Control and Cybernetics

vol. **37** (2008) No. 3

Joint investment strategies with a superadditive capitalization function*

by

Salvador Cruz Rambaud[1] and Aldo G. S. Ventre[2]

[1]Departamento de Dirección y Gestión de Empresas
Universidad de Almería, Spain

[2]Dipartimento di Cultura del Progetto and BENECON Research Center
Seconda Università degli Studi di Napoli, Italy

Abstract: Some financial investments offer different profitabilities according to the invested amounts. They are operations which differentiate rates of interest depending on the placed capital, i.e., operations whose underlying capitalization functions are not linear with respect to the invested sums. Usually, this differentiation is performed by assigning a variable rate that is an increasing function of the amounts at given jump points, and constant in each interval. As a result, the capitalization function is discontinuous with a finite number of jumps, once the investment term has been fixed. In this situation, an investor can take advantage of differentials in interest rates between two intervals and so it could be convenient, for a group of investors, to join their quantities of money because greater rates of interest can be achieved. The question is how to fairly distribute, among the individual agents, the obtained joint interest. Our answer is based on a modified sharing, according to the interests generated by a new continuous capitalization function which "covers" the discontinuities of the original function.

Keywords: current account, joint investment, sharing interest, capitalization function, superadditivity.

1. Introduction and problem statement

Consider n investors who have at their disposal the amounts $C_1, C_2, \ldots, C_n$, respectively. Suppose that they individually invest their money quantities during the same time period, obtaining profitabilities $i_1, i_2, \ldots, i_n$, respectively, where a greater profitability corresponds to a greater amount. This situation can be due, among other reasons, to:

*Submitted: January 2007; Accepted: July 2008.

1. The existence of different opportunities of investment, performed by each of the investors, in different financial markets.
2. The existence of a financial investment (progressive current account, etc.) with differentials in interest rates between two consecutive amount intervals.
3. The general climate of business confidence. Indeed, when the confidence of the economic agents in a country is high, the financial institutions offer lower interest rates in order to satisfy the increasing demand of this economy. The opposite occurs when the confidence of the economic agents is low, giving rise not only to greater but also different interest rates.

In this case, it would be convenient that the investors join their capitals in order to reach a larger amount and so take advantage of the new profitability from their joint investment, greater than that separately obtained by each of them. The problem arising is how to distribute the total profit or the obtained interest among the n investors "in a fair way".

Obviously, from our point of view, three principles must rule this sharing interest:

1. Each investor must earn more than before, when investing alone, and, moreover, the investor contributing more money must earn even more.
2. Each investor must obtain a profitability greater than before and, additionally, the investor contributing a greater amount must obtain also a greater profitability.
3. The profitability obtained by an investor is continuous with respect to the invested amount.

In effect, there are many financial investments whose underlying capitalization functions are not linear with respect to the invested amounts (Cruz, 1996). A remarkable particular case is when the capitalization function is superadditive with respect to the deposited quantity, i.e.

$$F\left(\sum_{i=1}^{n} C_i, t, p\right) \geq \sum_{i=1}^{n} F(C_i, t, p), \tag{1}$$

$F(C, t, p)$ being the expression of the capitalization function or financial law, defining the projection of the amount C, with the initial time of investment t, onto the time point (instant) p (Gil, 1992; see also Cruz and Ventre, 1998).

These functions offer the investors the possibility to obtain a greater joint profitability r^* if they all together decide to invest their savings, as a unique agent, instead of investing separately. A remarkable case is that of a *progressive current account* that is a bank transactions in which:

1. The set of possible positive amounts, $(0, +\infty)$, have been partitioned into n intervals open from the left-hand side and closed from the right-hand side:

$$(0, B_1], (B_1, B_2], \ldots, (B_{n-1}, +\infty).$$

2. The underlying capitalization function is the simple interest.
3. The interest rate increases with respect to the amounts $B_1, B_2, \dots, B_{n-1}$, and is constant in each interval.

Mathematically,

$$F(C,t,p) = \begin{cases} C[1+i_1(p-t)], & \text{if} \quad 0 < C \leq B_1 \\ C[1+i_2(p-t)], & \text{if} \quad B_1 < C \leq B_2 \\ \vdots & \quad \vdots \\ C[1+i_n(p-t)], & \text{if} \quad B_{n-1} < C \end{cases} \tag{2}$$

where $B_1, B_2, \dots, B_{n-1}$ are the threshold values, and $i_1, i_2, \dots, i_n$ are interest rates; $i_1 < i_2 < \dots < i_n$.

The problem arising in this simple financial strategy is that of interest sharing. At a first glance, it seems acceptable that sharing should be proportional to the amount C installed by each agent. But, considering more carefully the problem, it does not seem right that:

1. Two quantities in the same interval $(B_{k-1}, B_k]$, where $k = 1, 2, \dots, n-1$ and $B_0 = 0$, obtain the same increase of profitability when the amounts near the upper endpoint B_k contribute to the total sum more than those near the lower endpoint B_{k-1}; see Fig. 1.

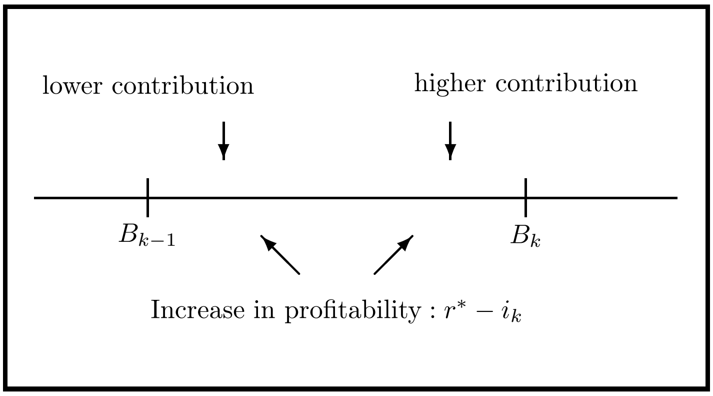

Figure 1. Same increase in profitability and different contributions in a generic interval.

