_0 \sin(\Theta) + v \right) \tag{50}$$

and the corresponding optimal control

$$\cos(\Theta) = -\frac{\lambda_1}{\sqrt{\lambda_1^2 + (\lambda_2 + \mu)^2}}, \quad \sin(\Theta) = -\frac{\lambda_2 + \mu}{\sqrt{\lambda_1^2 + (\lambda_2 + \mu)^2}}. \tag{51}$$

The adjoint equations remain unchanged, see Eqs. (44). On the regular subarcs, we have $\mu = 0$, whereas on the singular subarcs, we have $S(x,y) = y = 0$, and $S^{(1)}(x,y,\Theta) = v_0 \sin\Theta = 0$, so that $\lambda_2 + \mu = 0$, $\sin\Theta = 0$, and $\cos\Theta = -1$. If we choose the data and boundary conditions as in (46), we may expect a solution with one singular state subarc $[t_1, t_2]$. Due to Theorem 2, a solution of this nonsmooth optimal control problem must satisfy the same boundary value problem (44) as before, however, augmented by the following jump and switching conditions

$$\lambda_1(t_j^+) = \lambda_1(t_j^-), \quad j = 1,2, \tag{52a}$$

$$\lambda_2(t_1^+) = \lambda_2(t_1^-) + \kappa_1, \qquad \lambda_2(t_2^+) = \lambda_2(t_2^-), \tag{52b}$$

$$y(t_1) = 0, \qquad \mathcal{H}[t_j^+] = \mathcal{H}[t_j^-], \quad j = 1,2. \tag{52c}$$

Additional parameters of the boundary value problem are the switching times t_1, t_2, and the Lagrange multiplier κ_1. They are determined by the switching conditions (52c).

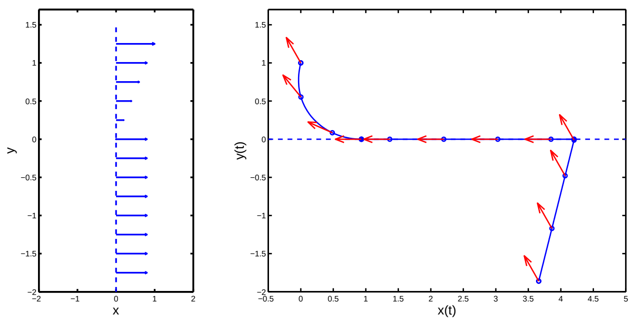

Figure 11a. Problem (Z): Minimum time path for the nonsmooth wind field (49).

In Fig. 11a the optimal flight path for the nonsmooth wind field is shown. The resulting minimal flight time is $t_f \doteq 7.3819697$. The scaled optimal state, adjoint and control variables are given in Fig. 11b.

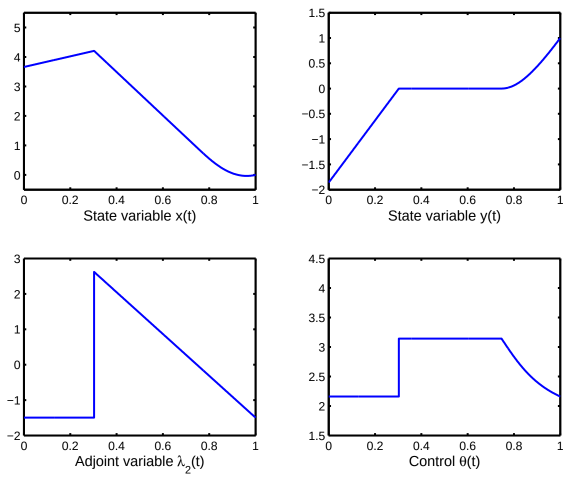

Figure 11b. Problem (Z): Corresponding optimal state, adjoint, and control functions.

7. Conclusions

In this paper optimal control problems with nonsmooth state differential equations are considered. Two solution types are distinguished. In the first part of the paper regular solutions have been considered. The regularity is characterized by the assumption that the switching function changes sign only at isolated points. In the second part so called singular state subarcs are admitted. These are nontrivial subarcs, where the switching function vanishes identically. For both situations necessary conditions are derived from the classical (smooth) optimal control theory. In addition, these necessary conditions have been applied to two classical nonsmooth OCPs. The first one describes the optimal control of an electric circuit containing a diode. The second example is the classical Zermelo's navigation problem with a nonsmooth wind field.

References

Arrow, K.J. (1949) On the Use of Winds in Flight Planning. *Journal of Meteorology* **6**, 150–159.

Augustin, D. and Maurer, H. (2000) Second Order Sufficient Conditions and Sensitivity Analysis for Optimal Multiprocess Control Problems. *Control and Cybernetics* **29**, 11–31.

Baumann, H. (2002) Treibstoffminimale luftunterstützte Orbittransfers mit Flugbahnebenenwechsel. Doctoral Thesis, University of Hamburg.

Bell, D.J. and Jacobson, D.H. (1975) *Singular Optimal Control Problems.* Academic Press, New York.

Bryson, A.E. and Ho, Y.C. (1969) *Applied Optimal Control.* Ginn and Company, Waltham, Massachusetts.

Chudej, K. (1995) Verallgemeinerte notwendige Bedingungen für zustandsbeschränkte Optimalsteuerungsaufgaben mit stückweise definierten Modellfunktionen. *Zeitschrift für Angewandte Mathematik und Mechanik* **75**, 587–588.

Clarke, F.H. (1983) *Optimization and Nonsmooth Analysis.* Wiley, New York.

Clarke, F.H. and Vinter, R.B.(1989) Application of Optimal Multiprocesses. *SIAM J. on Control and Optimization* **27**, 1048-1071, and 1071-1091.

Hestenes, M.R. (1966) *Calculus of Variations and Optimal Control Theory.* Wiley, New York.

Moyer, H.G. (2002) *Deterministic Optimal Control.* Trafford Publishing: Victoria, BC, Canada.

Oberle, H.J. and Rosendahl, R. (2006) Numerical Computation of a Singular-State Subarc in an Economic Optimal Control Problem. *Optimal Control Application and Methods* **27**, 211–235.

Oberle, H.J. and Grimm, W. (1989) BNDSCO – A Program for the numerical solution of optimal control problems. Report No. 515, Institut for

Flight Systems Dynamics, Oberpfaffenhofen, German Aerospace Research Establishment DLR.

ROSENDAHL, R. (2008) Sufficient Optimality Conditions for Nonsmooth Optimal Control Problems. Doctoral Thesis, University of Hamburg.

STOER, J. and BULIRSCH, R. (1996) *Introduction to Numerical Analysis.* Texts in Applied Mathematics, Springer, New York.

ZERMELO, E. (1930) Über die Navigation in der Luft als Problem der Variationsrechnung. *Annual report of the Deutsche-Mathematiker-Vereinigung* **39**, 44-48.

ZERMELO, E. (1931) Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung. *ZAMM* **11**, 114–124.