Control and Cybernetics

vol. **37** (2008) No. 4

Generation of analytic semi-groups in L^2 for a class of second order degenerate elliptic operators*

by

Piermarco Cannarsa[1], Dario Rocchetti[1] and Judith Vancostenoble[2]

[1]Dipartimento di Matematica, Università di Roma “Tor Vergata”
Via della Ricerca Scientifica 1, 00133 Roma, Italy

[2]Institut de Mathématiques de Toulouse, UMR CNRS 5219
Université Paul Sabatier Toulouse III
118 route de Narbonne, 31 062 Toulouse Cedex 4, France

e-mail: cannarsa@mat.uniroma2.it, rocchett@@mat.uniroma2.it,
judith.vancostenoble@math.univ-toulouse.fr

Abstract: We study the generation of analytic semigroups in the L^2 topology by second order elliptic operators in divergence form, that may degenerate at the boundary of the space domain. Our results, that hold in two space dimensions, guarantee that the solutions of the corresponding evolution problems support integration by parts. So, this paper provides the basis for deriving Carleman type estimates for degenerate parabolic operators.

Keywords: degenerate parabolic equation, weighted Sobolev spaces, normal trace theorem, Hardy type inequality.

1. Introduction

In this work we give some wellposedness results, using the semigroup approach, for a class of second order parabolic problems, where the characteristic form of the related second order operator, A, can be degenerate at the boundary of the domain. We will study equations of the form

$$u_t = \underbrace{\operatorname{div}(a(x)\nabla u)}_{Au} + c(t,x)u + f(t,x), \quad (t,x) \in Q := (0,T)\times\Omega, \quad T > 0, \tag{1.1}$$

where

(1) Ω is a bounded open subset of $\mathbb{R}^2$ with boundary $\Gamma := \partial\Omega$ of class $\mathcal{C}^2$;

*Submitted: January 2008; Accepted: June 2008.

(2) $a(x) := \big(a_{ij}(x)\big)_{i,j=1}^{2}$ are symmetric matrices such that $a_{ij} \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ for each $i, j = 1, 2$ and

$$a(x)\xi \cdot \xi = \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j > 0 \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^2;$$

(3) $c \in L^\infty(Q)$ and $f \in L^2(Q)$.

The above equation is associated with the initial condition

$$u(0, x) = u_0(x) \in L^2(\Omega), \tag{1.2}$$

and, in the *weakly degenerate* case defined in Section 2.2, the boundary conditions of Dirichlet type

$$u = 0 \qquad \text{in } (0, T) \times \Gamma, \tag{1.3}$$

or of Neumann type

$$\nu \cdot a\nabla u = 0 \qquad \text{in } (0, T) \times \Gamma. \tag{1.4}$$

In the *strongly degenerate* case, (1.1) is associated with boundary conditions of Neumann type (1.4) only (see also Section 2.2).

In Section 2.2, we shall impose further conditions on $a(x)$. A model example of such a degenerate coefficient $a(x)$ is a matrix-valued function such that the corresponding differential operator Au, in a suitable local coordinate system preserving the boundary distance $d(x)$, namely

$$x = (s, \delta), \quad \delta = d(x),$$

takes the form

$$\mathcal{A}u = \frac{\partial}{\partial s}\left(\beta^{-1}\delta^{\kappa_1}\frac{\partial u}{\partial s}\right) + \frac{\partial}{\partial \delta}\left(\beta\, \delta^{\kappa_2}\frac{\partial u}{\partial \delta}\right), \tag{1.5}$$

where κ_1, κ_2 are non negative constants and β is a suitable strictly positive function which depends on Γ. We note that a similar class of operators— though not exactly the same one—was considered in several papers that studied spectral properties using pseudo-differential calculus, see, e.g., Egorov and Shubin (1994).

We will use the weighted Sobolev spaces $H^1_a(\Omega)$, $H^2_a(\Omega)$ (see Section 3) that are, for an operator of the form (1.5), given to

$$\begin{cases} \mathcal{H}_1(\mathcal{S}^+) := \left\{ v \in L^2(\mathcal{S}^+) \mid \delta^{\kappa_1/2}\dfrac{\partial v}{\partial s}\,,\ \delta^{\kappa_2/2}\dfrac{\partial v}{\partial \delta} \in L^2(\mathcal{S}^+) \right\}, \\ \mathcal{H}_2(\mathcal{S}^+) := \big\{ u \in \mathcal{H}_1(\mathcal{S}^+) \mid \mathcal{A}u \in L^2(\mathcal{S}^+) \big\}, \\ \mathcal{S}^+ := (-s_0, s_0) \times (0, +\infty), \quad s_0 > 0\,. \end{cases}$$

Similar $H^1_a(\Omega)$ spaces were introduced in the sixties by Fichera (1956) and Oleinik and Radkewitch (1973) to study second order operators with nonnegative characteristic. So, in some sense, the subject of this paper can be regarded as a special case of the theory developed in Fichera (1956), Oleinik (1966), Oleinik and Radkewitch (1973). On the other hand, due to the specific features of the problem under investigation, the results we obtain here are much stronger than the ones obtained for general degenerate operators.

For instance, we show that, in the above spaces, the analogues of standard extension results for traces and normal traces hold true thanks to a suitable Hardy type inequality (see, e.g., Alabau-Boussouira et al., 2006; Davies, 1995; Martinez and Vancostenoble, 2006). Then, we derive a semigroup generation result that, in turn, yields that problem {(1.1),(1.2)}, associated (1.3) or (1.4), is well-posed. Finally, we provide maximal regularity estimates for the solution of a such problem.

The main motivation of this work is to provide wellposedness results in spaces that are suitable for integration by parts (see Section 4). Therefore, this paper can also be viewed as a preliminary step to the analysis of null controllability for degenerate parabolic operators in arbitrary space dimension. Indeed, as is well-known for uniformly parabolic operators and for degenerate operators in dimension 1 (see Alabau-Boussouira et al., 2006; Cannarsa et al., 2004, 2007; Martinez and Vancostenoble, 2006), a key tool for such an analysis are Carleman estimates, whose deduction heavily relies on integration by parts.

2. Assumptions

2.1. Assumptions on Ω

In the following, Ω is a bounded open set in $\mathbb{R}^2$ with boundary, Γ, of class $\mathcal{C}^r$, $r \geq 2$, and $d(x)$ represents the distance from Γ, that is,

$$d(x) := \min_{y\in\Gamma} |x - y| \qquad x \in \mathbb{R}^2 .$$

Moreover we name for every $\delta \geq 0$

$$\Omega_\delta := \{x \in \Omega \mid d(x) < \delta\}, \quad \Omega^\delta := \Omega \setminus \Omega_\delta, \quad \Gamma^\delta := \partial\Omega^\delta.$$

Since Γ is compact and at least of class $\mathcal{C}^2$, for some number $\delta_0 \in (0, 1)$ we have that

$$\forall x \in \Omega_{\delta_0} \quad \exists! y_x \in \Gamma \quad \text{such that} \quad d(x) = |x - y_x| .$$

We will set $y_x = p_\Gamma x$ (the projection of x onto Γ). Further, as is well-known,

$$\nabla d(x) = -\nu(p_\Gamma x) \qquad \forall x \in \Omega_{\delta_0}$$

where $\nu(p_\Gamma x)$ denotes the outward unit normal to Ω at $p_\Gamma x$ (see Section 7.1 for more details).

2.2. Assumptions on a

First, we give some preliminary notations. Let $\mathcal{O}$ be a subset of $\mathbb{R}^2$ with sufficiently smooth boundary we define

$$\mathcal{M}_2(\mathbb{R}) := \{2 \times 2 \text{ real matrices } m = (m_{ij})_{i,j=1}^2\},$$
$$\mathcal{S}_2(\mathbb{R}) := \{m \in \mathcal{M}_2(\mathbb{R}) \mid m_{ij} = m_{ji}\},$$
$$C^r(\mathcal{O}; \mathcal{M}_2(\mathbb{R})) := \{\mathcal{O} \ni x \mapsto m(x) \in \mathcal{M}_2(\mathbb{R}) \mid m_{ij} \in C^r(\mathcal{O})\},\ r \geq 0.$$

We also denote by $\lambda_i(x,m)$, $E_i(x,m)$, $(i = 1,2)$ the eigenvalues and associated eigenspaces of a matrix-valued function $x \mapsto m(x) \in S_2(\mathbb{R})$ defined on $\mathcal{O}$. We recall that, if $m(x) \in S_2(\mathbb{R})$, we can choose at least two different orthonormal eigenbases

$$\varepsilon_1(x,m) \in E_1(x,m),\ \varepsilon_2(x,m) \in E_2(x,m)$$

preserving the orientation of $\mathbb{R}^2$ [1]. In the following, we shall call a *determination of unit eigenvectors* one choice among all these bases. Further, for simplicity we will set

$$\lambda_i(x) := \lambda_i(x,a), \quad \varepsilon_i(x) := \varepsilon_i(x,a), \quad i = 1,2$$

if a is the coefficient of the second order operator in (1.1).

We shall assume that

(1) $a \in C^0(\overline{\Omega}; \mathcal{S}_2(\mathbb{R})) \cap C^1(\Omega; \mathcal{S}_2(\mathbb{R}))$;

(2) $a(x)\xi \cdot \xi > 0 \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^2 \quad$ (i.e. $a(x) > 0\ \forall x \in \Omega$);

(3) for all $x \in \Omega_{\delta_0} \cup \Gamma$ there exists a determination of unit eigenvectors $\varepsilon_1(x), \varepsilon_2(x)$ such that

$$\varepsilon_2(x) = -\nu(p_\Gamma x);$$

(4) there exists a constant $C \in (0,1]$ such that

$$a(x)\xi \cdot \xi \geq C\, a(x_\delta)\xi \cdot \xi \qquad \forall \xi \in \mathbb{R}^2\,,$$

where

$$x \in \Omega_{\delta_0}, \quad x_\delta := x + \delta\, \nu(p_\Gamma x), \quad \delta \in [0, d(x)]\,.$$

In this work we shall admit two types of degeneracy for $a(x)$:

– *The weakly degenerate case (WD):*

$\exists\, \alpha \in [0,1)$, $c_0 > 0$ such that

$$a(x)\xi \cdot \xi \geq c_0\, d(x)^\alpha |\xi|^2 \quad \forall\, \xi \in \mathbb{R}^2,\ x \in \Omega_{\delta_0};$$

[1] Let $v_1, v_2 \in \mathbb{R}^2$ a orthonormal basis of $\mathbb{R}^2$ and $[v_1|v_2] \in \mathcal{M}_2(\mathbb{R})$ such that the j-th column is equal to v_j. So, (v_1, v_2) preserves the orientation of $\mathbb{R}^2$ iff $\det\,[v_1|v_2] = 1$.

if $\alpha > 0$, $\exists\, 0 < \vartheta \le \alpha$ such that, for all $x \in \Omega_{\delta_0}$, the function

$$[0, d(x)) \ni \delta \longmapsto \lambda_2(x_\delta)/d(x_\delta)^\vartheta$$

is nondecreasing.

– *The strongly degenerate case (SD):*

$\exists C_0 > 0$ such that $\lambda_2(x) \le C_0\, d(x)\ \forall x \in \Omega_{\delta_0}$.

Notice that the (WD) case subsumes the nondegenerate case ($a(x) > 0\ \forall x \in \overline{\Omega}$).

Now, we give some remarks concerning the above assumptions.

(i) We recall that, if $a(x) \in \mathcal{S}_2(\mathbb{R})$ then, choosing an orthonormal eigenbasis $\varepsilon_1(x), \varepsilon_2(x)$ of $a(x)$ we can write[2]

$$a(x) = \sum_{i=1}^{2} \lambda_i(x)\, \varepsilon_i(x) \otimes \varepsilon_i(x)\,.$$

Notice that this representation formula does not depend on the particular eigenbasis.

Moreover, since $a(x) \in \mathcal{S}_2(\mathbb{R})$ and it satisfies assumption (2), then there exists a unique $\sigma : \overline{\Omega} \to \mathcal{S}_2(\mathbb{R})$ such that

$$a(x) = \sigma(x)\sigma(x) \quad \forall x \in \overline{\Omega}, \quad \sigma(x) > 0 \quad \forall x \in \Omega\,.$$

Indeed, as well known, we can choose

$$\sigma(x) = \sum_{i=1}^{2} \sqrt{\lambda_i(x)}\, \varepsilon_i(x) \otimes \varepsilon_i(x)\,.$$

One can prove (see, e.g., Bellman, 1960; Freidlin, 1985) that the element $\sigma_{ij}(x)$ possess the same regularity in $x \in \Omega$ as do the elements of the matrix $a(x)$. Of course, since $a \in C^0(\overline{\Omega}; \mathcal{S}_2(\mathbb{R}))$ and it satisfies assumption (2), then $\sqrt{\lambda_i(x)}$ are continuous in $\overline{\Omega}$. So, we have that

$$\sigma \in C^0(\overline{\Omega}; \mathcal{S}_2(\mathbb{R})) \cap C^1(\Omega; \mathcal{S}_2(\mathbb{R}))\,.$$

(ii) Assumption (4) is needed to prove the density result of Proposition 3.1. A class of coefficients with this property is given by all maps a satisfying properties (1),(2),(3) and such that there exist two strictly positive functions

$$\{b_1, b_2\} \subset C^0(\Omega_{\delta_0} \cup \Gamma),$$

such that for each $x \in \Omega_{\delta_0}$, $i \in \{1, 2\}$, the functions

$$[0, d(x)] \ni \delta \longmapsto b_i(x_\delta)$$

[2]The tensor product of two vectors p, q of $\mathbb{R}^2$ is defined as $(p \otimes q)(x) := p\,(q \cdot x)\ \forall x \in \mathbb{R}^2$.

are nonincreasing and such that

$$b_i(x_\delta) \le \lambda_i(x_\delta) \le C^{-1} b_i(x_\delta).$$

Indeed, it follows that

$$\lambda_i(x) \ge b_i(x) \ge b_i(x_\delta) \ge C\lambda_i(x_\delta), \quad i = 1, 2,$$

for all $x \in \Omega_{\delta_0}$, $\delta \in [0, d(x)]$.

So, denoting by $(\xi_1(x), \xi_2(x))$ the coordinates of $\xi \in \mathbb{R}^2$ relative to the orthonormal eigenbasis $\varepsilon_1(x), \varepsilon_2(x)$, one has

$$\begin{aligned} a(x)\xi \cdot \xi &= \lambda_1(x)|\xi_1(x)|^2 + \lambda_2(x)|\xi_2(x)|^2 \\ &\ge C\big(\lambda_1(x_\delta)|\xi_1(x_\delta)|^2 + \lambda_2(x_\delta)|\xi_2(x_\delta)|^2\big) = Ca(x_\delta)\xi \cdot \xi. \end{aligned}$$

(iii) Another interesting class of coefficients is given by all matrices $a(x)$ satisfying properties (1),(2),(3) such that

$$\lambda_1(x) = p_1(x)d(x)^{\kappa_1}, \ \lambda_2(x) = p_2(x)d(x)^{\kappa_2} \quad \text{on } \Omega_{\delta_0} \cup \Gamma,$$

where p_1, p_2 are strictly positive smooth functions and $\kappa_1, \kappa_2 \ge 0$. Obviously, in this case property (4) is satisfied and

$$\begin{cases} a(x) \text{ is (WD)} \ \Leftrightarrow \quad \kappa_1, \kappa_2 \in [0, 1), \\ a(x) \text{ is (SD)} \ \Leftrightarrow \quad \kappa_2 \ge 1. \end{cases}$$

2.3. Some examples

Let $\Omega = \mathcal{B}(0, 1)$ be the unitary ball in $\mathbb{R}^2$ and define

$$\lambda_1(x) := d(x)^{\kappa_1}, \quad \lambda_2(x) := d(x)^{\kappa_2}, \quad \kappa_1, \kappa_2 \ge 0.$$

Observe that, in polar coordinates (ρ, θ), we can write $\lambda_i(\rho, \theta) = (1 - \rho)^{\kappa_i}$.

