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On some nonconventional problem of a state filtration*

by

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Abstract: In the paper nonconventional linear equations of state filtration are derived. They are useful for some two-level hierarchical control system structure with coordinator and local controllers having different information. It is assumed that the system considered is described by a linear output equation and a linear state equation with control being a random variable for the coordinator generated by decision rules of the local controllers. The approach to state filtration is based on modified innovations and orthogonality principle. A simple numerical example is presented.

Keywords: hierarchical control structure, nonconventional state filtration, modified innovations, orthogonality principle.

1. Introduction

The paper deals with control with a quadratic cost for a stochastic system composed of interacting linear subsystems. Quality of control depends on the assumed information and control structures. In a one level structure a central decision maker determines values of control on the basis of available information collected from all subsystems. However, in large scale systems the process of transmission and transformation of information in a centralized manner can be difficult to realize. This leads to decentralization of information and control structures.

Control problems with decentralized measurement information are studied in a team decision theory, as well as in the hierarchical control (see Aoki, 1973; Chong and Athans, 1971; Ho, 1980). The problems may be complicated, especially in the case of the so called nonclassical information pattern, in which controllers do not have identical information.

Control and optimization for large scale systems are usually based on a decomposition of a global system into subsystems so as to decrease computational requirements and decrease the amount of information to be transmitted to and

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processed by decision makers. A conflict between local controllers is softened by the coordinator on the upper level.

Gessing (1988) formulated and solved a stochastic control problem with a quadratic cost for a system composed of interacting linear subsystems. Control is realized in a two-level structure with a coordinator on the upper level and local controllers on the lower level. It is assumed that the local controllers have essential information for their subsystems, while the coordinator has aggregate information on the whole system. An elastic constraint as a coordination equation is proposed in which decisions of the coordinator are a conditional expectation of decision rules of the local controllers.

In order to realize the controls of the decision makers, the current determination of the state estimates, performed by the coordinator, and the local controllers is needed. A solution to this problem was proposed in Gessing and Duda (1990).

In the present paper a new approach to the filtration problem is presented. It is based on modified innovations and orthogonality principle.

2. Control problem formulation and its solution

Consider a large scale dynamic system composed of M subsystems described by the equation

$$\begin{aligned} x_{n+1}^i &= A_n^{ii}x_n^i + B_n^i u_n^i + \sum_{j\neq i}^{M} A_n^{ij}x_n^j + w_n^i = \\ &= A_n^{ii}x_n^i + B_n^i u_n^i + v_n^i + w_n^i, \quad i = 1,2,..,M \end{aligned} \tag{1}$$

where $x_n^i,\ u_n^i,\ w_n^i,\ v_n^i$ are vectors of state, control, disturbance and interaction of the ith subsystem; B_n^i and $A_n^{ij},\ \ i,j = 1,2,...,M,\ \ n = 0,1...$ are appropriate matrices.

The model of measurements has the form

$$y_n^i = C_n^i x_n^i + r_n^i \tag{2}$$

where y_n^i and r_n^i are the vectors of the measurements and measurement errors, respectively.

It is assumed that the processes $w_n = [w_n^{1T}, ..., w_n^{MT}]^T$ and $r_n = [r_n^{1T}, ..., r_n^{MT}]^T$ are white noises, mutually independent. The initial state $x_0 = [x_0^{1T}, ...x_0^{MT}]^T$ is also random, independent of the above vectors. Additionally, we assume that $Er_n = 0$, $Ew_n = 0$, Ex_0 and the covariance matrices $W_n = Ew_n w_n^T$, $R_n = Er_n r_n^T$, $X_0 = E(x_0 - Ex_0)(x_0 - Ex_0)^T$ are finite and given.

The problem is to find the control laws $u_n^i = a_n^i(.), i = 1,2,...,M, n = 0,1,...N$ as functions of available information that minimize a performance index

$$I = E\sum_{n=0}^{N}\sum_{i=1}^{M}[x_n^{iT}Q_n^i x_n^i + a_n^{iT}(.)H_n^i a_n^i(.)] \tag{3}$$

where E denotes mean operation, Q_n^i and H_n^i are symmetric , non negative and positive-definite matrices.

The complexity and the effectiveness of a solution depends on the assumed information and control structures.

Gessing (1988) assumed that control is realized in a two level hierarchical structure with a coordinator on the upper level and local controllers on the lower one.

The ith local controller receives from the appropriate subsystem the measurement y_n^i, which is aggregated to the form

$$m_n^i = D_n^i y_n^i = F_n^i x_n^i + D_n^i r_n^i \tag{4}$$

and transmitted to the coordinator. Owing to the low dimension of the vector m_n^i, the amount of information transmitted and converted by the coordinator may be decreased.

Notice that for D_n^i equal to a unit matrix ($D_n^i = \mathbf{1}$), all information is transmitted to the coordinator. It is the case of classical control realized by a central decision maker. For $D_n^i = \mathbf{0}$ no information is transmitted from the i-th subsystem to the coordinator.

At time n a posteriori measurement information of the ith local controller and the coordinator is defined by $\vec{y}_n^i = [y_0^{iT}, ..., y_n^{iT}]^T$ and $\vec{m}_n = [\vec{m}_n^{1T}, ..., \vec{m}_n^{MT}]^T$, $\vec{m}_n^i = [m_0^{iT}, ..., m_n^{iT}]^T$, respectively.

As admissible control laws of the ith controller (a_n^i) and of the coordinator (b_n) we assume

$$\begin{aligned} u_n^i &= a_n^i(\hat{x}_n^i, \bar{x}_n^i, p_n^i) \\ p_n &= b_n(\vec{m}_n) \end{aligned} \tag{5}$$

where $\hat{x}_n^i$ is an estimate of the state x_n^i determined by the ith local controller, $\bar{x}_n^i$ is the estimate of the state x_n^i determined by the coordinator and sent to the ith controller; $p_n = [p_n^{1T}, ..., p_n^{MT}]^T$ is a vector of coordinating variables determined by the coordinator.

Additionally, it is assumed that the control laws fulfill an elastic constraint

$$E_{|\vec{m}_n} a_n^i(\hat{x}_n^i, \bar{x}_n^i, p_n^i) = p_n^i \tag{6}$$

where $E_{|\vec{m}_n}$ denotes conditional mean operation given $\vec{m}_n$.

