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# Robust $H_\infty$ control for a class of uncertain neutral systems with both state and control input time-varying delays via a unified LMI optimization approach*

by

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**Abstract:** The robust $H_\infty$ control problem is considered for a class of uncertain neutral system involving both state and control input time-varying delays. The uncertainties under consideration are nonlinear time-varying parameter perturbations. The methodology is based on the Lyapunov functional combined with a unified LMI approach, and a new delay-dependent criterion is proposed to guarantee the stabilization and disturbance attenuation of systems. Moreover, a convex optimization approach is used to solve the robust $H_\infty$ control disturbance attenuation problem. Finally, a numerical example is illustrated to show the validity of this paper. The simulation results reveal significant improvement over the recent results.

**Keywords:** robust $H_\infty$ control, neutral systems, unified LMI approach, convex optimization approach, delay-dependent criterion.

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## 1. Introduction

The phenomena of time delay are often encountered in various practical systems, such as AIDS epidemic, aircraft stabilization, chemical engineering systems, inferred grinding model, manual control, neural networks, nuclear reactors, population dynamics models, rolling mills, ship stabilization, and systems with lossless transmission lines. Moreover, time delay is frequently a source of instability and a source of generation of oscillation in many systems (Hale and Verduyn Lunel, 1993; Kolmanovskii and Myshkis, 1992). Hence, stability analysis and stabilization problems for time-delay systems received considerable attention.

In some systems, system models can be described by functional differential equation of neutral type, these models depending on the state delay but also on the state derivatives. Physical examples for neutral system include distributed networks, population ecology, processes including steam or heat exchange. Stability and stabilization in various neutral time-delay systems have been considered in recent year (Baser, 2003; He et al., 2004; Lien and Chen, 2003; Xu, Lam and Yang, 2002; Chen, 2004).

In practical systems, the analysis of a mathematical model is usually an important work for a control engineer, aiming to control the system. However, mathematical model always contains some uncertain elements. Therefore, under such imperfect knowledge of the mathematical model, design of a robust control such that the system responses can meet desired properties is an important topic. Hence, many robust control problems are analysed for a class of uncertain time-delay systems (Li and de Souza, 1997; Moon et al., 2001; Nian and Feng, 2003; Su, Su and Chu, 2003; Xu, Lam and Yang, 2002; Roh, 2002; Chen, 2004a,b).

Depending on whether the stability and stabilization criterion itself contains the magnitude of delays, criteria for time-delay systems can be classified into two categories, namely delay-independent criteria (Baser, 2003; Lien and Chen, 2003; Xu, Lam and Yang, 2002) and delay-dependent criteria (Gahinet et al., 1995; Gu, 2000; He et al., 2004; Li and Souza, 1997; Lien and Chen, 2003; Moon et al., 2001; Nian and Feng, 2003; Su, Su and Chu, 2003; Su, Lu and Tsai, 2001; Roh, 2002;Chen, 2004a,b). Generally speaking, the latter ones are less conservative than the former ones when the time-delay values are small.

On the other hand, the $H_\infty$ control concept was proposed to reduce the effect of the disturbance input on the regulated output to remain within a prescribed level. Recently, many researchers have been considering the $H_\infty$ control problem for time-delay systems, but their results are restricted to delay-independent criteria for neutral systems (Baser, 2003; Xu, Lam and Yang, 2002), or delay-dependent criteria for retarded system (Fridman and Shaked, 2003; Su, Su and Chu, 2003; Su, Lu and Tsai, 2001). To our best knowledge, the robust $H_\infty$ control for a class of uncertain neutral system with state and control input time-varying delay systems has never been considered in the past. This motivated us to the study reported in this paper.

In this paper, the delay-dependent $H_\infty$ control problem will be considered for a wider class of neutral state-control input delayed systems with nonlinear time-varying parameter perturbations. The presented systems are more general than the ones considered in Baser (2003), Fridman and Shaked (2003), Gu (2000), He et al. (2004), Li and de Souza (1997), Lien and Chen (2003), Moon et al. (2001), Nian and Feng (2003), Su, Su and Chu (2003), or Su, Lu and Tsai (2001), Xu, Lam and Yang (2002), Roh (2002), and Chen (2004a,b), where no input-delay term is considered in Baser (2003), He et al. (2004), Lien and Chen (2003), or Xu, Lam and Chang (2002), no neutral-delay term is discussed in Roh (2002) and Chen (2004a), no neutral perturbation term is developed in Chen (2004b), and no both input and neutral delays appear in Fridman and Shaked (2003), Gu (2000), Li and de Souza (1997), Moon et al. (2001), Nian and Feng (2003), Su, Su and Chu (2003), or Su, Lu and Tsai (2001), respectively. The objective is to apply the LMI optimization tool to find the $H_\infty$ control and minimize the $H_\infty$ norm bound. Both Lyapunov-Krasovskii theory and LMI technique are used. A new delay-dependent stabilizability criterion is proposed to finish the $H_\infty$ control design. A numerical example is given to illustrate the use of the proposed result.

