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Supertypes of voters in a model of general elections*†

by

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Abstract: The paper examines a model of general elections with electorate composed of infinitely many voters classified into a finite number of types. We focus on the problem of aggregation of different types of voters into supertypes for two cases of voting: with and without the possibility of abstention.

Keywords: general elections, preference-indifference relations, supertypes of voters.

1. Introduction

In the present paper we examine a model of general elections with electorate composed of infinitely many voters classified into a finite number of types. It is a continuation of the paper, Ekes (2003), where we have analysed the model of general elections, introduced in Ekes (1999). The model we describe here can be considered as an application of the concept of large games with small players having finite sets of strategies, introduced in Wieczorek (2004). Here we focus on the problem of aggregation of different types of voters in order to simplify and clarify examination of the model. We give full classification of supertypes in case of voting for one of two candidates without the possibility of abstention. We also examine the case of voting with the possibility of abstention.

Models of elections with infinitely many voters have also been considered in Wiszniewska-Matyszkiel (2002).

2. Description of the model

In the present paper we deal with the model of general elections (such as referendum or presidential elections) in which the electorate has to choose, by voting, one of a fixed number of options, possibly one of them being abstention.

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Formally, the electorate is choosing an element out of the set $K = \{1, \dots, k\}$ or $K = \{0, 1, \dots, k\}$ if abstention, denoted by 0, is allowed and taken into consideration. The electorate generates a distribution $(P_1, \dots, P_k)$ or $(P_0, P_1, \dots, P_k)$ on K and the *winner* is the unique element of $\{1, \dots, k\}$ with largest corresponding P_j; if there is no such unique element, we say that the elections end up with a *draw* and we denote this result by D, so the set of all outcomes is $\mathcal{O} = \{D, 1, \dots, k\}$. Since the electorate may generate any distribution, we must see it as infinite. Moreover we assume that each single voter is negligible, i.e. that his individual decision has no influence on the result of elections. As usual, members of the electorate should have some preferences, which do not apply only to the results of the elections but also to their individual behaviour, so that each member of the electorate has a pre-ordering relation on the set being the product of the set of all options and the set of all outcomes, i.e. on $K \times \mathcal{O}$. Obviously, this set has $(k+1)^2$ or $k(k+1)$ elements, depending on whether abstention is permitted or not. The number of possible pre-ordering relations is then very large; even for $k = 2$ there are $4,683$ pre-ordering relations in case without abstention and $7,087,261$ relations in case with abstention allowed.

In Ekes (2003) we have chosen a few "reasonable" preference-indifference relations among electorate and we have examined the behaviour of voters characterised by those preferences. In the present paper we are interested in aggregating voters of, presumably, different preferences according to their behaviour at equilibrium. Hence, the whole electorate is divided into n populations, differing in their preferences; the size of the i-th population $(i = 1, \dots, n)$, having a preference-indifference relation $\succsim_i$, is denoted by $q_i \geq 0$ (as usually we denote by $\succ_i$ the preference relation and by $\sim_i$ the indifference relation, both generated by $\succsim_i$). The i-th population generates in the course of elections a distribution p^i on K. Formally, p^i is an element of the standard simplex of dimension k or $k-1$, depending on the case (this simplex is denoted by $\Delta_{|K|}$). Consequently, a sequence of distributions of the decisions of all respective types, $\mathbf{p} = \left(p^1, \dots, p^n\right)$, which is a sequence of n elements of $\Delta_{|K|}$, generates a distribution of the votes in the whole electorate. For the j-th option to be chosen ($j = 1, \dots, k$ if abstention is not allowed and $j = 0, 1, \dots, k$ if abstention is allowed), we have then $P_j = Q^{-1} \cdot \sum_{i=1}^{n} q_i p_j^i$, where Q denotes $\sum_{i=1}^{n} q_i$. We say that the j-th option is winning at the elections if $P_j > P_l$ for all $l = 1, \dots, k$, $l \neq j$. If there exist at least two different options j and j' such that $P_j = P_{j'} = \max_{l=1,\dots,k} P_l$, then the elections end up with a draw. Observe that each sequence of distributions of the voters' decisions $\mathbf{p}$ uniquely determines the outcome of the elections, denoted by $x_{\mathbf{p}} \in \mathcal{O}$. We say that the sequence of distributions $\mathbf{p}$ is at *equilibrium* whenever, for $i = 1, \dots, n$ and $j \in \left\{m \in K \mid p_m^i > 0\right\}$ the following condition

$$(j, x_{\mathbf{p}}) \succsim_i (l, x_{\mathbf{p}})$$

holds for all $l \in K$, which informally means that no voters could improve their satisfaction by changing their individual decision on how to vote.