Gessing (1988) showed that the optimal control law a_n^{io} for the ith controller results from the local minimization and has the form

$$u_n^{io} = a_n^{io}(\hat{x}_n^i, \bar{x}_n^i, p_n^i) = p_n^i - L_n^i(\hat{x}_n^i - \bar{x}_n^i) \tag{7}$$

where p_n^i is the coordinating variable sent by the coordinator, and L_n^i is a matrix determined in an appropriate way.

The coordinating variables p_n result from the global optimization and have the form

$$p_n = L_n \bar{x}_n \tag{8}$$

where L_n is a matrix determined in an appropriate way and $\bar{x}_n = [\bar{x}_n^{1T}, ..., \bar{x}_n^{MT}]^T$ is an estimate of the state of the whole system determined by the coordinator.

In order to realize the controls (7) and (8) the current determination of the state estimates performed by the coordinator and local controllers is needed. The filtering problem was solved in Gessing and Duda (1990) under the following assumptions:

1. The estimate of the interaction $v_n^i = \sum_{j \neq i}^M A_n^{ij} x_n^j$, determined by the ith local controller, is equal to the estimate determined by the coordinator
2. The estimate $\bar{x}_{n+1}^i = E_{|\vec{m}_{n+1}} x_{n+1}^i$, determined by the coordinator, is sent to the ith local controller and used to determine the estimate $\hat{x}_{n+1}^i$
3. The correction

$$\hat{x}_{n+1|n+}^i - \hat{x}_{n+1|n}^i = E_{|\vec{y}_n^i, \bar{x}_{n+1}^i} x_{n+1}^i - E_{|\vec{y}_n^i} x_{n+1}^i \tag{9}$$

determined by the ith local controller, is equal to the correction

$$\bar{x}_{n+1}^i - \bar{x}_{n+1|n}^i = \bar{x}_{n+1}^i - E_{|\vec{m}_n} x_{n+1}^i \tag{10}$$

determined by the coordinator
4. Random variables x_0^i, r_n^i and w_n^i are mutually independent Gaussian white noises.

It is shown that the estimate $\hat{x}_{n+1}^i$, determined by the ith local controller, has the form

$$\begin{aligned} \hat{x}_{n+1}^i &= \hat{x}_{n+1|n+}^i + \hat{K}_{n+1}^i (y_{n+1}^i - C_{n+1}^i \hat{x}_{n+1|n+}^i) \\ \hat{x}_{n+1|n+}^i &= \bar{x}_{n+1}^i + G_n^i (\hat{x}_n^i - \bar{x}_n^i). \end{aligned} \tag{11}$$

The filtering equations of the coordinator have the form

$$\begin{aligned} \bar{x}_{n+1} &= \bar{x}_{n+1|n} + \bar{K}_{n+1}(m_{n+1} - F_{n+1}\bar{x}_{n+1|n}) \\ \bar{x}_{n+1|n} &= A_n \bar{x}_n + B_n p_n. \end{aligned} \tag{12}$$

The matrices G_n^i, $\hat{K}_{n+1}^i$, F_{n+1}, A_n, B_n, and $\bar{K}_{n+1}$ are determined in an appropriate way.

In the present paper a new approach to the filtration problem, based on modified innovations, is presented. The assumptions 2, 3 and 4 are not required.

3. New approach to the filtration problem

We restrict the estimate of the state $\hat{x}_n$ to be a linear estimate

$$\hat{x}_n = \alpha_0 + \sum_{j=1}^{n} \alpha_j y_j \tag{13}$$

that minimizes the mean square error (MSE)

$$MSE = E(x_n - \hat{x}_n)^T (x_n - \hat{x}_n). \tag{14}$$

Calculation of a recursive form of the estimate will be based on Theorem 1 and modified innovations.

THEOREM 1. *Let x and y be jointly distributed random vectors and let $\hat{x} = \alpha_0 + \alpha_1 y$ be the estimate of x given y. Then $\hat{x}$ minimizes the MSE if the error $(x - \hat{x})$ is orthogonal to y, i.e.,*

$$E(x - \hat{x})y^T = 0 \tag{15}$$

and $\hat{x}$ is unbiased.

Proof. If $\hat{x}$ is unbiased and the error $(x - \hat{x})$ is orthogonal to y then

$$E(x - \alpha_0 - \alpha_1 y) = 0 \tag{16}$$

$$E(x - \alpha_0 - \alpha_1 y)y^T = 0. \tag{17}$$

Thus

$$\alpha_0 = Ex - \alpha_1 Ey \tag{18}$$

$$P_{xy} + ExEy^T - \alpha_0 Ey^T - \alpha_1 (P_{yy} + EyEy^T) = 0. \tag{19}$$

As a result we have

$$\hat{x} = Ex + P_{xy} P_{yy}^{-1} (y - Ey). \tag{20}$$

The same result can be found directly by minimization of (14). ■

3.1. Modified innovations

Classical innovations are defined by Kamen and Su (1999):

$$e_n = y_n - \hat{y}_{n|n-1} \tag{21}$$

where $\hat{y}_{n|n-1}$ is the LMMSE estimate of y_n given $\vec{y}_{n-1} = [y_1^T, ..., y_{n-1}^T]^T$ i.e.,

$$\hat{y}_{n|n-1} = \sum_{j=1}^{n-1} \alpha_{n-1,j} y_j = \alpha_{n-1} \vec{y}_{n-1}. \tag{22}$$

In this paper the estimate (22) is modified to the form

$$\hat{y}_{n|n-1} = \alpha_0 + \sum_{j=1}^{n-1} \alpha_{n-1,j} y_j = \alpha_0 + \alpha_{n-1} \vec{y}_{n-1} \tag{23}$$

where $\alpha_{n-1} = [\alpha_{n-1,1}, ..., \alpha_[n-1, n-1]$.

The estimate $\hat{y}_{n|n-1}$ should be unbiased and the error $(y_n - \hat{y}_{n|n-1})$ should be orthogonal to $\vec{y}_{n-1}$.

The innovations e_j are endowed with definite properties.

Orthogonality 1. The innovation e_j is orthogonal to $y_i, \quad i = 1, ..., j-1$.

This results from Theorem 1. Namely

$$E(y_j - \hat{y}_{j|j-1}) \vec{y}_{j-1}^T = 0 \tag{24}$$

that is equivalent to

$$E(y_j - \hat{y}_{j|j-1}) y_i^T = E e_j y_i^T = 0, \quad i = 1, ..., j-1. \tag{25}$$

Orthogonality 2. The innovations are orthogonal to each other.