2. There is a jump in profitability between the last quantities in an interval and the first ones in the following, in spite of the fact that they contribute "almost the same quantity" to the total sum. In effect, with pure proportional profit sharing, defined in Section 3, the last amounts in an interval will receive a proportional increase in profitability greater than the increase of profitability performed by the first amounts in the next interval; see Fig. 2.

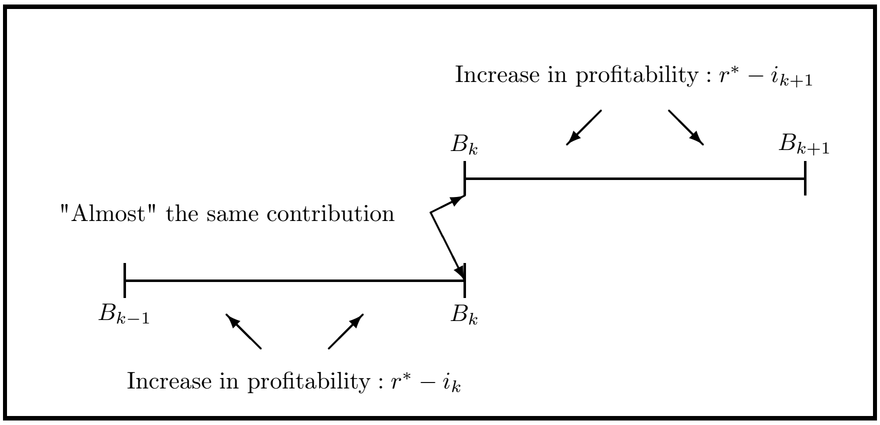

Figure 2. Different increases in profitability and "almost" the same contributions in two consecutive intervals.

Therefore, in order to design a new sharing interest procedure, in each interval we can distinguish two parts (see Fig. 3):

1. The part near the left endpoint of the interval: If two amounts belong to this part, logically the greater one must obtain a greater profitability, but without any "influence" of the greater profitabilities in the following interval. In other words, the amounts near the left are not influenced by a greater interest rate in the following interval.
2. The part near the right endpoint of the interval: If an amount belongs to this part, it would take advantage from the greater interest rate in the following interval and, therefore, it should have an increase in profitability greater than that of amounts in the first part, but less than the increase in profitability of the amounts in the following interval.

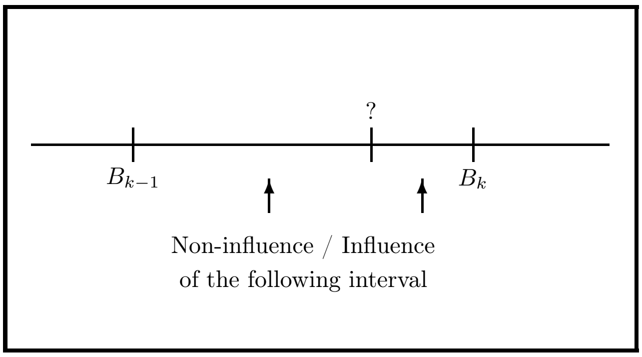

Figure 3. The two parts of a generic interval.

The problem is to find the point separating the two parts in the interval and, to do this, we are going to base on the own "potentiality" of the used capitalization function. Thus, a possible solution could be to divide the time interval $[t,p]$ into infinite parts so that an amount can be capitalized "many times" instead of only once. So the amounts near the right endpoint of each interval can reach this value and take some advantage of the greater interest rate of the following interval.

In effect, our approach will be to divide the time interval into infinite subperiods (Maravall, 1970) and successively to apply the capitalization function, since the simple interest is favourable to the splitting property of the time (Fürst, 1960). In the case of a differentiable and homogeneous capitalization function, this methodology leads to a new capitalization function, $G(C,t,p)$ (Cruz and Ventre, 1998 and 1999):

$$G(C,t,p) = C \cdot e^{\int_t^p \frac{\partial F(x,z)}{\partial z}\big|_{z=x} dx}, \tag{3}$$

called the *additive capitalization function associated to* $F(C,t,p)$; see Appendix 1.

Formula (3) will be used in the next Section of the paper. In order to apply this expression, we have to calculate first the partial derivative of function G with respect to the second variable z at $z=x$, and then its integral with respect to the first variable x between t and p. A more detailed description for building a function G and a justification of its name can be seen in Appendix 1.

It can be proved that function G presents two advantages: its profitability is continuous and strictly increasing with respect to the invested amount. Thus, the jumps of profitability in the initial capitalization function are eliminated and the discontinuities of function $F(C,t,p)$ are linked up, which guarantees a more than proportional continuous increase in profitability for all amounts. Therefore, by adopting this procedure, the quantities in the first part of each interval would obtain the same increase in profitability, and the quantities in the second part of each interval would reach a uniform increase in profitability.

This paper is organized as follows: Section 2 presents the additive capitalization function associated to the simple interest. Then we obtain the non-additive capitalization function underlying a progressive current account. After justifying the use of pure proportional sharing, in Section 3, this new capitalization function will be the framework for our proposal of the sharing model. Section 4 describes some numerical examples of the method introduced here and, finally, Section 5 summarizes and concludes.