In the sequel we consider two cases - voting with and without the possibility of abstention. We will describe the method of aggregating different types of voters into supertypes by skipping some voters' characteristics which are not necessary to describe their behaviour at equilibrium.

3. The case of voting for one of two candidates without abstention

Consider the case of voting for one of two candidates, who are denoted here by A and B for convenience. Each voter has to decide which candidate to vote for; abstention is not allowed. Therefore the set K has the form $K = \{A, B\}$. The set of outcomes is then $\mathcal{O} = \{D, A, B\}$, where D denotes draw, A denotes that A is the winner of elections and B denotes that B is the winner. In this setting there are six pairs consisting of an individual decision and an outcome of the elections. We enumerate them in the following way:

1. vote for A & A wins;
2. vote for A & a draw;
3. vote for A & B wins;
4. vote for B & A wins; (1)
5. vote for B & a draw;
6. vote for B & B wins.

Assume that all voters are characterised by strict preferences. Then, if there are no additional assumptions, we have to consider $6! = 720$ types of voters. Let $\mathbf{p} = \left(p^1, \ldots, p^{720}\right)$ be a sequence of distributions of decisions of voters of all types, i.e. $p^i = \left(p^i_A, p^i_B\right)$, $p^i_A + p^i_B = 1$, p^i_A, $p^i_B \geq 0$ for $i = 1, \ldots, 720$. Denote by P_A the fraction of votes won by the candidate A, which is the number $P_A = Q^{-1} \cdot \sum_{i=1}^{720} q_i p^i_A$ and, similarly, $P_B = Q^{-1} \cdot \sum_{i=1}^{720} q_i p^i_B$, where $Q = \sum_{i=1}^{720} q_i$ denotes the size of the whole electorate. Therefore, one of the following cases may be the *result* of the elections:

(A) $P_A > P_B$, i.e. the candidate A wins the elections;
(B) $P_A < P_B$, i.e. the candidate B wins the elections; (2)
(D) $P_A = P_B$, i.e. a draw occurs.

In order to find equilibria we need to check what are optimal decisions of voters of type i ($i = 1, \ldots, 720$), in cases $(A), (B), (D)$, respectively. Consider the case (A). Observe that if voters of a given type prefer the pair **1** to **4** then they all vote for A in this case. In other case (that is if they prefer the pair **4** to **1**) they all vote for B. If we consider the case (B), that is if the candidate B wins, then the behaviour of voters at equilibrium depends on whether they prefer the pair **3** rather than **6** or the opposite. And finally in the case (D) the behaviour of voters is defined by the ordering of pairs **2** and **5**. Therefore, when characterising equilibria, we have to take into consideration the following eight different sets of conditions:

$$
\begin{array}{rl}
\mathrm{I} & (\mathbf{1} \succ_i \mathbf{4}) \wedge (\mathbf{2} \succ_i \mathbf{5}) \wedge (\mathbf{3} \succ_i \mathbf{6}); \\
\mathrm{II} & (\mathbf{1} \succ_i \mathbf{4}) \wedge (\mathbf{2} \succ_i \mathbf{5}) \wedge (\mathbf{6} \succ_i \mathbf{3}); \\
\mathrm{III} & (\mathbf{1} \succ_i \mathbf{4}) \wedge (\mathbf{5} \succ_i \mathbf{2}) \wedge (\mathbf{3} \succ_i \mathbf{6}); \\
\mathrm{IV} & (\mathbf{1} \succ_i \mathbf{4}) \wedge (\mathbf{5} \succ_i \mathbf{2}) \wedge (\mathbf{6} \succ_i \mathbf{3}); \\
\mathrm{V} & (\mathbf{4} \succ_i \mathbf{1}) \wedge (\mathbf{2} \succ_i \mathbf{5}) \wedge (\mathbf{3} \succ_i \mathbf{6}); \\
\mathrm{VI} & (\mathbf{4} \succ_i \mathbf{1}) \wedge (\mathbf{2} \succ_i \mathbf{5}) \wedge (\mathbf{6} \succ_i \mathbf{3}); \\
\mathrm{VII} & (\mathbf{4} \succ_i \mathbf{1}) \wedge (\mathbf{5} \succ_i \mathbf{2}) \wedge (\mathbf{3} \succ_i \mathbf{6}); \\
\mathrm{VIII} & (\mathbf{4} \succ_i \mathbf{1}) \wedge (\mathbf{5} \succ_i \mathbf{2}) \wedge (\mathbf{6} \succ_i \mathbf{3}).
\end{array}
\tag{3}
$$