For $j > i \geq 1$ we have

$$E e_i e_j^T = E(y_i - \hat{y}_{i|i-1}) e_j^T = E y_i e_j^T - E(\alpha_0 + \alpha_{i-1} \vec{y}_{i-1}) e_j^T = 0. \tag{26}$$

Uncorrelatedness. The innovations are uncorrelated with each other.

This results from Orthogonality 2 and the fact that $E e_j = 0$ (unbiased estimate). Therefore, innovations are white noise.

Equivalent information. The measurement y_{n+1} can be obtained from linear combination of $e_i, \quad i = 1, ..., n+1$.

Let $\hat{y}_{1|0}$ take a given value $\bar{y}$, i.e. $\hat{y}_{1|0} = \bar{y}$, so that $e_1 = y_1 - \bar{y}$ and $y_1 = \bar{y} + e_1$. For e_2 we have

$$e_2 = y_2 - \hat{y}_{2|1} = y_2 - \alpha_2 - \alpha_{11} y_1. \tag{27}$$

We can find y_2 via

$$y_2 = \alpha_2 + \alpha_{11} \bar{y} + \alpha_{11} e_1 + e_2 = \alpha_2^* + e_2 + \alpha_{11} e_1. \tag{28}$$

We can continue this process indefinitely for any y_{n+1}

$$y_{n+1} = \alpha_{n+1}^* + e_{n+1} + \sum_{i=1}^{n} \alpha_{n,i} e_i. \tag{29}$$

3.2. Recursive form of x_{n+1}

Let

$$\hat{x}_{n+1} = \alpha_0 + \sum_{i=1}^{n+1} \alpha_{n+1,i} y_i = \beta_0 + \sum_{i=1}^{n+1} \beta_{n+1,i} e_i \tag{30}$$

be the LMMSE estimate of x_{n+1}.

This estimate should be unbiased and orthogonal to $\vec{e}_{n+1} = [e_1^T, ..., e_{n+1}^T]^T$, i.e.,

$$E\hat{x}_{n+1} = Ex_{n+1} = \beta_0 \tag{31}$$

$$E(x_{n+1} - \hat{x}_{n+1})e_i^T = 0, \quad i = 1, ..., n+1. \tag{32}$$

From (32), (31) and (30) we have

$$Ex_{n+1}e_i^T = E(\beta_0 + \sum_{j=1}^{n+1} \beta_{n+1,j} e_j)e_i^T = \beta_{n+1,i} Ee_i e_i^T. \tag{33}$$

Therefore, the matrix $\beta_{n+1,i}$ is

$$\beta_{n+1,i} = Ex_{n+1}e_i^T (Ee_i e_i^T)^{-1}. \tag{34}$$

Substituting (31) and (34) into (30) gives

$$\hat{x}_{n+1} = Ex_{n+1} + \sum_{i=1}^{n+1} Ex_{n+1}e_i^T (Ee_i e_i^T)^{-1} e_i. \tag{35}$$

By a derivation analogous to that presented above we find that the LMMSE estimate of x_{n+1} given $\vec{y}_n$ has the form

$$\begin{aligned} \hat{x}_{n+1|n} &= \alpha_0 + \sum_{i=1}^{n} \alpha_{n,i} y_i = \beta_0 + \sum_{i=1}^{n} \beta_{n,i} e_i = \\ &= Ex_{n+1} + \sum_{i=1}^{n} Ex_{n+1}e_i^T (Ee_i e_i^T)^{-1} e_i. \end{aligned} \tag{36}$$

Then, (35) can be written in a recursive form

$$\hat{x}_{n+1} = \hat{x}_{n+1|n} + Ex_{n+1}e_{n+1}^T (Ee_{n+1}e_{n+1}^T)^{-1} e_{n+1}. \tag{37}$$

It is a classical form of the LMMSE estimate.

Let y_n be described by

$$y_n = C_n x_n + r_n \tag{38}$$

where r_n is a white noise with $Er_n = 0$ and a covariance matrix R_n.

Therefore

$$e_{n+1} = C_{n+1}\tilde{x}_{n+1|n} + r_{n+1} \tag{39}$$

where $\tilde{x}_{n+1|n} = x_{n+1} - \hat{x}_{n+1|n}$.

It is assumed that $\hat{x}_{n+1|n}$ is the LMMSE estimate of x_{n+1} given information $\vec{y}_n$. Then $\hat{x}_{n+1|n}$ is unbiased and $\tilde{x}_{n+1|n}$ is orthogonal to $\vec{y}_n$.

From (39) we have

$$Ee_{n+1}e_{n+1}^T = C_{n+1}P_{n+1|n}C_{n+1}^T + R_{n+1} \tag{40}$$

where $P_{n+1|n} = E\tilde{x}_{n+1|n}\tilde{x}_{n+1|n}^T$.

To find $Ex_{n+1}e_{n+1}^T$ we use (39) again

$$\begin{aligned} Ex_{n+1}e_{n+1}^T &= Ex_{n+1}(C_{n+1}\tilde{x}_{n+1|n} + r_{n+1})^T = Ex_{n+1}\tilde{x}_{n+1|n}^T C_{n+1}^T = \\ &= E(\tilde{x}_{n+1|n} + \hat{x}_{n+1|n})\tilde{x}_{n+1|n}^T C_{n+1}^T = \\ &= E\tilde{x}_{n+1|n}\tilde{x}_{n+1|n}^T C_{n+1}^T + E\hat{x}_{n+1|n}\tilde{x}_{n+1|n}C_{n+1}^T. \end{aligned} \tag{41}$$

The last term in (41) is equal to zero because of orthogonality of the vectors $\hat{x}_{n+1|n}$ and $\tilde{x}_{n+1|n}$. Therefore

$$Ex_{n+1}e_{n+1}^T = P_{n+1|n}C_{n+1}^T. \tag{42}$$

Substituting (41) and (42) into (37) we find that

$$\hat{x}_{n+1} = \hat{x}_{n+1|n} + K_{n+1}(y_{n+1} - \hat{y}_{n+1|n}) = \hat{x}_{n+1|n} + K_{n+1}(y_{n+1} - C_{n+1}\hat{x}_{n+1|n}) \tag{43}$$

where K_{n+1} is a Kalman gain defined by

$$K_{n+1} = P_{n+1|n}C_{n+1}^T(C_{n+1}P_{n+1|n}C_{n+1}^T + R_{n+1})^{-1}. \tag{44}$$

4. Linear Kalman filter

Now we consider the LMMSE filtration problem of the ith local controller and the coordinator for the system described by (1) and (2) in the structure presented in the Section 2.

