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Some results for the reflection problems in Hilbert spaces*

by

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Abstract: This work is concerned with existence and uniqueness of a solution of a stochastic variational inequality on closed convex bounded subsets with nonempty interior and smooth boundary of a Hilbert space H (the reflection problem).

Keywords: stochastic variational inequality, Wiener processes in Hilbert spaces, convex sets, strong solutions.

1. Introduction

Let H be a separable Hilbert space (with scalar product $(\cdot,\cdot)$ and norm denoted by $|\cdot|$). We are given a closed convex subset K of H with non empty interior $\overset{\circ}{K}$. We denote by I_K the indicator function of K,

$$I_K(x) = \begin{cases} 0 \quad \text{if } x \in K. \\ \\ +\infty \quad \text{if } x \notin K. \end{cases}$$

and by ∂I_K the sub-differential of I_K (see, e.g., Barbu, 1993),

$$\partial I_K(x) = \{z \in H: \ (z, x-y) \geq 0, \quad \forall\, y \in K\}, \quad x \in H.$$

We have

$$\partial I_K(x) \begin{cases} = \{0\} \quad \text{if } x \in \overset{\circ}{K}, \\ = N_K(x) \quad \text{if } x \in \partial K, \\ = \varnothing \quad \text{if } x \notin K, \end{cases}$$

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where $N_K(x)$ is the normal cone of K at x. We are concerned here with the stochastic variational inequality

$$\begin{cases} dX(t) + (AX(t) + F(X(t)) + \partial I_K(X(t)))\,dt \ni \sqrt{Q}\,dW(t) \\ X(0) = x \in K. \end{cases} \tag{1}$$

Formally, equation (1) can be represented as

$$\begin{cases} dX(t) + (AX(t) + F(X(t))dt = \sqrt{Q}\,dW(t) \text{ in } \ \{t \in [0,T]: \ X(t) \in \mathring{K}\}, \\ dX(t) + (AX(t) + F(X(t)) + \zeta(t))dt = \sqrt{Q}\,dW(t) \\ \qquad\qquad\qquad\qquad \text{in } \ \{t \in [0,T]: \ X(t) \in \partial K\}, \\ X(0) = x \in K, \end{cases} \tag{2}$$

where $\zeta(t) \in N_K(X(t))$ for all $t \in [0,T]$. For a precise definition of solution see Definition 1 below.

HYPOTHESIS 1 (i) *$A : D(A) \subset H \to H$ is a linear self-adjoint operator on H such that A^{-1} is compact and $(Ax,x) \geq \delta |x|^2$, $\forall\, x \in D(A)$ for some $\delta > 0$.*

(ii) *$Q : H \to H$ is a linear, bounded, positive and self-adjoint operator on H such that $Qe^{tA} = e^{tA}Q$ for all $t \geq 0$.*

(iii) *$Q(H) \subset D(A)$ and* $\mathrm{Tr}\,[AQ] < \infty$.

(iv) *$F : H \to H$ is a Lipschitzian mapping such that for some $\gamma > 0$ we have*

$$(F(x),x) \geq -\gamma, \quad \forall\, x \in H.$$

(v) *W is a cylindrical Wiener process on H of the form*

$$W(t) = \sum_{k=1}^{\infty} \mu_k \beta_k(t) e_k, \quad t \geq 0,$$

where $\{\beta_k\}$ is a sequence of mutually independent real Brownian motions on a filtered probability spaces $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ (see Da Prato, 2004) and $\{e_k\}$ is an orthonormal basis in H, which will be taken as a system of eigen-functions for A for simplicity, i.e.

$$Ae_k = \alpha_k e_k, \quad \forall\, k \in \mathbb{N}.$$

(vi) $0 \in \mathring{K}$.

In most specific examples $H = L^2(\mathcal{O})$, A is an elliptic operator on $\mathcal{O}$ with appropriate boundary conditions, F is a Nemitski operator on $L^2(\mathcal{O})$ (see Section 5 below).