### Notation

Notation that will be used throughout the paper is as follows:

| | | |
|---|---|---|
| $C_0$ | := | set of continuous functions from $[-H,0]$ to $\Re^n$, |
| $\Re^n$ | := | $n$-dimensional real space, |
| $\Re^{m\times n}$ | := | set of all real $m$ by $n$ matrices, |
| $A^T$ | := | transpose of matrix $A$, |
| $\|x\|$ | := | Euclidean norm of vector $x$, |
| $\|A\|$ | := | spectral norm of matrix $A$, |
| $\|f(t)\|_2$ | := | $\sqrt{\int_0^\infty \|f(t)\|^2 dt}, \quad f(t) \in L_2[0,\infty)$, |
| $\|x_t\|_W$ | := | $\sqrt{\left(\|x(t)\|^2 + \int_{-H}^0 \|\dot{x}(t+s)\|^2 ds\right)}$, |
| $\|x_t\|_S$ | := | $\sup_{-H\leqslant s\leqslant 0} \|x(t+s)\|$, |
| $L_2[0,\infty)$ | := | space of square integrable vector functions on $[0,\infty)$, |
| $A \leqslant B$ | := | $B-A$ is a positive semi-definite symmetric matrix, |
| $P>0$ | := | $P$ is a positive definite symmetric matrix, |
| $P<0$ | := | $P$ is a negative definite symmetric matrix, |
| $I$ | := | unit matrix. |

## 2. Problem formulation and preliminaries

In this paper, we consider the following uncertain neutral system that has state and control input time-varying delays:

$$\dot{x}(t) = A_0x(t) + A_1x(t-h(t)) + A_2\dot{x}(t-\tau(t)) + B_0u(t) + B_1u(t-\eta(t))$$
$$+f_0(x(t),t) + f_1(x(t-h(t)),t) + f_2(\dot{x}(t-\tau(t)),t) + f_3(u(t-\eta(t)),t)$$
$$+B_ww(t),\ t \geqslant 0, \tag{1a}$$
$$z(t) = Cx(t) + Du(t) \tag{1b}$$
$$x(t) = \phi(t), \quad t \in [-H,0], \tag{1c}$$

where $x \in \Re^n$, $u \in \Re^m$, and $z \in \Re^q$ are system input, control input, and regulated output, respectively; $x_t$ is the state at time $t$ defined by $x_t(\theta) := x(t+\theta)$, $\forall\theta \in [-H,0]$, $w \in \Re^l$ is the disturbance input; the delays $h(t)$, $\tau(t)$ and $\eta(t)$ are three time-varying functions satisfying $0 \leqslant h(t) \leqslant h_M$, $0 \leqslant \tau(t) \leqslant \tau_M$, $0 \leqslant \eta(t) \leqslant \eta_M$, $\dot{h}(t) \leqslant h_D < 1$, $\dot{\tau}(t) \leqslant \tau_D < 1$ and $\dot{\eta}(t) \leqslant \eta_D < 1$, $H = \max\{h_M, \tau_M, \eta_M\}$. The matrices $A_0$, $A_1$, $A_2 \in \Re^{n\times n}$, $B_0 \in \Re^{n\times m}$, $B_1 \in \Re^{n\times m}$, $C \in \Re^{q\times n}$, $D \in \Re^{q\times m}$, are known, and the initial vector $\phi \in C_0$. The uncertainties $f_0(x(t),t)$, $f_1(x(t-h(t)),t)$, $f_2(\dot{x}(t-\tau(t)),t)$, and $f_3(u(t-\eta(t)),t)$ are nonlinear time-varying parameter perturbations with $f_0(0,t) = 0$, $f_1(0,t) = 0$, $f_2(0,t) = 0$ and $f_3(0,t) = 0$, respectively, satisfying the following quadratic inequalities

$$f_0^T(x(t),t)f_0(x(t),t) \leqslant \beta_0^2 \cdot x^T(t)x(t), \tag{2a}$$
$$f_1^T(x(t-h(t)),t)f_1(x(t-h(t)),t) \leqslant \beta_1^2 \cdot x^T(t-h(t))x(t-h(t)), \tag{2b}$$
$$f_2^T(\dot{x}(t-\tau(t)),t)f_2(\dot{x}(t-\tau(t)),t) \leqslant \beta_2^2 \cdot \dot{x}^T(t-\tau(t))\dot{x}(t-\tau(t)), \tag{2c}$$
$$f_3^T(u(t-\eta(t)),t)f_3(u(t-\eta(t)),t) \leqslant \beta_3^2 \cdot u^T(t-\eta(t))u(t-\eta(t)), \tag{2d}$$

where $\beta_0$, $\beta_1$, $\beta_2$, and $\beta_3$ are nonnegative constants.