4.1. Filtering equations for the ith local controller

Let us consider the model of the system (1).

In eqn. (1) control u_n^i has the form (7) and can by written as

$$u_n^i = p_n^{*i} - L_n^i\hat{x}_n^i = a_n^{*i}(\vec{y}_n^i, \vec{m}_n) \tag{45}$$

where $p_n^{*i} = p_n^i + L_n^i\bar{x}_n^i = b_n^i(\vec{m}_n)$.

The value of the coordinating variable p_n^{*i} is determined by the coordinator and sent to the ith controller.

The whole system (1) and the model of the ith measurement (2) can be described by the equations

$$x_{n+1} = A_n x_n + B_n^d a_n^* + w_n \tag{46}$$

$$y_n^i = C_n^{*i} x_n + r_n^i \tag{47}$$

where $x_n = [x_n^{1T}, ..., x_n^{MT}]^T$, $a_n^* = [a_n^{*1T}(.), .., a_n^{*MT}(.)]^T$, $w_n = [w_n^{1T}, ..., w_n^{MT}]^T$, $C_n^{*i} = [0, ..0, C_n^i, 0, ..0]$ and the forms of the matrices A_n, B_n^d result from (1).

Let $\hat{x}_{n+1}$ be a LMMSE estimate of x_{n+1} given $\{\overbrace{\vec{y}_n^i, \vec{m}_n}^{\vec{y}_n^{*i}}, y_{n+1}^i\}$.
According to (43) we have

$$\hat{x}_{n+1} = \hat{x}_{n+1|n} + K_{n+1}^{*i}(y_{n+1}^i - C_{n+1}^{*i}\hat{x}_{n+1|n}) \tag{48}$$

or

$$\hat{x}_{n+1}^i = \hat{x}_{n+1|n}^i + K_{n+1}^i(y_{n+1}^i - C_{n+1}^i\hat{x}_{n+1|n}^i), \quad i = 1, ...M. \tag{49}$$

The matrix K_{n+1}^i in (49) is an appropriate block of the matrix K_{n+1}^{*i}.
The error $\tilde{x}_{n+1|n}$ should be orthogonal to $\vec{y}_n^{*i}$, i.e.

$$E(x_{n+1} - \hat{x}_{n+1|n})\vec{y}_n^{*iT} = 0. \tag{50}$$

Let $\hat{x}_{n|n}$ be a LMMSE estimate of x_n given $\vec{y}_n^{*i}$.
Thus, from (46) and since $x_n = \hat{x}_{n|n} + \tilde{x}_{n|n}$, we have

$$x_{n+1} = A_n\hat{x}_{n|n} + A_n\tilde{x}_{n|n} + B_n^d a_n^* + w_n. \tag{51}$$

Then, (50) becomes

$$E(A_n\hat{x}_{n|n} + B_n^d a_n^* - \hat{x}_{n+1|n})\vec{y}_n^{*iT} + E(A_n\tilde{x}_{n|n})\vec{y}_n^{*iT} + Ew_n\vec{y}_n^{*iT} = 0. \tag{52}$$

Since $\hat{x}_{n|n}$ is LMMSE estimate of x_n given $\vec{y}_n^{*i}$ and w_n is independent of $\vec{y}_n^{*i}$, the last two terms in (52) are equal to zero.
Therefore

$$E(A_n\hat{x}_{n|n} + B_n^d a_n^* - \hat{x}_{n+1|n})\vec{y}_n^{*iT} = 0. \tag{53}$$

From (53) we have for $j = 1, ..., M$

$$E(A_n^{jj}\hat{x}_{n|n}^j + \sum_{k\neq j}^{M} A_n^{jk}\hat{x}_{n|n}^k + B_n^j a_n^{*j} - \hat{x}_{n+1|n}^j)\vec{y}_n^{*iT} = 0. \tag{54}$$

For $j = i$ we obtain

$$E(A_n^{ii}\hat{x}_{n|n}^i + \sum_{k\neq i}^{M} A_n^{ik}\hat{x}_{n|n}^k + B_n^i a_n^{*i} - \hat{x}_{n+1|n}^i)\vec{y}_n^{*iT} = \tag{55}$$

$$= E[E_{|\vec{y}_n^{*i}}(A_n^{ii}\hat{x}_{n|n}^i + \sum_{k\neq i}^{M} A_n^{ik}\hat{x}_{n|n}^k + B_n^i \overbrace{a_n^{*i}(\vec{y}_n^{*i})}^{u_n^i} - \hat{x}_{n+1|n}^i)\vec{y}_n^{*iT}] = 0. \tag{56}$$

Eqn. (56) is satisfied when

$$\hat{x}^i_{n+1|n} = A^{ii}_n \hat{x}^i_{n|n} + \sum_{k\neq i}^{M} A^{ik}_n \hat{x}^k_{n|n} + B^i_n u^i_n = A^{ii}_n \hat{x}^i_{n|n} + \hat{v}^i_n + B^i_n u^i_n \tag{57}$$

for every realization of $\vec{y}^{*i}_n$.

Let us assume that the estimate $\hat{x}^i_{n+1|n}$ determined by the ith local controller is based on eqn. (57), where $\hat{x}^i_{n|n}$ is replaced by $\hat{x}^i_n$ and $\hat{v}^i_n$ by $\bar{v}^i_n$- an LMMSE estimate of the interaction v^i_n based on $\vec{m}_n$. This estimate is determined by the coordinator and sent to the ith local controller.

Let us remind that $\hat{x}^i_{n+1}$ is the LMMSE estimate of x_{n+1} given $\{\overbrace{\vec{y}^i_n, \vec{m}_n}^{\vec{y}^{*i}_n}, y^i_{n+1}\}$, while $\hat{x}^i_{n+1|n+1}$ is the LMMSE estimate of x_{n+1} given $\vec{y}^{*i}_{n+1}$.