2. Financial process associated to a progressive current account

Consider an arbitrary interval $(B_{k-1}, B_k]$, where $k = 1, 2, \ldots, n-1$ and $B_0 = 0$, whose amounts are capitalized at a rate i_k of simple interest. Assume that

$$e^{i_k a} < \frac{B_k}{B_{k-1}}.$$

Let us denote by $G(C, a)$ the function of two variables C and $a := p - t$. From Equation (3), the corresponding additive capitalization function is (see Appendix 2):

$$G(C, a) = C \cdot e^{i_k \cdot a}, \tag{4}$$

for every $C \in \left(B_{k-1}, \frac{B_k}{e^{i_k \cdot a}}\right]$, because we assume that any other amount $C \in \left(\frac{B_k}{e^{i_k \cdot a}}, B_k\right]$ will be capitalized, during a part of a, at the rate i_k, and, during the other part, at the rate i_{k+1}; see Fig. 4. In effect, the successive capitalization of any amount in the interval $\left(B_{k-1}, \frac{B_k}{e^{i_k \cdot a}}\right]$ by means of function G falls always in the interval $(B_{k-1}, B_k]$, while the successive capitalization of any amount belonging to the interval $\left(\frac{B_k}{e^{i_k \cdot a}}, B_k\right]$ reaches the right endpoint B_k at the rate i_k and later it is assumed to be capitalized at the rate i_{k+1}. Observe that this methodology makes sense because $B_{k-1} < \frac{B_k}{e^{i_k \cdot a}}$.

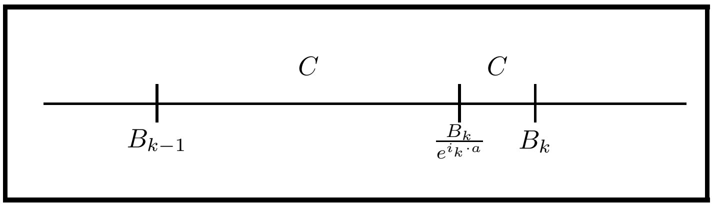

Figure 4. The two parts of a generic interval.

With respect to the first part of each interval, an increase in profitability occurs, equal to the quotient of profitabilities, i.e. to the quotient of interests:

$$\frac{r^*}{r} = \frac{\frac{I^*}{C \cdot a}}{\frac{I}{C \cdot a}} = \frac{I^*}{I} = \frac{Ce^{i_k \cdot a} - C}{C \cdot i_k \cdot a} = \frac{e^{i_k \cdot a} - 1}{i_k \cdot a} > 1,$$

where r^* and r are profitabilities, and I^* and I are interests. Thus, we have that the relative increment in profitabilities is:

$$\frac{r^* - r}{r} = \frac{r^*}{r} - 1 > 0.$$

Likewise, between the first parts of two consecutive intervals, the quotient of profitabilities increases, because $f(x) = \frac{e^{xa} - 1}{xa}$ is an increasing function of x and so:

$$\frac{e^{i_k \cdot a} - 1}{i_k \cdot a} < \frac{e^{i_{k+1} \cdot a} - 1}{i_{k+1} \cdot a}, \tag{5}$$

for every $k = 1, 2, \ldots, n - 1$.

On the other hand, with respect to the second part of each interval, for every $C \in \left(\frac{B_k}{e^{i_k \cdot a}}, B_k\right]$, we can find a real number $z = z(C)$, $0 < z < a$, such that C is capitalized at the rate of interest i_k during z and at the rate of interest i_{k+1} during the remaining subperiod $a - z$. Thus, we assume that this number must satisfy the following condition:

$$G(C, z) = C \cdot e^{i_k \cdot z} = B_k \tag{6}$$

and for the entire investment period a, taking into account the additivity of function $G(\cdot, \cdot)$, we have:

$$G(C, a) = G[G(C, z), (a - z)] = G(C, z) \cdot e^{i_{k+1} \cdot (a-z)} = B_k \cdot e^{i_{k+1} \cdot (a-z)}. \tag{7}$$

From Equation (6), we get:

$$z = z(C) = \frac{\ln B_k - \ln C}{i_k} \tag{8}$$

and, from (7), (8):

$$G(C, a) = B_k \cdot e^{\left(a - \frac{\ln B_k - \ln C}{i_k}\right) \cdot i_{k+1}}. \tag{9}$$

So, the expression for $G(C, a)$ is:

$$G(C, a) = \begin{cases} C \cdot e^{i_1 \cdot a}, & \text{if} \quad 0 < C \leq \frac{B_1}{e^{i_1 \cdot a}} \\ B_1 \cdot e^{\left(a - \frac{\ln B_1 - \ln C}{i_1}\right) \cdot i_2}, & \text{if} \quad \frac{B_1}{e^{i_1 \cdot a}} < C \leq B_1 \\ \vdots & \quad\ \vdots \\ C \cdot e^{i_k \cdot a}, & \text{if} \quad B_{k-1} < C \leq \frac{B_k}{e^{i_k \cdot a}} \\ B_k \cdot e^{\left(a - \frac{\ln B_k - \ln C}{i_k}\right) \cdot i_{k+1}}, & \text{if} \quad \frac{B_k}{e^{i_k \cdot a}} < C \leq B_k \\ \vdots & \quad\ \vdots \\ C \cdot e^{i_n \cdot a}, & \text{if} \quad B_{n-1} < C. \end{cases} \tag{10}$$

For every a, $G(C, a)$ is a continuous function of C. Indeed, from Eq. (10) it can be easily shown that:

$$G(C, a)|_{C=B_k^-} = G(C, a)|_{C=B_k^+} = B_k \cdot e^{i_{k+1} a},$$
$$G(C, a)|_{C=\left(\frac{B_k}{e^{i_k \cdot a}}\right)^-} = G(C, a)|_{C=\left(\frac{B_k}{e^{i_k \cdot a}}\right)^+} = B_k,$$

where by "-" and "+" we denoted the left and the right limits of the function $G(C, a)$ at points $C = \frac{B_k}{e^{i_k \cdot a}}$ and $C = B_k$.