By the Leibniz-Newton formula, it follows that

$$A_1\int_{t-h(t)}^{t}\dot{x}(s)ds = A_1x(t) - A_1x(t-h(t)), \quad \text{and}$$
$$B_1\int_{t-\eta(t)}^{t}\dot{u}(s)ds = B_1u(t) - B_1u(t-\eta(t)).$$

System (1a) with $u(t) = Kx(t)$ can be rewritten as:

$$\dot{x}(t) = (\hat{A} + \hat{B}K)x(t) + A_2\dot{x}(t-\tau(t)) - A_1\int_{t-h(t)}^{t}\dot{x}(s)ds \tag{3a}$$
$$-B_1\int_{t-\eta(t)}^{t}\dot{u}(s)ds + f_0(x(t),t) + f_1(x(t-h(t)),t)$$
$$+f_2(\dot{x}(t-\tau(t)),t) + f_3(u(t-\eta(t)),t) + B_ww(t), \quad t \geqslant 0,$$
$$z(t) = (C + DK)x(t), \tag{3b}$$
$$x(t) = \phi(t), \quad t \in [-H,0], \tag{3c}$$

where $\hat{A} = A_0 + A_1$, and $\hat{B} = B_0 + B_1$, so that the pair $(\hat{A},\ \hat{B})$ is stabilizable.

DEFINITION 1 *Consider the uncertain system (1) with (2) and the state feedback $u(t) = Kx(t)$. If the following conditions are satisfied:*

*(i) with $w(t) = 0$, the closed-loop system (1) with (2) and $u(t) = Kx(t)$ is asymptotically stable.*

*(ii) with zero initial condition (i.e. $\phi = 0$), the following condition is satisfied*

$$J = \int_0^\infty \left[z^T(t)z(t) - \gamma^2 \cdot w^T(t)w(t)\right] dt \leqslant 0, \quad (i.e. \sup_{w\neq 0, w(t)\in L_2[0,\infty)} \frac{\|z\|_2}{\|w\|_2} \leqslant \gamma),$$

*for some $\gamma > 0$. The control $u(t) = Kx(t)$ is said to be the $H_\infty$ control of system (1) with (2) and the disturbance attenuation $\gamma$. The parameter $\gamma$ is called the $H_\infty$-norm bound of the control.*

LEMMA 1 (YAKUBOWICH, 1977) *Let $\Omega_0(x)$ and $\Omega_1(x)$ be two arbitrary quadratic forms over $\Re^n$, then $\Omega_0(x) < 0$ for all $x \in \Re^n - \{0\}$ satisfying $\Omega_1(x) \leqslant 0$ if and only if there exist $\varepsilon \geqslant 0$ such that*

$$\Omega_0(x) - \varepsilon \cdot \Omega_1(x) < 0, \quad \forall x \in \Re^n - \{0\}.$$

## 3. Robust $H_\infty$ control design

Now we will solve for the controller gain $K$ of system (1) with (2) directly from the following LMIs optimization result.

THEOREM 1 *Consider system (1) with (2) and state feedback control $u(t) = Kx(t)$. Suppose that $\|A_2\| + \beta_2 < 1$ and if there exist non-negative constants $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$, some positive-definite symmetric matrices $X$, $Y_1$, $Y_2$, $Y_4$, $Q \in \Re^{nxn}$, $Y_3$, $Y_5 \in \Re^{mxm}$, and a matrix $W \in \Re^{m\times n}$, such that the following optimization problem is solved:*

$$\textit{minimize} \quad \rho, \tag{4a}$$

*subject to*

$$\begin{bmatrix}
\Lambda_{11} & 0 & \Lambda_{13} & 0 & B_w & \Lambda_{16} & \Lambda_{17} & I & I & I & I & \Lambda_{112} & \Lambda_{113} & \Lambda_{114} & X & W^T & \Lambda_{117} \\
0 & \Lambda_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{212} & \Lambda_{213} & \Lambda_{214} & 0 & 0 & 0 \\
\Lambda_{13}^T & 0 & \Lambda_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{312} & \Lambda_{313} & \Lambda_{314} & 0 & 0 & 0 \\
0 & 0 & 0 & \Lambda_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{412} & \Lambda_{413} & \Lambda_{414} & 0 & 0 & 0 \\
B_w^T & 0 & 0 & 0 & \Lambda_{55} & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{512} & \Lambda_{513} & \Lambda_{514} & 0 & 0 & 0 \\
\Lambda_{16}^T & 0 & 0 & 0 & 0 & -Y_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_{17}^T & 0 & 0 & 0 & 0 & 0 & -Y_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{88} & 0 & 0 & 0 & I & h_M \cdot I & \eta_M \cdot I & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{99} & 0 & 0 & I & h_M \cdot I & \eta_M \cdot I & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{1010} & 0 & I & h_M \cdot I & \eta_M \cdot I & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{1111} & I & h_M \cdot I & \eta_M \cdot I & 0 & 0 & 0 \\
\Lambda_{112}^T & \Lambda_{212}^T & \Lambda_{312}^T & \Lambda_{412}^T & \Lambda_{512}^T & 0 & 0 & I & I & I & I & -Y_2 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_{113}^T & \Lambda_{213}^T & \Lambda_{313}^T & \Lambda_{413}^T & \Lambda_{513}^T & 0 & 0 & h_M \cdot I & h_M \cdot I & h_M \cdot I & h_M \cdot I & 0 & -Y_4 & 0 & 0 & 0 & 0 \\
\Lambda_{114}^T & \Lambda_{214}^T & \Lambda_{314}^T & \Lambda_{414}^T & \Lambda_{514}^T & 0 & 0 & \eta_M \cdot I & \eta_M \cdot I & \eta_M \cdot I & \eta_M \cdot I & 0 & 0 & -Q & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Y_1 & 0 & 0 \\
W & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Y_3 & 0 \\
\Lambda_1 17^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0, \tag{4b}$$