Finally, the estimate $\hat{x}^i_{n+1}$ (used for real filtration) results from (49) with

$$\hat{x}^i_{n+1|n} = A^{ii}_n \hat{x}^i_n + B^i_n u^i_n + \bar{v}^i_n. \tag{58}$$

The form of K^{*i}_{n+1} results from (44)

$$K^{*i}_{n+1} = P^{*i}_{n+1|n} C^{*iT}_{n+1} (C^{*i}_{n+1} P^{*i}_{n+1|n} C^{*iT}_{n+1} + R^i_{n+1})^{-1} \tag{59}$$

where $R^i_{n+1} = E r^i_{n+1} r^{iT}_{n+1}$ and results from R_{n+1}.

Let us assume that $\hat{x}_{n+1|n}$ is the LMMSE estimate given $\vec{y}_n$. In this case we have

$$\tilde{x}_{n+1|n} = A_n \tilde{x}_n + w_n \tag{60}$$

and

$$P_{n+1|n} = E\tilde{x}_{n+1|n}\tilde{x}^T_{n+1|n} = A_n P_n A^T_n + W_n. \tag{61}$$

Basing on (61) it is proposed to use $P^{*i}_{n+1|n}$ in (59) as

$$P^{*i}_{n+1|n} = A_n P^{*i}_n A^T_n + W_n \tag{62}$$

where $P^{*i}_n = E\tilde{x}_n\tilde{x}^T_n = E(x_n - \hat{x}_n)(x_n - \hat{x}_n)^T$.

By substracting both sides of (48) from the identity $x_{n+1} = x_{n+1}$ we obtain

$$\begin{aligned} \tilde{x}_{n+1} &= \tilde{x}_{n+1|n} - K^{*i}_{n+1}(y^i_{n+1} - \hat{y}^i_{n+1|n}) = \\ &= (\mathbf{1} - K^{*i}_{n+1} C^{*i}_{n+1})\tilde{x}_{n+1|n} - K^{*i}_{n+1} r^i_{n+1}. \end{aligned} \tag{63}$$

Thus the covariance matrix P^{*i}_{n+1} has the form:

$$\begin{aligned} P^{*i}_{n+1} &= E\tilde{x}_{n+1}\tilde{x}^T_{n+1} = \\ &= (\mathbf{1} - K^{*i}_{n+1} C^{*i}_{n+1}) P^{*i}_{n+1|n} (\mathbf{1} - K^{*i}_{n+1} C^{*i}_{n+1})^T + K^{*i}_{n+1} R^i_{n+1} K^{*iT}_{n+1} = \\ &= P^{*i}_{n+1|n} - K^{*i}_{n+1} C^{*i}_{n+1} P^{*i}_{n+1|n} - P^{*i}_{n+1|n} C^{*iT}_{n+1} K^{*iT}_{n+1} + \\ &+ K^{*i}_{n+1}(C^{*i}_{n+1} P^{*i}_{n+1|n} C^{*iT}_{n+1} + R^i_{n+1}) K^{*iT}_{n+1} \end{aligned} \tag{64}$$

or, using (59),

$$P^{*i}_{n+1} = (\mathbf{1} - K^{*i}_{n+1} C^{*i}_{n+1}) P^{*i}_{n+1|n}. \tag{65}$$

Finally, $\hat{x}^i_{n+1}$ is computed from (49) and (58) with $\hat{x}^i_{0|-1} = Ex^i_0$ and K^i_{n+1} - an appropriate block matrix in K^{*i}_{n+1}. The value of control u^i_n is known and $\bar{v}^i_n$ is sent by the coordinator.

The matrix K^{*i}_{n+1} is calculated as follows.

For $P^{*i}_{0|-1} = X_0$ we can determine K^{*i}_0 from (59), P^{*i}_0 from (65), then $P^{*i}_{1|0}$ from (62), K^{*i}_1 from (59) and then P^{*i}_1 from (65), next $P^{*i}_{2|1}$ from (62), K^{*i}_2 from (59), and so on.

4.2. Filtering equations for the coordinator

Now we consider the LMMSE problem of the coordinator for the system described by the equation (46) and (4) for $i = 1, ..., M$.

Using (45) we can write (46) in the form

$$x_{n+1} = A_n x_n + B^d_n b_n - B^d_n L^d_n \hat{x}_n + w_n \tag{66}$$

where $b_n = [b^{1T}_n(\vec{m}_n), ..., b^{MT}_n(\vec{m}_n)]^T$ and $L^d_n = diag\{L^1_n, ..., L^M_n\}$.

It is assumed that an estimate of x_{n+1} at time $(n+1)$, determined by the coordinator, is based on measurement information $\vec{m}_{n+1}$.

Using (58) (1), (2) and (45) we can write (49) as

$$\begin{aligned}\hat{x}^i_{n+1} &= (A^{ii}_n + B^i_n L^i_n - K^i_{n+1} C^i_{n+1} A^{ii}_n)\hat{x}^i_n + K^i_{n+1} C^i_{n+1} A^{*i}_n x_n + B^i_n b^i_n + \\ &+ (\mathbf{1} - K^i_{n+1} C^i_{n+1})\bar{v}^i_n + K^i_{n+1} C^i_{n+1} w^i_n + K^i_{n+1} r^i_{n+1}, \quad i = 1, ..., M\end{aligned} \tag{67}$$

where $A^{*i}_n = [A^{i1}_n, , , , , A^{iM}_n]$.

Therefore, the estimate $\hat{x}_{n+1}$ is

$$\hat{x}_{n+1} = \hat{A}^x_n x_n + \hat{A}^{d\hat{x}}_n \hat{x}_n + B^d_n b_n(\vec{m}_n) + B^{d\bar{v}}_n \bar{v}_n + \Gamma^{dw}_n w_n + \Gamma^{dr}_{n+1} r_{n+1} \tag{68}$$

where $\hat{A}^x_n = vec\{K^i_{n+1} C^i_{n+1} A^{*i}_n\}$, $\hat{A}^{d\hat{x}}_n = diag\{A^{ii}_n + B^i_n L^i_n - K^i_{n+1} C^i_{n+1} A^{ii}_n\}$, $B^{d\bar{v}}_n = diag\{\mathbf{1} - K^i_{n+1} C^i_{n+1}\}$, $\Gamma^{dw}_n = diag\{K^i_{n+1} C^i_{n+1}\}$, $\Gamma^{dr}_{n+1} = diag\{K^i_{n+1}\}$ for $i = 1, ..., M$.