It is useful to take a look at the shape of $G(C, a)$ as a function of C, for any fixed a. Obviously, the graph corresponding to the first part of each interval is a straight line with slope $e^{i_k \cdot a}$. But, to deduce the shape of $G(C, a)$ in the second part of each interval, it is sufficient to calculate the first and the second partial derivatives and observe their signs. Indeed, we have the inequalities:

$$\frac{\partial G(C,a)}{\partial C} = G(C,a)\frac{i_{k+1}}{i_k \cdot C} > 0$$

and

$$\frac{\partial^2 G(C,a)}{\partial C^2} = G(C,a)\frac{i_{k+1}(i_{k+1}-i_k)}{i_k^2 \cdot C^2} > 0,$$

so the function $G(C,a)$ (where a is fixed) is increasing and convex, so that the function increases more rapidly for the last parts of each interval. This is correct, because these quantities are closer to the first quantities in the following interval and so they receive the influence of a greater rate of interest.

Finally, it can be shown that $G(C,a)$ is not differentiable at $C = \frac{C_k}{e^{i_k \cdot a}}$, since:

$$\begin{cases} \left.\dfrac{\partial G(C,a)}{\partial C}\right|_{C=\left(\frac{C_k}{e^{i_k \cdot a}}\right)^-} & = & e^{i_k \cdot a} \\ \left.\dfrac{\partial G(C,a)}{\partial C}\right|_{C=\left(\frac{C_k}{e^{i_k \cdot a}}\right)^+} & = & \frac{i_{k+1}}{i_k}e^{i_k \cdot a} > e^{i_k \cdot a}. \end{cases}$$

Analogously, $G(C,a)$ is not differentiable at $C = C_k$, since:

$$\begin{cases} \left.\dfrac{\partial G(C,a)}{\partial C}\right|_{C=C_k^-} & = & \frac{i_{k+1}}{i_k}e^{i_{k+1} \cdot a} > e^{i_{k+1} \cdot a} \\ \left.\dfrac{\partial G(C,a)}{\partial C}\right|_{C=C_k^+} & = & e^{i_{k+1} \cdot a}. \end{cases}$$

The illustration of $G(C,a)$, for a given value of a, is presented in Fig. 5. Observe that the slope of the successive segments in Fig. 5 is increasing.

Remark 1 In Section 2, we have applied a tacit assumption that for every $k = 1, 2, \ldots, n-1$, the following condition is satisfied:

$$B_{k-1} < \frac{B_k}{e^{i_k \cdot a}};$$

where B_{k-1} and B_k are the threshold values of the capitalization function $F(C,t,p) = F(C,a)$, a is the duration of investment period (i.e. $a = p - t$), and i_k the interest rate.

Of course, considering that the values of B_k, i_k $(k = 1, 2, \ldots, n-1)$ as well as a, are fixed exogenously, the above assumption not always holds. For example, for $i_k = 0.20$, $a = 5$ years, $B_{k-1} = 1,000,000$ and $B_k = 2,000,000$, we have:

$$\frac{B_k}{e^{i_k \cdot a}} = \frac{2,000,000}{e^{0.20 \cdot 5}} = 735,758.88 < 1,000,000 = B_{k-1}.$$

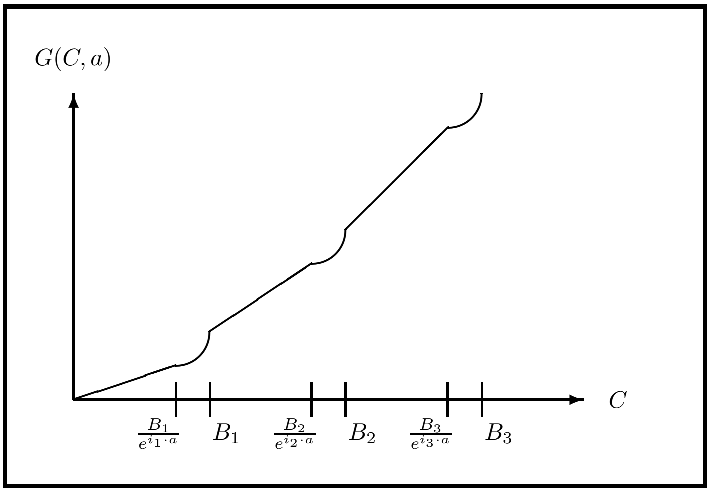

Figure 5. A progressive current account with three intervals.

Thus, for the cases as shown above, the proposed new capitalization function $G(C,a)$ given by Eq. (10) is not well defined, as the subinterval $\left(B_{k-1}, \frac{B_k}{e^{i_k \cdot a}}\right]$ is an empty set.

However, the tacit assumption mentioned above, stipulating non-existence of empty subintervals $\left(B_{k-1}, \frac{B_k}{e^{i_k \cdot a}}\right]$ is not restrictive but realistic, because, in financial practice, the values of i_k and a are small enough to verify that

$$\frac{B_k}{e^{i_k \cdot a}} \leq B_{k-1}.$$

Take into account that this is a short-term financial investment ($a < 1$) and that the distance between two consecutive threshold values, B_{k-1} and B_k, is very high.

Remark 2 For $C \in (B_{k-1}, B_k]$, we have divided the investment period a into two parts: z and $a - z$, where the value of z follows from the condition

$$G(C,z) = C \cdot e^{i_k \cdot z} = B_k.$$

Further on, we have assumed that the investor's amount C is capitalized at the interest rate i_k during subperiod z and at the interest rate i_{k+1} during the remaining subperiod $a - z$. Thus, for the entire period a, after some transformations, we have:

$$G(C,a) = B_k \cdot e^{\left(a - \frac{\ln B_k - \ln C}{i_k}\right) \cdot i_{k+1}}.$$

The above means that for capitalization of the values C belonging to the right subinterval $\left(\frac{B_k}{e^{i_k \cdot a}}, B_k\right]$, the interest rate i_{k+1} corresponding to the next $(B_{k-1}, B_k]$ interval is also (i.e. apart from i_k) taken into account.