$$\begin{bmatrix} -2X+Q & W^T \\ W & -Y_5 \end{bmatrix} < 0, \tag{4c}$$

*where*

$\Lambda_{11} = \hat{\Lambda}_{11} + \varepsilon_0 \cdot \beta_0^2 \cdot I,\ \ \hat{\Lambda}_{11} = \hat{A}X + X\hat{A}^T + \hat{B}W + W^T\hat{B}^T,\ \ \Lambda_{13} = A_2Y_2,$
$\Lambda_{16} = -A_1Y_4,\ \ \Lambda_{17} = -B_1Y_5,\ \ \Lambda_{112} = XA_0^T + W^TB_0^T,\ \ \Lambda_{113} = h_M \cdot \Lambda_{112},$
$\Lambda_{114} = \eta_M \cdot \Lambda_{112},\ \ \Lambda_{117} = XC^T + W^TD^T,\ \ \Lambda_{22} = -(1-h_D)\cdot Y_1 + \varepsilon_1 \cdot \beta_1^2 \cdot I,$
$\Lambda_{212} = Y_1A_1^T,\ \ \Lambda_{213} = h_M \cdot \Lambda_{212},\ \ \Lambda_{214} = \eta_M \cdot \Lambda_{212},$
$\Lambda_{33} = -(1-\tau_D)\cdot Y_2 + \varepsilon_2 \cdot \beta_2^2 \cdot I,\ \ \Lambda_{312} = Y_2A_2^T,\ \ \Lambda_{313} = h_M \cdot \Lambda_{312},$
$\Lambda_{314} = \eta_M \cdot \Lambda_{312},\ \ \Lambda_{44} = -(1-\eta_D)\cdot Y_3 + \varepsilon_3 \cdot \beta_3^2 \cdot I,\ \ \Lambda_{412} = Y_3B_1^T,$
$\Lambda_{413} = h_M \cdot \Lambda_{412},\ \ \Lambda_{414} = \eta_M \cdot \Lambda_{412},\ \ \Lambda_{55} = -\rho \cdot I,;\ \Lambda_{512} = B_w^T,$
$\Lambda_{513} = h_M \cdot \Lambda_{512},\ \ \Lambda_{514} = \eta_M \cdot \Lambda_{512},\ \ \Lambda_{88} = -\varepsilon_0 \cdot I,$
$\Lambda_{99} = -\varepsilon_1 \cdot I,\ \ \Lambda_{1010} = -\varepsilon_2 \cdot I,\ \ \Lambda_{1111} = -\varepsilon_3 \cdot I.$

*Then, the control $u(t) = Kx(t) = WX^{-1}x(t)$ is the $H_\infty$ control of system (1) with (2) and the disturbance attenuation $\gamma = \sqrt{\rho}$.*

*Proof.* Define the Lyapunov functional

$$\begin{aligned} V(x_t) &= x^T(t)Px(t) + \int_{t-h(t)}^{t} x^T(s)R_1x(s)ds + \int_{t-\tau(t)}^{t} \dot{x}^T(s)R_2\dot{x}(s)ds \\ &+ \int_{t-\eta(t)}^{t} x^T(s)K^TR_3Kx(s)ds + h_M \cdot \int_{t-h_M}^{t} (s-(t-h_M))\dot{x}^T(s)R_4\dot{x}(s)ds \\ &+ \eta_M \cdot \int_{t-\eta_M}^{t} (s-(t-\eta_M))\dot{x}^T(s)K^TR_5K\dot{x}(s)ds, \end{aligned} \tag{5}$$

where $P > 0$, $R_i > 0$, $i \in \{1,\ 2,\ 3,\ 4,\ 5\}$.

This functional $V(x_t)$ is a legitimate Lyapunov functional candidate, Kolmanovskii and Myshkis (1992). The time derivatives of $V_i(x_t)$, along the trajectories of system (3) satisfy

$$\begin{aligned} \dot{V}(x_t) &= x^T(t)\left[P(\hat{A}+\hat{B}K) + (\hat{A}+\hat{B}K)^TP\right]x(t) + 2x^T(t)PA_2\dot{x}(t-\tau(t)) \\ &-2x^T(t)PA_1\int_{t-h(t)}^{t}\dot{x}(s)ds - 2x^T(t)PB_1\int_{t-\eta(t)}^{t}\dot{u}(s)ds \\ &+2x^T(t)P\left[f_0(x(t),t) + f_1(x(t-h(t)),t) + f_2(\dot{x}(t-\tau(t)),t) + f_3(u(t-\eta(t)),t)\right] \\ &+2x^T(t)PB_ww(t) + x^T(t)R_1x(t) - (1-\dot{h}(t))x^T(t-h(t))R_1x(t-h(t)) \\ &+\dot{x}^T(t)R_2\dot{x}(t) - (1-\dot{\tau}(t))\dot{x}^T(t-\dot{\tau}(t)))R_2\dot{x}(t-\dot{\tau}(t))) + x^T(t)K^TR_3Kx(t) \end{aligned}$$