We can write (66) and (68) in block form as

$$x^u_{n+1} = A^u_n x^u_n + B^u_n b_n + B^{u\bar{v}}_n \bar{v}_n + \Gamma^{uw}_n w_n + \Gamma^{ur}_{n+1} r_{n+1} \tag{69}$$

where $x^u_n = [x^T_n, \hat{x}^T_n]^T$, $\bar{v}_n = [\bar{v}^{1T}_n, ..., \bar{v}^{MT}_n]^T$ and the forms of the matrices A^u_n, B^u_n, Γ^{uw}_n and Γ^{ur}_{n+1} result from (66) and (68).

The model of measurements for the coordinator results from (4) and can be written in the form

$$m_n = D_n y_n = D_n C_n x_n + D_n r_n = [D_n C_n \quad 0] x^u_n + D_n r_n = F^u_n x^u_n + D_n r_n \tag{70}$$

where $m_n = [m_n^{1T}, ..., m_n^{MT}]^T$, $y_n = [y_n^{1T}, ..., y_n^{mT}]^T$, $D_n = diag\{D_n^1, ..., D_n^M\}$, $C_n = diag\{C_n^1, .., C_n^M\}$.

The problem is to determine the LMMSE estimate $\bar{x}_n^u$ of the augmented state x_n^u given information $\vec{m}_n$.

From (43) we have

$$\bar{x}_{n+1}^u = \bar{x}_{n+1|n}^u + K_{n+1}^u(m_{n+1} - F_{n+1}^u \bar{x}_{n+1|n}^u). \tag{71}$$

The error $\tilde{x}_{n+1|n}^u$ should be orthogonal to $\vec{m}_n = [m_1^T, ..., m_n^T]^T$, i.e.

$$E(x_{n+1}^u - \bar{x}_{n+1|n}^u)\vec{m}_n^T = 0. \tag{72}$$

From (69) and since $x_n^u = \bar{x}_n^u + \tilde{x}_n^u$ we have

$$x_{n+1}^u = A_n^u \bar{x}_n^u + A_n^u \tilde{x}_n^u + B_n^u b_n + B_n^{u\bar{v}} \bar{v}_n + \Gamma_n^{uw} w_n + \Gamma_{n+1}^{ur} r_{n+1}. \tag{73}$$

Then (72) becomes

$$\begin{aligned} &E(A_n^u \bar{x}_n^u + B_n^u b_n + B_n^{u\bar{v}} \bar{v}_n - \bar{x}_{n+1|n}^u)\vec{m}_n^T + E(A_n^u \tilde{x}_n^u)\vec{m}_n^T + E(\Gamma_n^{uw} w_n)\vec{m}_n^T + \\ &+ E(\Gamma_{n+1}^{ur} r_{n+1})\vec{m}_n^T = 0. \end{aligned} \tag{74}$$

Since $\bar{x}_n^u$ is an LMMSE estimate of x_n^u given $\vec{m}_n$, w_n is independent of $\vec{m}_n$ and r_{n+1} is independent of $\vec{m}_n$, the last three terms in (74) are equal to zero. Therefore

$$\begin{aligned} &E(A_n^u \bar{x}_n^u + B_n^u b_n + B_n^{u\bar{v}} \bar{v}_n - \bar{x}_{n+1|n}^u)\vec{m}_n^T = EE_{|\vec{m}_n}\{[A_n^u \bar{x}_n^u + B_n^u \overbrace{b_n(\vec{m}_n)}^{p_n^*} + \\ &+ B_n^{u\bar{v}} \bar{v}_n - \bar{x}_{n+1|n}^u]\vec{m}_n^T\} = 0. \end{aligned} \tag{75}$$

Eqn. (75) is satisfied when

$$\bar{x}_{n+1|n}^u = A_n^u \bar{x}_n^u + B_n^u p_n^* + B_n^{u\bar{v}} \bar{v}_n \tag{76}$$

for every realization of $\vec{m}_n$.

From (44) and (70) we have

$$K_{n+1}^u = P_{n+1|n}^u F_{n+1}^{uT}(F_{n+1}^u P_{n+1|n}^u F_{n+1}^{uT} + D_{n+1} R_{n+1} D_{n+1}^T)^{-1} \tag{77}$$

where R_{n+1} is a covariance matrix of r_{n+1}.

Now from (69) and (76) we have

$$\tilde{x}_{n+1|n}^u = A_n^u \tilde{x}_n^u + \Gamma_n^{uw} w_n + \Gamma_{n+1}^{ur} r_{n+1}. \tag{78}$$

Thus

$$P_{n+1|n}^u = E\tilde{x}_{n+1|n}^u \tilde{x}_{n+1|n}^{uT} = A_n^u P_n^u A_n^{uT} + \Gamma_n^{uw} W_n \Gamma_n^{uwT} + \Gamma_{n+1}^{ur} R_{n+1} \Gamma_{n+1}^{urT} \tag{79}$$

where $P_n^u = E\tilde{x}_n^u \tilde{x}_n^{uT}$.

From (71) and (70) we have that

$$\tilde{x}_{n+1}^u = x_{n+1}^u - \bar{x}_{n+1}^u = \tilde{x}_{n+1|n}^u - K_{n+1}^u(m_{n+1} - \bar{m}_{n+1|n}) = \\ = (\mathbf{1} - K_{n+1}^u F_{n+1}^u)\tilde{x}_{n+1|n}^u - K_{n+1}^u D_{n+1} r_{n+1}. \tag{80}$$

The covariance matrix P_{n+1}^u has the form

$$P_{n+1}^u = (\mathbf{1} - K_{n+1}^u F_{n+1}^u) P_{n+1|n}^u (\mathbf{1} - K_{n+1}^u F_{n+1}^u)^T + \\ + K_{n+1}^u D_{n+1} R_{n+1} D_{n+1}^T K_{n+1}^{uT} - \\ - (\mathbf{1} - K_{n+1}^u F_{n+1}^u) E\tilde{x}_{n+1|n}^u r_{n+1}^T D_{n+1}^T K_{n+1}^{uT} - \\ - K_{n+1}^u D_{n+1} E r_{n+1} \tilde{x}_{n+1|n}^{uT} (\mathbf{1} - K_{n+1}^u F_{n+1}^u)^T. \tag{81}$$