We also define for each $(\rho, \theta) \in (0, 1] \times [0, 2\pi)$ the following vector fields

$$\varepsilon_1(\rho, \theta) := (-\sin\theta, \cos\theta), \quad \varepsilon_2(\rho, \theta) := -(\cos\theta, \sin\theta). \tag{2.1}$$

With some computations one has

$$a(\rho, \theta) := \sum_{i=1}^{2} (1 - \rho)^{\kappa_i} \varepsilon_i(\rho, \theta) \otimes \varepsilon_i(\rho, \theta) =$$

$$\begin{bmatrix} (1-\rho)^{\kappa_1}(\cos\theta)^2 + (1-\rho)^{\kappa_2}(\sin\theta)^2 & \sin\theta\cos\theta\big((1-\rho)^{\kappa_2} - (1-\rho)^{\kappa_1}\big) \\ \sin\theta\cos\theta\big((1-\rho)^{\kappa_2} - (1-\rho)^{\kappa_1}\big) & (1-\rho)^{\kappa_2}(\cos\theta)^2 + (1-\rho)^{\kappa_1}(\sin\theta)^2 \end{bmatrix}. \tag{2.2}$$

Since

$$\begin{cases} \rho(x_1, x_2) = |x| = (x_1^2 + x_2^2)^{1/2} \\ \cos\theta(x_1, x_2) = x_1|x|^{-1} \\ \sin\theta(x_1, x_2) = x_2|x|^{-1} \end{cases} \tag{2.3}$$

we have that, if $\kappa_1 \neq \kappa_2$, then the matrix in (2.2) can be degenerate on $\Gamma = \partial\mathcal{B}(0,1)$, it is bounded in $\overline{\mathcal{B}(0,1)}$, the versor fields

$$\varepsilon_1(x) = (-x_2|x|^{-1}, x_1|x|^{-1}), \quad \varepsilon_2(x) = -(x_1|x|^{-1}, x_2|x|^{-1})$$

are for all $x \in \overline{\mathcal{B}(0,1)} \setminus \{0\}$ a determination of the eigenvectors of $a(x)$, but, of course, a it is not defined at the origin.

Instead, in the case of $\kappa_1 = \kappa_2 = \kappa$, using (2.2), (2.3) we can write

$$a_\kappa(x) = \begin{bmatrix} (1-|x|)^\kappa & 0 \\ 0 & (1-|x|)^\kappa \end{bmatrix}, \quad x \in \overline{\Omega}.$$

This matrix-valued function has the same regularity as $d(x)^\kappa$, and the basis $(\varepsilon_1, \varepsilon_2)$ is a determination of the eigenvectors of $a(x)$ for all $x \in \overline{\mathcal{B}(0,1)} \setminus \{0\}$.

Now, we want to give an example of a matrix-valued function on $\overline{\mathcal{B}(0,1)}$ satisfying assumptions (1),(2),(3),(4). For this purpose we consider a smooth function $\chi : \mathcal{B}(0,1) \to [0,1]$ such that, for some $0 < \delta_0 < 1$, $0 < \delta_1 < 1 - \delta_0$,

$$\chi(x) = \begin{cases} 1 & x \in \mathcal{B}(0, \delta_1), \\ 0 & x \in \mathcal{B}(0,1) \setminus \mathcal{B}(0, 1-\delta_0). \end{cases}$$

Choosing $\varepsilon_1(\rho,\theta)$, $\varepsilon_2(\rho,\theta)$ as in (2.1), we define

$$\begin{aligned} a(\rho,\theta) &:= \chi(\rho,\theta) \sum_{i=1}^{2} \varepsilon_i(\rho,\theta) \otimes \varepsilon_i(\rho,\theta) \\ &\qquad + (1-\chi(\rho,\theta)) \sum_{i=1}^{2} (1-\rho)^{\kappa_i} \varepsilon_i(\rho,\theta) \otimes \varepsilon_i(\rho,\theta) \\ &= \chi(\rho,\theta) I_2 \ + \ (1-\chi(\rho,\theta)) a(\rho,\theta). \end{aligned}$$

Then, χI_2, $(1-\chi)a \in C^0\big(\overline{\Omega}; \mathcal{S}_2(\mathbb{R})\big) \cap C^\infty\big(\Omega; \mathcal{S}_2(\mathbb{R})\big)$ and a is positive definite in Ω.

We can generalize the previous example, by taking Ω as in Section 2.1 and choosing orthogonal versor fields $\varepsilon_1, \varepsilon_2$ on $\mathbb{R}^2$ such that

$$\varepsilon_2(x) = -\nu(p_\Gamma x) \quad x \in \Omega_{\delta_0} \cup \Gamma$$

and functions $\lambda_1, \lambda_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ such that λ_1, λ_2 are strictly positive in Ω and for each $x \in \Omega_{\delta_0}$ the functions

$$[0, d(x)] \ni \delta \longmapsto \lambda_i(x_\delta)$$

are nonincreasing.

Let $\chi \in \mathcal{D}(\Omega; [0,1])$ be a cut-off function such that

$$\chi(x) = \begin{cases} 1 & x \in \Omega^{\delta_1} \\ 0 & x \in \Omega_{\delta_2}\,, \end{cases}$$

where $0 < \delta_2 < \delta_1 < \delta_0$. Then, the matrix valued function defined by

$$a := \chi \sum_{i=1}^{2} \varepsilon_i \otimes \varepsilon_i + (1-\chi) \sum_{i=1}^{2} \lambda_i \varepsilon_i \otimes \varepsilon_i = \chi I_2 + (1-\chi) \sum_{i=1}^{2} \lambda_i \varepsilon_i \otimes \varepsilon_i$$

satisfies assumptions (1),(2),(3),(4) of Section 2.2.

3. Main functional spaces

Definition 3.1 $H^1_a(\Omega) := \{v \in L^2(\Omega) \;\mid\; a\nabla v \cdot \nabla v \in L^1(\Omega)\}$ *is endowed with the norm*

$$\|v\|^2_{H^1_a(\Omega)} := \|v\|^2_{L^2(\Omega)} + \|a\nabla v \cdot \nabla v\|_{L^1(\Omega)}. \tag{3.1}$$

Here, ∇v *is the distributional gradient of* v.

An equivalent definition of $H^1_a(\Omega)$ is the following

$v \in H^1_a(\Omega)$ *iff* $v \in L^2(\Omega)$ *and there exists* $h = (h_1, h_2) \in L^1_{loc}(\Omega)^2$ *such that*

$$\int_\Omega v \, \frac{\partial \varphi}{\partial x_i} \, dx = - \int_\Omega h_i \, \varphi \, dx \quad \forall \varphi \in \mathcal{D}(\Omega), \; i = 1, 2$$

and

$$\int_\Omega a(x) h \cdot h \, dx < \infty.$$

Observe that $H^1(\Omega) \subset H^1_a(\Omega) \subset H^1_{\mathrm{loc}}(\Omega)$. Moreover, we will prove the following result (see Section 7.2):

Proposition 3.1 $H^1_a(\Omega)$ *is a Hilbert space. Furthermore,* $\mathcal{C}^\infty(\overline{\Omega})$ *is dense in* $H^1_a(\Omega)$.

Definition 3.2 $H^1_{a,0}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\,\|\cdot\|_{H^1_a(\Omega)}}$.

Let us mention that, in the (SD) case, one can prove that $H^1_{a,0}(\Omega) = H^1_a(\Omega)$. On the other hand, in the (WD) case the space $H^1_{a,0}(\Omega)$ may also be explicitly characterized (see later on in Propositions 5.2, 5.4). Lastly we define the following pre-Hilbert space:

DEFINITION 3.3 *$H^2_a(\Omega) := \{u \in H^1_a(\Omega) \mid div(a\nabla u) \in L^2(\Omega)\}$ endowed with the norm*

$$\|u\|^2_{H^2_a(\Omega)} := \|u\|^2_{H^1_a(\Omega)} + \|div(a\nabla u)\|^2_{L^2(\Omega)}. \tag{3.2}$$

An equivalent definition of $H^2_a(\Omega)$ is the following

$u \in H^2_a(\Omega)$ iff $u \in H^1_a(\Omega)$ and there exists $g \in L^2(\Omega)$ such that

$$\int_\Omega a\nabla u \cdot \nabla\varphi\, dx = -\int_\Omega g\,\varphi\, dx \quad \forall\varphi \in \mathcal{D}(\Omega).$$

In Section 7.3 we will prove the following

LEMMA 3.1 *$H^2_a(\Omega)$ is a Hilbert space. Moreover, $H^2_a(\Omega) \subset H^2_{loc}(\Omega)$.*

We observe that in the (WD) case, we may have $H^2(\Omega) \not\subseteq H^2_a(\Omega)$. Indeed, by choosing, for example

$$\lambda_1(x) = 1, \quad \lambda_2(x) = d(x)^{1/2}, \quad x \in \Omega_{\delta_0},$$

and a function $v \in C^2(\Omega)$ such that

$$v(x) = 1 + d(x), \quad x \in \Omega_{\delta_0},$$

one has that $v \notin H^2_a(\Omega)$.

4. Trace operators

In this section we recall the standard theory of trace and normal trace operators (for details see, e.g., Adams, 1975; Baiocchi and Capelo, 1983; Lions and Magenes, 1972; Necas, 1967; Showalter, 1977; Temam, 1977), and we extend the standard Normal Trace Theorem to more general function spaces.

4.1. Standard trace theory

Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain with boundary Γ and let $\varphi \in C^\infty(\overline{\Omega})$. Then, since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, the map $\varphi \mapsto \varphi_{|\Gamma}$ can be extended to a continuous map $\gamma \in \mathcal{L}(H^1(\Omega); L^2(\Gamma))$. Moreover, by defining the *trace space* $H^{1/2}(\Gamma)$ by

$$H^{1/2}(\Gamma) := \gamma(H^1(\Omega)), \quad \|\phi\|_{H^{1/2}(\Gamma)} := \inf\{\|v\|_{H^1(\Omega)} \mid v \in H^1(\Omega),\ \gamma v = \phi\},$$

one has that $H^{1/2}(\Gamma)$ is a Banach space, the injection of $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$ is continuous with dense range, and, by definition, $\gamma \in \mathcal{L}(H^1(\Omega); H^{1/2}(\Gamma))$. Furthermore, we will denote by $(H^1(\Omega))'$ the dual of $H^1(\Omega)$, and by $H^{-1/2}(\Gamma)$ the dual of $H^{1/2}(\Gamma)$.

4.2. Standard normal trace theory

Let us consider the Hilbert space

$$H_{\rm div}(\Omega) := \{w \in L^2(\Omega)^2 \ \mid \ {\rm div}\,(w) \in L^2(\Omega)\},$$

and, for $w \in H_{\rm div}(\Omega)$, the linear functional

$$\mathcal{T}_w\, v := \int_\Omega {\rm div}\,(w) v + w \cdot \nabla v \, dx\,, \quad v \in H^1(\Omega).$$

By standard theory, we have :

$$\forall w \in H_{\rm div}(\Omega), \qquad \mathcal{T}_w \in (H^1(\Omega))'.$$

Moreover, since $C^\infty(\overline{\Omega})^2$ is dense in $H_{div}(\Omega)$, there exists a unique normal trace operator $\gamma_\nu \in \mathcal{L}(H_{\rm div}(\Omega); H^{-1/2}(\Gamma))$ such that

$$\begin{cases} \gamma_\nu w = (\nu \cdot w)_{|\Gamma} & \text{if } w \in \mathcal{C}^\infty(\overline{\Omega})^2; \\ \mathcal{T}_w\, v = \langle \gamma_\nu w, \gamma v\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} & \text{for all } w \in H_{\rm div}(\Omega),\ v \in H^1(\Omega). \end{cases}$$

4.3. Extension to a more general space

We introduce the pre-Hilbert space

$$H_{{\rm div},a}(\Omega) := \{w \in L^2_{a^{-1}}(\Omega) \ \mid \ {\rm div}\,(w) \in L^2(\Omega)\},$$

where

$$L^2_{a^{-1}}(\Omega) := \{w \in L^2(\Omega)^2 \ \mid \ a^{-1} w \cdot w \ \in L^1(\Omega)\},$$

endowed by the norm

$$\|w\|^2_{H_{div,a}(\Omega)} := \left\|a^{-1} w \cdot w\right\|_{L^1(\Omega)} + \|div(w)\|^2_{L^2(\Omega)}\,. \tag{4.1}$$

Obviously, one has that $H_{{\rm div},a}(\Omega) \subset H_{\rm div}(\Omega)$. Hence, for all $w \in H_{{\rm div},a}(\Omega)$, $\mathcal{T}_w$ is defined. We will prove (see Section 7.4) the following

LEMMA 4.1 *$H_{div,a}(\Omega)$ is a Hilbert space. Moreover, if $w \in H_{div,a}(\Omega)$, then $\mathcal{T}_w \in (H^1_a(\Omega))'$ where $(H^1_a(\Omega))'$ denotes the dual space of $H^1_a(\Omega)$. Furthermore, the following integration by parts formula holds :*

$$\forall (w, v) \in H_{div,a}(\Omega) \times H^1_a(\Omega),$$
$$\int_\Omega w \cdot \nabla v \, dx = - \int_\Omega div\,(w) v \, dx + \mathcal{T}_w\, v.$$

Finally, the number $\mathcal{T}_w\, v$ is characterized by

$$\mathcal{T}_w\, v = \lim_{\delta \to 0} \langle \gamma^\delta_\nu w, \gamma^\delta v\rangle_{H^{-1/2}(\Gamma^\delta), H^{1/2}(\Gamma^\delta)},$$

where, for all $\delta \in (0, \delta_0)$, γ^δ and γ_ν^δ are, respectively, defined as follows:

$$\begin{cases} \gamma^\delta := \gamma_\delta \circ r_\delta \text{ where} \\ r_\delta \ : H_a^1(\Omega) \ni \ v \mapsto v_{|\Omega^\delta} \in H^1(\Omega^\delta) \text{ is the restriction operator and} \\ \gamma_\delta \in \mathcal{L}(H^1(\Omega^\delta); H^{1/2}(\Gamma^\delta)) \text{ is the standard trace operator,} \end{cases}$$

and

$$\begin{cases} \gamma_\nu^\delta := \gamma_{\nu,\delta} \circ R_\delta \text{ where} \\ R_\delta \ : H_{div,a}(\Omega) \ni \ v \mapsto v_{|\Omega^\delta} \in H_{div}(\Omega^\delta) \text{ is the restriction operator and} \\ \gamma_{\nu,\delta} \in \mathcal{L}(H_{div}(\Omega^\delta); H^{-1/2}(\Gamma^\delta)) \text{ is the standard normal trace operator.} \end{cases}$$

Observe that using the above definition of $H_{\mathrm{div},a}(\Omega)$, space $H_a^2(\Omega)$ may also be characterized by

$$H_a^2(\Omega) = \{u \in H_a^1(\Omega) \ \mid \ a\nabla u \in H_{\mathrm{div},a}(\Omega)\}.$$

Thus, we have :

Corollary 4.1 *For all $u \in H_a^2(\Omega)$, $\mathcal{T}_{a\nabla u} \in (H_a^1(\Omega))'$ and the following integration by parts formula holds :*

$$\forall (u,v) \in H_a^2(\Omega) \times H_a^1(\Omega),$$
$$\int_\Omega a\nabla u \cdot \nabla v \, dx = -\int_\Omega div\,(a\nabla u) v \, dx + \mathcal{T}_{a\nabla u}\, v.$$

We recall that, in the general case, $\mathcal{T}_{a\nabla u}\, v$ is given by

$$\mathcal{T}_{a\nabla u}\, v = \lim_{\delta \to 0} \langle \gamma_\nu^\delta(a\nabla u), \gamma^\delta v\rangle_{H^{-1/2}(\Gamma^\delta), H^{1/2}(\Gamma^\delta)}.$$

Let us mention that in the (WD) case, $\mathcal{T}_{a\nabla u}\, v$ may also be characterized by

$$\mathcal{T}_{a\nabla u}\, v = \langle \gamma_\nu^a(a\nabla u), \gamma^a v\rangle_{H_a^{-1/2}(\Gamma), H_a^{1/2}(\Gamma)},$$

where γ^a and γ_ν^a are the operators defined below (see the following section).

