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On the lower semicontinuity of functionals involving Lebesgue or improper Riemann integrals in infinite horizon optimal control problems *

by

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Abstract: This paper deals with infinite horizon optimal control problems, which are formulated in weighted Sobolev spaces $W_p^{1,n}(\mathbf{R}^+,\nu)$ and weighted L_p-spaces $L_p^r(\mathbf{R}^+,\nu)$. We ask for the consequences of the interpretation of the integral within the objective as a Lebesgue or an improper Riemann integral. In order to justify the use of both types of integrals, various applications of infinite horizon problems are presented. We provide examples showing that lower semicontinuity may fail for objectives involving Lebesgue as well as improper Riemann integrals. Further we prove a lower semicontinuity theorem for an objective with Lebesgue integral under more restrictive growth conditions on the integrand.

Keywords: optimal control, infinite horizon, weighted Sobolev spaces, lower semicontinuity, Lebesgue integral, improper Riemann integral.

1. Introduction

a) Optimal control with infinite horizon

In the present paper we investigate infinite horizon optimal control problems. The motivation for studying them arises primarily from economics and biology where the infinite time horizon is a very natural phenomenon. Starting with the paper of Halkin (1974), the research in the field of optimal control problems with infinite horizon has dramatically increased. From the extensive literature, we mention only a few references, namely the textbook of Carlson, Haurie and Leizarowitz (1991) and the related chapters in Feichtinger and Hartl (1986) as well as some examples for papers with background in economics (Benveniste and

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Scheinkman, 1982; Magill, 1982), continuum mechanics (Leizarowitz and Mizel, 1989; Zaslavski, 1995) and biology (Colonius and Kliemann, 1989; Goh, Leitmann and Vincent, 1974). Throughout the development of the theory, necessary and sufficient optimality conditions as well as existence results were obtained. However, it must be emphasized that, with very few exceptions (e.g. Blot and Hayek, 1996, 2000; Blot and Michel, 1996; as well as Dmitruk and Kuz'kina, 2005), no attention was given to the important question whether the interpretation of the objective as a Lebesgue or as an improper Riemann integral is appropriate to the problem formulation.

The first aim of our paper is to demonstrate that different integral types can be useful in applications but lead to completely different theoretical results. Further, we point out that, for a correct setting of the problem, the choice of an appropriate state space is essential. Let us mention that the Lagrange multipliers associated with the constraints belong to the dual of the space wherein the constraint set has a nonempty interior.

Our paper is organized as follows. In order to demonstrate how *Lebesgue and improper Riemann integrals* as well as *weighted Lebesgue and Sobolev spaces* come into consideration, we start in this section with the presentation of some typical applications. Then we provide the general formulation of the infinite horizon problem. In Section 2, we prepare some analytical tools concerning weighted Lebesgue and Sobolev spaces and the Nemytskij operator. In Section 3, we state the main results of the paper.

In the case of the Lebesgue integral, we provide an example where lower semicontinuity is violated. Under additional assumptions, we succeed in proving lower semicontinuity of the objective with respect to an appropriate weak topology.

In the case of the improper Riemann integral, we provide two examples. The first one demonstrates that, for the same data, the different interpretations of the integral lead to different feasible sets. In the second one, the weak lower semicontinuity of the objective fails.

Consequently, in the cases where semicontinuity of the objective is missing it is impossible to develop an existence theory via the Weierstrass theorem. By this observation, one is led to treat infinite horizon problems by means of duality theory (see Klötzler, 1979), as outlined in Pickenhain and Lykina (2006).

b) Application: Optimal economic growth

Since Ramsey's pioneering work, the problem of optimal economic growth has been treated with an infinite time horizon. In a recent version, the problem can be formulated as follows (see Carlson, Haurie and Leizarowitz, 1991, pp. 6 ff.):

$$J(K,Z,C) \,=\, \int_0^\infty e^{-\varrho\, t}\, U(\, C(t)\,)\, dt \longrightarrow \text{Max}\,!\,; \tag{1.1}$$

$$F(\,K(t)\,) \,=\, Z(t) \,+\, C(t)\,; \tag{1.2}$$

$$\dot{K}(t) = Z(t) - \mu K(t)\,; \tag{1.3}$$
$$K(0) = k_0\,. \tag{1.4}$$

Here the production function F and the utility function U are given, while the investment, respectively consumption rates Z and C and the capital stock K are optimization variables. Under certain assumptions on the data, it can be shown that there exists a constant capital level $\overline{k}$ such that, "for any nonnegative value of ϱ the optimal trajectory over an infinite time horizon exists and converges toward $\overline{k}$, and this is true for any initial state k_0" (Carlson, Haurie and Leizarowitz, 1991, pp. 8). From this property it is clear that the function K cannot belong to any usual Sobolev space but to a weighted space as introduced below.

c) Application: Production-inventory model

This model has been presented in Sethi and Thompson (2000), pp. 154 ff.:

$$J(I,P) = \int_0^\infty e^{-\varrho\, t} \Big(\frac{h}{2} \big(I(t) - \hat{I}(t) \big)^2 + \frac{c}{2} \big(P(t) - \hat{P}(t) \big)^2 \Big)\, dt \longrightarrow \text{Min}\,!\,; \tag{2.1}$$
$$\dot{I}(t) = P(t) - S(t)\,;\ I(0) = i_0\,. \tag{2.2}$$

Here $\hat{I}$ and $\hat{P}$ are given goal levels for inventory and production, S is the given sales rate, h and c are given positive coefficients, and the current inventory and production rates I and P are optimization variables.

Again, the optimal trajectory of the problem belongs to a weighted Sobolev space. Since the objective in this problem is similar to the norm in the weighted space $W_p^1(\mathbf{R}^+, \widetilde{\nu})$ with $\widetilde{\nu}(t) = e^{-\varrho\, t}$ (see Section 2.b) below), it seems to be very natural to choose $W_p^1(\mathbf{R}^+, \widetilde{\nu})$ as the state space. We mention that here and in the preceding example, the integrals have to be understood in the Lebesgue sense. In the following example, however, the appropriate integral notion is not a priori determined.

d) Application: Pest control

Let X and Y denote the population numbers of two interacting species where X is a pest and Y is its natural predator. Then X and Y obey the following dynamics (see Carlson, Haurie and Leizarowitz, 1991, pp. 4 ff., and Goh, Leitmann and Vincent, 1974):

$$\dot{X}(t) = X(t)\big(1 - Y(t)\big)\,,\ X(0) = x_0\,; \tag{3.1}$$
$$\dot{Y}(t) = Y(t)\big(X(t) - 1\big)\,,\ Y(0) = y_0\,. \tag{3.2}$$

It is well known that this system admits a nontrivial stationary point, namely $\hat{x} = \hat{y} = 1$, while the trajectories in the state space circle around the equilibrium.