From (78) it results that

$$E\tilde{x}_{n+1|n}^u r_{n+1}^T = \Gamma_{n+1}^{ur} R_{n+1}. \tag{82}$$

Thus the covariance matrix P_{n+1}^u fulfils the equation

$$P_{n+1}^u = (\mathbf{1} - K_{n+1}^u F_{n+1}^u) P_{n+1|n}^u (\mathbf{1} - K_{n+1}^u F_{n+1}^u)^T + \\ + K_{n+1}^u D_{n+1} R_{n+1} D_{n+1}^T K_{n+1}^{uT} - \\ - (\mathbf{1} - K_{n+1}^u F_{n+1}^u) \Gamma_{n+1}^{ur} R_{n+1} D_{n+1}^T K_{n+1}^{uT} - \\ - K_{n+1}^u D_{n+1} R_{n+1} \Gamma_{n+1}^{urT} (\mathbf{1} - K_{n+1}^u F_{n+1}^u)^T. \tag{83}$$

We substitute (77) into the second term in (83). Thus

$$P_{n+1}^u = (\mathbf{1} - K_{n+1}^u F_{n+1}^u) P_{n+1|n}^u - \\ - (\mathbf{1} - K_{n+1}^u F_{n+1}^u) \Gamma_{n+1}^{ur} R_{n+1} D_{n+1}^T K_{n+1}^{uT} - \\ - K_{n+1}^u D_{n+1} R_{n+1} \Gamma_{n+1}^{urT} (\mathbf{1} - K_{n+1}^u F_{n+1}^u)^T. \tag{84}$$

Finally, $\bar{x}_{n+1}^u$ is computed from (71) and (76) with $\bar{x}_{0|-1}^u = Ex_0^u$ and K_{n+1}^u defined by (77). The values of p_n^* and $\bar{v}_n$ are known.

The matrix K_{n+1}^u is calculated as follows.

For $P_{0|-1}^u = diag\{X_0, X_0\}$ we can determine K_0^u from (77), P_0^u from (84) and $P_{1|0}^u$ from (79), K_1^u from (77) and next P_1^u from (84), $P_{2|1}^u$ from (79), K_2^u from (77), and so on.

According to the notation $x_n^u = [x_n^T, \hat{x}_n^T]^T$ only $\bar{x}_n$in $\bar{x}_n^u$ is needed for real filtration.

5. Example

Let us consider an autonomous, stationary system composed of two subsystems with

$$A_n^{11} = \begin{bmatrix} 0.083 & -033 & -0.045 \\ 0.450 & 0.790 & -0.030 \\ 0.300 & 0.922 & -0.001 \end{bmatrix}, \quad A_n^{12} = \begin{bmatrix} 0.119 & -0.016 & -0.106 & 0.053 \\ -0.120 & -0.160 & 0.142 & 0.022 \\ -0.002 & 0.026 & -0.080 & -0.092 \end{bmatrix}, \tag{85}$$

$$A_n^{21} = \begin{bmatrix} -0.035 & -0.072 & 0.040 \\ 0.043 & 0.029 & 0.036 \\ 0.063 & -0.020 & 0.065 \\ -0.080 & 0.035 & 0.033 \end{bmatrix}, \quad A_n^{22} = \begin{bmatrix} 0.272 & -0.230 & -0.033 & -0.002 \\ 0.597 & 0.867 & -0.020 & -0.001 \\ 0.359 & 0.952 & 0.993 & -0.001 \\ 0.130 & 0.488 & 0.998 & 0.999 \end{bmatrix}, \tag{86}$$

$$C_n^1 = \begin{bmatrix} 0 & 0 & 0.005 \\ 0 & 0.85 & 0 \end{bmatrix}, \quad C_n^2 = \begin{bmatrix} 0 & 0 & 0 & 0.003 \\ 0 & 0.045 & 0 & 0.001 \\ 0 & 0 & 0.15 & 0 \end{bmatrix}, \tag{87}$$

$$R_n = \begin{bmatrix} 0.05 & 0.015 & 0 & 0 & 0 \\ 0.015 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0.019 & 0.002 & 0.002 \\ 0 & 0 & 0.002 & 0.040 & 0.004 \\ 0 & 0 & 0.002 & 0.004 & 0.030 \end{bmatrix}, \tag{88}$$

$$W_n = \begin{bmatrix} 0.272 & 0.102 & 0.029 & 0 & 0 & 0 & 0 \\ 0.102 & 0.107 & 0.045 & 0 & 0 & 0 & 0 \\ 0.029 & 0.045 & 0.022 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.800 & 0.356 & 0.112 & 0.027 \\ 0 & 0 & 0 & 0.356 & 0.316 & 0.128 & 0.036 \\ 0 & 0 & 0 & 0.112 & 0.128 & 0.058 & 0.017 \\ 0 & 0 & 0 & 0.027 & 0.036 & 0.017 & 0.005 \end{bmatrix}, \tag{89}$$

$$X_0 = \begin{bmatrix} 0.4 & 0 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0.03 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.62 & 0 & -0.41 & 0 \\ 0 & 0 & 0 & 0 & 0.41 & 0 & 0.77 \\ 0 & 0 & 0 & -0.41 & 0 & 0.77 & 0 \\ 0 & 0 & 0 & 0 & -0.77 & 0 & 2.03 \end{bmatrix}. \tag{90}$$

Let us assume that for the system

$$x_{n+1} = \begin{bmatrix} x_{n+1}^1 \\ x_{n+1}^2 \end{bmatrix} = \begin{bmatrix} A_n^{11} & A_n^{12} \\ A_n^{21} & A_n^{22} \end{bmatrix} \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} + \begin{bmatrix} w_n^1 \\ w_n^2 \end{bmatrix} \tag{91}$$

$$y_n = \begin{bmatrix} y_n^1 \\ y_n^2 \end{bmatrix} = \begin{bmatrix} C_n^1 & \mathbf{0_1} \\ \mathbf{0_2} & C_n^2 \end{bmatrix} \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} + \begin{bmatrix} r_n^1 \\ r_n^2 \end{bmatrix} \tag{92}$$

the estimates result from a classical Kalman filter i.e.

$$\hat{x}_{n+1} = \hat{x}_{n+1|n} + K_{n+1}(y_{n+1} - \hat{y}_{n+1|n}). \tag{93}$$

Thus, the estimates $\hat{x}_n^1$ and $\hat{x}_n^2$ are based on the information $[y_0^T, ..., y_n^T]^T$ and result from (93).