5. Trace extensions on $H_a^1(\Omega)$

Proposition 5.1 (Trace in $H_a^1(\Omega)$) *In the (WD) case, there exists a unique trace operator $\gamma^a \in \mathcal{L}(H_a^1(\Omega); L^2(\Gamma))$ that extends the standard one $\gamma \in \mathcal{L}(H^1(\Omega); L^2(\Gamma))$.*

On the contrary, we observe that in the (SD) case, in general it is not possible to define the notion of trace. Indeed, upon choosing, for example,

$$\lambda_1(x) = 1, \quad \lambda_2(x) = d(x), \quad x \in \Omega_{\delta_0},$$

any function $v \in C^1(\Omega)$ such that

$$v(x) = \log\Big(\big|\log\big(d(x)\big)\big|\Big), \quad x \in \Omega_{\delta_0},$$

satisfies $v \in H^1_a(\Omega)$ but $v = \infty$ on Γ.

PROPOSITION 5.2 *In the (WD) case, there exists a constant $C_H > 0$ such that*

$$\forall v \in Ker\{\gamma^a\}, \quad \int_\Omega v^2 \frac{\lambda_2(x)}{d(x)^2}\,dx \;\leq\; C_H \int_\Omega a(x)\nabla v \cdot \nabla v\,dx. \tag{5.1}$$

Moreover, the space $H^1_{a,0}(\Omega)$ may be characterized by

$$H^1_{a,0}(\Omega) = Ker\{\gamma^a\} := \{v \in H^1_a(\Omega) \;\mid\; \gamma^a v = 0\}.$$

Notice that the Hardy-type inequality (5.1) extends the Poincaré inequality to the space $H^1_a(\Omega)$. As a consequence, the Hilbert space $H^1_{a,0}(\Omega)$ may be endowed with the following norm

$$\forall v \in H^1_{a,0}(\Omega), \qquad \|v\|_{H^1_{a,0}(\Omega)} := \|a\nabla v \cdot \nabla v\|_{L^1(\Omega)},$$

that is equivalent on $H^1_{a,0}(\Omega)$ to the previous norm $\|\cdot\|_{H^1_a(\Omega)}$.

DEFINITION 5.1 (TRACE SPACE)

$$H^{1/2}_a(\Gamma) := \gamma^a(H^1_a(\Omega))$$

is a Banach space endowed with the norm

$$\forall \phi \in H^{1/2}_a(\Gamma), \qquad \|\phi\|_{H^{1/2}_a(\Gamma)} := inf\,\{\|v\|_{H^1_a(\Omega)} \;\mid\; v \in H^1_a(\Omega),\ \gamma^a v = \phi\}.$$

We also denote by $H^{-1/2}_a(\Gamma)$ the dual space of $H^{1/2}_a(\Gamma)$.

PROPOSITION 5.3 *In the (WD) case , there exists a unique normal trace operator $\gamma^a_\nu \in \mathcal{L}(H_{div,a}(\Omega); H^{-1/2}_a(\Gamma))$ such that*

$$\begin{cases} \gamma^a_\nu w = (\nu \cdot w)_{|\Gamma} & \textit{if } w \in \mathcal{C}^\infty(\overline{\Omega})^2; \\ \mathcal{T}_w\, v = \langle \gamma^a_\nu w, \gamma^a v\rangle_{H^{-1/2}_a(\Gamma), H^{1/2}_a(\Gamma)} & \textit{for all } w \in H_{div,a}(\Omega),\ v \in H^1_a(\Omega). \end{cases}$$

Notice that, since $H_{div,a}(\Omega) \subset H_{div}(\Omega)$, then

$$\gamma^a_\nu w = \gamma_\nu w \quad \forall w \in H_{div,a}(\Omega)\,.$$

PROPOSITION 5.4 *In the (SD) case one has that*

(1) $H_{div,a}(\Omega) \subseteq Ker\{\gamma_\nu\} := \{w \in H_{div}(\Omega) \;\mid\; \gamma_\nu w = 0\}$*;*

(2) $\forall\,(u,v) \in H^2_a(\Omega) \times H^1_a(\Omega)$,

$$\int_\Omega a\nabla u \cdot \nabla v\,dx = -\int_\Omega div\,(a\nabla u)v\,dx\,;$$

(3) $H^1_a(\Omega) = H^1_{a,0}(\Omega)$.

6. Well-posedness

6.1. The degenerate problem

Let us fix $T > 0$ and introduce the notations $Q := (0,T)\times\Omega$ and $\Sigma := (0,T)\times\Gamma$. We are interested in the following evolution equation

$$u_t - \text{ div } (a(x)\nabla u) + c(t,x)u = f(t,x) \qquad \text{in } Q, \tag{6.1}$$

where f is given in $L^2(Q)$ and $c \in L^\infty(Q)$. We associate with this equation the initial condition

$$u(0,x) = u_0(x) \in L^2(\Omega), \tag{6.2}$$

and boundary conditions of Dirichlet type

$$\gamma^a u = 0 \qquad \text{in } \Sigma, \tag{6.3}$$

or of Neumann type

$$\gamma_\nu(a\nabla u) = 0 \qquad \text{in } \Sigma. \tag{6.4}$$

The choice of the boundary conditions depends on the way $a(x)$ degenerates at the boundary. If $a(x)$ is (WD), it is possible to consider both boundary conditions. Indeed, by the results established in the previous sections we know that, as in theory of uniformly parabolic equations, we can define trace operators γ^a, γ^a_ν. So, by standard methods well-posedness follows. On the other hand, trace operator γ^a does not make sense when $a(x)$ is (SD). Moreover, by Proposition 5.4 we know that $H^1_a(\Omega) = H^1_{a,0}(\Omega)$; so, $H^1_{a,0}(\Omega)$ is not a suitable space to deal with homogeneous Dirichlet boundary conditions. Hence, when $a(x)$ is (SD), we only consider the Neumann boundary condition (6.4). We now give the main result of the paper:

THEOREM 6.1 *One has that*

(1) *in both (WD) and (SD) cases, the operator* $(A_1, D(A_1))$ *given by*

$$A_1 u = div(a\nabla u), \quad D(A_1) = \{u \in H^2_a(\Omega) \ | \ \gamma_\nu(a\nabla u) = 0\},$$

is m-dissipative and self-adjoint. Moreover, in the (SD) case,

$$D(A_1) = H^2_a(\Omega).$$

(2) *In the (WD) case, the operator* $(A_2, D(A_2))$ *given by*

$$A_2 u = div(a\nabla u), \quad D(A_2) = H^2_a(\Omega) \cap H^1_{a,0}(\Omega)$$

is m-dissipative and self-adjoint. Moreover, A_2 *is strictly dissipative, that is*

$$\sup \Big\{ < A_2 u, u >_{L^2(\Omega)} \ | \ u \in D(A_2), \ \|u\|_{L^2(\Omega)} = 1 \Big\} < 0.$$

As a consequence, both A_1 and A_2 are the infinitesimal generators of the strongly continuous semigroups denoted by e^{tA_1}, e^{tA_2} respectively. We will also show (see Section 7.9) that e^{tA_1}, e^{tA_2} are analytic. Moreover, the family of operators in $\mathcal{L}(L^2(\Omega))$ given by

$$C(t)u := c(t,.)u, \quad t \in (0,T), \quad u \in L^2(\Omega)$$

can be seen as a family of bounded perturbation of A_1 (resp. A_2). Thus, using standard techniques (see, e.g., Bensoussan et al., 1993; Cazenave and Haraux, 1998; Showalter, 1977), one can prove the following well-posedness results.

THEOREM 6.2 *In both the (WD) and the (SD) case, for all $f \in L^2(Q)$ and $u_0 \in L^2(\Omega)$, there exists a unique weak solution $u \in C^0\left([0,T];L^2(\Omega)\right)\cap L^2\left(0,T;H^1_a(\Omega)\right)$ of $\{(6.1),(6.2)\}$ with homogeneous Neumann boundary conditions* (6.4). *Moreover, one has*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(Q)} + \int_0^T \|u(t)\|^2_{H^1_a(\Omega)}dt \le C\left(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f(t)\|^2_{L^2(\Omega)}dt\right),$$

for some constant $C > 0$.

THEOREM 6.3 *In the (WD) case, for all $f \in L^2(Q)$ and $u_0 \in L^2(\Omega)$, there exists a unique weak solution $u \in C^0\left([0,T];L^2(\Omega)\right) \cap L^2\left(0,T;H^1_{a,0}(\Omega)\right)$ of $\{(6.1),(6.2)\}$ with homogeneous Dirichlet boundary conditions* (6.3). *Moreover, one has*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(Q)} + \int_0^T \|u(t)\|^2_{H^1_{a,0}(\Omega)}dt \le C\left(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f(t)\|^2_{L^2(\Omega)}dt\right),$$

for some constant $C > 0$.

6.2. Space regularity of solutions

Now, we give some L^2-estimates for the first and second derivatives of functions u in $D(A_1)$ or in $D(A_2)$.

In order to estimate solutions near the boundary, we first introduce the notions of (distributional) *directional derivatives along the unit eigenvectors of a*:

$$\partial_{\varepsilon_i}u(x) := \varepsilon_i(x)\cdot\nabla u(x), \quad x \in \Omega_{\delta_0}, \quad i = 1,2\,.$$

For simplicity, we reduce our analysis to the particular case

$$\lambda_1(x) = d(x)^{\kappa_1}, \quad \lambda_2(x) = d(x)^{\kappa_2}, \quad x \in \Omega_{\delta_0},$$

where $\kappa_1, \kappa_2 \ge 0$.

PROPOSITION 6.1 *Let $\Gamma \in \mathcal{C}^r$ with $r \geq 2$. Then every $u \in D(A_1)$ satisfies*

$$d^{\kappa_1/2}\partial_{\varepsilon_1}u,\ d^{\kappa_2/2}\partial_{\varepsilon_2}u \in L^2(\Omega_{\delta_0/2}).$$

Moreover, for $r \geq 3$ the second order derivatives can be estimated by distinguishing the two following cases:

(1) *if $\kappa_1 = 0$ and $\kappa_2 \geq 0$, then*

$$\partial^2_{\varepsilon_1}u,\ d^{\kappa_2/2}\partial^2_{\varepsilon_2,\varepsilon_1}u,\ \partial_{\varepsilon_2}(d^{\kappa_2}\partial_{\varepsilon_2}u)\ \in L^2(\Omega_{\delta_0/2});$$

(2) *if $\kappa_1 > 0$ and $\kappa_2 \geq 0$, then*

$$d^{(\kappa_1+\theta)/2}\partial^2_{\varepsilon_1}u,\ d^{(\kappa_2+\theta)/2}\partial^2_{\varepsilon_2,\varepsilon_1}u,\ d^{(\theta-\kappa_1)/2}\partial_{\varepsilon_2}(d^{\kappa_2}\partial_{\varepsilon_2}u)\ \in L^2(\Omega_{\delta_0/2}),$$

where

$$\theta := \max\{\kappa_1, \kappa_1 + (2-\kappa_2)\}.$$

Notice that if $u \in D(A_2)$, one can prove that u satisfies properties (1) or (2) of Proposition 6.1, as arguing in Section 7.10.

7. Proofs

7.1. Notations and preliminary results

We start summarizing some properties of the oriented boundary distance, that will be used in most proofs (for details see, e.g., Cannarsa and Sinestrari, 2004; Delfour and Zolesio, 1994; Gilbarg and Trudinger, 1983).

PROPOSITION 7.1 *If Γ is a compact $\mathcal{C}^r$- differentiable curve with $r \geq 2$, then there exists $\delta_0 > 0$ such that for all $x \in \mathcal{B}(\Gamma, \delta_0) := \{x \in \mathbb{R}^2 \mid d(x) < \delta_0\}$ there exists a unique $p_\Gamma x$ in Γ such that*

(1) $d(x) = |x - p_\Gamma x|$,

(2) $x = p_\Gamma x - \nu(p_\Gamma x)d_\Gamma(x)$,

where ν is the outward unit normal to Ω and d_Γ is the oriented boundary distance defined by

$$d_\Gamma(x) := \begin{cases} d(x), & x \in \Omega, \\ -d(x), & x \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Moreover, one has that

(3) $p_\Gamma \in C^{r-1}(\mathcal{B}(\Gamma, \delta_0))^2$,

(4) $d_\Gamma \in C^r(\mathcal{B}(\Gamma, \delta_0))$,

(5) $\nabla d_\Gamma(x) = -\nu(p_\Gamma x) \quad \forall x \in \mathcal{B}(\Gamma, \delta_0)$.

We observe that, by taking $\varepsilon_i(x) := (\varepsilon_i \circ p_\Gamma)(x)$ for $i = 1, 2$, where $\varepsilon_1(x)$, $\varepsilon_2(x)$ is a determination satisfying assumption (3) on $a(x)$, one can extend these vector fields to $\mathcal{B}(\Gamma, \delta_0)$. As a consequence, one has

$$D^2 d_\Gamma(x)\varepsilon_1(x) = -\frac{k(p_\Gamma x)}{1 - k(p_\Gamma x)d_\Gamma(x)}\,\varepsilon_1(x), \quad D^2 d_\Gamma(x)\varepsilon_2(x) = 0, \quad \forall x \in \mathcal{B}(\Gamma, \delta_0)\,.$$

Here, $k(p_\Gamma x)$ is the curvature of Γ at $p_\Gamma x$. Thus, using the notion of tensor product,

$$D^2 d_\Gamma(x) = -\frac{k(p_\Gamma x)}{1 - k(p_\Gamma x)d_\Gamma(x)}\,\varepsilon_1(x) \otimes \varepsilon_1(x), \quad x \in \mathcal{B}(\Gamma, \delta_0)\,. \tag{7.1}$$

Notice that the function

$$\beta(x) := 1 - k(p_\Gamma x)d_\Gamma(x) \tag{7.2}$$

is strictly positive in $\mathcal{B}(\Gamma, \delta_0)$.

Next, we introduce the following map:

$$\begin{aligned} X(\gamma_p) : \mathcal{R}_p &\longrightarrow \mathbb{R}^2 \\ (s, \delta') &\mapsto X(\gamma_p)(s, \delta') := \gamma_p(s) - \nu(\gamma_p(s))\delta', \end{aligned} \tag{7.3}$$

where

$$\gamma_p : (-s_p, s_p) \to \mathbb{R}^2, \; s_p > 0$$

is a suitable $\mathcal{C}^r$ local parametrization of Γ such that

$$\gamma_p(0) = p, \quad \gamma_p'(s) = \varepsilon_1(\gamma_p(s)), \quad s \in (-s_p, s_p)$$

and

$$\mathcal{R}_p := (-s_p, s_p) \times (-\delta_0, \delta_0) \subset \mathbb{R}^2\,.$$

Notice that, since $\Gamma \in \mathcal{C}^r$, $\nu \in C^{r-1}(\Gamma)^2$. Thus $X(\gamma_p) \in C^{r-1}(\mathcal{R}_p)^2$. For simplicity, in the following we will set $X_p := X(\gamma_p)$.