$$
\begin{aligned}
&-(1-\dot{\eta}(t))u^T(t-\eta(t))R_3u(t-\eta(t))+h_M^2\cdot\dot{x}^T(t)R_4\dot{x}(t)\\
&-h_M\cdot\int_{t-h_M}^{t}\dot{x}^T(s)\,R_4\dot{x}(s)\,ds+\eta_M^2\cdot\dot{x}^T(t)K^TR_5K\dot{x}(t)\\
&-\eta_M\cdot\int_{t-\eta_M}^{t}\dot{u}^T(s)R_5\dot{u}(s)ds.
\end{aligned}
$$

By the following inequality, Gu (2000):

$$
\begin{aligned}
&-h_M\cdot\int_{t-h_M}^{t}\dot{x}^T(s)R_4\dot{x}(s)ds\leqslant-\left(\int_{t-h_M}^{t}\dot{x}(s)ds\right)^TR_4\left(\int_{t-h_M}^{t}\dot{x}(s)ds\right)\\
&\leqslant-\left(\int_{t-h(t)}^{t}\dot{x}(s)ds\right)^TR_4\left(\int_{t-h(t)}^{t}\dot{x}(s)ds\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&-\eta_M\cdot\int_{t-\eta_M}^{t}\dot{u}^T(s)R_5\dot{u}(s)ds\leqslant-\left(\int_{t-\eta_M}^{t}\dot{u}(s)ds\right)^TR_5\left(\int_{t-\eta_M}^{t}\dot{u}(s)ds\right)\\
&\leqslant-\left(\int_{t-\eta(t)}^{t}\dot{u}(s)ds\right)^TR_5\left(\int_{t-\eta(t)}^{t}\dot{u}(s)ds\right).
\end{aligned}
$$

Define a functional by

$$
J(x_t,w(t))=\dot{V}(x_t)+z^T(t)\,z(t)-\gamma^2w^T(t)\,w(t)\,.
$$

From the time derivatives of $V(x_t)$, we have

$$
\Omega(x_t,w(t))=\dot{V}(x_t)+z^T(t)\,z(t)-\gamma^2w^T(t)\,w(t)\leqslant\varsigma^T(t)\psi_0\varsigma(t), \tag{6a}
$$

where

$$
\begin{aligned}
\varsigma^T(t)=[\,&x^T(t)\quad x^T(t-h(t))\quad \dot{x}^T(t-\tau(t))\quad u^T(t-\eta(t))\quad w^T(t)\\
&\int_{t-h(t)}^{t}\dot{x}^T(s)ds\quad \int_{t-\eta(t)}^{t}\dot{u}^T(s)ds\quad f_0^T(x(t),t)\quad f_1^T(x(t-h(t)),t)\\
&f_2^T(\dot{x}(t-\tau(t)),t)\quad f_3^T(u(t-\eta(t)),t)\,],
\end{aligned}
$$

$$\psi_0 = \begin{bmatrix} \tilde{\Lambda}_1 1 & 0 & PA_2 & 0 & PB_w & -PA_1 & -PB_1 & P & P & P & P \\ 0 & -(1-h_D)\cdot R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_2^T P & 0 & -(1-\tau_D)\cdot R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-\eta_D)\cdot R_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_w^T P & 0 & 0 & 0 & -\gamma^2\cdot I & 0 & 0 & 0 & 0 & 0 & 0 \\ -A_1^T P & 0 & 0 & 0 & 0 & -R_4 & 0 & 0 & 0 & 0 & 0 \\ -B_1^T P & 0 & 0 & 0 & 0 & 0 & -R_5 & 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$+ \begin{bmatrix} (A_0+B_0K)^T \\ A_1^T \\ A_2^T \\ B_1^T \\ B_w^T \\ 0 \\ 0 \\ I \\ I \\ I \\ I \end{bmatrix} \cdot (R_2 + h_M^2\cdot R_4 + \eta_M^2\cdot K^T R_5 K)\cdot \begin{bmatrix} (A_0+B_0K)^T \\ A_1^T \\ A_2^T \\ B_1^T \\ B_w^T \\ 0 \\ 0 \\ I \\ I \\ I \\ I \end{bmatrix}^T, \qquad (6b)$$

where $\tilde{\Lambda}_{11} = P(\hat{A}+\hat{B}K)+(\hat{A}+\hat{B}K)^T P+R_1+K^T R_3 K+(C+DK)^T(C+DK)$. Pre-multiplying and post-multiplying the matrix $\psi_0$ in (6b) by $\Theta^T$ and $\Theta$, where

$$\Theta = \begin{bmatrix} P^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_2^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_3^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_4^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R_5^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

and applying a change of variables, $W = KP^{-1}$, $X = P^{-1}$, and $Y_i = R_i^{-1}$,