The estimates $\hat{x}_n^1$ and $\hat{x}_n^2$, determined from the decentralized Kalman filter, result from (49) and depend on local measurement information.

For illustration we present the estimation errors $(x_n^{11} - \hat{x}_n^{11})$ (Fig.1) and $(x_n^{21} - \hat{x}_n^{21})$ (Fig.2) for the first state components of both subsystems $(x_n^{i1}, i = 1, 2)$ resulting from the classical Kalman filter and the decentralized one for $D = \mathbf{1}$.

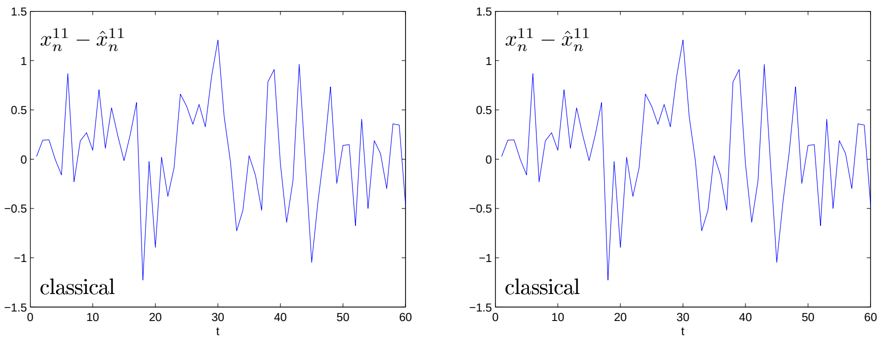

Figure 1. Estimation errors for classical and decentralized Kalman filters for the first subsystem

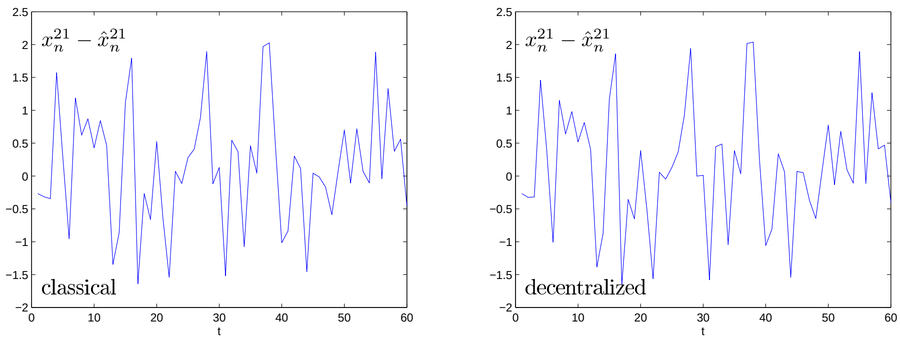

Figure 2. Estimation errors for classical and decentralized Kalman filters for the second subsystem

The results show that the assumption of replacement of interaction estimate $\hat{v}_n^i$ by $\bar{v}_n^i$ does not significantly worsen the estimation.

Figs. 3 and 4 show the estimation errors $(x_n^{i1} - \bar{x}_n^{i1})$, $(x_n^{i1} - \hat{x}_n^{i1})$ for $i = 1, 2$ and for $D_n^1 = [0\ 1]$, $D_n^2 = [\ 1\ 0\ 0]$.

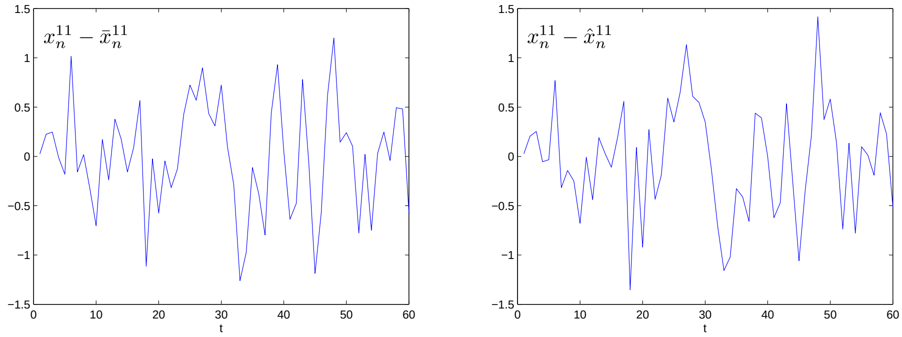

Figure 3. Estimation errors for the first subsystem and $D_n^1 = [0\ 1]$, $D_n^2 = [1\ 0\ 0]$.

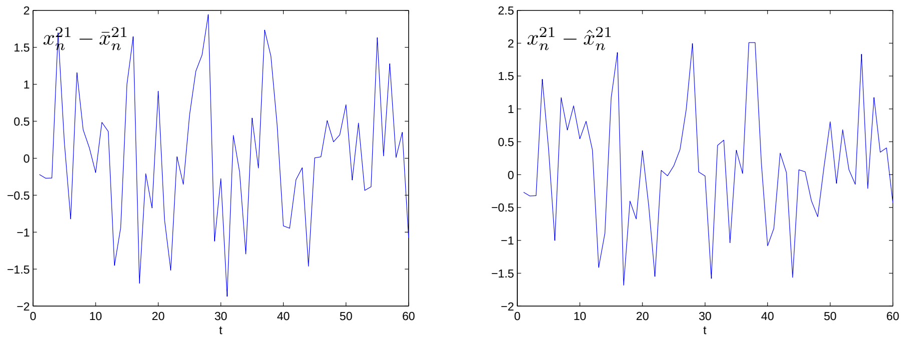

Figure 4. Estimation errors for the second subsystem and $D_n^1 = [0\ 1]$, $D_n^2 = [1\ 0\ 0]$.

Comparing Fig. 1 to Fig. 3 and Fig. 2 to Fig. 4 we can analyse the influence of aggregation on filtration quality for the subsystems. For given data the proposed decentralized filter may be accepted.

Conclusion

In this paper the linear state Kalman filter for the dynamic system controlled in the two-level control and information structures is derived. The approach is based on the modified innovations and leads to the decentralized filters for the local controllers and augmented optimal filter for the coordinator, whose implementation can be computationally intensive. Some methods reducing the computational cost are known in the literature (see Chien and Fu, 1999) and may be applied to the problem considered.

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