Lemma 7.1 *If $\Gamma \in \mathcal{C}^r$ with $r \geq 2$, then X_p is a $\mathcal{C}^{r-1}$- diffeomorphism of $\mathcal{R}_p$ onto $X_p(\mathcal{R}_p)$. Moreover, the following results hold:*

(1) *$\det DX_p(s, \delta') = \beta(X_p(s, \delta'))$;*

(2) *$DX_p^{-1}(x)\varepsilon_2(x) = e_2$;*

(3) *$DX_p^{-1}(x)\varepsilon_1(x) = \beta(x)^{-1}e_1$;*

(4) *$\overline{a}(x) := DX_p^{-1}(x)a(x)(DX_p^{-1}(x))^* = diag\{\lambda_1(x)\beta(x)^{-2}, \lambda_2(x)\}$.*

Here, (e_1, e_2) is the standard basis of $\mathbb{R}^2$ and β is defined as in (7.2).

Proof of Lemma 7.1. Since $\gamma_p : (-s_p, s_p) \to \mathbb{R}^2$ is one to one, it follows that X_p is invertible and its inverse is

$$X_p^{-1} : X_p(\mathcal{R}_p) \ni x \mapsto (\gamma_p^{-1}(p_\Gamma x), d_\Gamma(x)) \in \mathcal{R}_p.$$

Thus, by Proposition 7.1 it follows that

$$\text{for all } r \geq 2, \quad \Gamma \in \mathcal{C}^r \quad \Rightarrow \quad X_p \text{ is a } \mathcal{C}^{r-1}\text{- diffeomorphism}.$$

Let us compute the jacobian of X_p:

$$\begin{cases} \dfrac{\partial X_p}{\partial s}(s,\delta') = \dfrac{d\gamma_p}{ds}(s) - \dfrac{d}{ds}\Big(\nu\big(\gamma_p(s)\big)\Big)\delta', \\ \dfrac{\partial X_p}{\partial \delta'}(s,\delta') = -\nu(\gamma_p(s)). \end{cases} \tag{7.4}$$

Moreover, by the chain rule one obtains

$$\frac{d}{ds}\Big(\nu\big(\gamma_p(s)\big)\Big) = D(\nu(p_\Gamma x))_{|x=\gamma_p(s)}\varepsilon_1(x)_{|x=\gamma_p(s)}$$

and by point (5) of Proposition 7.1 it follows that

$$D(\nu(p_\Gamma x))\varepsilon_1(x) = -D^2 d_\Gamma(x)\varepsilon_1(x).$$

Since

$$D^2 d_\Gamma(x)\varepsilon_1(x) = -k(x)\varepsilon_1(x) \quad \forall x \in \Gamma$$

(see (7.1)), by (7.4), we obtain that

$$DX_p(s,\delta') = \big[\varepsilon_1(\gamma_p(s))\beta(X_p(s,\delta')) \mid -\nu(\gamma_p(s))\big]. \tag{7.5}$$

By assumption (3) on $a(x)$ we know that $\varepsilon_2(x) = -\nu(p_\Gamma x)$. So,

$$det\,[\varepsilon_1(x)| - \nu(p_\Gamma x)] = 1$$

whence

$$det\, DX_p(s,\delta') = \beta(X_p(s,\delta'))\; det\,\big[\varepsilon_1(\gamma_p(s)) \mid -\nu(\gamma_p(s))\big] = \beta(X_p(s,\delta')).$$

Now, with easy algebraic computations, by (7.5) we obtain

$$DX_p^{-1}(x) = \big[\varepsilon_1(x)\beta(x)^{-1} \mid \varepsilon_2(x)\big]^*.$$

Proof of (2):

$$\begin{aligned} DX_p^{-1}(x)\varepsilon_2(x) &= \big[\varepsilon_1(x)\beta(x)^{-1} \mid \varepsilon_2(x)\big]^*\varepsilon_2(x) \\ &= (\beta(x)^{-1}\varepsilon_1(x)\cdot\varepsilon_2(x), \varepsilon_2(x)\cdot\varepsilon_2(x)) = e_2. \end{aligned}$$

Proof of (3):

$$\begin{aligned} DX_p^{-1}(x)\varepsilon_1(x) &= \big[\varepsilon_1(x)\beta(x)^{-1} \mid \varepsilon_2(x)\big]^*\varepsilon_1(x) \\ &= (\beta(x)^{-1}\varepsilon_1(x)\cdot\varepsilon_1(x), \varepsilon_2(x)\cdot\varepsilon_1(x)) = \beta(x)^{-1}e_1. \end{aligned}$$

Proof of (4):

$$\begin{aligned}\overline{a}(x) &= DX_p^{-1}(x)\Big(\lambda_1(x)\varepsilon_1(x)\otimes\varepsilon_1(x)+\lambda_2(x)\varepsilon_2(x)\otimes\varepsilon_2(x)\Big)(DX_p^{-1}(x))^*\\ &= \lambda_1(x)DX_p^{-1}(x)\varepsilon_1(x)\otimes DX_p^{-1}(x)\varepsilon_1(x)\\ &+ \lambda_2(x)DX_p^{-1}(x)\varepsilon_2(x)\otimes DX_p^{-1}(x)\varepsilon_2(x)\\ &= \lambda_1(x)\beta(x)^{-2}\,e_1\otimes e_1+\lambda_2(x)\,e_2\otimes e_2\\ &= diag\big\{\lambda_1(x)\beta(x)^{-2},\lambda_2(x)\big\}\,. \end{aligned}$$ ■

Now, we construct a *system of localizations* for the functions of $H^1_a(\Omega)$. Since Γ is compact, there exists a finite number of points $p_1,\dots,p_m\in\Gamma$ such that $\Gamma\subset\cup_{k=1}^m X_{p_k}(\mathcal{R}_{p_k})$; here, the sets $\mathcal{R}_{p_k}$ and the maps X_{p_k} satisfy the assumptions of Lemma 7.1. For $k\in\{1,\dots,m\}$, we denote

$$\mathcal{B}_k:=(-s_{p_k},s_{p_k}),\quad \mathcal{R}_k:=\mathcal{R}_{p_k},\quad U_k:=X_{p_k}(\mathcal{R}_{p_k})$$

and set $U_0:=\Omega^{\delta_0/2}$, so that one has that $\Omega\subset\cup_{k=0}^m U_k$.

Furthermore, choose $\chi_k\in\mathcal{D}(U_k;[0,1])$ such that $\sum_{k=0}^m\chi_k=1$ on a neighborhood of $\overline{\Omega}$ and set for simplicity

$$X_k:=X_{p_k},\quad X_0:=id.$$

Then, it is possible to rewrite $v\in H^1_a(\Omega)$ as

$$\begin{cases} v=\sum_{k=0}^m v_k\circ X_k^{-1}\ \text{on }\Omega,\\ v_k:=\chi_k v\circ X_k.\end{cases}\tag{7.6}$$

We will call the functions v_k , $k\in\{0,\dots,m\}$, the *associated functions* of v.

Last, we give the following

Lemma 7.2 *If $f\in C^0(\Omega)\cap L^1(\Omega)$, then for all $\delta\in(0,\delta_0/2)$ one has that*

$$\int_{\Omega_\delta}f(x)dx=\int_0^\delta\left(\int_{\Gamma^{\delta'}}f\,dr\right)d\delta'.$$

Proof of Lemma 7.2. Using the previous setting it is possible to rewrite f in the following form

$$\begin{cases} f=\sum_{k=1}^m f_k\circ X_k^{-1}\ \text{on }\Omega_\delta,\\ f_k:=\chi_k f\circ X_k.\end{cases}$$

Thus, because $s\mapsto X_k(s,\delta')$ is a local parametrization of $\Gamma^{\delta'}$, by definition of line integral one has

$$\int_{\Gamma^{\delta'}}f\,dr=\int_{\Gamma^{\delta'}}\sum_{k=1}^m f_k\circ X_k^{-1}\,dr=\sum_{k=1}^m\int_{\mathcal{B}_k}f_k(X_k(s,\delta'))\beta(X_k(s,\delta'))ds.\tag{7.7}$$

On the other hand, one has that

$$\int_{\Omega_\delta} f(x)\,dx = \sum_{k=1}^{m} \int_0^\delta \int_{\mathcal{B}_k} f_k(X_k(s,\delta'))\beta(X_k(s,\delta'))ds d\delta'. \qquad \blacksquare$$

7.2. Proof of Proposition 3.1

Let, for $i = 1, 2$, $\lambda_i(x)$ and $\varepsilon_i(x)$ be respectively, the eigenvalues and the unit eigenvectors of the symmetric matrices $a(x) = \sigma(x)\sigma(x)$. We observe that by assumptions on $a(x)$, it follows that $\lambda_i(x) > 0$ and $\sigma^{-1}(x)$ is defined $\forall x \in \Omega$. Moreover, for all $\delta \in (0, \delta_0)$, we can write

$$\begin{aligned}\int_\Omega a(x)\nabla v \cdot \nabla v\,dx &= \int_\Omega \sigma(x)\nabla v \cdot \sigma(x)\nabla v\,dx \\ &= \int_{\Omega^\delta} \sigma(x)\nabla v \cdot \sigma(x)\nabla v\,dx + \sum_{i=1}^{2} \int_{\Omega_\delta} |\varepsilon_i \cdot \nabla v|^2 \lambda_i(x)\,dx.\end{aligned} \tag{7.8}$$

First step. We want to prove that $H_a^1(\Omega)$ is complete for the norm defined in (3.1). For simplicity, we define the weighted space

$$L_a^2(\Omega) := \left\{ w \in L_{loc}^1(\Omega)^2 \mid \int_\Omega a(x) w \cdot w\,dx < \infty \right\}.$$

Let $(v_n)_n$ be a Cauchy sequence in $H_a^1(\Omega)$. Then there exist $v \in L^2(\Omega)$, $g = (g_1, g_2) \in L^2(\Omega)^2$ such that

$$v_n \longrightarrow v \text{ in } L^2(\Omega), \quad \sigma\nabla v_n \longrightarrow g \text{ in } L^2(\Omega)^2.$$

Therefore, if we show that $\nabla v = \sigma^{-1} g$, we obtain the conclusion. For this purpose, since

$$v_n \to v \text{ in } L^2(\Omega) \quad \Rightarrow \quad \nabla v_n \to \nabla v \text{ in } \mathcal{D}'(\Omega)$$

and the distributional limit is unique, it is sufficient to prove that

$$\nabla v_n \to \sigma^{-1} g \text{ in } L_a^2(\Omega) \quad \Rightarrow \quad \nabla v_n \to \sigma^{-1} g \text{ in } \mathcal{D}'(\Omega).$$

For all $\varphi \in \mathcal{D}(\Omega)$ one has that

$$\begin{aligned}&\left| \int_\Omega (\nabla v_n - \sigma^{-1} g)\varphi\,dx \right| \leq \|\varphi\|_{L^\infty(\Omega)} \int_{\text{supp}\{\varphi\}} |\nabla v_n - \sigma^{-1} g|\,dx \\ &\leq \|\varphi\|_{L^\infty(\Omega)} |\Omega|^{1/2} \left(\int_{\text{supp}\{\varphi\}} (\nabla v_n - \sigma^{-1} g) \cdot (\nabla v_n - \sigma^{-1} g)\,dx \right)^{1/2} \\ &\leq \|\varphi\|_{L^\infty(\Omega)} |\Omega|^{1/2} \sup_{x \in \text{supp}\{\varphi\}} \left\| a^{-1}(x) \right\|^{1/2} \left\| \nabla v_n - \sigma^{-1} g \right\|_{L_a^2(\Omega)} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Second step. Now, we prove that $H^1(\Omega)$ is dense in $H^1_a(\Omega)$; we observe that, since the injection of $H^1(\Omega)$ into $H^1_a(\Omega)$ is continuous, it directly follows from this result that $C^\infty(\overline{\Omega})$ is dense in $H^1_a(\Omega)$. Let $v \in H^1_a(\Omega)$ be given and define the family $(v_\delta)_\delta$, with $\delta \in (0, \delta_0/2)$, in the following way:

$$v_\delta(x) := \begin{cases} v(x), & x \in \Omega^\delta, \\ v\left(p_{\Gamma^{2\delta}} x + \nu(p_\Gamma x) d(x)\right), & x \in \Omega_\delta, \end{cases}$$

where $p_{\Gamma^{2\delta}} x$ is the projection of x onto $\Gamma^{2\delta}$. We want to show that

(i) $v_\delta \in H^1(\Omega)$ for all $\delta \in (0, \delta_0/2)$;

(ii) $v_\delta \to v$ in $H^1_a(\Omega)$ as $\delta \to 0$.

For this purpose, we first state a preliminary lemma:

Lemma 7.3 *Let $\delta \in (0, \delta_0/2)$, if $\Gamma \in \mathcal{C}^r$ with $r \geq 2$, then the map*

$$\rho_\delta : x \longmapsto \rho_\delta x := p_{\Gamma^{2\delta}} x + \nu(p_\Gamma x) d(x),$$

is a $\mathcal{C}^{r-1}$- diffeomorphism of Ω_δ onto $\Omega^\delta \setminus \Omega^{2\delta}$. Moreover, for all $x \in \Omega_\delta$, one has that

(1) $\rho_\delta \rho_\delta x = x$;

(2) $\det D\rho_\delta(x) = -\beta(\rho_\delta x)\beta(x)^{-1}$;

(3) $D\rho_\delta(x)\varepsilon_2(x) = -\varepsilon_2(x)$;

(4) $D\rho_\delta(x)\varepsilon_1(x) = \beta(\rho_\delta x)\beta(x)^{-1}\,\varepsilon_1(x)$.

Proof of Lemma 7.3. As a direct consequence of the definition it follows that ρ_δ is a one to one map of Ω_δ onto $\Omega^\delta \setminus \Omega^{2\delta}$ of class $\mathcal{C}^{r-1}$ and satisfying (1). Using Proposition 7.1 and (7.1) we compute the jacobian of ρ_δ.

$$\begin{aligned} D\rho_\delta(x) &= D\big(x - \nu(p_\Gamma x)(2\delta - d(x)) + \nu(p_\Gamma x) d(x)\big) \\ &= I_2 + 2D\big(\nu(p_\Gamma x)(d(x) - \delta)\big) \\ &= I_2 + 2\nu(p_\Gamma x) \otimes \nabla(d(x) - \delta) + 2(d(x) - \delta) D\nu(p_\Gamma x) \\ &= I_2 - 2\nu(p_\Gamma x) \otimes \nu(p_\Gamma x) + 2(d(x) - \delta) D^2 d(x) \\ &= \sum_{i=1}^{2} \varepsilon_i(x) \otimes \varepsilon_i(x) - 2\varepsilon_2(x) \otimes \varepsilon_2(x) - \frac{2(d(x) - \delta)k(p_\Gamma x)}{1 - k(p_\Gamma x) d(x)}\, \varepsilon_1(x) \otimes \varepsilon_1(x) \\ &= \beta(\rho_\delta x)\beta(x)^{-1} \varepsilon_1(x) \otimes \varepsilon_1(x) - \varepsilon_2(x) \otimes \varepsilon_2(x)\,. \end{aligned}$$ ■

Now let us complete the proof of Proposition 3.1.