Assume now that the population X is supressed to a rate $0 \leq U \leq U_{\max}$ by treatment with some pesticide. Then Y will be influenced with a rate cU as well, and the dynamics of the controlled system become

$$\dot{X}(t) = X(t)\big(1 - Y(t) - U(t)\big), \ X(0) = x_0\,; \tag{4.1}$$

$$\dot{Y}(t) = Y(t)\big(X(t) - 1 - c\,U(t)\big), \ Y(0) = y_0\,. \tag{4.2}$$

The objective for the finite time interval $[\,0\,,\,T\,]$ is

$$J(X,Y,U) = \int_0^T \Big(X(t) + \alpha\,U(t)\Big)\,dt \longrightarrow \text{Min}\,!\,, \tag{5}$$

balancing the damage caused by the pest and the cost of its controlling with a constant $\alpha > 0$. However, "there is no natural reason for bounding the time interval on which the system has to be controlled" (Carlson, Haurie and Leizarowitz, 1991, p. 5). In the infinite horizon, the Lebesgue integral

$$\int_0^\infty \Big(X(t) + \alpha\,U(t)\Big)\,dt \tag{6}$$

becomes infinite for any admissible control U. When normalizing with respect to X, i. e., replacing the integral (6) by

$$\int_0^\infty \Big(\big(X(t) - \hat{x}\big) + \alpha\,U(t)\Big)\,dt\,, \tag{7}$$

the integral has to be understood in the Riemann sense while the absolute convergence cannot be guaranteed.

e) General formulation of the infinite horizon problems

As mentioned before, the infinite horizon control problem

$$(\mathrm{P})_\infty: \quad J(x,u) = \int_0^\infty r(t,x(t),u(t))\,\widetilde{\nu}(t)\,dt \longrightarrow \text{Min}\,!\,; \tag{8.1}$$

$$(x,u) \in W_p^{1,n}(\mathbf{R}^+,\nu) \times L_p^r(\mathbf{R}^+,\nu)\,; \tag{8.2}$$

$$\dot{x}(t) = f(t,x(t),u(t)) \quad \text{a. e. on } \mathbf{R}^+\,; \ x(0) = x_0\,; \tag{8.3}$$

$$u(t) \in \mathrm{U} \subset \mathbf{R}^r \quad \text{a. e. on } \mathbf{R}^+ \tag{8.4}$$

is not well defined since the interpretation of the integral within the objective is ambiguous. In order to make this point precise, we denote the set of pairs (x,u) satisfying (8.2) – (8.4) by $\mathcal{A}$ and formulate the following basic problems:

$$(\mathrm{P})_\infty^L: \quad J_L(x,u) = L\text{-}\!\int_0^\infty r(t,x(t),u(t))\,\widetilde{\nu}(t)\,dt \longrightarrow \text{Min}\,!\,; \tag{9.1}$$

$$(x,u) \in \mathcal{A} \cap \mathcal{A}_L \tag{9.2}$$

where the integral in the objective is understood as a Lebesgue integral, and $\mathcal{A}_L$ consists of all processes $(x,u) \in \mathcal{A}$, which make the Lebesgue integral in (9.1) convergent. In the second problem,

$$(\mathrm{P})_\infty^R: \quad J_R(x,u) = R\text{-}\int_0^\infty r(t,x(t),u(t))\,\widetilde{\nu}(t)\,dt \longrightarrow \mathrm{Min}\,!\,; \tag{10.1}$$

$$(x,u) \in \mathcal{A} \cap \mathcal{A}_R\,, \tag{10.2}$$

the integral in the objective is understood as an improper Riemann integral, and $\mathcal{A}_R$ consists of all processes $(x,u) \in \mathcal{A}$, which make the improper Riemann integral in (10.1) (at least conditionally) convergent.

The function ν is a density function in the sense explained below. The function $\widetilde{\nu}$ is assumed to be nonnegative, but not necessarily a density function. The weighted spaces $W_p^{1,n}(\mathbf{R}^+,\nu)$ and $L_p^r(\mathbf{R}^+,\nu)$ will be defined in Section 2.b) below.

f) Consequences of the distinction between Lebesgue and improper Riemann integrals

Let us remind that

$$R\text{-}\int_0^\infty f(t)\,dt := \lim_{T\to\infty} R\text{-}\int_0^T f(t)\,dt \tag{11}$$

where $f:\ \mathbf{R}^+ \to \mathbf{R}$ has to be R-integrable over any closed interval $[\,0\,,\,T\,] \subset \mathbf{R}^+$. If, under this assumption, the Lebesgue integral converges absolutely, i. e.

$$L\text{-}\int_0^\infty |\,f(t)\,|\,dt < \infty\,, \tag{12}$$

then the Lebesgue and the improper Riemann integral coincide,

$$L\text{-}\int_0^\infty f(t)\,dt = R\text{-}\int_0^\infty f(t)\,dt = \lim_{T\to\infty} L\text{-}\int_0^T f(t)\,dt \tag{13}$$

(see Elstrodt, 1996, p. 151 f., Theorem 6.3.). It may happen, however, as in the famous example with $f(t) = \sin t/t$, that the improper Riemann integral

$$R\text{-}\int_0^\infty \frac{\sin t}{t}\,dt\,, \tag{14}$$

converges conditionally (i.e., the corresponding series converges non-absolutely, see Fichtenholz, 1990, p. 520 f.) while the Lebesgue integral over the same domain does not exist (see Elstrodt, 1996, p. 152). As a consequence of these facts, the feasible domains $\mathcal{A}_L$ and $\mathcal{A}_R$ are, in general, incomparable. Applying the Lebesgue integral notion, we exclude from $\mathcal{A}$ all feasible trajectories, which make the improper Riemann integral non-absolutely convergent. On the other hand, by applying the improper Riemann integral, we lose all trajectories from $\mathcal{A}$, which are Lebesgue integrable but not Riemann integrable even on compact sets. For these reasons, it is very important to formulate an infinite horizon problem with the proper integral notion reflecting the situation behind the model in an appropriate way. As we will see in Section 3 below, the problems with distinct integral types require a completely different mathematical treatment.

2. Some background from functional analysis

a) Basic notations

Let us write $[\,0\,,\,\infty\,) = \mathbf{R}^+$. We denote by $M^n(\mathbf{R}^+)$, $L_p^n(\mathbf{R}^+)$ and $C^{0,n}(\mathbf{R}^+)$ the spaces of all vector functions $x:\ \mathbf{R}^+ \to \mathbf{R}^n$ with Lebesgue measurable, in the pth power Lebesgue integrable or continuous components, respectively (see Dunford and Schwartz, 1988, p. 146 and pp. 285 ff., Elstrodt, 1996, pp. 228 ff.). The Sobolev space $W_p^{1,n}(\mathbf{R}^+)$ is defined then as the space of all vector functions $x:\ \mathbf{R}^+ \to \mathbf{R}^n$, whose components belong to $L_p(\mathbf{R}^+)$ and admit distributional derivatives $\dot{x}_i$ (see Yosida, 1980, p. 49) belonging to $L_p(\mathbf{R}^+)$ as well. For $n = 1$, we suppress the superscript in the labels of the spaces. The interpretation of the integrals will be made precise by the symbols $L\text{-}\int$ for the Lebesgue and $R\text{-}\int$ for the Riemann integral.