Under Hypothesis 1 the stochastic convolution $W_A(t)$,

$$W_A(t) = \int_0^t e^{-A(t-s)} \sqrt{Q}\, dW(s), \quad \forall\, t \geq 0,$$

is a well defined mean square continuous process in $V = D(A^{1/2})$ and (see Da Prato, 2004),

$$\mathbb{E} \sup_{t\in[0,T]} \|W_A(t)\|^2 < +\infty. \tag{3}$$

Assumption (iii) is, of course, quite restrictive but it is essential for our approach since it implies continuity of $W_A : [0, +\infty) \to D(A^{1/2})$.

The existence and uniqueness of a strong solution X to equation (1) was an open problem except for the finite-dimensional case (Barbu and Da Prato, 2008; Cépa, 1994, 1998) and few special cases, for instance $H = L^2(0, 1)$, $A = -\Delta$, $K = \{x \in H : \ x \geq -\sigma$ a.e. on $(0, 1)\}$ where $\sigma \geq 0$ (see Haussmann and Pardoux, 1989; Nualart and Pardoux, 1992; Zambotti, 2001; as well as Barbu and Rascanu, 1997, Rascanu, 1996, and Zhang, 1997).

In this paper we prove the existence and uniqueness of a solution of 1 under Hypothesis 1.

Then we consider the transition semigroup $P_t : C_b(H) \to C_b(H)$

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad x \in K, \tag{4}$$

where $X = X(t, x)$ is the solution of (1) and $C_b(H)$ denotes the space of all mappings from H into $\mathbb{R}$ which are uniformly continuous and bounded. We prove existence and, in some cases, uniqueness of an invariant measure ν of P_t.

Finally we consider the infinitesimal generator N of P_t in $L^2(H, \nu)$, i.e.

$$D(N) = \left\{\varphi \in L^2(H, \nu) :\ \exists \lim_{t\to 0} \frac{1}{t}\, (P_t\varphi - \varphi) \quad \text{in} \ \ L^2(H, \nu)\right\},$$

and

$$N\varphi = \lim_{t\to 0} \frac{1}{t}\, (P_t\varphi - \varphi) \text{ in } \ L^2(H, \nu), \quad \forall\, \varphi \in D(N).$$

It is an interesting problem to see the relationship between the abstract operator N and the differential operator

$$N_0\varphi = \frac{1}{2}\, \text{Tr}\, [QD^2\varphi] - (x, AD\varphi) - (F(x), D\varphi), \quad \forall\, \varphi \in D(N_0), \tag{5}$$

with the domain

$$D(N_0) = \{\varphi \in D(N) \cap C_b^2(H) :\ AD\varphi \in C_b^1(H),\ (D_\varphi(x), N_K(x)) = 0,\ \forall\, x \in \partial K\},$$

where $N_K(x)$ is the normal cone of K at x.

In Section 4 we prove that N_0 is a section of N(Theorem 2). It would be interesting to show that $D(N_0)$ is a core for N. This is indeed the case if H is finite-dimensional (see Barbu and Da Prato, 2008). The problem is open when H is infinite-dimensional as in the present case.

Notations

We shall denote by $C([0,T];H)$ the space of all continuous functions from $[0,T]$ to H and by $BV([0,T];H)$ the space of all functions with bounded variation from $[0,T]$ to H.

We set $V = D(A^{1/2})$ with norm $\|\cdot\|$ and denote by V' the dual of V in the pairing induced by the scalar product $(\cdot,\cdot)$ of H. We have $V \subset H \subset V'$ algebraically and topologically.

By $C_W([0,T];H)$, $L^2_W([0,T];V)$, $L^2_W([0,T];V')$ we shall denote standard spaces of adapted processes on $[0,T]$ (see Da Prato, 2004, 2006; Da Prato and Zabczyk, 1996). Π_K is the projection on K.

By $D\varphi$ and $D^2\varphi$ we shall denote the Gâteaux derivatives of a function $\varphi : H \to \mathbb{R}$ of first and second order.

2. Existence and uniqueness for equation (1)

We shall assume here that Hypothesis 1 holds.