Proof of (i) **:** preliminarily, we observe that since $H^1_a(\Omega) \subset H^1_{loc}(\Omega)$, then $v \circ \rho_\delta \in H^1(\Omega_\delta)$ and

$$\nabla(v \circ \rho_\delta)(x) = (D\rho_\delta(x))^* \nabla v(\rho_\delta x), \quad x\text{-a.e. in } \Omega_\delta. \tag{7.9}$$

As a consequence, for all $\varphi \in \mathcal{D}(\Omega)$ one has

$$\begin{aligned}\int_\Omega v_\delta \nabla\varphi\, dx &= \int_{\Omega^\delta} v\nabla\varphi\, dx + \int_{\Omega_\delta} v\circ\rho_\delta \nabla\varphi\, dx \\ &= -\int_{\Omega^\delta} \nabla v\,\varphi\, dx + \int_{\Gamma^\delta} \gamma^\delta v\,\gamma^\delta\varphi\,(\nu\circ p_\Gamma)\, dr \\ &\quad -\int_{\Omega_\delta} \nabla(v\circ\rho_\delta)\varphi\, dx - \int_{\Gamma^\delta} \gamma^\delta(v\circ\rho_\delta)\,\gamma^\delta\varphi\,(\nu\circ p_\Gamma)\, dr \\ &= -\int_\Omega \nabla v_\delta\,\varphi\, dx\end{aligned} \tag{7.10}$$

where

$$\nabla v_\delta = \begin{cases}\nabla v & \text{in} \quad \Omega^\delta \\ \nabla(v\circ\rho_\delta) & \text{in} \quad \Omega_\delta\,.\end{cases}$$

Proof of (ii) **:** one has that

$$\|v_\delta - v\|^2_{H^1_a(\Omega)} = \|v_\delta - v\|^2_{H^1_a(\Omega_\delta)} \le 2\,\|v\|^2_{H^1_a(\Omega_\delta)} + 2\,\|v_\delta\|^2_{H^1_a(\Omega_\delta)}\,,$$

where $\|v\|^2_{H^1_a(\Omega_\delta)} \to 0$ as $\delta \to 0$. So, it is sufficient to estimate the term $\|v_\delta\|^2_{H^1_a(\Omega_\delta)}$. By assumptions (1),(2),(3) on $a(x)$, (7.8), (7.9) and Lemma 7.3, one has that

$$\begin{aligned}&\int_{\Omega_\delta} v_\delta^2(x) + a(x)\nabla v_\delta(x)\cdot\nabla v_\delta(x)\, dx \\ &= \int_{\Omega_\delta} v_\delta^2(x) + |D\rho_\delta(x)\varepsilon_2(x)\cdot\nabla v(\rho_\delta x)|^2\lambda_2(x) + |D\rho_\delta(x)\varepsilon_1(x)\cdot\nabla v(\rho_\delta x)|^2\lambda_1(x)\, dx \\ &= \int_{\Omega_\delta} v_\delta^2(x) + |\varepsilon_2(x)\cdot\nabla v(\rho_\delta x)|^2\lambda_2(x) + \beta(\rho_\delta x)^2\beta(x)^{-2}|\varepsilon_1(x)\cdot\nabla v(\rho_\delta x)|^2\lambda_1(x)\, dx \\ &= \int_{\Omega^\delta\setminus\Omega^{2\delta}} \Big(v^2(y) + |\varepsilon_2(y)\cdot\nabla v(y)|^2\lambda_2(\rho_\delta y)\Big)\,\beta(\rho_\delta y)\beta(y)^{-1}\, dy \\ &+ \int_{\Omega^\delta\setminus\Omega^{2\delta}} \beta(\rho_\delta y)^3\beta(y)^{-3}\,|\varepsilon_1(y)\cdot\nabla v(y)|^2\lambda_1(\rho_\delta y)\, dy\,.\end{aligned} \tag{7.11}$$

Since β is a bounded strictly positive function, by (7.11) and assumption (4) on $a(x)$ one has

$$\int_{\Omega_\delta} v_\delta^2(x) + a(x)\nabla v_\delta(x)\cdot\nabla v_\delta(x)\, dx \le C\int_{\Omega^\delta\setminus\Omega^{2\delta}} v^2(y) + a(y)\nabla v\cdot\nabla v\, dy$$

for some positive constant C. As we pass to the limit as $\delta \to 0$, the conclusion follows.

7.3. Proof of Lemma 3.1

We want to show that $H^2_a(\Omega)$ is complete for the norm defined in (3.2). Let us consider a Cauchy sequence $(u_n)_n$ in $H^2_a(\Omega)$. Then there exist $u \in H^1_a(\Omega)$, $h \in L^2(\Omega)$ such that

$$u_n \longrightarrow u \ \text{ in } \ H^1_a(\Omega), \quad div(a\nabla u_n) \longrightarrow h \ \text{ in } \ L^2(\Omega).$$

Since the distributional limit is unique, to obtain the conclusion it suffices to prove that

$$u_n \to u \ \text{ in } \ H^1_a(\Omega) \quad \Rightarrow \quad div(a\nabla u_n) \to div(a\nabla u) \ \text{ in } \mathcal{D}'(\Omega).$$

For all $\varphi \in \mathcal{D}(\Omega)$ one has that

$$\left|\int_\Omega \big(div(a\nabla u_n) - div(a\nabla u)\big)\varphi\, dx\right| = \left|-\int_\Omega (a\nabla u_n - a\nabla u)\cdot\nabla\varphi\, dx\right|$$
$$\le \|\nabla\varphi\|_{L^\infty(\Omega)^2} \sup_{x\in\Omega}\|\sigma(x)\| \int_\Omega |\sigma(\nabla u_n - \nabla u)|\, dx$$
$$\le \|\nabla\varphi\|_{L^\infty(\Omega)^2} \sup_{x\in\Omega}\|\sigma(x)\| \ |\Omega|^{1/2}\, \|u_n - u\|_{H^1_a(\Omega)} \quad \stackrel{n\to\infty}{\longrightarrow} 0.$$

7.4. Proof of Lemma 4.1

First step. We want to prove that $H_{div,a}(\Omega)$ is complete for the norm defined in (4.1). Let us consider a Cauchy sequence $(w_n)_n$ in $H_{div,a}(\Omega)$. It follows that $\sigma^{-1}w_n$ and $div(w_n)$ are also Cauchy sequences, respectively, in $L^2(\Omega)^2$ and in $L^2(\Omega)$; then, $\sigma^{-1}w_n$ converges to some limit u in $L^2(\Omega)^2$, that is, w_n converges to $w := \sigma u$ in $L^2_{a^{-1}}(\Omega)$, and $div(w_n)$ converges to some limit $g \in L^2(\Omega)$. Furthermore, one has that

$$w_n \to w \text{ in } L^2_{a^{-1}}(\Omega) \quad \Rightarrow \quad div(w_n) \to div(w) \text{ in } \mathcal{D}'(\Omega)\,.$$

By the uniqueness of distributional limit, $g = div(w)$.

Second step. We begin by proving that if w belongs to $H_{div,a}(\Omega)$, then $\mathcal{T}_w$ is in $(H^1_a(\Omega))'$. Let $v \in H^1_a(\Omega)$, then

$$|\mathcal{T}_w v| = \left|\int_\Omega div(w)v + \sigma\sigma^{-1}w\cdot\nabla v\, dx\right|$$
$$\le \ \|div(w)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \left\|\sigma^{-1}w\right\|_{L^2(\Omega)^2}\|\sigma\nabla v\|_{L^2(\Omega)^2}$$
$$\le \ \left(\left\|\sigma^{-1}w\right\|_{L^2(\Omega)^2} + \|div(w)\|_{L^2(\Omega)}\right)\left(\|v\|_{L^2(\Omega)} + \|\sigma\nabla v\|_{L^2(\Omega)^2}\right)$$
$$\le \ \|w\|_{H_{div,a}(\Omega)}\|v\|_{H^1_a(\Omega)}\,.$$

Now, since $\mathcal{T}_w \in (H^1_a(\Omega))'$, by the absolute continuity of Lebesgue's integral one has

$$\lim_{\delta\to 0}\int_{\Omega^\delta} div(w)v + w\cdot\nabla v\,dx = \mathcal{T}_w v.$$

On the other hand, by the standard normal trace theory, for all $\delta \in (0,\delta_0)$ we have

$$\int_{\Omega^\delta} div(w)v + w\cdot\nabla v\,dx = \langle\gamma^\delta_\nu w, \gamma^\delta v\rangle_{H^{-1/2}(\Gamma^\delta),H^{1/2}(\Gamma^\delta)}\,.$$

7.5. Proof of Proposition 5.1

As a first step we prove the following

LEMMA 7.4 *If $\varphi \in C^\infty(\overline{\Omega})$, then, for $k = 1,\dots,m$, the associated functions $\varphi_k = \chi_k\varphi\circ X_k$ satisfy*

$$\int_{\mathcal{B}_k}|\varphi_k(s,0)|^2\,ds \le \frac{\delta_0^{1-\alpha}}{c_0(1-\alpha)}\int_{\mathcal{B}_k}\int_0^{\delta_0}\overline{a}(X_k(s,\delta'))\nabla\varphi_k\cdot\nabla\varphi_k\,ds\,d\delta'.$$

Proof of Lemma 7.4. One has

$$\varphi_k(s,0) = -\int_0^{\delta_0}\left(\frac{\lambda_2(X_k(s,\delta'))}{\lambda_2(X_k(s,\delta'))}\right)^{1/2}\frac{\partial\varphi_k}{\partial\delta'}(s,\delta')\,d\delta',$$

thus, by Holder's inequality it follows that

$$|\varphi_k(s,0)|^2 \le \int_0^{\delta_0}\frac{d\delta'}{\lambda_2(X_k(s,\delta'))}\int_0^{\delta_0}\lambda_2(X_k(s,\delta'))\left|\frac{\partial\varphi_k}{\partial\delta'}(s,\delta')\right|^2 d\delta'. \tag{7.12}$$

Now, since $a(x)$ is (WD), we have that $\lambda_2(x) \ge c_0\,d(x)^\alpha$ for all $x\in\Omega$ and for some fixed $\alpha\in(0,1)$. Thus from (7.12), we obtain

$$|\varphi_k(s,0)|^2 \le c_0^{-1}\int_0^{\delta_0}\frac{d\delta'}{(\delta')^\alpha}\int_0^{\delta_0}\lambda_2(X_k(s,\delta'))\left|\frac{\partial\varphi_k}{\partial\delta'}(s,\delta')\right|^2 d\delta'. \tag{7.13}$$

By property (4) of Lemma 7.1, for $k = 1,\dots,m$ one has

$$\begin{aligned}&\overline{a}(X_k(s,\delta'))\nabla\varphi_k\cdot\nabla\varphi_k\\ &= \beta(X_k(s,\delta'))^{-2}\lambda_1(X_k(s,\delta'))\left|\frac{\partial\varphi_k}{\partial s}\right|^2 + \lambda_2(X_k(s,\delta'))\left|\frac{\partial\varphi_k}{\partial\delta'}\right|^2,\end{aligned} \tag{7.14}$$

thus, upon integrating both members of (7.13) on $\mathcal{B}_k$, the conclusion follows. ■

Let us now complete the proof of Proposition 5.1. For any $\varphi \in C^\infty(\overline{\Omega})$, using Lemmas 7.1, 7.4 and (7.6), one has

$$\begin{aligned}\int_\Gamma \varphi^2\, dr &= \int_\Gamma \big|\sum_{k=1}^m \varphi_k \circ X_k^{-1}\big|^2\, dr \le C\sum_{k=1}^m \int_\Gamma \big|\varphi_k \circ X_k^{-1}\big|^2\, dr\\ &= C\sum_{k=1}^m \int_{\mathcal{B}_k} |\varphi_k(s,0)|^2\, ds\\ &\le C\sum_{k=1}^m \int_{\mathcal{B}_k}\int_0^{\delta_0} \overline{a}(X_k(s,\delta'))\nabla\varphi_k\cdot\nabla\varphi_k\, ds\, d\delta'\\ &= C\sum_{k=1}^m \int_{\Omega\cap U_k} a(x)\nabla(\varphi_k\circ X_k^{-1})\cdot\nabla(\varphi_k\circ X_k^{-1})\beta(x)^{-1}\, dx\\ &\le C\int_\Omega \varphi^2 + a(x)\nabla\varphi\cdot\nabla\varphi\, dx.\end{aligned}$$

Here C is a suitable positive constant. Hence, the standard trace operator is continuous from $\big(C^\infty(\overline{\Omega}), \|.\|_{H^1_a(\Omega)}\big)$ into $L^2(\Gamma)$. Finally, by the density of Proposition 3.1 we obtain the conclusion.

7.6. Proof of Proposition 5.2

As a first step, we introduce the following inequality.

Lemma 7.5 *In the (WD) case , there exists $C'_H > 0$ such that for all $\delta \in (0,\delta_0]$ one has*

$$\int_{\Omega_\delta} v^2 \frac{\lambda_2(x)}{d(x)^2}\, dx \le C'_H \int_{\Omega_\delta} a(x)\nabla v\cdot\nabla v\, dx, \quad \forall v \in Ker\{\gamma^a\}. \tag{7.15}$$

Proof of Lemma 7.5. We first prove that for all $\varphi \in C^\infty(\overline{\Omega})$, $\delta \in (0,\delta_0]$ and $k = 1,\dots,m$, there exists a positive constant C such that

$$\begin{aligned}&\int_{\mathcal{B}_k}\int_0^\delta |\hat\varphi_k(s,\delta') - \hat\varphi_k(s,0)|^2 \frac{\lambda_2(X_k(s,\delta'))}{(\delta')^2}\, ds d\delta'\\ &\qquad \le C\int_{\mathcal{B}_k}\int_0^\delta \overline{a}(X_k(s,\delta))\nabla\hat\varphi_k\cdot\nabla\hat\varphi_k\, ds d\delta',\end{aligned} \tag{7.16}$$

where $\hat\varphi_k := \varphi\circ X_k$. Fix $\mu \in (\vartheta, 1)$, for all $s \in \mathcal{B}_k$ we have

$$\begin{aligned}&\int_0^\delta |\hat\varphi_k(s,\delta') - \hat\varphi_k(s,0)|^2 \frac{\lambda_2(X_k(s,\delta'))}{(\delta')^2}\, d\delta'\\ &= \int_0^\delta \Big|\int_0^{\delta'} t^{\mu/2}\frac{\partial\hat\varphi_k}{\partial\delta'}(s,t)t^{-\mu/2}dt\Big|^2 \frac{\lambda_2(X_k(s,\delta'))}{(\delta')^2}\, d\delta'\\ &\le \int_0^\delta \left(\int_0^{\delta'} t^\mu\Big|\frac{\partial\hat\varphi_k}{\partial\delta'}(s,t)\Big|^2 dt\int_0^{\delta'} t^{-\mu}dt\right)\frac{\lambda_2(X_k(s,\delta'))}{(\delta')^2}\, d\delta'.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\int_0^\delta |\hat\varphi_k(s,\delta')-\hat\varphi_k(s,0)|^2\,\frac{\lambda_2(X_k(s,\delta'))}{(\delta')^2}\,d\delta'\\&\le\frac{1}{1-\mu}\int_0^\delta\left(\int_0^{\delta'}t^\mu\Big|\frac{\partial\hat\varphi_k}{\partial\delta'}(s,t)\Big|^2dt\right)\frac{\lambda_2(X_k(s,\delta'))}{(\delta')^{1+\mu}}\,d\delta'.\end{aligned}$$

Applying Fubini's Theorem, we obtain

$$\begin{aligned}&\int_0^\delta |\hat\varphi_k(s,\delta')-\hat\varphi_k(s,0)|^2\,\frac{\lambda_2(X_k(s,\delta'))}{(\delta')^2}\,d\delta'\\&\le\frac{1}{1-\mu}\int_0^\delta t^\mu\Big|\frac{\partial\hat\varphi_k}{\partial\delta'}(s,t)\Big|^2\left(\int_t^\delta\frac{\lambda_2(X_k(s,\delta'))}{(\delta')^{1+\mu}}\,d\delta'\right)dt.\end{aligned}\tag{7.17}$$

Now, because $a(x)$ is (WD), one has that for all fixed $s\in\mathcal{B}_k$ the function $\delta'\mapsto\lambda_2(X_k(s,\delta'))/(\delta')^\vartheta$ is nonincreasing on $(0,\delta_0]$. So, one has that

$$\begin{aligned}\int_t^\delta\frac{\lambda_2(X_k(s,\delta'))}{(\delta')^{1+\mu}}\,d\delta'&\le\frac{\lambda_2(X_k(s,t))}{t^\vartheta}\int_t^\delta(\delta')^{\vartheta-1-\mu}\,d\delta'\\&\le\frac{1}{\mu-\vartheta}\frac{\lambda_2(X_k(s,t))}{t^\mu}\,.\end{aligned}$$

Using this last inequality, integrating both members of (7.17) on $\mathcal{B}_k$ and recalling (7.14), we deduce (7.16) with $C=[(1-\mu)(\mu-\vartheta)]^{-1}$.