b) Weighted Lebesgue and Sobolev spaces

A continuous function $\nu:\ \mathbf{R}^+ \to \mathbf{R}^+$ with positive values is called a *density function* iff it is Lebesgue integrable over $\mathbf{R}^+$:

$$L\text{-}\int_0^\infty \nu(t)\,dt \,<\, \infty \tag{15}$$

(see Kufner, 1985, p. 18, 3.4.). By means of a density function $\nu \in C^0(\mathbf{R}^+)$, we define for any $1 \leq p < \infty$ the *weighted Lebesgue space*

$$L_p^n(\mathbf{R}^+\!,\nu) = \Big\{x \in M^n(\mathbf{R}^+)\,\Big|\,\|\,x\,\|_{L_p^n(\mathbf{R}^+\!,\nu)} = \Big(L\text{-}\int_0^\infty |\,x(t)\,|^p \nu(t)\,dt\Big)^{1/p} < \infty\Big\}$$

as well as (16)

$$L_\infty^n(\mathbf{R}^+\!,\nu) = \Big\{x \in M^n(\mathbf{R}^+)\,\Big|\,\|\,x\,\|_{L_\infty^n(\mathbf{R}^+\!,\nu)} = \operatorname*{ess\,sup}_{0\,\leq\,t\,<\,\infty}\big|\,x(t)\,\nu(t)\,\big| \,<\, \infty\,\Big\} \tag{17}$$

and the *weighted Sobolev space*

$$W_p^{1,n}(\mathbf{R}^+\!,\nu) = \big\{\,x \in M^n(\mathbf{R}^+)\ \big|\ x \in L_p^n(\mathbf{R}^+\!,\nu)\,,\ \dot{x} \in L_p^n(\mathbf{R}^+\!,\nu)\,\big\} \tag{18}$$

(see Kufner, 1985, pp. 11 f.). Equipped with the norm

$$\| x \|_{W_p^{1,n}(\mathbf{R}^+,\nu)} = \| x \|_{L_p^n(\mathbf{R}^+,\nu)} + \| \dot{x} \|_{L_p^n(\mathbf{R}^+,\nu)} , \tag{19}$$

$W_p^{1,n}(\mathbf{R}^+,\nu)$ becomes a Banach space (this can be confirmed analogously to Kufner, 1985, p. 19, Theorem 3.6.). Any linear, continuous functional $\varphi : L_p(\mathbf{R}^+,\nu) \to \mathbf{R}$ can be represented by a function $y \in L_q(\mathbf{R}^+,\nu)$ with $p^{-1}+q^{-1} = 1$ if $1 < p < \infty$ and $q = \infty$ if $p = 1$:

$$\langle \varphi , x \rangle = \int_0^\infty y(t)\, x(t)\, \nu(t)\, dt \quad \forall\, x \in L_p^1(\mathbf{R}^+,\nu) . \tag{20}$$

We may apply Elstrodt (1996), p. 287, Theorem 3.2, since the measure generated by the density function ν is σ-finite on $\mathbf{R}^+$. Note that in the case $1 < p < \infty$, the representation (20) remains valid even if ν misses the property (15) of ν does not hold.

c) Compact imbedding for weighted Sobolev spaces

Let us recall first an imbedding result for non-weighted Sobolev spaces over unbounded domains. We state it as

Lemma 2.1. (Adams and Fournier, 2007, p. 194, Example 6.48, together with p. 197 f., Theorem 6.52) *Assume that $1 \leq p < \infty$, and a positive, nonincreasing, continuously differentiable function $\nu : \mathbf{R}^+ \to \mathbf{R}$ with bounded derivative is given. By means of ν, we define the open set*

$$\Omega_\nu = \big\{ (t,\xi) \in \mathbf{R}^2 \bigm| 0 < t\,,\ 0 < \xi < \nu(t) \big\} . \tag{21}$$

Then the imbedding $W_p^1(\Omega_\nu) \to L_p(\Omega_\nu)$ is compact iff the function ν satisfies the condition

$$\lim_{t\to\infty} \frac{\nu(t+\varepsilon)}{\nu(t)} = 0 \tag{22}$$

for every fixed $\varepsilon > 0$.

For weighted Sobolev spaces, we mention the following theorem recently proved by Antoci (2003):

Theorem 2.1. (Antoci, 2003, p. 63, Theorem 4.3.) *Assume that $1 \leq p < \infty$, and a continuous density function $\nu : \mathbf{R}^+ \to \mathbf{R}$ is given. By means of ν, we define the open set*

$$\Omega_\nu = \big\{ (t,\xi) \in \mathbf{R}^2 \bigm| 0 < t\,,\ 0 < \xi < \nu(t) \big\} . \tag{23}$$

If the imbedding $W_p^1(\Omega_\nu) \to L_p(\Omega_\nu)$ is compact then the imbedding $W_p^1(\mathbf{R}^+,\nu) \to L_p(\mathbf{R}^+,\nu)$ for the weighted spaces is compact as well.

d) Properties of the Nemytskij operator

For a given Carathéodory function $g(t,\xi):\ \mathbf{R}^+ \times \mathbf{R}^s \to \mathbf{R}$ (i. e., $g(\,\cdot\,,\xi)$ is Lebesgue measurable for all $\xi \in \mathbf{R}^s$, and $g(t,\,\cdot\,)$ is continuous for almost all $t \in \mathbf{R}^+$), the insertion of a s-vector function $x(t):\ \mathbf{R}^+ \to \mathbf{R}^s$ into g defines a Nemytskij operator $N(\,\cdot\,)$ with

$$(Nx)(t) \;=\; g(t, x(t))\,. \tag{24}$$

Note that the following theorem, which has been stated in Vainberg (1964), is valid for Lebesgue spaces on unbounded domains as well.

Theorem 2.2. (Vainberg, 1964, p. 162, Theorem 19.2., together with pp. 154 f., Theorem 19.1.) *Let $g(t,\xi):\ \mathbf{R}^+ \times \mathbf{R}^s \to \mathbf{R}$ be a Carathéodory function. Then the Nemytskij operator N associated with g by (24) is a bounded and continuous operator between the spaces $L_p^s(\mathbf{R}^+)$ and $L_{p'}(\mathbf{R}^+)$ with $1 \le p < \infty$ and $1 \le p' < \infty$ iff g satisfies the growth condition*

$$\big|\, g(t,\xi) \,\big| \;\le\; A(t) \,+\, B \cdot \sum_{i=1}^{s} |\,\xi_i\,|^{p/p'} \quad \forall\, (t,\xi) \in \mathbf{R}^+ \times \mathbf{R}^s \tag{25}$$

with a function $A \in L_{p'}(\mathbf{R}^+)$ and a constant $B > 0$.

3. Semicontinuity of functionals with integrals over $[\,0\,,\,\infty\,)$

a) A semicontinuity theorem in the case of Lebesgue integrals

We state now a semicontinuity theorem for the objective (9.1) involving the Lebesgue integral.

Theorem 3.1. *Let $1 < p < \infty$.*
Consider a nonnegative integrand $r(t,\xi,v):\ \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R}$, a positive function $\widetilde{\nu}(t):\ \mathbf{R}^+ \to \mathbf{R}$ and a density $\nu(t):\ \mathbf{R}^+ \to \mathbf{R}$ under the following assumptions:

1) The function $r(t,\xi,v)$ is continuous with respect to t, continuously differentiable with respect to ξ and v, and convex with respect to v.