DEFINITION 1 *The adapted process $X \in C_W([0,T];H) \cap L^2_W(0,T;V)$ is said to be a strong solution to* (1) *if there are functions $Y \in C_W([0,T];H) \cap L^2_W(0,T;V)$ and $\eta \in BV([0,T];H)$ such that,*

(i) *We have*

$$X(t) = Y(t) + W_A(t), \quad \text{a.e. in } \Omega \times [0,T] \times H, \quad \mathbb{P}\text{-a.s.}\,. \tag{6}$$

(ii) $X(t) \in K$ *for all* $t \in [0,T]$.

(iii) *We have*

$$Y(t) + \int_0^t (AY(s) + F(X(s)))ds + \eta(t) = x, \quad \forall\, t \in [0,T],\ \mathbb{P}\text{-a.e.}. \tag{7}$$

(iv) *For all $t \in [0,T]$ and $Z \in C([0,T];K)$*

$$\int_0^t (d\eta(s), X(s) - Z(s))ds \geq 0,\ \mathbb{P}\text{-a.e.}. \tag{8}$$

In (8) $\int_0^t (d\eta(s), X(s) - Z(s))ds$ is the Stieltjes integral with respect to η. Note that by Hypothesis 1 it follows that V is compactly embedded in H.

THEOREM 1 *Under Hypothesis* 1 *there is a unique strong solution to equation* (1).

Proof. *Existence*. We start with the approximating equation

$$\begin{cases} dX_\epsilon + (AX_\epsilon + F(X_\epsilon) + \beta_\epsilon(X_\epsilon))dt = \sqrt{Q}\, dW \\ X_\epsilon(0) = x, \end{cases} \tag{9}$$

where β_ϵ is the Yosida approximation of ∂I_K,

$$\beta_\epsilon(x) = \frac{1}{\epsilon}\,(x - \Pi_K(x)), \quad \forall\, x \in H,\ \epsilon > 0.$$

Equation (9) has a unique strong solution $X_\epsilon \in C_W([0,T];H)$ such that $Y_\epsilon := X_\epsilon - W_A$ belongs to $L^2_W(0,T;H)$. We can rewrite (9) as

$$\begin{cases} \dfrac{dY_\epsilon}{dt} + AY_\epsilon + F(X_\epsilon) + \beta_\epsilon(X_\epsilon) = 0, \\ Y_\epsilon(0) = x, \end{cases} \tag{10}$$

which is considered for a fixed $\omega \in \Omega$. Since by Hypotheses 1(vi), $0 \in \mathring{K}$ there is $\rho > 0$ such that

$$(\beta_\epsilon(x), x - \rho\theta) \geq 0, \quad \forall\, \theta \in H,\ |\theta| = 1.$$

This yields

$$\rho|\beta_\epsilon(x)| \leq (\beta_\epsilon(x), x), \quad \forall\, x \in H. \tag{11}$$

Step 1. There exists $C = C(\omega) > 0$ such that

$$|Y_\epsilon(t)|^2 + \int_0^t \|Y_\epsilon(s)\|^2 ds + \int_0^t |\beta_\epsilon(X_\epsilon(s))| ds \leq C. \tag{12}$$

Indeed, multiplying (10) by $Y_\epsilon(s)$, integrating over $(0,t)$ and taking into account (11), yields

$$\begin{aligned} &\frac{1}{2}\,|Y_\epsilon(t)|^2 + \int_0^t \|Y_\epsilon(s)\|^2 ds + \rho \int_0^t |\beta_\epsilon(X_\epsilon(s))| ds \\ &\leq \frac{1}{2}\,|x|^2 + \gamma \int_0^t |X_\epsilon(s)|^2 ds + \int_0^t (F(X_\epsilon(s)) + \beta_\epsilon(X_\epsilon(s)), W_A(s)) ds. \end{aligned} \tag{13}$$

In order to estimate the last term in (13), we recall (3) and choose a decomposition $0 < t_1 < \cdots < t_N = t$ of $[0,t]$ such that for $t, s \in [t_{i-1}, t_i]$ we have

$$|W_A(t) - W_A(s)| \leq \frac{\rho}{2}.$$

Then we write

$$\begin{aligned} \int_0^t (\beta_\epsilon(X_\epsilon(s)), W_A(s)) ds \quad &= \quad \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (\beta_\epsilon(X_\epsilon(s)), W_A(s) - W_A(t_i)) ds \\ &\quad + \sum_{i=1}^N \left(W_A(t_i), \int_{t_{i-1}}^{t_i} \beta_\epsilon(X_\epsilon(s)) ds \right) . \end{aligned}$$