Consider, for example, conditions given by I. If preferences of voters of a type i satisfy them, then at equilibrium in all cases $(A), (B), (D)$ all voters of this type vote for A - we could call such voters strict supporters of the candidate A. Similarly conditions given by VIII describe strict supporters of the candidate B. If we consider voters with preferences satisfying conditions given by IV, then we observe that at equilibrium they vote for A if A wins and they vote for B if B wins or if there is a draw, so we could call them opportunists.

Note that conditions I - VIII divide the set of types of voters into subsets, which are pairwise disjoint and sum up to the whole set of 720 types. Moreover, the behaviour of voters in a given subset at equilibrium is identical. Therefore, we will aggregate types of voters into eight *supertypes,* described by the conditions I - VIII. Denote by $\mathbf{p}$ the sequence of distributions of decisions of voters of all supertypes, i.e.
$\mathbf{p} = (p^I, p^{II}, ..., p^{VIII})$. Then we have a following theorem:

THEOREM 1 *A sequence of distributions* $\mathbf{p}$ *is at equilibrium if and only if*

$$p_A^I = p_A^{II} = p_A^{III} = p_A^{IV} = p_B^V = p_B^{VI} = p_B^{VII} = p_B^{VIII} = 1$$

and the case (A) *occurs*

or

$$p_A^I = p_A^{III} = p_A^V = p_A^{VII} = p_B^{II} = p_B^{IV} = p_B^{VI} = p_B^{VIII} = 1$$

and the case (B) *occurs*

or

$$p_A^I = p_A^{II} = p_A^V = p_A^{VI} = p_B^{III} = p_B^{IV} = p_B^{VII} = p_B^{VIII} = 1$$

and the case (D) *occurs.*

Denote by $q^s = (q_I^s, ..., q_{VIII}^s)$ the vector of sizes of respective supertypes (or rather the vector of shares of respective supertypes in the whole electorate). After some transformations, made by using formulas for P_A and P_B and together with (2), we obtain the following conditions, describing the sizes of populations of voters in the electorate:

$$
\begin{array}{ll}
(A') & q_I^s + q_{II}^s + q_{III}^s + q_{IV}^s > q_V^s + q_{VI}^s + q_{VII}^s + q_{VIII}^s; \\
(B') & q_{II}^s + q_{IV}^s + q_{VI}^s + q_{VIII}^s > q_I^s + q_{III}^s + q_V^s + q_{VII}^s; \\
(D') & q_I^s + q_{II}^s + q_V^s + q_{VI}^s = q_{III}^s + q_{IV}^s + q_{VII}^s + q_{VIII}^s.
\end{array}
$$

If q^s satisfies inequality (A'), then equilibrium at which candidate A wins is obtainable. Similarly, if q^s satisfies inequality (B'), then it is possible that the candidate B wins at equilibrium. And finally if (D') holds, then a draw may arise at equilibrium. Let us take for example $q^s_I = q^s_{III} = \frac{4}{32}$, $q^s_{II} = q^s_{IV} = \frac{6}{32}$, $q^s_V = q^s_{VII} = \frac{1}{32}$, $q^s_{VI} = q^s_{VIII} = \frac{5}{32}$. This distribution of sizes satisfies all inequalities (A'), (B') and (D'), so each result of elections is possible at equilibrium. On the other hand, if we take $q^s_I = q^s_{II} = q^s_{III} = q^s_{IV} = q^s_{VI} = q^s_{VIII} = \frac{3}{32}$, $q^s_V = \frac{6}{32}$, $q^s_{VII} = \frac{8}{32}$, then no equilibrium is possible.