2) The integrand r satisfies the following growth condition with respect to its second and third arguments:

$$\begin{aligned} &\Big|\, r\Big(t,\, \frac{\xi_1}{\nu(t)^{1/p}},\, \ldots,\, \frac{\xi_n}{\nu(t)^{1/p}},\, \frac{v_1}{\nu(t)^{1/p}},\, \ldots,\, \frac{v_r}{\nu(t)^{1/p}}\Big) \cdot \widetilde{\nu}(t) \,\Big| \\ &\quad \le\; A_1(t) \,+\, B_1 \cdot \sum_{i=1}^{n} \frac{|\,\xi_i\,|^{p/q}}{\nu(t)^{1/q}} \,+\, B_1 \cdot \sum_{k=1}^{r} \frac{|\,v_k\,|^{p/q}}{\nu(t)^{1/q}} \quad \forall\, (t,\xi,v) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^r \end{aligned} \tag{26}$$

with a function $A_1 \in L_1(\mathbf{R}^+)$, a constant $B_1 > 0$, and $p^{-1} + q^{-1} = 1$.

3) The gradient $\nabla_v r$ satisfies the following growth condition with respect to its second and third arguments:

$$\Big| \nabla_v r\big(t, \frac{\xi_1}{\nu(t)^{1/p}}, \dots, \frac{\xi_n}{\nu(t)^{1/p}}, \frac{v_1}{\nu(t)^{1/p}}, \dots, \frac{v_r}{\nu(t)^{1/p}}\big) \cdot \frac{\widetilde{\nu}(t)}{\nu(t)^{1/p}} \Big| \tag{27}$$
$$\leq A_2(t) + B_2 \cdot \sum_{i=1}^{n} \frac{|\xi_i|^{p/q}}{\nu(t)^{1/q}} + B_2 \cdot \sum_{k=1}^{r} \frac{|v_k|^{p/q}}{\nu(t)^{1/q}} \quad \forall (t, \xi, v) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^r$$

with a function $A_2 \in L_q(\mathbf{R}^+)$, a constant $B_2 > 0$ and $p^{-1} + q^{-1} = 1$.

4) The density function ν is monotonically decreasing and continuously differentiable with a bounded derivative. Moreover, the condition

$$\lim_{t \to \infty} \frac{\nu(t+\varepsilon)}{\nu(t)} = 0 \tag{28}$$

holds for arbitrary $\varepsilon > 0$.

Consider two weakly convergent sequences $\{x^N\} \rightharpoonup x_0$ and $\{u^N\} \rightharpoonup u_0$ with x^N, $x_0 \in W_p^{1,n}(\mathbf{R}^+, \nu)$ and $u^N, u_0 \in L_p^r(\mathbf{R}^+, \nu)$. Then the following lower semicontinuity relation holds:

$$J_L(x_0, u_0) = L\text{-}\int_0^\infty r(t, x_0(t), u_0(t))\, \widetilde{\nu}(t)\, dt$$
$$\leq \liminf_{N \to \infty} L\text{-}\int_0^\infty r(t, x^N(t), u^N(t))\, \widetilde{\nu}(t)\, dt = \liminf_{N \to \infty} J_L(x^N, u^N). \tag{29}$$

Proof. • **Step 1:** *Compactness of the imbedding $W_p^{1,n}(\mathbf{R}^+, \nu) \to L_p^n(\mathbf{R}^+, \nu)$.* By assumption 4), we can apply Lemma 2.1 in order to ensure the compactness of the imbedding $W_p^{1,n}(\Omega_\nu) \to L_p^n(\Omega_\nu)$ where $\Omega_\nu \subset \mathbf{R}^2$ is defined by

$$\Omega_\nu = \big\{ (t, \xi) \in \mathbf{R}^2 \;\big|\; 0 < t,\; 0 < \xi < \nu(t) \big\}. \tag{30}$$

Then from Theorem 2.1 we get the compactness of the imbedding $W_p^{1,n}(\mathbf{R}^+, \nu) \to L_p^n(\mathbf{R}^+, \nu)$. Consequently, from the weak convergence of the sequence $\{x^N\}$ in the space $W_p^{1,n}(\mathbf{R}^+, \nu)$ follows its convergence in $L_p^n(\mathbf{R}^+, \nu)$-norm.

• **Step 2:** *A lower estimate for $J_L(x^N, u^N)$.* From differentiability and convexity of r with respect to v, we derive

$$r(t, x^N(t), u^N(t)) \geqslant r(t, x^N(t), u_0(t)) \tag{31}$$
$$+ \nabla_v r(t, x^N(t), u_0(t))^{\mathrm{T}} \big(u^N(t) - u_0(t) \big) \implies$$

$$r(t,x^N(t),u^N(t))\,\widetilde{\nu}(t) \;\geqslant\; r(t,x^N(t),u_0(t))\,\widetilde{\nu}(t) \tag{32}$$
$$+\nabla_v\, r(t,x_0(t),u_0(t))^{\mathrm{T}}\,\big(\,u^N(t)-u_0(t)\,\big)\,\widetilde{\nu}(t)$$
$$+\big(\,\nabla_v\, r(t,x^N(t),u_0(t)) - \nabla_v\, r(t,x_0(t),u_0(t))\,\big)^{\mathrm{T}}\,\big(\,u^N(t)-u_0(t)\,\big)\,\widetilde{\nu}(t) \;\Longrightarrow$$
$$L\text{-}\!\int_0^\infty r(t,x^N(t),u^N(t))\,\widetilde{\nu}(t)\,dt \tag{33}$$
$$\geqslant\; J_1(x^N,u_0) \;+\; J_2(x_0,u^N,u_0) \;+\; J_3(x^N,x_0,u^N,u_0)$$

with

$$J_1(x^N,u_0) \;=\; L\text{-}\!\int_0^\infty r(t,x^N(t),u_0(t))\,\widetilde{\nu}(t)\,dt\,; \tag{34.1}$$
$$J_2(x_0,u^N,u_0) \;=\; L\text{-}\!\int_0^\infty \nabla_v\, r(t,x_0(t),u_0(t))^{\mathrm{T}}\,\big(\,u^N(t)-u_0(t)\,\big)\,\widetilde{\nu}(t)\,dt\,; \tag{34.2}$$
$$J_3(x^N,x_0,u^N,u_0) \;=\; L\text{-}\!\int_0^\infty \big(\,\nabla_v\, r(t,x^N(t),u_0(t)) \tag{34.3}$$
$$-\,\nabla_v\, r(t,x_0(t),u_0(t))\,\big)^{\mathrm{T}}\cdot\big(\,u^N(t)-u_0(t)\,\big)\,\widetilde{\nu}(t)\,dt\,.$$

The existence of the integrals $J_1(x^N,u_0)$, $J_2(x_0,u^N,u_0)$ and $J_3(x^N,x_0,u^N,u_0)$ on the right-hand side of (33) will be confirmed in Step 4 below.

• **Step 3:** *Consequences of the growth conditions.*

LEMMA 3.1. *The Nemytskij operator $N(\,\cdot\,,\,\cdot\,)$ with*

$$(N(x^N,u_0))(t) \;=\; r(t,x^N(t),u_0(t)) \tag{35}$$

is a continuous map between the spaces $L_p^{n+r}(\mathbf{R}^+,\nu)$ and $L_1(\mathbf{R}^+,\widetilde{\nu})$.