In principle, if the investment period is long enough, the interest rates i_{k+2}, i_{k+3}, etc. should also have an impact on the capitalization process of the value of C, when the capitalized amounts exceed the right endpoints of the consecutive intervals. The above could be attained by dividing the investment period a into more than two subintervals; and then the successive use of additive property of capitalization function of the form $G(C,a) = C \cdot e^{i_k \cdot a}$.

Nevertheless, this situation is unrealistic because, in the financial practice, the values of i_k and a are small enough and the distance between two consecutive threshold values, B_{k-1} and B_k, is very high.

3. Sharing model

The problem is the fair distribution, among the individual agents, of the joint interest obtained with a superadditive capitalization function. From the point of view of economic theory, the considered problem should be attacked using cooperative Game Theory (cooperation among all the players is allowed): each individual participant is willing to maximize his quota, but coalitions within subgroups of players are not allowed (for example, several participants individually decide to join their amounts when this idea is promoted by an external agent unconnected with the group of investors, or this initiative is the result of a strategy within the companies of a group). In this way, a solution which is declared as fair should be analyzed within such a framework (Cruz and Valls, 2003 and 2005).

Most of sharing profit formulae arise from the same problem of the best approximation of profit sharing:

Proposition 1 (Quesada and Navas, 1998) *Assume $k_i \geq 0$, $\omega_i > 0$ and $\sum_{j=1}^{n} k_j \leq K$. For every $q > 1$, the function*

$$\Phi_q : \mathbb{R}^n \longrightarrow \mathbb{R},$$

defined by:

$$\Phi_q(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \frac{1}{\omega_j}(x_j - k_j)^q, \tag{11}$$

subject to the conditions:

$$\sum_{j=1}^{n} x_j = K,\ x_i \geq k_i,\ i = 1, 2, \ldots, n, \tag{12}$$

has an absolute minimum at

$$\hat{x}_i = k_i + \frac{\omega_i^{\frac{1}{q-1}}}{\sum_{j=1}^{n} \omega_j^{\frac{1}{q-1}}} \left(K - \sum_{j=1}^{n} k_j \right),\ i = 1, 2, \ldots, n. \tag{13}$$

Let $C_1, C_2, \ldots, C_n$ be the amounts contributed by investors $1, 2, \ldots, n$, respectively, and let us put

$$k_i = F(C_i, t, p)$$

and

$$K = F\left(\sum_{j=1}^{n} C_j, t, p\right).$$

Finally, let $V_i(C_1, C_2, \ldots, C_n)$ denote the final amount allocated to the i-th investor. In the Theory of Games context, $V_i(C_1, C_2, \ldots, C_n)$ can be interpreted as the payoff function of the i-th investor considered as a player. The function V_i depends not only on the individual decision C_i of the i-th investor (player), but also on the decision variables of all other investors. Let us put

$$x_i = V_i(C_1, C_2, \ldots, C_n).$$

The most interesting sharing models are obtained when $q = 2$. In this case, the problem of the best approximation to the profit sharing question coincides with the least squares method.

A condition to be verified by the proposed sharing model is the independence of how the collusion has been performed, that is ($n = 3$),

$$\begin{aligned} V_1(C_1, C_2, C_3) &= V_1(C_1, C_2 + C_3, 0) = V_1(C_1, 0, C_3 + C_2) \\ V_2(C_1, C_2, C_3) &= V_2(C_1 + C_3, C_2, 0) = V_2(0, C_2, C_3 + C_1) \\ V_3(C_1, C_2, C_3) &= V_3(C_1 + C_2, 0, C_3) = V_3(0, C_2 + C_1, C_3). \end{aligned} \tag{14}$$

It can be shown (Quesada and Navas, 1998) that, among all sharing methods, only the pure proportional sharing:

$$k_i = 0, \ \omega_i = C_i, \ i = 1, 2, \ldots, n,$$

satisfies the last condition. Then the solution of the optimization problem is given by:

$$\hat{V}_i(C_1, C_2, \ldots, C_n) = \frac{C_i}{\sum_{j=1}^{n} C_j} F\left(\sum_{j=1}^{n} C_j, t, p\right). \tag{15}$$

Observe that, in this case, $x_i = V_i(C_1, C_2, \ldots, C_n)$, $\omega_i = C_i$, $k_i = 0$ and $q = 2$, from Eq. (11) we have the following goal function to be minimized:

$$\Phi = \sum_{j=1}^{n} \frac{V_j^2}{C_j}.$$

The interpretation of this optimization problem is as follows. The sum of the square deviations from zero (notice that, in the pure proportional sharing, there is no previous sharing) related to the invested amounts, is a minimum.