4. The case of voting for one of two candidates with abstention

Now each voter can vote for the candidate A, or the candidate B or may abstain from voting. Therefore the set K has the form $K = \{A, B, 0\}$, where 0 denotes abstention. The set of outcomes has not changed; it is $\mathcal{O} = \{D, A, B\}$, where D denotes a draw, A denotes that A is the winner of elections and B denotes that B is the winner. In this framework there are nine pairs consisting of an individual decision and an outcome of the elections. We enumerate them in the following way:

1. vote for A & A wins;
2. vote for A & a draw;
3. vote for A & B wins;
4. vote for B & A wins; (4)
5. vote for B & a draw;
6. vote for B & B wins;
7. abstain from voting & A wins;
8. abstain from voting & a draw;
9. abstain from voting & B wins.

In this case there exist $9! = 362,880$ strict preferences, which can be represented in the electorate. As before, let us denote by $\mathbf{p}$ a sequence of distributions of decisions of voters of all types, i.e. $p^i = (p^i_A, p^i_B, p^i_0)$, $p^i_A + p^i_B + p^i_0 = 1$, p^i_A, p^i_B, $p^i_0 \geq 0$ for all types i. Denote $P_A = Q^{-1} \cdot \sum_{i=1}^{9!} q_i p^i_A$, similarly, $P_B = Q^{-1} \cdot \sum_{i=1}^{9!} q_i p^i_B$ and $P_0 = Q^{-1} \cdot \sum_{i=1}^{9!} q_i p^i_0$, where $Q = \sum_{i=1}^{9!} q_i$. The result of elections is described in the same way as before, that is

(A) $P_A > P_B$, i.e. the candidate A wins the elections;
(B) $P_B > P_A$, i.e. the candidate B wins the elections; (5)
(D) $P_A = P_B$, i.e. a draw occurs.

The behaviour of different types of voters at equilibria can be examined in the same way as in the previous section. We conclude, that it depends on the ordering of the following triples of pairs defined in (4): $(\mathbf{1}, \mathbf{4}, \mathbf{7})$; $(\mathbf{2}, \mathbf{5}, \mathbf{8})$ and $(\mathbf{3}, \mathbf{6}, \mathbf{9})$. In order to describe the equilibrium behaviour of voters we only need to know which pair is most preferred in each triple, therefore we have 27 different possibilities, which describe supertypes in this case. We will not give the full description of equilibria here - it can be done in the very same way as in the previous section.

Note that the size of population of voters who abstained has no influence on the result of voting in this setting. It can be modified, e.g. by making the outcome of the elections dependent on the percentage of electorate casting votes in the elections. We should then assume that if the number P_0, denoting the fraction of voters in the whole electorate who decide to abstain, exceeds a given threshold, then the elections will not be decisive (there will be no winner). The outcome, denoted by D and called a draw, can also describe this situation. Therefore we can choose a threshold t and define the result of elections as follows:

(A) $(P_A > P_B) \wedge (P_0 < tQ)$, i.e. the candidate A wins the elections;
(B) $(P_B > P_A) \wedge (P_0 < tQ)$, i.e. the candidate B wins the elections;
(D) $(P_A = P_B) \vee (P_0 \geq tQ)$, i.e. a draw occurs.

5. The case of preference-indifference relations

Till now we have considered only strict preferences, but our results can be easily applied to the larger set of preferences. Observe that in case without abstention the equilibrium behaviour of voters whose preferences allow for indifference (are not strict) can be identical to the behaviour of voters of some of previously defined supertypes. The only restriction is that for a voter the pair **1** cannot be indifferent to the pair **4**, the pair **2** cannot be indifferent to the pair **5** and finally the pair **3** cannot be indifferent to the pair **6**. If preferences satisfy these conditions, then we can include such type of voters to the corresponding supertype.

If a voter is indifferent to at least one of pairs discussed above, then his behaviour at equilibrium cannot be precisely determined. Consider for example a following ordering of alternatives: $\mathbf{1} \sim \mathbf{4} \succ \mathbf{2} \succ \mathbf{3} \succ \mathbf{5} \succ \mathbf{6}$. Then, at equilibrium where the candidate A wins the elections, a voter with such preferences can either vote for the candidate A or for the candidate B. Observe that we can also aggregate different types of voters into supertypes in this case. We obtain 19 additional supertypes (12 different supertypes with one indifference among three concerned pairs, 6 different supertypes with two indifferences and one with three indifferences). Formally new supertypes have to satisfy the following conditions:

$$\begin{array}{rl}
\text{IX} & (\mathbf{1} \sim_i \mathbf{4}) \wedge (\mathbf{2} \succ_i \mathbf{5}) \wedge (\mathbf{3} \succ_i \mathbf{6}); \\
\text{X} & (\mathbf{1} \sim_i \mathbf{4}) \wedge (\mathbf{2} \succ_i \mathbf{5}) \wedge (\mathbf{6} \succ_i \mathbf{3}); \\
\text{XI} & (\mathbf{1} \sim_i \mathbf{4}) \wedge (\mathbf{5} \succ_i \mathbf{2}) \wedge (\mathbf{3} \succ_i \mathbf{6}); \\
\text{XII} & (\mathbf{1} \sim_i \mathbf{4}) \wedge (\mathbf{5} \succ_i \mathbf{2}) \wedge (\mathbf{6} \succ_i \mathbf{3}); \\
\text{XIII} & (\mathbf{2} \sim_i \mathbf{5}) \wedge (\mathbf{1} \succ_i \mathbf{4}) \wedge (\mathbf{3} \succ_i \mathbf{6}); \\
\text{XIV} & (\mathbf{2} \sim_i \mathbf{5}) \wedge (\mathbf{1} \succ_i \mathbf{4}) \wedge (\mathbf{6} \succ_i \mathbf{3}); \\
\text{XV} & (\mathbf{2} \sim_i \mathbf{5}) \wedge (\mathbf{4} \succ_i \mathbf{1}) \wedge (\mathbf{3} \succ_i \mathbf{6});
\end{array} \qquad (6)$$

XVI	$(\mathbf{2} \sim_i \mathbf{5}) \wedge (\mathbf{4} \succ_i \mathbf{1}) \wedge (\mathbf{6} \succ_i \mathbf{3})$;
XVII	$(\mathbf{3} \sim_i \mathbf{6}) \wedge (\mathbf{1} \succ_i \mathbf{4}) \wedge (\mathbf{2} \succ_i \mathbf{5})$;
XVIII	$(\mathbf{3} \sim_i \mathbf{6}) \wedge (\mathbf{1} \succ_i \mathbf{4}) \wedge (\mathbf{5} \succ_i \mathbf{2})$;
XIX	$(\mathbf{3} \sim_i \mathbf{6}) \wedge (\mathbf{4} \succ_i \mathbf{1}) \wedge (\mathbf{2} \succ_i \mathbf{5})$;
XX	$(\mathbf{3} \sim_i \mathbf{6}) \wedge (\mathbf{4} \succ_i \mathbf{1}) \wedge (\mathbf{5} \succ_i \mathbf{2})$;
XXI	$(\mathbf{1} \sim_i \mathbf{4}) \wedge (\mathbf{2} \sim_i \mathbf{5}) \wedge (\mathbf{3} \succ_i \mathbf{6})$;
XXII	$(\mathbf{1} \sim_i \mathbf{4}) \wedge (\mathbf{2} \sim_i \mathbf{5}) \wedge (\mathbf{6} \succ_i \mathbf{3})$;
XXIII	$(\mathbf{1} \sim_i \mathbf{4}) \wedge (\mathbf{3} \sim_i \mathbf{6}) \wedge (\mathbf{2} \succ_i \mathbf{5})$;
XXIV	$(\mathbf{1} \sim_i \mathbf{4}) \wedge (\mathbf{3} \sim_i \mathbf{6}) \wedge (\mathbf{5} \succ_i \mathbf{2})$;
XXV	$(\mathbf{2} \sim_i \mathbf{5}) \wedge (\mathbf{3} \sim_i \mathbf{6}) \wedge (\mathbf{1} \succ_i \mathbf{4})$;
XXVI	$(\mathbf{2} \sim_i \mathbf{5}) \wedge (\mathbf{3} \sim_i \mathbf{6}) \wedge (\mathbf{4} \succ_i \mathbf{1})$;
XXVII	$(\mathbf{1} \sim_i \mathbf{4}) \wedge (\mathbf{2} \sim_i \mathbf{5}) \wedge (\mathbf{3} \sim_i \mathbf{6})$.