$i = \{1, 2, 3, 4, 5\}$, we can obtain the following result:

$$\psi_1 = \begin{bmatrix} \tilde{\Lambda}_{11} & 0 & A_2Y_2 & 0 & B_w & -A_1Y_4 & -B_1Y_5 & I & I & I & I \\ 0 & -(1-h_D)\cdot Y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Y_2A_2^T & 0 & -(1-\tau_D)\cdot Y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-\eta_D)\cdot Y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_w^T & 0 & 0 & 0 & -\gamma^2\cdot I & 0 & 0 & 0 & 0 & 0 & 0 \\ -Y_4A_1^T & 0 & 0 & 0 & 0 & -Y_4 & 0 & 0 & 0 & 0 & 0 \\ -Y_5B_1^T & 0 & 0 & 0 & 0 & 0 & -Y_5 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} X(A_0+B_0K)^T \\ Y_1A_1^T \\ Y_2A_2^T \\ Y_3B_1^T \\ B_w^T \\ 0 \\ 0 \\ I \\ I \\ I \\ I \end{bmatrix} \cdot (Y_2^{-1} + h_M^2 \cdot Y_4^{-1} + \eta_M^2 \cdot K^TY_5^{-1}K) \cdot \begin{bmatrix} X(A_0+B_0K)^T \\ Y_1A_1^T \\ Y_2A_2^T \\ Y_3B_1^T \\ B_w^T \\ 0 \\ 0 \\ I \\ I \\ I \\ I \end{bmatrix}^T, \tag{6c}$$

where $\tilde{\Lambda}_{11} = \hat{\Lambda}_{11} + XY_1^{-1}X + W^TY_3^{-1}W + (CX + DW)^T(CX + DW)$.

In view of Lemma 1 with (4c), we can obtain the following result:

$$-2X + Q + W^TY_5^{-1}W < 0.$$

By taking $W = KX$, and the following equality:

$$(X - Q)Q^{-1}(X - Q) = XQ^{-1}X - 2X + Q \geqslant 0,$$

the above condition is equivalent to the following result:

$$K^TY_5^{-1}K < Q^{-1}.$$

Hence we have

$$\Omega\left(x_t, w\left(t\right)\right) \leqslant \varsigma^T(t)\psi_2\varsigma(t), \tag{6d}$$

where

$$\psi_2 = \begin{bmatrix} \tilde{\Lambda}_{11} & 0 & A_2Y_2 & 0 & B_w & -A_1Y_4 & -B_1Y_5 & I & I & I & I \\ 0 & -(1-h_D)\cdot Y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Y_2A_2^T & 0 & -(1-\tau_D)\cdot Y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-\eta_D)\cdot Y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_w^T & 0 & 0 & 0 & -\gamma^2\cdot I & 0 & 0 & 0 & 0 & 0 & 0 \\ -Y_4A_1^T & 0 & 0 & 0 & 0 & -Y_4 & 0 & 0 & 0 & 0 & 0 \\ -Y_5B_1^T & 0 & 0 & 0 & 0 & 0 & -Y_5 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} X(A_0+B_0K)^T \\ Y_1A_1^T \\ Y_2A_2^T \\ Y_3B_1^T \\ B_w^T \\ 0 \\ 0 \\ I \\ I \\ I \\ I \end{bmatrix} \cdot \left(Y_2^{-1} + h_M^2\cdot Y_4^{-1} + \eta_M^2\cdot Q^{-1}\right)\cdot \begin{bmatrix} X(A_0+B_0K)^T \\ Y_1A_1^T \\ Y_2A_2^T \\ Y_3B_1^T \\ B_w^T \\ 0 \\ 0 \\ I \\ I \\ I \\ I \end{bmatrix}^T . \tag{6e}$$

By Theorem 3.1.6 of Kolmanovskii and Myshkis (1992) with (4c) and (6e), suppose that $\|A_2\| + \beta_2 < 1$ and if there exist some positive-definite symmetric matrices $X$, $Y_1$, $Y_2$, $Y_4$, $Q \in \Re^{nxn}$, $Y_3$, $Y_5 \in \Re^{mxm}$, and a matrix $W \in \Re^{m\times n}$, then a sufficient condition for asymptotic stability is

$$\varsigma^T(t)\psi_2(t)\varsigma(t) < 0. \tag{7}$$

From (7), it follows that

$$\Omega\left(x_t, w\left(t\right)\right) = \dot{V}\left(x_t\right) + z^T\left(t\right)z\left(t\right) - \gamma^2 w^T\left(t\right)w\left(t\right) \leqslant \varsigma^T(t)\psi_0\varsigma(t) < 0,$$
$$\text{for all } \varsigma\left(t\right) \neq 0. \tag{8}$$