But, in our case, as indicated in the introduction, this solution does not lead to a fair sharing. So, assuming that coalitions of the form (14) within subgroups of investors are not allowed, we propose the following solution:

$$\hat{V}_i(C_1, C_2, \ldots, C_n) = \frac{G(C_i, a)}{\sum_{j=1}^n G(C_j, a)} F\left(\sum_{j=1}^n C_j, t, p\right), \tag{16}$$

where

$$a = p - t,\ k_i = 0,\ \omega_i = G(C_i, a),\ i = 1, 2, \ldots, n.$$

4. Numerical examples

4.1. Sharing interests

Consider three investors ($n = 3$) who have amounts of $C_1 = 500,000$; $C_2 = 1,200,000$, and $C_3 = 1,500,000$ monetary units, at their disposal. Suppose that they decide to invest them in a progressive current account (see the definition in Section 1) for $a = 2$ years, according to the following expression ($a = p - t$, as usual):

$$F(C, a) = \begin{cases} C(1 + 0.10a), & \text{if} \quad 0 < C \leq 1,000,000 \\ C(1 + 0.20a), & \text{if} \quad 1,000,000 < C \leq 2,000,000 \\ C(1 + 0.30a), & \text{if} \quad 2,000,000 < C. \end{cases} \tag{17}$$

If each investor decides to invest on his own, according to (17), the obtained amounts will be:

$$F(500,000; 2) = 500,000(1 + 0.10 \cdot 2) = 600,000;$$
$$F(1,200,000; 2) = 1,200,000(1 + 0.20 \cdot 2) = 1,680,000$$

and

$$F(1,500,000; 2) = 1,500,000(1 + 0.20 \cdot 2) = 2,100,000.$$

Thus, the sum of the obtained amounts will be:

$$F(500,000; 2) + F(1,200,000; 2) + F(1,500,000; 2) = 4,380,000.$$

If the three investors decided to invest together ($C_1 + C_2 + C_3 = 3,200,000$), from (17), the jointly obtained amount would be:

$$F(3,200,000; 2) = 3,200,000(1 + 0.30 \cdot 2) = 5,120,000$$

and the resulting surplus (i.e. additional profit) would be equal to:

$$5,120,000 - 4,380,000 = 740,000.$$

The individual final amounts obtained by using the new capitalization function, $G(C, a)$ given by Eq. (10) for $k = 1, 2$ and $B_1 = 1,000,000$, $B_2 = 2,000,000$, can be calculated as follows. Taking into account that:

$$0 < C_1 = 500,000 \leq \frac{1,000,000}{e^{0.10 \cdot 2}} = 818,730.75;$$
$$1,000,000 < C_2 = 1,200,000 \leq \frac{2,000,000}{e^{0.20 \cdot 2}} = 1,340,640.09;$$

and

$$1,340,640.09 = \frac{2,000,000}{e^{0.20 \cdot 2}} < C_3 = 1,500,000 \leq 2,000,000;$$

we have:

$$G(500,000; 2) = 500,000 \cdot e^{0.10 \cdot 2} = 610,701.38;$$
$$G(1,200,000; 2) = 1,200,000 \cdot e^{0.20 \cdot 2} = 1,790,189.64$$

and

$$G(1,500,000; 2) = 2,000,000 e^{\left(2 - \frac{\ln 2,000,000 - \ln 1,500,000}{0.20}\right) 0.30} = 2,367,001.75.$$

The results are summarized in Table 1, where:

- Individual (respectively modified individual) interests, I_i (respectively I_i^*), have been calculated by applying the formula

$$I_i = F(C_i, a) - C_i$$
$$(\text{resp. } I_i^* = \frac{G(C_i, a)}{\sum_{j=1}^{n} G(C_j, a)} F\left(\sum_{j=1}^{n} C_j, t, p\right) - C_i).$$

- Profitabilities, r_i (resp. r_i^*), have been obtained from the formula $r_i = \frac{I_i}{C_i \cdot a}$ (resp. $r_i^* = \frac{I_i^*}{C_i \cdot a}$).
- Obviously, the increase in profitability is $\Delta r_i = r_i^* - r_i$.

The example considered in this Section can be used to illustrate the fact that the deduced capitalization function $G(C, a)$ is not decomposable. This is due to the discontinuity of F. A capitalization function $F(C, a)$ is said to be *decomposable* or *additive* (Cruz and Ventre, 1999) if

$$F(F(C, a), b) = F(C, a + b), \tag{18}$$

for every C, a and b.

Table 1. Sharing interests

Amounts	Individual interest	Profitability (%)	Modified interest	Profitability* (%)	Increase in profitability
500,000	100,000	10.00	155,801.47	15.58	5.58
1,200,000	480,000	20.00	722,394.52	30.10	10.10
1,500,000	600,000	20.00	1,041,804.01	34.73	14.73
3,200,000	1,180,000		1,920,000		

Indeed, let be $C = 900,000$ and $a = b = 1$. As

$$900,000 < \frac{1,000,000}{e^{0.10}} = 905,114,$$

from formula (10), it is verified that:

$$G(900,000; 1) = 900,000 \cdot e^{0.10} = 994,350$$

and

$$G(G(900,000; 1), 1) = G(994,350; 1) =$$
$$= 1,000,000e^{\left(1-\frac{\ln 1,000,000-\ln 994,350}{0.10}\right)0.20} = 1,207,639,9.$$

On the other hand, taking into account that:

$$\frac{1,000,000}{e^{0.10\cdot 2}} = 818,730.75 < 900,000,$$

from formula (10) again, it is verified that:

$$G(900,000; 2) = 1,000,000e^{\left(2-\frac{\ln 1,000,000-\ln 900,000}{0.10}\right)0.20} = 1,208,378.01.$$

Thus, $G(G(900,000; 1); 1) \neq G(900,000; 2)$ and so $G(C, a)$ is not decomposable.

4.2. Sharing discounts

In Section 3, we have seen that the investors invest cooperatively their money $C_1, C_2, \ldots, C_n$ and, as result of this joint investment of the total amount $\sum_{j=1}^{n} C_j$, they are getting an additional profit due to superadditivity of the capitalization function:

$$F\left(\sum_{j=1}^{n} C_j, t, p\right) \geq \sum_{j=1}^{n} F(C_j, t, p).$$

Here, coalitions are not allowed (by assumption). An alternative approach based on discounting functions is given by the following case. Suppose now that three companies have debts of amounts $C_1 = 500,000$; $C_2 = 1,200,000$, and $C_3 = 1,500,000$ monetary units with another company that supplies a common service to these three companies. Assume that, because of their turnovers, a rebate will be applied by the creditor of 10%, 20%, and 20% per year, respectively, using the formula of linear discounting. As the three debts have their origin in a common service offered by the same company during $a = 1$ year, the debtor companies could contract the service together. Then the obtained discount could be greater, for instance, equal to 30%. How would be the total discount distributed?