Remark 1 *Observe that voters of the last supertype are in fact interested only in the result of elections; their own behaviour has no meaning for them.*

Therefore we have classified 4863 different types of voters into 27 supertypes. If we denote by $\mathbf{p}$ the sequence of distributions of decisions of voters of all supertypes, including new supertypes just defined, i.e. $\mathbf{p} = (p^I, p^{II}, ..., p^{XXVII})$, then we have the following theorem:

Theorem 2 *A sequence of distributions* $\mathbf{p}$ *is at equilibrium if and only if*

$$p_A^I = p_A^{II} = p_A^{III} = p_A^{IV} = p_A^{XIII} = p_A^{XIV} = p_A^{XVII} = p_A^{XVIII} = p_A^{XXV} = 1;$$
$$p_B^V = p_B^{VI} = p_B^{VII} = p_B^{VIII} = p_B^{XV} = p_B^{XVI} = p_B^{XIX} = p_B^{XX} = p_B^{XXVI} = 1;$$
$$p^{IX}, p^X, p^{XI}, p^{XII}, p^{XXI}, p^{XXII}, p^{XXIII}, p^{XXIV}, p^{XXVII} \text{ arbitrary}$$

and the case (A) *occurs or*

$$p_A^I = p_A^{III} = p_A^V = p_A^{VII} = p_A^{IX} = p_A^{XI} = p_A^{XIII} = p_A^{XV} = p_A^{XXI} = 1;$$
$$p_B^{II} = p_B^{IV} = p_B^{VI} = p_B^{VIII} = p_B^X = p_B^{XII} = p_B^{XIV} = p_B^{XVI} = p_B^{XXII} = 1;$$
$$p^{XVII}, p^{XVIII}, p^{XIX}, p^{XX}, p^{XXIII}, p^{XXIV}, p^{XXV}, p^{XXVI}, p^{XXVII} \text{ arbitrary}$$

and the case (B) *occurs or*

$$p_A^I = p_A^{II} = p_A^V = p_A^{VI} = p_A^{IX} = p_A^X = p_A^{XVII} = p_A^{XIX} = p_A^{XXIII} = 1;$$
$$p_B^{III} = p_B^{IV} = p_B^{VII} = p_B^{VIII} = p_B^{XI} = p_B^{XII} = p_B^{XVIII} = p_B^{XX} = p_B^{XXIV} = 1;$$
$$p^{XIII}, p^{XIV}, p^{XV}, p^{XVI}, p^{XXI}, p^{XXII}, p^{XXV}, p^{XXVI}, p^{XXVII} \text{ arbitrary.}$$

and the case (D) *occurs.*

Now equilibrium distribution is uniquely defined only for those voters, who are not indifferent to the result obtained in course of elections. Conditions, concerning sizes of populations of voters of different supertypes, implying the existence of a given kind of equilibrium depend now not only on the numbers

q_i^s but also on the actual distribution of decisions at equilibrium. Therefore we have:

(A'')

$$q_I^s + q_{II}^s + q_{III}^s + q_{IV}^s + q_{XIII}^s + q_{XIV}^s + q_{XVII}^s + q_{XVIII}^s + q_{XXV}^s + q_{IX}^s p_A^{IX} + \\ + q_X^s p_A^X + q_{XI}^s p_A^{XI} + q_{XII}^s p_A^{XII} + q_{XXI}^s p_A^{XXI} + q_{XXII}^s p_A^{XXII} + q_{XXIII}^s p_A^{XXIII} + \\ + q_{XXIV}^s p_A^{XXIV} + q_{XXVII}^s p_A^{XXVII} > \\ q_V^s + q_{VI}^s + q_{VII}^s + q_{VIII}^s + q_{XV}^s + q_{XVI}^s + q_{XIX}^s + q_{XX}^s + q_{XXVI}^s + \\ + q_{IX}^s p_B^{IX} + q_X^s p_B^X + q_{XI}^s p_B^{XI} + q_{XII}^s p_B^{XII} + q_{XXI}^s p_B^{XXI} + q_{XXII}^s p_B^{XXII} + \\ + q_{XXIII}^s p_B^{XXIII} + q_{XXIV}^s p_B^{XXIV} + q_{XXVII}^s p_B^{XXVII}$$