Consequently,

$$\int_0^t (\beta_\epsilon(X_\epsilon(s)), W_A(s))ds \le \frac{\rho}{2} \int_0^t |\beta_\epsilon(X_\epsilon(s))|ds$$
$$+\left|\sum_{i=1}^N \left(W_A(t_i), \int_{t_{i-1}}^{t_i} (AY_\epsilon(s) + F(X_\epsilon(s)))ds + Y_\epsilon(t_i) - Y_\epsilon(t_{i-1})\right)\right| .$$

Now, using the estimate

$$\left(W_A(t_i), \int_{t_{i-1}}^{t_i} AY_\epsilon(s)ds\right) \le C \int_{t_{i-1}}^{t_i} \|Y_\epsilon(s)\|^2 ds,$$

which follows from (1.3), we get after some computations the estimate (12).

Step 2. We prove existence of the limits of $Y_\epsilon(t)$ and $\eta_\epsilon(t)$ as $\epsilon \to 0$.

We first prove that the sequence $\{Y_\epsilon\}$ is equi-continuous in $C([0,T];H)$. Let $h > 0$, then

$$\frac{d}{dt}\,(Y_\epsilon(t+h) - Y_\epsilon(t)) + A(Y_\epsilon(t+h) - Y_\epsilon(t))$$
$$+F(X_\epsilon(t+h)) - F(X_\epsilon(t)) + \beta_\epsilon(X_\epsilon(t+h)) - \beta_\epsilon(X_\epsilon(t)) = 0.$$

Taking into account that W_A is $\mathbb{P}$-a.s. continuous in H (by (3)), we may assume that

$$\sup_{t\in[0,T]} |W_A(t+h) - W_A(t)| \le \delta(h) \to 0 \quad \text{as } h \to 0.$$

We deduce by the monotonicity of β_ϵ and because F is Lipschitz that

$$|Y_\epsilon(t+h) - Y_\epsilon(t)| \le C\delta(h), \quad \forall\, t \in [0,T],\ h > 0,\ \epsilon > 0. \tag{14}$$

So $\{Y_\epsilon\}$ is equi-continuous. To apply the Ascoli–Arzelà Theorem we have to prove that for each $t \in [0,T]$ the set $\{Y_\epsilon(t)\}_{\epsilon>0}$ is pre-compact in H. To prove this, choose for any $\epsilon > 0$ a sequence $\{f_n^\epsilon\} \subset L^2(0,T;V)$ such that

$$|f_n^\epsilon - \beta_\epsilon(Y_\epsilon + W_A)|_{L^1(0,T;H)} \le \frac{1}{n}, \quad n \in \mathbb{N}.$$

On the other hand, for each $n \in \mathbb{N}$ the set

$$M_n := \left\{\int_0^t e^{-A(t-s)} f_n^\epsilon ds + e^{-At}x :\ \epsilon > 0\right\}$$

is compact in H because $\{f_n^\epsilon\}$ is bounded in $L^2(0,T;H)$ for each $n \in \mathbb{N}$. This implies that for any $\delta > 0$ there are $N(n) \in \mathbb{N}$ and $\{u_i^n\}_{i=1,\dots,N(n)} \subset H$ such that

$$\bigcup_{i=1}^{N(n)} B(u_i^n, \delta) \supset M_n.$$

($B(u_i^n, \delta)$ is the ball with center u_i^n and radius δ.) Therefore

$$\left\{ Y_\epsilon(t) := \int_0^t e^{-A(t-s)} f_n^\epsilon ds + e^{-At} x : \ \epsilon > 0 \right\} \subset \bigcup_{i=1}^{N(n)} B(u_i^n, \delta + n^{-1}).$$