Proof. Since

$$(x^N,u_0)\in L_p^{n+r}(\mathbf{R}^+,\nu) \quad\Longleftrightarrow\quad \big(\,x^N\nu^{1/p},\;u_0\,\nu^{1/p}\,\big)\in L_p^{n+r}(\mathbf{R}^+) \tag{36}$$

and

$$r(\,\cdot\,,x^N(\,\cdot\,),u_0(\cdot))\in L_1(\mathbf{R}^+,\widetilde{\nu}) \quad\Longleftrightarrow\quad r(\,\cdot\,,x^N(\,\cdot\,),u_0(\cdot))\cdot\widetilde{\nu}(\,\cdot\,)\in L_1(\mathbf{R}^+)\,, \tag{37}$$

the growth condition from Theorem 2.2. reads as

$$\big|\,r\big(\,t,\,\frac{\xi_1}{\nu(t)^{1/p}},\;\dots,\;\frac{\xi_n}{\nu(t)^{1/p}},\,\frac{v_1}{\nu(t)^{1/p}},\;\dots,\;\frac{v_r}{\nu(t)^{1/p}}\,\big)\cdot\widetilde{\nu}(t)\,\big| \tag{38}$$
$$\leq\; A_1(t) \;+\; B_1\cdot\sum_{i=1}^{n}\frac{|\,\xi_i\,|^{p/q}}{\nu(t)^{1/q}} \;+\; B_1\cdot\sum_{k=1}^{r}\frac{|\,v_k\,|^{p/q}}{\nu(t)^{1/q}} \quad \forall\,(t,\xi,v)\in\mathbf{R}^+\times\mathbf{R}^n\times\mathbf{R}^r$$

with $A_1\in L_1(\mathbf{R}^+)$ and $B_1>0$. Since this condition was assumed, the Nemytskij operator (35) maps $L_p^{n+r}(\mathbf{R}^+,\nu)$ continuously into $L_1(\mathbf{R}^+,\widetilde{\nu})$. ∎

LEMMA 3.2. *The Nemytskij operator $N(\cdot,\cdot)$ with*

$$(N(x_0,u_0))(t) \;=\; \big|\,\nabla_v\, r(t,x_0(t),u_0(t))\,\big|\,\frac{\widetilde{\nu}(t)}{\nu(t)} \tag{39}$$

is a (continuous) map between the spaces $L_p^{n+r}(\mathbf{R}^+,\nu)$ and $L_q(\mathbf{R}^+,\nu)$.

Proof. Since

$$(x_0,u_0)\in L_p^{n+r}(\mathbf{R}^+,\nu) \quad\Longleftrightarrow\quad \big(x_0\,\nu^{1/p},\,u_0\,\nu^{1/p}\big)\in L_p^{n+r}(\mathbf{R}^+) \tag{40}$$

and

$$\begin{aligned}&\big|\,\nabla_v\, r(\cdot,x_0(\cdot),u_0(\cdot))\,\big|\,\frac{\widetilde{\nu}(\cdot)}{\nu(\cdot)}\in L_q(\mathbf{R}^+,\nu)\\ \Longleftrightarrow\quad &\big|\,\nabla_v\, r(\cdot,x_0(\cdot),u_0(\cdot))\,\big|\,\frac{\widetilde{\nu}(\cdot)}{\nu(\cdot)^{1/p}}\in L_q(\mathbf{R}^+)\,,\end{aligned} \tag{41}$$

the growth condition from Theorem 2.2. reads as

$$\begin{aligned}&\big|\,\nabla_v\, r\big(t,\,\frac{\xi_1}{\nu(t)^{1/p}},\,\dots,\,\frac{\xi_n}{\nu(t)^{1/p}},\,\frac{v_1}{\nu(t)^{1/p}},\,\dots,\,\frac{v_r}{\nu(t)^{1/p}}\big)\cdot\frac{\widetilde{\nu}(t)}{\nu(t)^{1/p}}\,\big|\\ &\le\; A_2(t)\;+\;B_2\cdot\sum_{i=1}^{n}\frac{|\,\xi_i\,|^{p/q}}{\nu(t)^{1/q}}\;+\;B_2\cdot\sum_{k=1}^{r}\frac{|\,v_k\,|^{p/q}}{\nu(t)^{1/q}}\quad\forall\,(t,\xi,v)\in\mathbf{R}^+\times\mathbf{R}^n\times\mathbf{R}^r\end{aligned} \tag{42}$$

with $A_2\in L_q(\mathbf{R}^+)$ and $B_2>0$. Since this condition was assumed, the Nemytskij operator (39) maps the space $L_p^{n+r}(\mathbf{R}^+,\nu)$ continuously into $L_q(\mathbf{R}^+,\nu)$. ∎

LEMMA 3.3. *The Nemytskij operator $N(\cdot,\cdot,\cdot)$ with*

$$(N(x^N,x_0,u_0))(t)=\big|\,\big(\,\nabla_v\, r(t,x^N(t),u_0(t))-\nabla_v\, r(t,x_0(t),u_0(t))\,\big)\cdot\frac{\widetilde{\nu}(t)}{\nu(t)}\,\big| \tag{43}$$

is a continuous map between the spaces $L_p^{n+n+r}(\mathbf{R}^+,\nu)$ and $L_q(\mathbf{R}^+,\nu)$.

Proof. We have (44)

$$(x^N,x_0,u_0)\in L_p^{n+n+r}(\mathbf{R}^+,\nu)\;\Longleftrightarrow\;\big(\,x^N\nu^{1/p},\,x_0\,\nu^{1/p},\,u_0\,\nu^{1/p}\,\big)\in L_p^{n+n+r}(\mathbf{R}^+)$$

and

$$\begin{aligned}&\big|\,\big(\,\nabla_v\, r(\cdot,x^N(\cdot),u_0(\cdot))-\nabla_v\, r(\cdot,x_0(\cdot),u_0(\cdot))\,\big)\cdot\frac{\widetilde{\nu}(\cdot)}{\nu(\cdot)}\,\big|\in L_q(\mathbf{R}^+,\nu) &&(45)\\ \Longleftrightarrow\quad&\big|\,\big(\,\nabla_v\, r(\cdot,x^N(\cdot),u_0(\cdot))-\nabla_v\, r(\cdot,x_0(\cdot),u_0(\cdot))\,\big)\cdot\frac{\widetilde{\nu}(\cdot)}{\nu(\cdot)^{1/p}}\,\big|\in L_q(\mathbf{R}^+)\,.\end{aligned}$$