From (8) with $w\left(t\right) = 0$, there exists a $\lambda > 0$ and the following result holds:

$$\dot{V}\left(x_t\right)\Big|_{w(t)=0} \leqslant -\lambda\cdot\left\|x\left(t\right)\right\|^2. \tag{9}$$

Hence, we conclude that the systems (1) and (3) with (2), under $w\left(t\right) = 0$, are both asymptotically stabilizable by $u\left(t\right) = WX^{-1}x\left(t\right)$. Integrating the inequality in (8) from 0 to $\infty$, we obtain

$$V(x_\infty) - V(x_0) + \|z(t)\|_2^2 - \gamma^2 \cdot \|w(t)\|_2^2 \leqslant 0.$$

With zero initial condition ($x_0 = 0$), we have

$$V(x_0) = 0, \quad V(x_\infty) \geqslant 0,$$

and

$$\|z(t)\|_2 \leqslant \gamma \cdot \|w(t)\|_2, \quad w(t) \in L_2[0, \infty).$$

By Definition 1, the control $u(t) = Kx(t) = WX^{-1}x(t)$ is the $H_\infty$ control of system (1) with (2) and the disturbance attenuation $\gamma = \sqrt{\rho}$.

Using Lemma 1, we further rewrite (7) and (2) as follows:

$$\begin{aligned}&\varsigma^T(t)\psi_2\varsigma(t) - \varepsilon_0 \cdot (f_0^T(x(t),t)f_0(x(t),t) - \beta_0^2 \cdot x^T(t)x(t))\\ &-\varepsilon_1 \cdot (f_1^T(x(t-h(t)),t)f_1(x(t-h(t)),t) - \beta_1^2 \cdot x^T(t-h(t))x(t-h(t)))\\ &-\varepsilon_2 \cdot (f_2^T(\dot{x}(t-\tau(t)),t)f_2(\dot{x}(t-\tau(t)),t) - \beta_2^2 \cdot \dot{x}^T(t-\tau(t))\dot{x}(t-\tau(t)))\\ &-\varepsilon_3 \cdot (f_3^T(u(t-\eta(t)),t)f_3(u(t-\eta(t)),t) - \beta_3^2 \cdot u^T(t-\eta(t))u(t-\eta(t))) < 0.\end{aligned} \tag{10}$$

By the Schur complement of Boyd et al. (1994), the linear matrix inequality of (4b) is equivalent to the condition (10). Therefore, we conclude that the systems (1) and (3) with (2) are both asymptotically stabilizable by $u(t) = WX^{-1}x(t)$ with the disturbance attenuation $\gamma = \sqrt{\rho}$. ∎

Remark 1 The condition $\|A_2\| + \beta_2 < 1$ in Theorem 1 will guarantee that the systems (1) satisfy the Lipschitz condition in the argument $\dot{x}(t-\eta(t))$ for a Lipschitz constant less than 1, Kolmanovskii and Myshkis (1992).

Remark 2 Note that for the entries of (4b) and (4c) are affined with matrices $X$, $Y_1$, $Y_2$, $Y_3$, $Y_4$, $Y_5$, $Q$, $W$, and constants $\varepsilon_0$ $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$. Hence the standard LMI optimization approach can be directly employed to solve the optimal problem (4a) with conditions (4b) and (4c). We can utilize Matlab's LMI Control Toolbox to find the solutions of matrices $W$, $X$, and the $H_\infty$ control is given by $u(t) = WX^{-1}x(t)$, Gahinet et al. (1995).

Remark 3 It is interesting to note that the LMI conditions (4b)-(4c) are dependent on $h_M$, $\eta_M$ and independent of $\tau_M$.

Remark 4 If $\gamma > 0$ is a known parameter, we can use the LMIs (4a)-(4b) with $\rho = \gamma^2$ to find the feasible solutions. The $H_\infty$ control problem can also be solved with Matlab without optimization.

## 4. Numerical example

To illustrate the effectiveness of the proposed method, Matlab LMI Toolbox was used to calculate the following example.

EXAMPLE 1 Consider the system (1) with (2) the following parameters:

$$
\begin{aligned}
&A_0 = \begin{bmatrix} -1 & 0 \\ 0.2 & -0.3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.02 & -0.01 \\ 0.01 & -0.02 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.2 \\ 0 & 0.2 \end{bmatrix}, \\
&B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
&\beta_0 = \beta_2 = 0.1, \quad \beta_1 = \beta_3 = 0.2, \quad h_M = 0.1, \quad \eta_M = 0.2 h_D = \tau_D = \eta_D = 0.1.
\end{aligned}
\tag{11}
$$