Next, let us consider $v\in Ker\{\gamma^a\}$ and let us prove (7.15). By Proposition 3.1 there exists a sequence $(\varphi_n)_n\subset C^\infty(\overline{\Omega})$ such that $\varphi_n\to v$ on $H^1_a(\Omega)$; denoting

$$\hat\varphi_{n,k}:=\varphi_n\circ X_k,\quad \Omega_{\delta,k}:=\Omega_\delta\cap U_k,\quad k=1,\dots,m$$

and using (7.16), one has that

$$\begin{aligned}&\int_{\Omega_\delta}|\varphi_n-\gamma^a\varphi_n|^2\frac{\lambda_2(x)}{d(x)^2}\,dx\\&\le\sum_{k=1}^m\int_{\Omega_{\delta,k}}|\varphi_n-\gamma^a\varphi_n|^2\frac{\lambda_2(x)}{d(x)^2}\,dx\\&=\sum_{k=1}^m\int_{\mathcal{B}_k}\int_0^\delta|\hat\varphi_{n,k}(s,\delta')-\hat\varphi_{n,k}(s,0)|^2\,\frac{\lambda_2(X_k(s,\delta'))}{(\delta')^2}\,\beta(X_k(s,\delta'))\,dsd\delta'\\&\le C'_H\sum_{k=1}^m\int_{\mathcal{B}_k}\int_0^\delta\overline{a}(X_k(s,\delta'))\nabla\hat\varphi_{n,k}\cdot\nabla\hat\varphi_{n,k}\,dsd\delta'\end{aligned}$$

$$= C'_H \sum_{k=1}^{m} \int_{\Omega_{\delta,k}} a(x)\nabla\varphi_n \cdot \nabla\varphi_n \,\beta(x)^{-1}\,dx$$
$$\le C'_H \int_{\Omega_\delta} a(x)\nabla\varphi_n \cdot \nabla\varphi_n \,dx.$$

Here C'_H is a suitable positive constant. So, passing to the limit as $n \to \infty$ the conclusion follows. ■

We are now ready to prove Proposition 5.2. First, we show that the inequality (5.1) follows from (7.15) and Poincaré's inequality by a cut-off function argument. Picking a smooth function χ such that

$$\begin{cases} 0 \le \chi(x) \le 1, & x \in \Omega, \\ \chi(x) = 1, & x \in \Omega^{\delta_0}, \\ \chi(x) = 0, & x \in \Omega_{\delta_0/2}, \end{cases}$$

using the Poincaré inequality and the Leibniz rule we obtain

$$\int_\Omega |\chi v|^2\,dx \le 2C_P \left(\int_\Omega \chi^2 |\nabla v|^2 dx + \int_\Omega |\nabla\chi|^2 v^2 dx \right) \quad \forall v \in \operatorname{Ker}\{\gamma^a\},$$

for a suitable positive constant C_P. Thus, by assumptions on $a(x)$, it follows that there exists $C = C(a, \delta_0, C_P) > 0$ such that

$$\int_{\Omega^{\delta_0}} |v|^2 \frac{\lambda_2(x)}{d(x)^2}\,dx \le C \left(\int_{\Omega^{\delta_0/2}} a(x)\nabla v \cdot \nabla v\,dx + \int_{\Omega_{\delta_0}} |v|^2 \frac{\lambda_2(x)}{d(x)^2}\,dx \right) \quad (7.18)$$

for all $v \in \operatorname{Ker}\{\gamma^a\}$. Finally, using (7.15), (7.18) we obtain (5.1).

Next, by Proposition 5.1 we know that the trace operator γ^a is continuous. Thus, $\operatorname{Ker}\{\gamma^a\}$ is a closed subspace of $H^1_a(\Omega)$. Furthermore $\gamma^a(\mathcal{D}(\Omega)) = 0$ and so, by the definition of $H^1_{a,0}(\Omega)$ one has that $H^1_{a,0}(\Omega) \subset \operatorname{Ker}\{\gamma^a\}$.

Last, it remains to show that $\operatorname{Ker}\{\gamma^a\} \subset H^1_{a,0}(\Omega)$. Let $v \in \operatorname{Ker}\{\gamma^a\}$ be given and define the family $(v_\delta)_\delta$, with $\delta \in (0, \delta_0)$, in the following way

$$\begin{cases} v_\delta(x) := \chi_\delta(x) v(x), \\ \chi_\delta(x) := (d(x)/\delta) \wedge 1. \end{cases}$$

Since the injection of $H^1(\Omega)$ into $H^1_a(\Omega)$ is continuous, it is sufficient to show that

(*i*) $v_\delta \in H^1_0(\Omega)$ for all $\delta \in (0, \delta_0)$;

(*ii*) $v_\delta \to v$ in $H^1_a(\Omega)$ as $\delta \to 0$.

Proof of (i) : one has that

$$\begin{aligned} &\int_\Omega v_\delta^2 + |\nabla v_\delta|^2\, dx \le \int_\Omega v^2\, dx + 2\int_\Omega |v\nabla\chi_\delta|^2 + |\chi_\delta\nabla v|^2\, dx \\ &= \int_\Omega v^2\, dx + \frac{2}{\delta^2}\int_{\Omega_\delta} v^2\, dx + 2\int_{\Omega^\delta} |\nabla v|^2\, dx + \frac{2}{\delta^2}\int_{\Omega_\delta} |d(x)\nabla v|^2\, dx \\ &\le \int_\Omega v^2\, dx + \frac{2}{\delta^2}\int_{\Omega_\delta} v^2\, dx + 2\int_{\Omega^\delta} |\nabla v|^2\, dx + c_0^{-1}\frac{2}{\delta^2}\int_{\Omega_\delta} a(x)\nabla v\cdot\nabla v\, dx. \end{aligned} \tag{7.19}$$

So, $(v_\delta)_\delta \subset H^1(\Omega)$. It remains to prove that $\gamma v_\delta = 0$ for all $\delta \in (0,\delta_0)$. Since $v \in Ker\{\gamma^a\} \subset H^1_a(\Omega)$, by Proposition 3.1, there exists $(\varphi_n)_n \subset C^\infty(\overline{\Omega})$ such that $\varphi_n \to v$ in $H^1_a(\Omega)$, we have

$$Ker\{\gamma^a\} \ni \chi_\delta\varphi_n \to \chi_\delta v = v_\delta \text{ in } H^1_a(\Omega),$$

then, because $Ker\{\gamma^a\}$ is closed, one has that $v_\delta \in Ker\{\gamma^a\}$. Thus, we have obtained that $\gamma(v_\delta) = \gamma^a(v_\delta) = 0$ for all $\delta \in (0,\delta_0)$.

Proof of (ii) : one has that

$$\begin{aligned} &\int_\Omega |v - v_\delta|^2 + |\sigma(x)\nabla(v - v_\delta)|^2\, dx \\ &= \int_{\Omega_\delta} (1-\chi_\delta)^2 v^2 + |-v\sigma(x)\nabla\chi_\delta + (1-\chi_\delta)\sigma(x)\nabla v|^2\, dx \\ &\le \int_{\Omega_\delta} v^2\, dx + 2\int_{\Omega_\delta} |\sigma(x)\nabla v|^2\, dx + \frac{2}{\delta^2}\int_{\Omega_\delta} v^2|\sigma(x)\nabla d(x)|^2\, dx. \end{aligned} \tag{7.20}$$

Finally, we estimate the last term of (7.20). By (7.15), one has

$$\begin{aligned} \frac{2}{\delta^2}\int_{\Omega_\delta} v^2|\sigma(x)\nabla d(x)|^2\, dx &= \frac{2}{\delta^2}\int_{\Omega_\delta} v^2\lambda_2(x)\, dx \le 2\int_{\Omega_\delta} v^2\frac{\lambda_2(x)}{d(x)^2}\, dx \\ &\le 2\, C'_H\int_{\Omega_\delta} |\sigma(x)\nabla v|^2\, dx. \end{aligned} \tag{7.21}$$

From (7.20), (7.21) we obtain

$$\int_\Omega |v - v_\delta|^2 + |\sigma(x)\nabla(v - v_\delta)|^2\, dx \le \int_{\Omega_\delta} v^2\, dx + 2(C'_H + 1)\int_{\Omega_\delta} |\sigma(x)\nabla v|^2\, dx.$$

By passing to the limit as $\delta \to 0$, the conclusion follows.

7.7. Proof of Proposition 5.3

In order to prove Proposition 5.3, we first give some regularity and density results.

LEMMA 7.6 *In the (WD) case there holds $H^1_a(\Omega) \subset W^{1,1}(\Omega)$.*

Proof of Lemma 7.6. Because $H^1_a(\Omega) \subset H^1_{loc}(\Omega)$, it is sufficient to prove that if $u \in H^1_a(\Omega)$, then $\partial_i u \in L^1(\Omega_\delta)$ for $i = 1, 2$ and $\delta \in (0, \delta_0/2)$. By Lemma 7.2 one has

$$\begin{aligned}\int_{\Omega_\delta} |\partial_i u|\, dx &\le \left(\int_{\Omega_\delta} d(x)^{-\alpha}\, dx \int_{\Omega_\delta} d(x)^\alpha |\partial_i u|^2 dx\right)^{\frac{1}{2}} \\ &= \left(\int_0^\delta d\delta' \int_{\Gamma^{\delta'}} d^{-\alpha}\, dr \int_{\Omega_\delta} d(x)^\alpha |\partial_i u|^2 dx\right)^{\frac{1}{2}} \\ &\le \left(c_0^{-1} \int_0^\delta (\delta')^{-\alpha} |\Gamma^{\delta'}|\, d\delta' \int_{\Omega_\delta} a(x)\nabla u \cdot \nabla u\, dx\right)^{\frac{1}{2}}.\end{aligned}$$

Here, $|\Gamma^{\delta'}|$ denotes the length of $\Gamma^{\delta'}$. So, since $\delta' \mapsto |\Gamma^{\delta'}|$ is a bounded continuous function on $(0, \delta_0/2)$ and $\alpha \in (0, 1)$, we obtain the conclusion. ■

LEMMA 7.7 *In the (WD) case, for $v \in H^1_a(\Omega)$, denote by $\widetilde{v}$ its trivial extension on $\mathbb{R}^2$:*

$$\widetilde{v}(x) := \begin{cases} v(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Then, if $\widetilde{v} \in W^{1,1}(\mathbb{R}^2)$, one has that $v \in H^1_{a,0}(\Omega)$.

Proof of Lemma 7.7. Let us consider the open covering $\{U_k\}_{k=0}^m$ of $\overline{\Omega}$ defined in Section 7.1. We observe that, since $\Gamma \in \mathcal{C}^r$ with $r \ge 2$, it is possible to find $\{U_k\}_{k=0}^m$ such that, for $k = 1, \dots, m$, the sets $\Omega_k := \Omega \cap U_k$ are *star-shaped* (see, e.g., Temam, 1977) with respect to one of their points. Moreover, let us consider the partition of unity $(\chi_k)_{k=0}^m$ subordinated to this covering and $v \in H^1_a(\Omega)$; we may write

$$v(x) = \sum_{k=0}^m \chi_k(x) v(x) \quad \text{on } \Omega.$$

Since the function $\chi_0 v$ has compact support in Ω, it belongs to $H^1_{a,0}(\Omega)$. Thus it remains to prove that all function $v_k := \chi_k v$, where $k \ge 1$, belongs to the same space. After a translation in $\mathbb{R}^2$ we can suppose that Ω_k is star-shaped with respect to $0 \in \Omega_k$. Let us consider the family

$$v_{k,\lambda}(x) := \widetilde{v_k}(\lambda x), \quad \lambda \ge 1,\ x \in \Omega,$$

where $\widetilde{v_k}$ is the trivial extension of v_k in $\mathbb{R}^2$. Since

$$H^1_a(\Omega_k) \cap W^{1,1}_0(\Omega_k) = H^1_{a,0}(\Omega_k)$$

and $v_k \in H^1_a(\Omega_k)$, to obtain the conclusion it is sufficient to prove that

(i) $v_{k,\lambda} \in W_0^{1,1}(\Omega_k)$;

(ii) $v_{k,\lambda} \to v_k$ in $W^{1,1}(\Omega_k)$ as $\lambda \to 1$.

We have that $\text{supp}\{v_{k,\lambda}\} \subset \lambda\Omega_k := \{x \in \Omega_k \mid \lambda x \in \Omega_k\}$, thus in order to prove (i), it is sufficient to show that $v_{k,\lambda} \in W^{1,1}(\Omega_k)$. Preliminarily, we observe that if $\widetilde{v_k} \in W^{1,1}(\mathbb{R}^2)$ then

(1) $(\nabla\widetilde{v_k})(x) = (\nabla v_k)(x)$, x-a.e. in Ω_k;

(2) $(\nabla v_{k,\lambda})(x) = \lambda(\nabla\widetilde{v_k})(\lambda x)$, x-a.e. in $\mathbb{R}^2$.