(B'')

$$q_I^s + q_{III}^s + q_V^s + q_{VII}^s + q_{IX}^s + q_{XI}^s + q_{XIII}^s + q_{XV}^s + q_{XXI}^s + q_{XVII}^s p_B^{XVII} + \\ + q_{XVIII}^s p_B^{XVIII} + q_{XIX}^s p_B^{XIX} + q_{XX}^s p_B^{XX} + q_{XXIII}^s p_B^{XXIII} + q_{XXIV}^s p_B^{XXIV} + \\ + q_{XXV}^s p_B^{XXV} + q_{XXVI}^s p_B^{XXVI} + q_{XXVII}^s p_B^{XXVII} > \\ q_{II}^s + q_{IV}^s + q_{VI}^s + q_{VIII}^s + q_X^s + q_{XII}^s + q_{XIV}^s + q_{XVI}^s + q_{XXII}^s + \\ + q_{XVII}^s p_A^{XVII} + q_{XVIII}^s p_A^{XVIII} + q_{XIX}^s p_A^{XIX} + q_{XX}^s p_A^{XX} + q_{XXIII}^s p_A^{XXIII} \\ + q_{XXIV}^s p_A^{XXIV} + q_{XXV}^s p_A^{XXV} + q_{XXVI}^s p_A + q_{XXVII}^s p_A^{XXVII}$$

(D'')

$$q_I^s + q_{II}^s + q_V^s + q_{VI}^s + q_{IX}^s + q_X^s + q_{XVII}^s + q_{XIX}^s + q_{XXIII}^s + q_{XIII}^s p_A^{XIII} + \\ + q_{XIV}^s p_A^{XIV} + q_{XV}^s p_A^{XV} + q_{XVI}^s p_A^{XVI} + q_{XXI}^s p_A^{XXI} + q_{XXII}^s p_A^{XXII} + \\ + q_{XXV}^s p_A^{XXV} + q_{XXVI}^s p_A^{XXVI} + q_{XXVII}^s p_A^{XXVII} = \\ = q_{III}^s + q_{IV}^s + q_{VII}^s + q_{VIII}^s + q_{XI}^s + q_{XII}^s + q_{XVIII}^s + q_{XX}^s + q_{XXIV}^s + \\ + q_{XIII}^s p_B^{XIII} + q_{XIV}^s p_B^{XIV} + q_{XV}^s p_B^{XV} + q_{XVI}^s p_B^{XVI} + q_{XXI}^s p_B^{XXI} + \\ + q_{XXII}^s p_B^{XXII} + q_{XXV}^s p_B^{XXV} + q_{XXVI}^s p_B^{XXVI} + q_{XXVII}^s p_B^{XXVII}.$$

Similar situation appears in case with abstention allowed. Now the condition for preference-indifference relation to belong to some of previously defined supertypes is that for each triple of pairs $(\mathbf{1}, \mathbf{4}, \mathbf{7})$; $(\mathbf{2}, \mathbf{5}, \mathbf{8})$ and $(\mathbf{3}, \mathbf{6}, \mathbf{9})$ a voter has to be able to chose one most preferred pair. If this is so, then we can include such type of voters in one of the already existing supertypes. The rest of preference orderings in this case can be classified into new supertypes, analogously to the case without abstention. There are 316 new supertypes, therefore in case with abstention allowed we are able to reduce $7,087,261$ different types to 343 supertypes.

6. Concluding remarks

The aggregation of types of voters into supertypes decreases significantly the size of the model considered. For a model with k alternatives without possibility of abstention there are $(k(k+1))!$ different strict preference relations, which we can aggregate to k^{k+1} supertypes. In case with abstention allowed there is $\left((k+1)^2\right)!$ different strict preference relation and $(k+1)^{k+1}$ supertypes. Since the behaviour of voters of a given supertype at equilibrium is exactly the same, the equilibrium analysis of the reduced model becomes more clear, although in fact we take into consideration all possible profiles of preferences existing in the electorate.

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References

EKES, M. (1999) *Models of economy with infinitely many consumers and producers classified into a finite number of types.* Thesis (in Polish), Warsaw School of Economics.

EKES, M. (2003) General elections modelled with infinitely many voters. *Control and Cybernetics* **32**, 163-173.

WIECZOREK, A. (2004) Large games with only small players and finite strategy sets. *Applicationes Mathematicae* **31**, 79-96.

WISZNIEWSKA-MATYSZKIEL, A. (2002) Discrete time dynamic games with a continuum of players I: decomposable games. *International Game Theory Review* **4**(3), 331-342.