Since n is arbitrary we infer that for each $\delta > 0$, the set $\{Y_\epsilon(t)\}_{\epsilon > 0}$ can be covered by a finite number of balls of radius δ and therefore it is precompact in H as claimed. Then by (14) and the Ascoli–Arzelà Theorem we infer that on a subsequence, $Y_\epsilon \to Y$ strongly in $C([0,T];H)$ and weakly in $L^2(0,T;V)$. Moreover, thanks to Helly's Theorem we have that there is $\eta \in BV([0,T];H)$ such that for $\epsilon \to 0$

$$\int_0^t \beta_\epsilon(X_\epsilon(s)) ds \to \eta(t) \text{ weakly in } \ H, \ \forall\, t \in [0,T], \tag{15}$$

which implies that

$$\int_0^t (\beta_\epsilon(X_\epsilon(s)), Z(s)) ds \to \int_0^t (d\eta(s), Z(s)), \quad \forall\, Z \in C([0,T];K). \tag{16}$$

Letting $\epsilon \to 0$ into the identity

$$Y_\epsilon(t) + \int_0^t (AY_\epsilon(s + F(Y_\epsilon(s))) ds + \int_0^t \beta_\epsilon(Y_\epsilon(s) + W_A(s))) ds = x,$$

we see that (Y, η) satisfy (7). Finally, by (16) and the monotonicity of β_ϵ we have (recall that $\beta_\epsilon(Z(s)) = 0$),

$$(\beta_\epsilon(Y_\epsilon(s) + W_A(s)), Y_\epsilon(s) + W_A(s) - Z(s)) \geq 0, \quad \forall\, Z \in C([0,T;K),$$

we see that (8) holds for $X = Y + W_A$, i.e.,

$$\int_0^t (d\eta(s), Y(s) + W_A(s) - Z(s)) \geq 0, \quad \forall\, Z \in C([0,T];K). \tag{17}$$

On the other hand, by Itô's formula in (9) we get

$$\mathbb{E}|X_\epsilon(t)|^2 + \int_0^t \mathbb{E}\|X_\epsilon(s)\|^2 ds \leq C, \forall\, \epsilon > 0,$$

which clearly implies that $X \in C_W([0,T];H) \cap L^2_W(0,T,V)$. This completes the proof of existence.

Uniqueness. Assume that (Y_1, η_1), (Y_2, η_2) are two strong solutions of (1). Then by condition (iv) in Definition 1 we have

$$\int_0^t (d(\eta_1(s) - \eta_2(s)), Y_1(s) - Y_2(s))ds \geq 0 \quad \forall\, t \in [0,T].$$

This yields

$$\int_0^t d(Y_1(s) - Y_2(s), \\ \int_0^s (A(Y_1(\tau) - Y_2(\tau)) + F(X_1(\tau) - F(X_2(\tau))d\tau, Y_1(s) - Y_2(s)) \leq 0$$

and by integration we obtain that (see Lemma 1 below)

$$\frac{1}{2}\, |Y_1(t) - Y_2(t)|^2 + \int_0^t (A(Y_1 - Y_2) + F(X_1) - F(X_2), Y_1 - Y_2)ds \leq 0,$$

$\forall\, t \in [0,T]$, which implies via Gronwall's lemma that $Y_1 = Y_2$.

In particular, the latter implies that the sequence $\{\epsilon\}$ founded before is independent of ω and so there is indeed a unique pair satisfying Definition 1. ■

Lemma 1 *Let $y \in C([0,T];H) \cap L^2(0,T;V)$ be such that*

$$y(t) + \int_0^t Ay(s)ds \in BV([0,T];H).$$

Then

$$\int_0^t (d(y(s) + \int_0^s A(y(\tau)d\tau), y(s)) \\ = \frac{1}{2}\, |y(t)|^2 - \frac{1}{2}\, |x|^2 + \int_0^t (Ay(s), y(s))ds, \quad \forall\, t \in [0,T]. \quad (18)$$

Proof. Of course (18) is true if $y \in C([0,T];V)$ which is not, however, the case here. Approximating y by a sequence $\{y_n\} \in C([0,T];V)$ which is strongly convergent in $C([0,T];H)$ and such that the functions

$$t \to y_n(t) + \int_0^t Ay_n(s)ds,$$

have uniform bounded variation from $[0,T]$ to H, we may get (18) by passing to limit in the corresponding equality for y_n. ■

Remark 1 It is not clear whether Theorem 1 remains valid in absence of Hypothesis 1(iii) or for if $Q = I$. (This happens, however, for the obstacle problem, see Nualart and Pardoux, 1992; Zambotti, 2001).