Then from assumption 3) we derive the following growth condition:

$$\begin{aligned}
&\Big|\, \nabla_v\, r\big(t,\, \frac{\widetilde{\xi}_1}{\nu(t)^{1/p}},\, \dots,\, \frac{\widetilde{\xi}_n}{\nu(t)^{1/p}},\, \frac{v_1}{\nu(t)^{1/p}},\, \dots,\, \frac{v_r}{\nu(t)^{1/p}}\,\big) \\
&\qquad - \nabla_v\, r\big(t,\, \frac{\xi_1}{\nu(t)^{1/p}},\, \dots,\, \frac{\xi_n}{\nu(t)^{1/p}},\, \frac{v_1}{\nu(t)^{1/p}},\, \dots,\, \frac{v_r}{\nu(t)^{1/p}}\,\big)\,\Big| \cdot \frac{\widetilde{\nu}(t)}{\nu(t)^{1/p}} \\
&\le \Big|\, \nabla_v\, r\big(t,\, \frac{\widetilde{\xi}_1}{\nu(t)^{1/p}},\, \dots,\, \frac{\widetilde{\xi}_n}{\nu(t)^{1/p}},\, \frac{v_1}{\nu(t)^{1/p}},\, \dots,\, \frac{v_r}{\nu(t)^{1/p}}\,\big)\,\Big| \cdot \frac{\widetilde{\nu}(t)}{\nu(t)^{1/p}} \qquad (46) \\
&\qquad + \Big|\, \nabla_v\, r\big(t,\, \frac{\xi_1}{\nu(t)^{1/p}},\, \dots,\, \frac{\xi_n}{\nu(t)^{1/p}},\, \frac{v_1}{\nu(t)^{1/p}},\, \dots,\, \frac{v_r}{\nu(t)^{1/p}}\,\big)\,\Big| \cdot \frac{\widetilde{\nu}(t)}{\nu(t)^{1/p}} \\
&\le 2\, A_2(t) + B_2 \cdot \sum_{i=1}^{n} \frac{|\, \widetilde{\xi}_i\,|^{p/q}}{\nu(t)^{1/q}} + B_2 \cdot \sum_{i=1}^{n} \frac{|\, \xi_i\,|^{p/q}}{\nu(t)^{1/q}} + 2\, B_2 \cdot \sum_{k=1}^{r} \frac{|\, v_k\,|^{p/q}}{\nu(t)^{1/q}} \qquad (47) \\
&\qquad \forall\, (t, \widetilde{\xi}, \xi, v) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^r
\end{aligned}$$

where $A_2 \in L_q(\mathbf{R}^+)$ and $2\,B_2 > 0$. Consequently, the Nemytskij operator (43) maps the space $L_p^{n+n+r}(\mathbf{R}^+, \nu)$ continuously into $L_q(\mathbf{R}^+, \nu)$. ∎

• **Step 4:** *The integrals* $J_1(x^N, u_0)$, $J_2(x_0, u^N, u_0)$ *and* $J_3(x^N, x_0, u^N, u_0)$. From Lemma 3.1. we conclude that the integrals $J_1(x^N, u_0)$ are finite for all $N \in \mathbf{N}$. Together with Step 1, we can further derive the limit relation

$$\liminf_{N\to\infty} J_1(x^N, u_0) \;=\; \lim_{N\to\infty} J_1(x^N, u_0) \;=\; L\text{-}\!\int_0^\infty r(t, x_0(t), u_0(t))\, \widetilde{\nu}(t)\, dt\,, \qquad (48)$$

and the last integral is finite as well. Next we estimate $J_2(x_0, u^N, u_0)$ by Hölder's inequality (see Elstrodt, 1996, p. 222, Theorem 1.5):

$$\begin{aligned}
\big|\, J_2(x_0, u^N, u_0)\,\big| &= \Big|\, L\text{-}\!\int_0^\infty \nabla_v\, r(t, x_0(t), u_0(t))^{\mathrm{T}} \,\big(\, u^N(t) - u_0(t)\,\big)\, \widetilde{\nu}(t)\, dt\,\Big| \\
&\le\; L\text{-}\!\int_0^\infty \big|\, \nabla_v\, r(t, x_0(t), u_0(t))\, \frac{\widetilde{\nu}(t)}{\nu(t)}\,\big| \cdot \big|\, u^N(t) - u_0(t)\,\big|\, \nu(t)\, dt \\
&\le\; \Big(\, L\text{-}\!\int_0^\infty \big|\, \nabla_v\, r(t, x_0(t), u_0(t))\, \frac{\widetilde{\nu}(t)}{\nu(t)}\,\big|^q\, \nu(t)\, dt\,\Big)^{1/q} \\
&\qquad \cdot\; \Big(\, L\text{-}\!\int_0^\infty \big|\, u^N(t) - u_0(t)\,\big|^p\, \nu(t)\, dt\,\Big)^{1/p} \\
&=\; \Big\|\, \nabla_v\, r(\,\cdot\,, x_0(\,\cdot\,), u_0(\,\cdot\,))\, \frac{\widetilde{\nu}(\,\cdot\,)}{\nu(\,\cdot\,)}\,\Big\|_{L_q(\mathbf{R}^+,\nu)} \cdot\; \big\|\, u^N - u_0\,\big\|_{L_p(\mathbf{R}^+,\nu)}\,. \qquad (49)
\end{aligned}$$

From Lemma 3.2. it follows that the first norm in (49) is finite, and $J_2(x_0, u^N, u_0)$ can be understood as the application of a linear, continuous functional to the difference $(u^N - u_0) \in L_p^r(\mathbf{R}^+, \nu)$. Then from the weak convergence $\{\, u^N \,\} \rightharpoonup u_0$

in $L_p^r(\mathbf{R}^+, \nu)$ it follows that

$$\liminf_{N\to\infty} J_2(x_0, u^N, u_0) = \lim_{N\to\infty} J_2(x_0, u^N, u_0) = 0\,. \tag{50}$$

In order to estimate $J_3(x^N, x_0, u^N, u_0)$, we apply Hölder's inequality again:

$$\begin{aligned}
&\left|\, J_3(x^N, x_0, u^N, u_0)\,\right| \\
&= \Big|\, L\text{-}\!\int_0^\infty \big(\nabla_v\, r(t, x^N(t), u_0(t)) - \nabla_v\, r(t, x_0(t), u_0(t))\,\big)^{\mathrm{T}} \big(\, u^N(t) - u_0(t)\,\big)\, \widetilde{\nu}(t)\, dt\,\Big| \\
&\le\ L\text{-}\!\int_0^\infty \Big|\, \big(\, \nabla_v\, r(t, x^N(t), u_0(t)) - \nabla_v\, r(t, x_0(t), u_0(t))\,\big)\, \frac{\widetilde{\nu}(t)}{\nu(t)}\,\Big| \\
&\qquad \cdot \left|\, u^N(t) - u_0(t)\,\right| \nu(t)\, dt \\
&\le \Big(L\text{-}\!\int_0^\infty \Big|\, \big(\, \nabla_v\, r(t, x^N(t), u_0(t)) - \nabla_v\, r(t, x_0(t), u_0(t))\,\big)\, \frac{\widetilde{\nu}(t)}{\nu(t)}\,\Big|^q\, \nu(t)\, dt \Big)^{1/q} \\
&\qquad \cdot \Big(L\text{-}\!\int_0^\infty \left|\, u^N(t) - u_0(t)\,\right|^p \nu(t)\, dt \Big)^{1/p} \\
&= \Big\|\, \big(\, \nabla_v\, r(\,\cdot\,, x^N(\,\cdot\,), u_0(\,\cdot\,)) - \nabla_v\, r(\,\cdot\,, x_0(\,\cdot\,), u_0(\,\cdot\,))\,\big)\, \frac{\widetilde{\nu}(\,\cdot\,)}{\nu(\,\cdot\,)}\,\Big\|_{L_q(\mathbf{R}^+, \nu)} \\
&\qquad \cdot \left\|\, u^N - u_0\,\right\|_{L_p^r(\mathbf{R}^+, \nu)}\,.
\end{aligned} \tag{51}$$