Note that the results proposed in Baser (2003), Fridman and Shaked (2003), Su, Su and Chu (2003), or Xu, Lam and Yang (2002), offer no feasible solutions to the above neutral system. By using Matlab in Theorem 1, we obtain a solution for the optimization problem:

$$
\begin{aligned}
&\rho = 4.9521, \quad X = \begin{bmatrix} 1.7183 & 0.0911 \\ 0.0911 & 0.5045 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 5524.2 & 3328.3 \\ 3328.3 & 2438.6 \end{bmatrix}, \\
&Y_2 = \begin{bmatrix} 774907468 & -4 \\ -4 & 1 \end{bmatrix}, \quad Y_3 = 21.8556, \quad Y_4 = \begin{bmatrix} 42.5421 & 37.8495 \\ 37.8495 & 43.0602 \end{bmatrix}, \\
&Y_5 = 0.0463, \quad Q = \begin{bmatrix} 3.3170 & -0.1280 \\ -0.1280 & 0.1346 \end{bmatrix}, W = \begin{bmatrix} -0.0703 & -0.2011 \end{bmatrix}, \\
&\varepsilon_0 = 9.1025, \quad \varepsilon_1 = 1538.4, \quad \varepsilon_2 = 48.6436, \quad \varepsilon_3 = 213.0819.
\end{aligned}
$$

The state feedback $H_\infty$ control of system (1) with (2) and (11) is given by

$$
u(t) = W X^{-1} x(t) = \begin{bmatrix} -0.02 & -0.395 \end{bmatrix} x(t),
$$

with the disturbance attenuation $\gamma = \sqrt{\rho} = 2.2253$.

## 5. Conclusion

In this paper, the state feedback $H_\infty$ control problem for a class of uncertain neutral systems has been studied, containing time-varying delays on both state and control input. Based on the unified LMI optimization, a new delay-dependent criterion has been proposed for the existence of memoryless $H_\infty$ state feedback control for such system. Furthermore, no parameters in Theorem 1 require to be tuned. A numerical example has illustrated that the usefulness of the main result.

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## References

BASER, U. (2003) Output feedback $H_\infty$ control problem for linear neutral systems: delay independent case. *ASME J. Dynam. Systems Meas. Control* **125**, 177-185.

BOYD, S., GHAOUI, L. EL, FERON, E. and BALAKRISHNAN, V. (1994) *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, PA.

CHEN, J.D. (2004A) Genetic Robust Controller Design for Uncertain Systems with Time-Delays in Both State and Control input via LMI Approach. *J. Chin. Inst. Eng.* **27**, 1055-1061.

CHEN, J.D. (2004B) Robust Control for Uncertain Neutral Systems with Time-Delays in State and Control Input via LMI and GAs. *Appl. Math. Comput.* **157**, 535-548.

FRIDMAN, E. and SHAKED, U. (2003) Delay-dependent stability and $H_\infty$ control: constant and time-varying delays. *Int. J. Control* **76**, 48-60.

GAHINET, P., NEMIROVSKI, A., LAUB, A. and CHILALI, M. (1995) *LMI Control Toolbox User's Guide.* The Mathworks, Natick, Massachusetts.

GU, K. (2000) An integral inequality in the stability problem of time-delay systems. *Proc. 39th IEEE CDC*, Sydney, Australia, 2805-2810.

HALE, J.K. and VERDUYN LUNEL, S. M. (1993) *Introduction to Functional Differential Equations.* Springer-Verlag, New York.

HE, Y., WU, M., SHE, J.H. and LIU, G.P. (2004) Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays. *Systems Control Lett.* **51**, 57-65.

KOLMANOVSKII, V.B. and MYSHKIS, A. (1992) *Applied Theory of Functional Differential Equations.* Kluwer Academic Publishers, Netherlands.

LI, X. and DE SOUZA, C.E. (1997) Delay-dependent robust stability and stabilization of uncertain linear delay systems: a linear matrix inequality approach. *IEEE Trans. Automat. Control* **42**, 1144-1148.

LIEN, C.H. and CHEN, J.D. (2003) Discrete delay-independent and discrete delay-dependent criteria for a class of neutral systems. *ASME J. Dynam. Systems Meas. Control* **125**, 33-41.

MOON, Y.S., PARK, P., KWON, W.H. and LEE, Y.S. (2001) Delay-dependent robust stabilization of uncertain state-delayed systems. *Int. J. Control* **74**, 1447-1455.

NIAN, X. and FENG, J. (2003) Guaranteed-cost control of a linear uncertain system with multiple time-varying Delays: an LMI Approach. *IEE Proc.*

*Control Theory Appl.* **150**, 17-22.

ROH, Y.H. (2002) Robust Stability of Predictor-Based Control Systems with Delayed Control. *Int. J. Syst. Sci.* **33**, 81-86.

SU, N.J., SU, H.Y. and CHU, J. (2003) Delay-dependent robust $H_\infty$ control for uncertain time-delay systems. *IEE Proc. Control Theory Appl.* **150**, 489-492.

SU, T.J., LU, C.Y. and TSAI, J.S.H. (2001) LMI approach to delay dependent robust stability for uncertain time-delay systems. *IEE Proc. Control Theory Appl.* **148**, 209-212.

XU, S., LAM, J. and YANG, C. (2002) Robust $H_\infty$ control for uncertain linear neutral delay systems. *Optim. Control Appl. Methods* **23**, 113-123.

YAKUBOVICH, V.A. (1977) S-procedure in nonlinear control theory. *Vestnik Leningrad Univ. Math.* **4**, 73-93 [English translation].