3. Invariant measures

Let $X = X(t,x)$ be the solution of (1) obtained above. We denote by $P_t : C_b(H) \to C_b(H)$ the corresponding transition semigroup,

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad x \in K. \tag{19}$$

PROPOSITION 1 *Assume that Hypothesis 1 holds. Then there is at least one invariant measure ν for P_t with support included in $K \cap V$. If in addition there exists $\gamma_1 > 0$ such that*

$$(A(u-v) + F(u) - F(v), u-v) \geq \gamma_1 |u-v|^2, \quad \forall\, u,v \in D(A), \tag{20}$$

the invariant measure ν is unique.

Proof. We come back to the approximating equation (9) and denote by P_t^ϵ the corresponding transition semigroup, i.e.,

$$P_t^\epsilon\varphi(x) = \mathbb{E}[\varphi(X_\epsilon(t,x))], \quad x \in H, \tag{21}$$

and by N_ϵ the corresponding Kolmogorov operator,

$$(N_\epsilon\varphi)(x) = \frac{1}{2}\,\mathrm{Tr}\,[QD^2\varphi] - (Ax + F(x) + \beta_\epsilon(x), D\varphi(x)).$$

It is known that under our conditions there is at least one invariant measure ν_ϵ for P_t^ϵ. We claim that $\{\nu_\epsilon\}_{\epsilon>0}$ is tight. Indeed, by the invariance of ν_ϵ it follows that $\int_H (N_\epsilon\varphi)(x)\nu_\epsilon(dx) = 0$, so that

$$\int_H (\|x\|^2 + (F_\epsilon(x) + \beta_\epsilon(x), x)\, \nu_\epsilon(dx) \leq \mathrm{Tr}\, Q. \tag{22}$$

Since V is compactly embedded in H, we infer that $\{\nu_\epsilon\}_{\epsilon>0}$ is tight. Let ν be a weak limit point of $\{\nu_\epsilon\}_{\epsilon>0}$; then one can easily check that ν is an invariant measure for P_t and

$$\int_H (\|x\|^2 + (F(x), x))\, \nu(dx) \leq \mathrm{Tr}\, Q, \tag{23}$$

which implies supp $\nu \subset V$.

On the other hand, we have

$$\begin{aligned}(\beta_\epsilon(x), x) &= (\beta_\epsilon(x), x - \Pi_K(x)) + (\beta_\epsilon(x), \Pi_K(x))\\ &\geq (\beta_\epsilon(x), x - \Pi_K(x)) = \frac{1}{\epsilon}\, |x - \Pi_K(x)|^2\end{aligned}$$

and, taking into account (22), this implies

$$\int_{K^c} |x - \Pi_K(x)|^2 \nu(dx) = 0$$

and therefore supp $\nu \subset K$ as claimed. Finally, if (20) holds the invariant measure is unique by a standard argument (see Da Prato and Zabczyk, 1996). ■

4. The infinitesimal generator

This section is devoted to study the relationship between the infinitesimal generator N of P_t and the differential operator N_0 defined by (5).

To this end we shall also assume that K has a special form precised in Hypothesis 2 below

HYPOTHESIS 2 *There exists $T \in L(H)$ self-adjoint, positive and $r > 0$ such that*

$$K = \{x \in H : \ (Tx, x) \le r^2, \quad \forall\, x \in H\}.$$

Then the boundary of K is given by

$$\partial K = \{x \in H : \ (Tx, x) = r^2, \quad \forall\, x \in H\},$$

while $N_K(x) = \{\lambda T x\}_{\lambda > 0}$ is the normal cone to K.