Now, the discounting function to be applied could be:

$$F(C,a) = \begin{cases} C(1-0.10a), & \text{if} \quad 0 < C \leq 1,000,000 \\ C(1-0.20a), & \text{if} \quad 1,000,000 < C \leq 2,000,000 \\ C(1-0.30a), & \text{if} \quad 2,000,000 < C. \end{cases} \tag{19}$$

Thus, according to (19), the discounted amounts, contracting each company the service individually, would be:

$$F(500,000;1) = 500,000(1-0.10) = 450,000;$$
$$F(1,200,000;1) = 1,200,000(1-0.20) = 960,000$$

and

$$F(1,500,000;1) = 1,500,000(1-0.20) = 1,200,000.$$

If the three companies decide to contract together the service, the jointly discounted amount to be paid would be:

$$F(3,200,000;1) = 3,200,000(1-0,30) = 2,240,000.$$

In this case, it can be shown that the expression of $G(C,a)$ is a formula analogous to (10):

$$G(C,a) = \begin{cases} C \cdot e^{-i_1 \cdot a}, & \text{if} \quad 0 < C \leq B_1 \\ B_1 \cdot e^{-\left(a - \frac{\ln C - \ln B_1}{i_2}\right) \cdot i_1}, & \text{if} \quad B_1 < C \leq B_1 \cdot e^{i_1 \cdot a} \\ C \cdot e^{-i_2 \cdot a}, & \text{if} \quad B_1 \cdot e^{i_2 \cdot a} < C \leq B_2 \\ \vdots & \qquad \vdots \\ B_{k-1} \cdot e^{-\left(a - \frac{\ln C - \ln B_{k-1}}{i_k}\right) \cdot i_{k-1}}, & \text{if} \quad B_{k-1} < C \leq B_{k-1} \cdot e^{i_k \cdot a} \\ C \cdot e^{-i_k \cdot a}, & \text{if} \quad B_{k-1} \cdot e^{i_k \cdot a} < C \leq B_k \\ \vdots & \qquad \vdots \\ B_{n-1} \cdot e^{-\left(a - \frac{\ln C - \ln B_{n-1}}{i_n}\right) \cdot i_{n-1}}, & \text{if} \quad B_{n-1} < C \leq B_{n-1} \cdot e^{i_n \cdot a} \\ C \cdot e^{-i_n \cdot a}, & \text{if} \quad B_{n-1} \cdot e^{i_n \cdot a} < C. \end{cases} \tag{20}$$

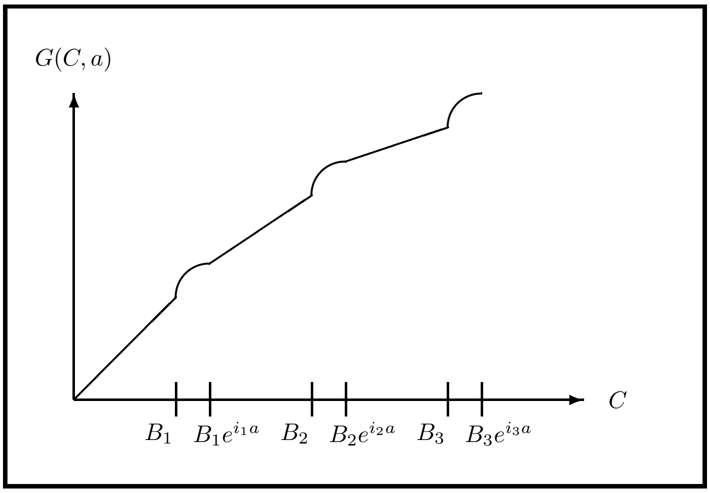

Figure 6. A progressive linear discounting with three intervals.

In this case, the graphic representation of $G(C,a)$, for a given value of a, is presented in Fig. 6.

Taking into account that now:

$$1,000,000 < 1,200,000 < 1,000,000 \cdot e^{0.20} = 1,221,402.76 < 1,500,000,$$

we can calculate, according to (20), the discounted amounts using the discounting function $G(C,a)$:

$$G(500,000;1) = 500,000 \cdot e^{-0.10} = 452,418.71;$$

$$G(1,200,000;1) = 1,000,000 \cdot e^{-\left(1-\frac{\ln 1,200,000 - \ln 1,000,000}{0.20}\right)0.10} = 991,199.73$$

and

$$G(1,500,000;1) = 1,500,000 \cdot e^{-0.20} = 1,228,096.13.$$

The difference of this subsection with respect to subsection 4.1 is that now $F(C,a)$ is subadditive with respect to the discounted amount, whereby the proportional sharing will be done using the discounts instead of the discounted amounts:

$$C_i - V_i(C_1, C_2, \ldots, C_n) = \frac{C_i - G(C_i,a)}{\sum_{j=1}^{n}[C_j - G(C_j,a)]}\left[\sum_{j=1}^{n} C_j - F\left(\sum_{j=1}^{n} C_j, t, p\right)\right]. \tag{21}$$

The results are summarized in Table 2, where:

- Individual (respectively modified individual) discounts, D_i (respectively D_i^*), have been calculated by applying the formula $D_i = C_i - F(C_i, a)$

$$\text{(respectively } D_i^* = \frac{C_i - G(C_i, a)}{\sum_{j=1}^{n}[C_j - G(C_j, a)]}\left[\sum_{j=1}^{n} C_j - F\left(\sum_{j=1}^{n} C_j, t, p\right)\right]\text{).}$$

- Discount rates, d_i (resp. d_i^*), have been obtained from the formula $d_i = \frac{D_i}{C_i \cdot a}$ (resp. $d_i^* = \frac{D_i^*}{C_i \cdot a}$).
- Obviously, the increase in discount rate is $\Delta d_i = d_i^* - d_i$.