Here, ∇ is the distributional gradient. One has that

$$\begin{aligned}
\|v_{k,\lambda}\|_{W^{1,1}(\Omega_k)} &= \int_{\Omega_k} |v_{k,\lambda}(x)|dx + \int_{\Omega_k} |\nabla_x v_{k,\lambda}(x)|dx \\
&\overset{(2)}{=} \int_{\lambda\Omega_k} |v_k(\lambda x)|dx + \int_{\lambda\Omega_k} \lambda|\nabla_x \widetilde{v_k}(\lambda x)|dx \\
&= \int_{\Omega_k} \lambda^{-1}|v_k(y)|dy + \int_{\Omega_k} |\nabla_y \widetilde{v_k}(y)|dy \\
&\overset{(1)}{=} \int_{\Omega_k} \lambda^{-1}|v_k(y)|dy + \int_{\Omega_k} |\nabla_y v_k(y)|dy.
\end{aligned} \tag{7.22}$$

So, by Lemma 7.6, (i) holds. Now we prove point (ii). By the definition of $v_{k,\lambda}$,

$$\begin{cases} v_{k,\lambda}(x) \to v_k(x) \quad \text{x-a.e. in } \Omega_k \quad \text{as } \lambda \to 1, \\ \|v_{k,\lambda}\|_{L^1(\Omega_k)} \to \|v_k\|_{L^1(\Omega_k)} \quad \text{as } \lambda \to 1, \end{cases}$$

thus it follows that

$$v_{k,\lambda} \to v_k \quad \text{in } L^1(\Omega_k) \quad \text{as } \lambda \to 1.$$

By the same argument we want to prove that

$$\int_{\Omega_k} |\nabla(v_{k,\lambda} - v_k)|\, dx \to 0 \quad \text{as } \lambda \to 1.$$

We know that

$$\nabla v_{k,\lambda}(x) \overset{(2)}{=} \lambda\nabla\widetilde{v_k}(\lambda x) \overset{(1)}{=} \lambda\nabla v_k(\lambda x) \quad \text{x-a.e. in } \Omega_k\,.$$

Then

$$\begin{cases} \nabla v_{k,\lambda}(x) \to \nabla v_k(x) \quad \text{x-a.e. in } \Omega_k \quad \text{as } \lambda \to 1, \\ \|\nabla v_{k,\lambda}\|_{L^1(\Omega_k)^2} = \|\nabla v_k\|_{L^1(\Omega_k)^2} \quad \forall\, \lambda \geq 1. \end{cases}$$

∎

Lemma 7.8 *In the (WD) case* $C^\infty(\overline{\Omega})^2$ *is dense in* $H_{div,a}(\Omega)$.

Proof of Lemma 7.8 . Let $f \in (H_{div,a}(\Omega))'$, by the Riesz Theorem there exists a unique $g = (g_1, g_2) \in H_{div,a}(\Omega)$ such that

$$< f, w >= \int_\Omega a^{-1} g \cdot w \, dx + \int_\Omega div(g) div(w) \, dx, \quad \forall w \in H_{div,a}(\Omega).$$

Set $h := div(g)$, if $f = 0$ on $C^\infty(\overline{\Omega})^2$ we have that

$$\int_\Omega a^{-1} g \cdot \varphi \, dx = - \int_\Omega h div(\varphi) \, dx \quad \forall \varphi \in C^\infty(\overline{\Omega})^2. \tag{7.23}$$

In particular, (7.23) holds for all $\varphi \in \mathcal{D}(\Omega)^2$, then $\nabla h = a^{-1} g$ in the sense of distributions, and thus $h \in H^1_a(\Omega)$.

Set

$$\widetilde{a^{-1}g} := \begin{cases} a^{-1}g & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega \end{cases}, \quad \widetilde{h} := \begin{cases} h & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega \end{cases},$$

by (7.23) one has that

$$\int_{\mathbb{R}^2} \widetilde{a^{-1}g} \cdot \psi \, dx = - \int_{\mathbb{R}^2} \widetilde{h} div(\psi) \, dx \quad \forall \psi \in \mathcal{D}(\mathbb{R}^2)^2. \tag{7.24}$$

Observe that by Lemma 7.6, $h \in W^{1,1}(\Omega)$. Then $\nabla h = a^{-1} g \in L^1(\Omega)^2$ and thus $\widetilde{a^{-1}g} \in L^1(\mathbb{R}^2)^2$. Moreover, it follows from (7.24) that $\nabla \widetilde{h} = \widetilde{a^{-1}g}$ in the sense of distributions; so, we have obtained that $\widetilde{h} \in W^{1,1}(\mathbb{R}^2)$. Finally, by Lemma 7.7 one has that $h \in H^1_{a,0}(\Omega)$.

In sum, we have shown that, if $f \in (H_{div,a}(\Omega))'$ is such that $< f, \varphi >= 0$ $\forall \varphi \in C^\infty(\overline{\Omega})^2$, then there exists a unique $h \in H^1_{a,0}(\Omega)$ such that

$$< f, w >= \int_\Omega \nabla h \cdot w + h \, div(w) \, dx \quad \forall w \in H_{div,a}(\Omega).$$

The last step to obtain the conclusion is to prove that $h = 0$. For this purpose we consider the sequence of functionals $(f_n)_n$ defined as

$$< f_n, w >:= \int_\Omega \nabla h_n \cdot w + h_n \, div(w) \, dx, \quad w \in H_{div,a}(\Omega),$$

where $(h_n)_n \subset \mathcal{D}(\Omega)$ is such that $h_n \to h$ in $H^1_a(\Omega)$. For showing that $h = 0$ it is sufficient to prove that

(*i*) $< f_n, w > \to < f, w > \quad \forall w \in H_{div,a}(\Omega)$;

(*ii*) $< f_n, w >= 0 \quad \forall n \in \mathbb{N}, \ \forall w \in H_{div,a}(\Omega)$.

Proof of (i) :

$$
\begin{aligned}
| < f_n - f, w > | &\leq \int_\Omega |\sigma\sigma^{-1}\nabla(h_n - h)\cdot w| + |h_n - h||div(w)|\,dx \\
&\leq \left(\int_\Omega a\nabla(h_n - h)\cdot\nabla(h_n - h)\,dx\right)^{\frac{1}{2}} \left(\int_\Omega a^{-1} w\cdot w\,dx\right)^{\frac{1}{2}} \\
&\quad + \left(\int_\Omega |h_n - h|^2\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |div(w)|^2\,dx\right)^{\frac{1}{2}} \quad \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Proof of (ii) : by the standard normal trace theory one has that

$$
\int_\Omega \nabla h_n \cdot w + h_n\, div(w)\,dx = \langle \gamma_\nu w, \gamma h_n\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0
$$

$\forall w \in H_{div,a}(\Omega),\ \forall n \in \mathbb{N}.$ ■

Finally, we are ready to give the proof of Proposition 5.3. Let $w \in C^\infty(\overline{\Omega})^2$ and $v \in H^1_a(\Omega)$. By Proposition 3.1 we know that $C^\infty(\overline{\Omega})$ is dense in $H^1_a(\Omega)$ and therefore by the standard normal trace theory one has that

$$
\int_\Gamma (w\cdot\nu)\gamma^a v\,dr = \mathcal{T}_w v.
$$

Furthermore, by Lemma 4.1 we obtain

$$
\left|\int_\Gamma (w\cdot\nu)\gamma^a v\,dr\right| \leq \|w\|_{H_{div,a}(\Omega)} \|v\|_{H^1_a(\Omega)}\,.
$$

Set $\gamma^a v = \phi$ for $\phi \in H^{1/2}_a(\Gamma)$, the last inequality holds for all $\Phi \in H^1_a(\Omega)$ such that $\gamma^a\Phi = \phi$. Then

$$
\begin{aligned}
\left|\int_\Gamma (w\cdot\nu)\phi\,dr\right| &\leq \|w\|_{H_{div,a}(\Omega)} \inf\left\{\|\Phi\|_{H^1_a(\Omega)} \mid \gamma^a\Phi = \phi\right\} \\
&= \|w\|_{H_{div,a}(\Omega)} \|\phi\|_{H^{1/2}_a(\Gamma)}\,.
\end{aligned}
$$

Hence, the functional

$$
\phi \longmapsto < J(w\cdot\nu)_{|\Gamma}, \phi > := \int_\Gamma (w\cdot\nu)\phi\,dr
$$

is continuous in $H^{1/2}_a(\Gamma)$ and the operator

$$
w \longmapsto J(w\cdot\nu)_{|\Gamma} \in H^{-1/2}_a(\Gamma),
$$

is continuous from $\left(C^\infty(\overline{\Omega})^2,\ \|\cdot\|_{H_{div,a}(\Omega)}\right)$ into $H^{-1/2}_a(\Gamma)$. Here, J is the Riesz isomorphism in $L^2(\Gamma)$, that is

$$
J:\ L^2(\Gamma) \ni h \longmapsto Jh \in H^{-1/2}_a(\Gamma), \quad < Jh, v > := \int_\Gamma hv\,dr.
$$

So, by the density result claimed in Lemma 7.8 we obtain the conclusion.

7.8. Proof of Proposition 5.4

As a first step we prove point (1) . Afterward we will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Setting for simplicity $< \gamma_\nu^\delta w, \gamma^\delta v > := \langle \gamma_\nu^\delta w, \gamma^\delta v \rangle_{H^{-1/2}(\Gamma^\delta), H^{1/2}(\Gamma^\delta)}$, preliminarily we introduce some technical results.

LEMMA 7.9 *Let $w \in H_{div}(\Omega)$, $v \in H^1(\Omega)$ be given. Then the functions*

$$[0, \delta_0/2] \ni \delta \longmapsto \int_{\Gamma^\delta} |\gamma^\delta v|^2 dr,$$
$$[0, \delta_0/2] \ni \delta \longmapsto < \gamma_\nu^\delta w, \gamma^\delta v >$$

are continuous.

Proof of Lemma 7.9. Let $\varphi \in C^\infty(\overline{\Omega})$ and δ_1, $\delta_2 \in [0, \delta_0/2]$, recalling the proof of Lemma 7.2, one has

$$\begin{aligned}
&\left| \int_{\Gamma^{\delta_1}} |\gamma^{\delta_1} \varphi|^2 dr - \int_{\Gamma^{\delta_2}} |\gamma^{\delta_2} \varphi|^2 dr \right| \\
&\le \sum_{k=1}^m \int_{\mathcal{B}_k} \Big| |\varphi(X_k(s, \delta_1))|^2 \beta(X_k(s, \delta_1)) - |\varphi(X_k(s, \delta_2))|^2 \beta(X_k(s, \delta_2)) \Big| \, ds \\
&\le m \max_k |\mathcal{B}_k| \left\| \frac{\partial}{\partial \delta'} \Big(|\varphi \circ X_k|^2 \beta \circ X_k \Big) \right\|_{L^\infty(\mathcal{R}_k)} |\delta_1 - \delta_2|.
\end{aligned}$$

So, by the density result of Proposition 3.1, it follows that $\int_{\Gamma^\delta} |\gamma^\delta v|^2 dr$ is continuous in δ. Now, by the Gauss-Green Formula one obtains

$$\begin{aligned}
\big| < \gamma_\nu^{\delta_1} w, \gamma^{\delta_1} v > - < \gamma_\nu^{\delta_2} w, \gamma^{\delta_2} v > \big| &\le sgn\{\delta_1 - \delta_2\} \int_{\Omega^{\delta_1} \setminus \Omega^{\delta_2}} |div(wv)| \, dx \\
&\le \|w\|_{H_{div}(\Omega^{\delta_1} \setminus \Omega^{\delta_2})} \|v\|_{H^1(\Omega^{\delta_1} \setminus \Omega^{\delta_2})} .
\end{aligned}$$

Thus, since $|\Omega^{\delta_1} \setminus \Omega^{\delta_2}|$ tends to zero as $\delta_1 - \delta_2$ tends to zero, the conclusion follows. ■

Proof of (1). Let $w \in C^\infty(\overline{\Omega})^2$, $v \in H^1(\Omega)$ and δ_1, $\delta_2 \in (0, \delta_0/2)$ such that $\delta_1 < \delta_2$. By Lemma 7.2 and Holder's inequality one has that

$$\begin{aligned}
&\int_{\delta_1}^{\delta_2} \big| < \gamma_\nu^\delta w, \gamma^\delta v > \big|^2 \frac{d\delta}{\delta} = \int_{\delta_1}^{\delta_2} \left| \int_{\Gamma^\delta} \Big(w \cdot \nu \circ p_\Gamma \Big) \gamma^\delta v \, dr \right|^2 \frac{d\delta}{\delta} \\
&\le \int_{\delta_1}^{\delta_2} \left(\int_{\Gamma^\delta} \big| w \cdot \nu \circ p_\Gamma \big|^2 dr \int_{\Gamma^\delta} |\gamma^\delta v|^2 dr \right) \frac{d\delta}{\delta} \\
&\le \max_{\delta \in [0, \delta_2]} \left(\int_{\Gamma^\delta} |\gamma^\delta v|^2 dr \right) \int_{\delta_1}^{\delta_2} \left(\int_{\Gamma^\delta} \big| w \cdot \nu \circ p_\Gamma \big|^2 dr \right) \frac{d\delta}{\delta} \\
&= M(\delta_2, v) \int_{\Omega^{\delta_1} \setminus \Omega^{\delta_2}} \frac{\big| w(x) \cdot \nu(p_\Gamma x) \big|^2}{d(x)} \, dx.
\end{aligned} \tag{7.25}$$

Since $C^\infty(\overline{\Omega})^2$ is dense in $H_{div}(\Omega)$, by (7.25) we obtain that for all $w \in H_{div}(\Omega)$, $v \in H^1(\Omega)$ one has

$$\int_{\delta_1}^{\delta_2} \big| < \gamma_\nu^\delta w, \gamma^\delta v > \big|^2 \frac{d\delta}{\delta} \leq M(\delta_2, v) \int_{\Omega^{\delta_1} \setminus \Omega^{\delta_2}} \frac{\big|w(x) \cdot \nu(p_\Gamma x)\big|^2}{d(x)}\, dx. \quad (7.26)$$

On the other hand, by assumptions (1),(2),(3) on $a(x)$ we have

$$\|w\|^2_{L^2_{a^{-1}}(\Omega_{\delta_0})} = \int_{\Omega_{\delta_0}} \frac{\big|w(x) \cdot \varepsilon_1(x)\big|^2}{\lambda_1(x)} + \frac{\big|w(x) \cdot \nu(p_\Gamma x)\big|^2}{\lambda_2(x)}\, dx, \quad (7.27)$$

thus, from (7.26), (7.27), because $a(x)$ is (SD), it follows that for all $w \in H_{div,a}(\Omega)$, $v \in H^1(\Omega)$ one has

$$\int_{0}^{\delta_2} \big| < \gamma_\nu^\delta w, \gamma^\delta v > \big|^2 \frac{d\delta}{\delta} \leq M(\delta_2, v)\, C_0\, \|w\|^2_{H_{div,a}(\Omega_{\delta_2})}. \quad (7.28)$$

On the other hand, by Lemma 7.9, $\delta \mapsto < \gamma_\nu^\delta w, \gamma^\delta v >$ is continuous in $\delta = 0$, thus by (7.28) one has that

$$\lim_{\delta \to 0} < \gamma_\nu^\delta w, \gamma^\delta v > = < \gamma_\nu w, \gamma v > = 0 \quad \forall w \in H_{div,a}(\Omega),\ v \in H^1(\Omega),$$

which proves point (1) of Proposition 5.4.

(1) $\Rightarrow$ (2). If $v \in H^1(\Omega)$ and $w = a\nabla u \in H_{div,a}(\Omega)$, then by the standard normal trace theory, one has that $\mathcal{T}_w v = < \gamma_\nu w, \gamma v >$. Thus, by (1) we have that $\mathcal{T}_w v = 0$ for all $v \in H^1(\Omega)$. Since $H^1(\Omega)$ is dense in $H^1_a(\Omega)$ the conclusion follows.

(2) $\Rightarrow$ (3). We prove that if $f \in (H^1_a(\Omega))'$ is such that $f = 0$ on $\mathcal{D}(\Omega)$, then f is the null functional. By the Riesz Theorem there exists a unique $g \in H^1_a(\Omega)$ such that

$$< f, v > = \int_\Omega gv + a\nabla g \cdot \nabla v\, dx \quad \forall v \in H^1_a(\Omega).$$

Thus, if $f = 0$ on $\mathcal{D}(\Omega)$, it follows that

$$\int_\Omega g\varphi\, dx = -\int_\Omega a\nabla g \cdot \nabla\varphi\, dx \quad \forall \varphi \in \mathcal{D}(\Omega),$$

that is

$$div(a\nabla g) = g \quad (7.29)$$

in the sense of distributions. By (7.29) it follows that $a\nabla g \in H_{div,a}(\Omega)$ and that

$$-\psi\, div(a\nabla g) + \psi g = 0 \quad \forall \psi \in C^\infty(\overline{\Omega}).$$

Then, by point (2) we have that

$$0 = \int_\Omega -\psi\, div(a\nabla g) + \psi g\, dx = < \psi, g >_{H^1_a(\Omega)} \quad \forall \psi \in C^\infty(\overline{\Omega}).$$

By the density result of Proposition 3.1, the conclusion follows.