THEOREM 2 *Assume that Hypotheses 1 and 2 are fulfilled. Let $\varphi \in C_b^2(H) \cap D(N)$ be such that $A\varphi \in C_b^1(H)$ and*

$$(D\varphi(x), Tx) = 0, \quad \text{on } \partial K.$$

Then $\varphi \in D(N)$ and

$$N\varphi = \frac{1}{2}\,\mathrm{Tr}\,[QD^2\varphi] - (x, AD\varphi) - (F(x), D\varphi). \tag{24}$$

Proof. Let $\varphi \in C_b^2(H) \cap D(N)$. By (9), applying Itô's formula, we have

$$\begin{aligned}
&\varphi(X_\epsilon(t)) - \varphi(x) + \int_0^t (AX_\epsilon(s) + F(X_\epsilon(s)), D\varphi(X_\epsilon(s)))ds\\
&+ \int_0^t (\beta_\epsilon(X_\epsilon(s)), D\varphi(X_\epsilon(s)))ds = \frac{1}{2}\int_0^t \mathrm{Tr}\,[QD^2\varphi(X_\epsilon(s))]ds\\
&+ \int_0^t (D\varphi(X_\epsilon(s)), \sqrt{Q}\, dW(s))ds.
\end{aligned} \tag{25}$$

Invoking (15) and (16) we have for $\epsilon \to 0$

$$\varphi(X_\epsilon(t)) \to \varphi(X(t)), \quad \text{uniformly in } t, \ \mathbb{P}\text{-a.s.}, \tag{26}$$

$$\begin{aligned}
&(AX_\epsilon(t) + F(X_\epsilon(t)), D\varphi(X_\epsilon(t))\\
&\to (X(t), AD\varphi(X(t))) + (F(X(t)), D\varphi(X(t))), \quad \mathbb{P}\text{-a.s.}.
\end{aligned} \tag{27}$$

$$\mathrm{Tr}\,[QD^2\varphi(X_\epsilon(t))] \to \mathrm{Tr}\,[QD^2\varphi(X(t))], \quad \mathbb{P}\text{-a.s.}. \tag{28}$$

$$\int_0^t (\beta_\epsilon(s), D\varphi(X_\epsilon(s)) \to \int_0^t (d\eta(s), D\varphi(X(s)), \quad \mathbb{P}\text{-a.s.}. \tag{29}$$

Then, letting $\epsilon \to 0$ in (25) we obtain by (27)–(29),

$$\begin{aligned}&\varphi(X(t)) - \varphi(x) + \int_0^t (AX(s) + F(X(s)), D\varphi(X(s)))ds \\ &+ \int_0^t (d\eta(s), D\varphi(X(s)))ds = \frac{1}{2} \int_0^t \text{Tr}\,[QD^2\varphi(X(s))]ds \\ &+ \int_0^t (D\varphi(X(s)), \sqrt{Q}\, dW(s))ds, \quad \mathbb{P}\text{-a.s..}\end{aligned} \tag{30}$$

We claim that

$$\int_0^t (d\eta(s), D\varphi(X(s)))ds = 0, \quad \forall\, t \in [0,T],\ \mathbb{P}\text{-a.s..} \tag{31}$$

Let $I = \{s \in (0,t) : X(s) \in \mathring{K}\}$ and $I^c = (0,t) \setminus I = \{s \in (0,t) : X(s) \in \partial K\}$. Then by (8) we see that

$$\int_0^t (d\eta(s), X(s) - \Pi_K(X(s) \pm \lambda D\varphi(X(s)))ds \geq 0, \quad \forall\, \lambda > 0, \tag{32}$$

which implies, for λ sufficiently small,

$$\mp \int_I (d\eta(s), D\varphi(X(s))) + \frac{1}{\lambda} \int_{I^c} (d\eta(s), X(s) - \Pi_K(X(s) \pm \lambda D\varphi(X(s))) \geq 0. \tag{33}$$

Now we want let $\lambda \to 0$ in the second term. For this we note that

$$\Pi_K(x) = \frac{rx}{|T^{1/2}x|}, \quad \forall\, x \in H \setminus K$$

and

$$D\Pi_K(x) = \frac{r}{|T^{1/2}x|} - r\frac{x \otimes Tx}{|T^{1/2}x|^3}, \quad \forall\, x \in H \setminus K.$$