Since the weakly convergent sequence $\{\, u^N \,\}$ is bounded, the second norm difference in (51) is bounded as well, and from Step 1 and Lemma 3.3. it follows that

$$\lim_{N\to\infty} \Big\|\, \big(\, \nabla_v\, r(\,\cdot\,, x^N(\,\cdot\,), u_0(\,\cdot\,)) - \nabla_v\, r(\,\cdot\,, x_0(\,\cdot\,), u_0(\,\cdot\,))\,\big)\, \frac{\widetilde{\nu}(\,\cdot\,)}{\nu(\,\cdot\,)}\,\Big\|_{L_q(\mathbf{R}^+, \nu)} = 0\,. \tag{52}$$

Consequently, we have

$$\liminf_{N\to\infty} J_3(x^N, x_0, u^N, u_0) = \lim_{N\to\infty} J_3(x^N, x_0, u^N, u_0) = 0\,. \tag{53}$$

• **Step 5 :** *The lower semicontinuity relation for J_L.* From (33), (48), (50) and (53), we get finally

$$\begin{aligned}
&\liminf_{N\to\infty} J_L(x^N, u^N) \\
&\quad \geqslant \liminf_{N\to\infty} J_1(x^N, u_0) + \liminf_{N\to\infty} J_2(x_0, u^N, u_0) + \liminf_{N\to\infty} J_3(x^N, x_0, u^N, u_0) \\
&\quad = \lim_{N\to\infty} J_1(x^N, u_0) + \lim_{N\to\infty} J_2(x_0, u^N, u_0) + \lim_{N\to\infty} J_3(x^N, x_0, u^N, u_0) \\
&\quad = L\text{-}\!\int_0^\infty r(t, x_0(t), u_0(t))\, \widetilde{\nu}(t)\, dt = J_L(x_0, u_0)\,,
\end{aligned} \tag{54}$$

and the proof of Theorem 3.1. is complete. ∎

Remark 3.1. We assumed $1 \le p < \infty$ in view of Theorem 2.1. used in the proof.

b) Counterexamples in the case of improper Riemann integrals

In the first example we confirm that the different interpretations of the integral within the objective of an infinite horizon problem lead to different feasible sets. Moreover, the problem has to be formulated in the framework of weighted spaces.

EXAMPLE 3.1. Let the integrand $r(t,v):\ \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$ be given by

$$r(t,v) \;=\; \begin{cases} \dfrac{\sin t}{t}\cdot v \;\big|\; 2k\pi \le t \le (2k+1)\pi\,; \\ 2\,\dfrac{\sin t}{t}\cdot v \;\big|\; (2k+1)\pi \le t \le (2k+2)\pi\,. \end{cases} \tag{55}$$

Consider the "loosely formulated" infinite horizon control problem

$$(\mathrm{P}_1)_\infty: \quad J(x,u) \;=\; -\int_0^\infty r(t,u(t))\,dt \longrightarrow \text{Min !}\,; \tag{56.1}$$

$$(x,u) \in W_p^1(\mathbf{R}^+,\nu) \times L_p(\mathbf{R}^+,\nu)\,; \tag{56.2}$$

$$\dot{x}(t) \;=\; u(t) \;\text{ a. e. on } \mathbf{R}^+,\; x(0)=0\,; \tag{56.3}$$

$$u(t) \in \mathrm{U} \;=\; [\,\tfrac{1}{2}\,,\,1\,] \;\text{ a. e. on } \mathbf{R}^+. \tag{56.4}$$

As in Section 1.e), we denote by $\mathcal{A}$ the set of pairs fulfilling (56.2) – (56.4), by $\mathcal{A}_L$ the subset of pairs $(x,u)\in\mathcal{A}$, which make $L\text{-}\!\int_0^\infty\big(-r(t,u(t))\big)\,dt$ convergent, and by $\mathcal{A}_R$ the subset of pairs $(x,u)\in\mathcal{A}$, which make $R\text{-}\!\int_0^\infty\big(-r(t,u(t))\big)\,dt$ convergent. Since the integral is of type (14) it is obvious that

$$\mathcal{A}_L \;=\; \emptyset \quad\text{and}\quad \mathcal{A}_R \neq \emptyset\,. \tag{57}$$

The optimal control in $(\mathrm{P}_1)_\infty^R$ is given through

$$u^*(t) \;=\; \begin{cases} 1 \;\big|\; 2k\pi \le t \le (2k+1)\pi\,; \\ \dfrac{1}{2} \;\big|\; (2k+1)\pi \le t \le (2k+2)\pi\,. \end{cases} \tag{58}$$

If we define the functions $\widetilde{\nu}$ and ν by

$$\widetilde{\nu}(t) \;\equiv 1\,,\;\; \nu(t) = e^{-\varrho\, t} \tag{59}$$

with $\varrho > 0$, then the corresponding state x^* according to (56.3) satisfies

$$x^* \notin W_p^1(\mathbf{R}^+) \quad\text{but}\quad x^* \in W_p^1(\mathbf{R}^+,\nu) \tag{60}$$

for any $1 < p < \infty$ what justifies the choice of a weighted Sobolev space as the state space.

The second example shows that an objective with an improper Riemann integral can fail to be weakly lower semicontinuous.

EXAMPLE 3.2. Consider the problem

$$(\mathrm{P}_2)_\infty: \quad J(x,u) \;=\; -\int_0^\infty \sin\big(x_1(t)\big)\,dt \longrightarrow \mathrm{Min}\,!\,; \tag{61.1}$$

$$(x,u) \in W_p^{1,2}(\mathbf{R}^+,\nu) \times \big(L_p(\mathbf{R}^+,\nu) \cap C^0(\mathbf{R}^+)\big)\,; \tag{61.2}$$

$$\dot x_1(t) \;=\; x_2(t) \;\text{ a. e. on } \mathbf{R}^+,\; x_1(0)=0\,; \tag{61.3}$$

$$\dot x_2(t) \;=\; u(t) \;\text{ a. e. on } \mathbf{R}^+,\; x_2(0)=0\,; \tag{61.4}$$

$$u(t) \in \mathrm{U} = [\,0\,,1\,] \;\text{ a. e. on } \mathbf{R}^+. \tag{61.5}$$