Table 2. Sharing discounts

Amounts	Individual discount	Discount rate (%)	Modified discount	Discount rate* (%)	Increase in discount rate
500,000	50,000	10.00	86,464.69	17.29	7.29
1,200,000	240,000	20.00	379,431.74	31.62	11.62
1,500,000	300,000	20.00	494,103.57	32.94	12.94
3,200,000	590,000		960,000		

5. Conclusion and possible further research

Some capitalization functions are superadditive with respect to the invested amount. This means that, for a group of investors, it could be very interesting to take a coalition strategy in order to obtain a greater profitability. The problem arising is that of interest sharing of the obtained joint interest in a fair way, among the individual investors. But, usually, the original capitalization function to be applied is a jumping function and a preliminary question is to "cover" its discontinuities. We have solved this problem with purely financial tools, using the additive capitalization function associated to the initial capitalization function. Later, this new function has been used to determine the weights in the proportional sharing of the jointly amount obtained by the group of investors. Our solution satisfies a fundamental rule: if an investor contributes more than another one, not only his interest but also his profitability must be greater.

As a possible further research we point out the search for a solution of the presented problem under the assumption that the modified (new) capitalization function, $G(C, t, p)$, is not only continuous, but also a differentiable function with respect to the variable C. As a result, we would obtain the smooth approximation of the relation between $G(C, t, p)$ and C.

Acknowledgements

We are very grateful for the comments and suggestions of an anonymous referee.

Appendix 1

A capitalization function $F(C,t,p)$ is said to be *homogeneous* if $F(C,t,p) = C \cdot F(1,t,p)$. Let us denote $F(1,t,p) := F(t,p)$. Thus, $F(C,t,p) = C \cdot F(t,p)$. If, moreover, $F(C,t,p)$ is *differentiable*, both partial derivatives $\frac{\partial F(t,p)}{\partial t}$ and $\frac{\partial F(t,p)}{\partial p}$ exist. Let us divide the time interval $[t,p]$ into infinite subperiods and successively apply the capitalization function. This process leads to a new capitalization function, $G(C,t,p)$:

$$G(C,t,p) = C \cdot \lim_{n\to\infty} \prod_{k=1}^{n} F\left(t + \frac{k-1}{n}(p-t), t + \frac{k}{n}(p-t)\right) =$$

$$= C \cdot e^{\lim_{n\to\infty} \sum_{k=1}^{n} \ln F\left(t+\frac{k-1}{n}(p-t), t+\frac{k}{n}(p-t)\right)}.$$

Let us denote $x = t + \frac{k-1}{n}(p-t)$ and $dx = \frac{1}{n}(p-t)$. Thus, in the exponent of the function given above, for infinitesimally small values of dx, we have:

$$\ln F(x, x+dx) = \ln F(x,x) + \frac{1}{F(x,x)} \left.\frac{\partial F(x,z)}{\partial z}\right|_{z=x} dx$$

and, from definition,

$$F(x,x) = 1.$$

So, it can be concluded that:

$$G(C,t,p) = C \cdot e^{\int_t^p \left.\frac{\partial F(x,z)}{\partial z}\right|_{z=x} dx}. \qquad \blacksquare$$

It can be easily proved that $G(G(C,t,p),p,q) = G(C,t,q)$. If F is continuous, the above equation justifies why function G is called *the additive capitalization function associated to* F.

Appendix 2

Let $F(C,t,p) = C \cdot [1 + i(p-t)]$ be the capitalization function of simple interest at rate i. As

$$\left.\frac{\partial F(x,z)}{\partial z}\right|_{z=x} = \left.\frac{\partial [1 + i(z-x)]}{\partial z}\right|_{z=x} = i,$$

according to Appendix 1, the new capitalization function $G(C,t,p)$ corresponding to $F(C,t,p)$ is:

$$G(C,t,p) = C \cdot e^{\int_t^p i dx} = C \cdot e^{i(p-t)},$$

which is the well known formula for capitalization function defined for the case of continuously compound interest rate i. $\blacksquare$

References

CRUZ, S. (1996) Financial laws with algebraic automata theory. *Estudios de Economía Aplicada* **6**, 65-102.

CRUZ, S. and VALLS, M.C. (2003) Progressive current accounts: Profit-sharing interest. *International Game Theory Review* **5** (2), 139-149.

CRUZ, S. and VALLS, M.C. (2005) Profit-sharing in transactions governed by a subadditive capitalization function. *International Journal of Theoretical and Applied Finance* **8** (1), 59-74.

CRUZ, S. and VENTRE, A.G.S. (1998) Decomposable financial laws and profitability. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **6** (4), 329-344.

CRUZ, S. and VENTRE, A.G.S. (1999) Decomposable financial processes associated to non-homogeneous laws. *Giornale dell'Istituto Italiano degli Attuari*, **LXII** (1-2), 79-88.

FÜRST, D. (1960) Funzioni e leggi di capitalizzazione subadditive. *Giornale dell'Istituto Italiano degli Attuari*, 56-77.

GIL, L. (1992) *Matemática de las Operaciones Financieras*. Ed. AC, Madrid.

MARAVALL, D. (1970) *Matemática Financiera*. Ed. Dossat, Madrid.

QUESADA, J.M.and NAVAS, J. (1998) Mejor I_p-aproximación y fórmulas de reparto del excedente en funciones financieras supraditivas. *7th Jornadas de ASEPUMA* Valencia (Spain), 451-463.