So, if $s \in I^c$ we have

$$\begin{aligned}&\lim_{\lambda \to 0} \frac{1}{\lambda}\, (X(s) - \Pi_K(X(s) \pm \lambda D\varphi(X(s))) = \mp D\Pi_K(x) \cdot D\varphi(X(s)) \\ &= \mp D\varphi(X(s)) \pm r^{-2}(TX(s), D\varphi(X(s))T^{1/2}X(s) = \mp D\varphi(X(s)),\end{aligned}$$

because $X(s) \in \partial K$ and $(D\varphi(X(s)), Tx) = 0$ on ∂K.

Now, letting $\lambda \to 0$ in (33) yields

$$\int_0^t (d\eta(s), D\varphi(X(s)))ds = 0,$$

and (25) follows.

Finally, (30) becomes

$$\begin{aligned}&\varphi(X(t))-\varphi(x)+\int_0^t(AX(s)+F(X(s)),D\varphi(X(s)))ds\\&=\frac{1}{2}\int_0^t \mathrm{Tr}\,[QD^2\varphi(X(s))]ds+\int_0^t(D\varphi(X(s)),\sqrt{Q}\,dW(s))ds,\quad \mathbb{P}\text{-a.s.}\end{aligned}$$

and since $AD\varphi(X(s))\in C([0,T]\ H)$ the latter yields

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t}\,(\mathbb{E}[\varphi(X(t,x))]-\varphi(x))\\&=-(x,AD\varphi(x))-(F(x),D\varphi(x))+\frac{1}{2}\,\mathrm{Tr}\,[QD^2\varphi(x)=N_0\varphi(x),\end{aligned}$$

as claimed. ∎

5. An example

Consider equation (1) in $H=L^2(\mathcal{O})$

$$\begin{cases}dX-\Delta X dt+f(X)dt+\partial I_K(X)dt\ni\sqrt{Q}\,dW(t),\\ X(0)=x\quad\text{in}\ \ \mathcal{O},\quad X=0\quad\text{on}\ \ \partial\mathcal{O},\end{cases}\tag{34}$$

where $\mathcal{O}$ is an open bounded domain of $\mathbb{R}^d$,

$$K=\{x\in L^2(\mathcal{O}):\ |x|_{L^2(\mathcal{O})}\le 1\},$$

$f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function such that $f(r)r\ge-\gamma,\ \gamma>0$, for all $r\in\mathbb{R}$ and $Q=-A^{-l},\ l>0$, $A=-\Delta$, $D(A)=H^2(\mathcal{O})\cap H_0^1(\mathcal{O})$.

Here W is a Wiener process in $L^2(\mathcal{O})$,

$$W(t)=\sum_{j=1}^\infty\beta_j(t)e_j,$$

where (e_j) is an orthonormal basis of eigenfunctions for A, $Ae_j=\lambda_j e_j$ and (β_j) is a system of independent Brownian motions in a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge 0},\mathbb{P})$.

In order to satisfy Hypothesis 1(iii) we shall assume also that

$$\sum_{j=1}^\infty\lambda_j^{1-l}<\infty.$$

For $d=1$ the latter holds if $l>1/2$.

Then Theorem 1 applies and (34) has a unique solution $X(t,x)$ in the sense of Definition 1. So, we can consider the transition semigroup $P_t\varphi(x)=$

$\mathbb{E}[\varphi(X(t,x))]$. By Theorem 2 if $\varphi \in D(N_0)$, that is if it is sufficiently regular and satisfy a Neumann condition on the boundary of K, then the infinitesimal generator N associated with semigroup P_t has the explicit form

$$(N\varphi)(x) = \frac{1}{2}\sum_{j=1}^{\infty}\lambda_j^{-l}(D^2\varphi(x)e_j, e_j) - \sum_{j=1}^{\infty}\lambda_j(D\varphi(x), e_j) - (f(x), D\varphi(x)), \quad (35)$$

(Here $(\cdot,\cdot)$ is the scalar product in $H = L^2(\mathcal{O})$.)

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