Let the functions $\widetilde{\nu}$ and ν be defined as in (59). Again, we denote by $\mathcal{A}$ the set of pairs fulfilling (61.2) – (61.5), by $\mathcal{A}_L$ the subset of pairs $(x,u) \in \mathcal{A}$, which make $L\text{-}\!\int_0^\infty \big(-\sin\big(x_1(t)\big)\big)\,dt$ convergent, and by $\mathcal{A}_R$ the subset of pairs $(x,u) \in \mathcal{A}$, which make $R\text{-}\!\int_0^\infty \big(-\sin\big(x_1(t)\big)\big)\,dt$ convergent. In this problem, we get $\mathcal{A}_L \neq \emptyset$, $\mathcal{A}_R \neq \emptyset$ since $(x_0,u_0) \equiv \big(\binom{0}{0},0\big) \in \mathcal{A}_L \cap \mathcal{A}_R$. Consider now the sequence $\{u^N\}$ of controls

$$u^N(t) \;=\; \frac{2}{N} \tag{62}$$

admissible in $(\mathrm{P}_2)_\infty^R$. Then the corresponding states x_1^N and x_2^N according to (61.3)-(61.4) belong to the weighted Sobolev space $W_p^1(\mathbf{R}^+,\nu)$ for any $1<p<\infty$, and

$$\lim_{N\to\infty} \big\| x^N - x_0 \big\|_{W_p^{1,2}(\mathbf{R}^+,\nu)} \;=\; 0 \quad \text{as well as} \tag{63.1}$$

$$\lim_{N\to\infty} \big\| u^N - u_0 \big\|_{L_p(\mathbf{R}^+,\nu)} \;=\; 0\,. \tag{63.2}$$

However, the lower semicontinuity of the objective fails along the sequence $\{(x^N,u^N)\} \rightharpoonup (x_0,u_0) \in W_p^{1,2}(\mathbf{R}^+,\nu) \times L_p(\mathbf{R}^+,\nu)$. We calculate (see Fichtenholz, 1990, p. 554, Nr. 491, Examples 3 and 4)

$$R\text{-}\!\int_0^\infty \sin\big(x_1^N(t)\big)\,dt \;=\; R\text{-}\!\int_0^\infty \sin\Big(\frac{t^2}{N}\Big)\,dt \;=\; \frac{\sqrt{N}}{2}\cdot R\text{-}\!\int_0^\infty \frac{\sin s}{\sqrt{s}}\,ds$$

$$=\; \frac{\sqrt{N}}{2}\cdot\sqrt{\frac{\pi}{2}}\,. \tag{64}$$

Consequently, we have

$$J_R(x_0,u_0) \;=\; 0 \;>\; \lim_{N\to\infty} J_R(x^N,u^N)$$

$$=\; \lim_{N\to\infty} R\text{-}\!\int_0^\infty \Big(-\sin\big(x_1^N(t)\big)\Big)\,dt \;=\; -\infty\,, \tag{65}$$

and the functional with the improper Riemann integral is not weakly lower semicontinuous.

c) Counterexample in the case of Lebesgue integrals

This example was considered by Halkin (1974) in order to demonstrate that the adjoint function y of this problem does not satisfy the natural transversality condition $\lim_{T\to\infty} y(T) = 0$, corresponding to the terminal condition that $x(\infty)$ is free.

EXAMPLE 3.3. Consider the problem

$$(\mathrm{P}_3)^L_\infty : \quad J_L(x,u) \,=\, L\text{-}\!\int_0^\infty \big(-(1-x(t))\,u(t)\big)\,dt \longrightarrow \mathrm{Min}\,!\,; \tag{66.1}$$

$$(x,u) \in W^1_p(\mathbf{R}^+\!,\nu) \times L_p(\mathbf{R}^+\!,\nu)\,; \tag{66.2}$$

$$\dot{x}(t) \,=\, (1-x(t))\,u(t) \;\text{ a. e. on } \mathbf{R}^+,\; x(0)=0\,; \tag{66.3}$$

$$u(t) \in \mathrm{U} = [\,0\,,1\,] \;\text{ a. e. on } \mathbf{R}^+. \tag{66.4}$$

Let the functions $\widetilde{\nu}$ and ν be defined as in (59). Again, we denote by $\mathcal{A}$ the set of pairs fulfilling (66.2) – (66.4), and by $\mathcal{A}_L$ the subset of pairs $(x,u)\in\mathcal{A}$, which make $L\text{-}\!\int_0^\infty (1-x(t))\,u(t)\,dt$ convergent. Integrating the separated differential equation (66.3) with the initial condition $x(0)=0$, we obtain $x(t) = 1-e^{-F(t)}$ with $F(t) = \int_0^t u(s)\,ds$. We study the following sequence of feasible processes $(x^N, u^N)\in\mathcal{A}_L$:

$$u^N(t) \,=\, \begin{cases} 0 \;\big|\; 0\le t< N\,; \\ \dfrac{1}{N} \;\big|\; N\le t<\infty \end{cases} \tag{67}$$

and

$$x^N(t) \,=\, \begin{cases} 0 & \big|\; 0\le t< N\,; \\ -e^{1-t/N}+1 & \big|\; N\le t<\infty\,. \end{cases} \tag{68}$$

We see again that

$$u^N \notin L_p(\mathbf{R}^+) \quad\text{but}\quad u^N \in L_p(\mathbf{R}^+\!,\nu) \tag{69}$$

and

$$x^N \notin W^1_p(\mathbf{R}^+) \quad\text{but}\quad x^N \in W^1_p(\mathbf{R}^+\!,\nu) \tag{70}$$

for any $1\le p<\infty$, what justifies the choice of a weighted Sobolev space as the state space. Moreover, $\{\,(x^N,u^N)\,\}$ converges to $(x_0,u_0)\equiv(0,0)$ in $W^1_p(\mathbf{R}^+\!,\nu)\times L_p(\mathbf{R}^+\!,\nu)$-norm, since

$$\lim_{N\to\infty} \big\|\, x^N \,\big\|_{L_p(\mathbf{R}^+,\nu)} \,=\, 0\,; \tag{71.1}$$

$$\lim_{N\to\infty} \big\|\, \dot{x}^N \,\big\|_{L_p(\mathbf{R}^+,\nu)} \,=\, 0\,; \tag{71.2}$$

$$\lim_{N\to\infty} \big\|\, u^N \,\big\|_{L_p(\mathbf{R}^+,\nu)} \,=\, 0 \tag{71.3}$$

for all $1 \leq p < \infty$. Finally, we obtain

$$\begin{aligned} J_L(x^N, u^N) &= L\text{-}\int_0^{\infty} \big(-(1 - x^N(t))\, u^N \big)\, dt \\ &= L\text{-}\int_N^{\infty} \big(-(1 - x^N(t))\, u^N \big)\, dt = -1\,, \end{aligned} \tag{72}$$

but the insertion of $(x_0, u_0) \equiv (0, 0)$ gives

$$J_L(x_0, u_0) = L\text{-}\int_0^{\infty} 0\, dt = 0\,. \tag{73}$$

Consequently, we arrive at

$$J_L(x_0, u_0) = 0 > \lim_{N \to \infty} J_L(x^N, u^N) = -1\,, \tag{74}$$

and the functional with the Lebesgue integral is neither strongly nor weakly lower semicontinuous within the spaces $W_p^1(\mathbf{R}^+, \nu) \times L_p(\mathbf{R}^+, \nu)$, $1 \leq p < \infty$. In this example, the growth conditions (26) and (27) of Theorem 3.1. are not satisfied, and the theorem is not applicable. Let us finally note that the replacement of the control set $\mathrm{U} = [\,0\,,\,1\,]$ by an interval $[\,\alpha\,,\,1\,]$ with $0 < \alpha < 1$ leads to a problem where the objective is constant and, therefore, continuous in any topology on the feasible domain.

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