7.9. Proof of Theorem 6.1

In this section we prove the main result of this work; for this purpose we use the Lions method (see, e.g., Baiocchi and Capelo, 1983; Bensoussan et al., 1993; Lions and Magenes, 1972; Necas, 1967; Showalter, 1977) based on the Lax-Milgram Lemma. As the first step we give some results related to the bilinear form $q(.,.)$ associated to the operator $div(a\nabla u)$.

LEMMA 7.10 *The following results holds:*

(1) *the injection* $i : H^1_a(\Omega) \to L^2(\Omega)$ *is continuous with dense range;*

(2) *the bilinear form*

$$q(u,v) := \int_\Omega a\nabla u \cdot \nabla v\, dx, \quad u, v \in H^1_a(\Omega),$$

is continuous, positive, symmetric and there exist $\lambda \in \mathbb{R}$, $\alpha_0 > 0$ *such that*

$$q(v,v) + \lambda \|v\|^2_{L^2(\Omega)} \geq \alpha_0 \|v\|^2_{H^1_a(\Omega)} \ \forall\, v \in H^1_a(\Omega);$$

moreover, the previous inequality holds for all $\lambda > 0$ *;*

(3) *in the (WD) case,* $q(.,.)$ *is also coercive on* $H^1_{a,0}(\Omega)$ *.*

Proof of Lemma 7.10. Obviously, by definition of $\|.\|_{H^1_a(\Omega)}$, it follows that i is continuous. Thus, since $C^\infty(\overline{\Omega})$ is dense in $H^1_a(\Omega)$ by Proposition 3.1, it is also dense in $L^2(\Omega)$. Now, we prove point (2). One has that

$$|q(u,v)| = \left|\int_\Omega \sigma\nabla u \cdot \sigma\nabla v\, dx\right| = \left|\int_\Omega \sigma\nabla v \cdot \sigma\nabla u\, dx\right| \leq \|\sigma\nabla v\|_{L^2(\Omega)^2} \|\sigma\nabla u\|_{L^2(\Omega)^2} .$$

Furthermore, by the definition of $\|.\|_{H^1_a(\Omega)}$, for every $\lambda > 0$, $v \in H^1_a(\Omega)$, one has

$$q(v,v) + \lambda \|v\|^2_{L^2(\Omega)} \geq (1 \wedge \lambda) \|v\|^2_{H^1_a(\Omega)} .$$

Thus, for some $\lambda \in (0,1)$ we obtain that $q(v,v) \geq 0$ for all $v \in H^1_a(\Omega)$. It remains to prove point (3). By Hardy's inequality (5.1),

$$q(v,v) \geq C_H \int_\Omega v^2 \frac{\lambda_2(x)}{d(x)^2}\, dx \geq C \int_\Omega v^2\, dx, \quad \forall v \in H^1_{a,0}(\Omega),$$

where C is a suitable positive constant. Then, one has

$$q(v,v) \geq \frac{1 \wedge C}{2} \|v\|^2_{H^1_a(\Omega)}, \quad \forall v \in H^1_{a,0}(\Omega). \quad \blacksquare$$

Now, we establish the following Green Formula:

Lemma 7.11 *If* $(u,v) \in D(A_1) \times H^1_a(\Omega)$ *or* $(u,v) \in D(A_2) \times H^1_{a,0}(\Omega)$, *one has*

$$\int_\Omega a\nabla u \cdot \nabla v \, dx = -\int_\Omega div\,(a\nabla u) v \, dx. \tag{7.30}$$

Proof of Lemma 7.11. By Corollary 4.1, it is sufficient to check that $\mathcal{T}_{a\nabla u} v = 0$. If $u \in D(A_1)$, then

$$\mathcal{T}_{a\nabla u} v = \langle \gamma_\nu(a\nabla u), \gamma v\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 0 \quad \forall v \in H^1(\Omega).$$

Besides, $H^1(\Omega)$ is dense in $H^1_a(\Omega)$ by Proposition 3.1. Hence

$$\mathcal{T}_{a\nabla u} v = 0 \quad \forall (u,v) \in D(A_1) \times H^1_a(\Omega).$$

If $(u,v) \in D(A_2) \times H^1_{a,0}(\Omega)$, then

$$\mathcal{T}_{a\nabla u} v = \langle \gamma^a_\nu(a\nabla u), \gamma^a v\rangle_{H_a^{-1/2}(\Gamma), H_a^{1/2}(\Gamma)} = 0,$$

since we have proved in Proposition 5.2 that $H^1_{a,0}(\Omega) = Ker\{\gamma^a\}$. ■

Now, we are ready to prove Theorem 6.1. From Lemma 7.11 and point (2) of Lemma 7.10, it follows that A_1 and A_2 are dissipative: indeed for $i = 1, 2$ one has

$$< A_i u, u >_{L^2(\Omega)} = \int_\Omega div(a\nabla u) u \, dx = -\int_\Omega a\nabla u \cdot \nabla u \, dx \leq 0 \quad \forall u \in D(A_i).$$

We observe that by point (3) of Lemma 7.10 it directly follows that A_2 is strictly dissipative, and, by point (2), that A_1, A_2 are generators of analytic semigroups. Moreover, by Lemma 7.11 again, it follows that A_1 and A_2 are also self-adjoint. Now, we want to prove that A_1 is maximal. Let us consider the bilinear form defined in Lemma 7.10. By the Lax-Milgram Lemma, one has that for all $f \in L^2(\Omega)$ there exists a unique $u \in H^1_a(\Omega)$ such that

$$\int_\Omega a\nabla u \cdot \nabla v \, dx + \lambda \int_\Omega uv \, dx = \int_\Omega fv \, dx, \quad \forall v \in H^1_a(\Omega). \tag{7.31}$$

In particular, u satisfies the variational problem (7.31) for all $v \in \mathcal{D}(\Omega)$; then $div(a\nabla u) = -f + \lambda u$ in the sense of distribution and thus $u \in H^2_a(\Omega)$. Now, by (7.30), (7.31), we have that for all $f \in L^2(\Omega)$ there exists a unique $u \in D(A_1)$ such that

$$< (A_1 - \lambda I)u + f, v >_{L^2(\Omega)} = 0 \quad \forall v \in H^1_a(\Omega) \quad \Leftrightarrow \quad R(A_1 - \lambda I) = L^2(\Omega),$$

where $R(A_1 - \lambda I)$ is the range of $A_1 - \lambda I$. In an analogous way one proves the maximality of A_2. Last, it remains to check that, in the (SD) case, the domain $D(A_1)$ is equal to $H^2_a(\Omega)$. Since u belongs to $H^2_a(\Omega)$, one has that $a\nabla u$ belongs to $H_{div,a}(\Omega)$. Thus, by point (1) of Proposition 5.4, one obtains that $\gamma_\nu(a\nabla u) = 0$.

7.10. Proof of Proposition 6.1

We start by observing that $u \in D(A_1)$ if and only if there exists $f \in L^2(\Omega)$ such that $u \in H^1_a(\Omega)$ solves the variational problem (7.31). Hence, picking a smooth function χ such that

$$\begin{cases} 0 \le \chi(x) \le 1, & x \in \Omega, \\ \chi(x) = 0, & x \in \Omega^{\delta_0}, \\ \chi(x) = 1, & x \in \Omega_{\delta_0/2}, \end{cases}$$

by the assumptions made on $a(x)$, it follows that $u \in D(A_1)$ solves

$$\int_{\Omega_{\delta_0}} \lambda_1(x)\partial_{\varepsilon_1} u\, \partial_{\varepsilon_1}(\chi v) + \lambda_2(x)\partial_{\varepsilon_2} u\, \partial_{\varepsilon_2}(\chi v)\, dx + \lambda \int_{\Omega_{\delta_0}} u\chi v\, dx = \int_{\Omega_{\delta_0}} f\chi v\, dx,$$

for all $v \in H^1_a(\Omega)$. So, choosing $v = u$, with some computations we obtain that $\lambda_1^{1/2}\partial_{\varepsilon_1} u, \lambda_2^{1/2}\partial_{\varepsilon_2} u$ belong to $L^2(\Omega_{\delta_0/2})$. Now, we want to estimate the distributional derivatives $\partial^2_{\varepsilon_1} u$, $\partial^2_{\varepsilon_2,\varepsilon_1} u$ and $\partial_{\varepsilon_2}(d^{\kappa_2}\partial_{\varepsilon_2} u)$. Set

$$\beta_k(s,\delta) := (\beta \circ X_k)(s,\delta), \tag{7.32}$$

for $k = 1, \dots, m$, and let $z \in H^2_a(\Omega)$ be given, with simple computations one has that

$$\begin{cases} \partial_{\varepsilon_1} z = \beta_k^{-1} \dfrac{\partial}{\partial s}(z \circ X_k), \\ \partial_{\varepsilon_2} z = \dfrac{\partial}{\partial \delta}(z \circ X_k), \\ \partial^2_{\varepsilon_1} z = \beta_k^{-1} \dfrac{\partial}{\partial s}\big(\beta_k^{-1} \dfrac{\partial}{\partial s}(z \circ X_k)\big), \\ \partial_{\varepsilon_2}(d^{\kappa_2}\partial_{\varepsilon_2} z) = \dfrac{\partial}{\partial \delta}\big(\delta^{\kappa_2} \dfrac{\partial}{\partial \delta}(z \circ X_k)\big), \\ \partial^2_{\varepsilon_2,\varepsilon_1} z = \dfrac{\partial}{\partial \delta}\big(\beta_k^{-1} \dfrac{\partial}{\partial s}(z \circ X_k)\big). \end{cases} \tag{7.33}$$

Let us, for example, compute $\partial^2_{\varepsilon_1} z$, using Lemma 7.1. Since

$$z = z \circ X_k \circ X_k^{-1}$$

we have that

$$\begin{aligned}\partial_{\varepsilon_1} z :=& \nabla z \cdot \varepsilon_1 = (DX_k^{-1})^* \nabla(z \circ X_k) \cdot \varepsilon_1 = \nabla(z \circ X_k) \cdot DX_k^{-1}\varepsilon_1 \\ =& \nabla(z \circ X_k) \cdot \beta_k^{-1} e_1 = \beta_k^{-1} \frac{\partial}{\partial s}(z \circ X_k) .\end{aligned}$$

Thus

$$\begin{aligned}\partial^2_{\varepsilon_1} z &= \nabla(\nabla z \cdot \varepsilon_1) \cdot \varepsilon_1 = (DX_k^{-1})^* \nabla(\nabla(z \circ X_k) \cdot \varepsilon_1) \cdot \varepsilon_1 \\ &= \nabla(\nabla(z \circ X_k) \cdot \varepsilon_1) \cdot DX_k^{-1}\varepsilon_1 = \nabla\Big(\beta_k^{-1} \frac{\partial}{\partial s}(z \circ X_k)\Big) \cdot \beta_k^{-1} e_1 \\ &= \beta_k^{-1} \frac{\partial}{\partial s}\Big(\beta_k^{-1} \frac{\partial}{\partial s}(z \circ X_k)\Big).\end{aligned}$$

Since $\Gamma \in \mathcal{C}^r$ with $r \geq 3$, one has that $\beta \in C^1(\overline{\Omega}_{\delta_0})$ and X_k is a $\mathcal{C}^2$- diffeomorphism. Also, β is strictly positive on $\overline{\Omega}_{\delta_0}$. So β_k, $\beta_k^{-1} \in C^1(\overline{\mathcal{R}_k}^+)$, where

$$\mathcal{R}_k^+ := \mathcal{B}_k \times (0, \delta_0).$$

Consequently, in order to obtain the conclusion it is sufficient to estimate the first and second derivatives of the functions

$$\hat{u}_k := u \circ X_k, \quad k = 1, \dots, m .$$

Set $\Omega_k := \Omega \cap U_k = X_k(\mathcal{R}_k^+)$ where $\{U_k\}_{k=1}^m$ is defined as in Section 7.1, one has that $u \in D(A_1)$ solves

$$\int_{\Omega_k} d(x)^{\kappa_1} \partial_{\varepsilon_1} u \, \partial_{\varepsilon_1} v + d(x)^{\kappa_2} \partial_{\varepsilon_2} u \, \partial_{\varepsilon_2} v \, dx + \lambda \int_{\Omega_k} uv \, dx = \int_{\Omega_k} fv \, dx, \quad (7.34)$$

for all $v \in H_a^1(\Omega_k)$ which vanish in a neighborhood of ∂U_k .
By defining the Hilbert space

$$\mathcal{H}_1(\mathcal{R}_k^+) := \left\{ v \in L^2(\mathcal{R}_k^+) \;\Big|\; \delta^{\kappa_1/2} \frac{\partial v}{\partial s} , \, \delta^{\kappa_2/2} \frac{\partial v}{\partial \delta} \in L^2(\mathcal{R}_k^+) \right\},$$

endowed by the norm

$$\|v\|^2_{\mathcal{H}_1(\mathcal{R}_k^+)} := \int_{\mathcal{R}_k^+} |v|^2 + \delta^{\kappa_1} \left|\frac{\partial v}{\partial s}\right|^2 + \delta^{\kappa_2} \left|\frac{\partial v}{\partial \delta}\right|^2 \, ds d\delta,$$

using the diffeomorphism X_k on (7.34) and setting $\hat{f}_k := f \circ X_k$, it is easy to see that $\hat{u}_k$ solves

$$\int_{\mathcal{R}_k^+} \beta_k^{-1} \delta^{\kappa_1} \frac{\partial \hat{u}_k}{\partial s} \frac{\partial v}{\partial s} + \beta_k \delta^{\kappa_2} \frac{\partial \hat{u}_k}{\partial \delta} \frac{\partial v}{\partial \delta} ds d\delta + \lambda \int_{\mathcal{R}_k^+} \hat{u}_k v \, \beta_k \, ds d\delta = \int_{\mathcal{R}_k^+} \hat{f}_k v \, \beta_k \, ds d\delta \quad (7.35)$$

for all $v \in \mathcal{H}_1^0(\mathcal{R}_k^+) := \{v \in \mathcal{H}_1(\mathcal{R}_k^+)$ which vanish in a neighborhood of $\partial \mathcal{R}_k\}$. We note that $\hat{u}_k$ satisfies the variational problem (7.35) for all $v \in \mathcal{D}(\mathcal{R}_k^+)$. Thus, it solves the partial differential equation

$$\mathcal{A}_k \hat{u}_k := \frac{\partial}{\partial s} \left(\beta_k^{-1} \delta^{\kappa_1} \frac{\partial \hat{u}_k}{\partial s} \right) + \frac{\partial}{\partial \delta} \left(\beta_k \delta^{\kappa_2} \frac{\partial \hat{u}_k}{\partial \delta} \right) = (\lambda \hat{u}_k - \hat{f}_k) \beta_k, \quad (7.36)$$

in the sense of distributions.

Now, we want to extend the classical difference quotients method for strongly elliptic problems (see, e.g., Agmon, 1965; Necas, 1967) to problem (7.35). Let v be a function defined on $\mathcal{R}_k^+$ such that the distance of the support of v from $\partial\mathcal{R}_k$ is positive. For each $h \in \mathbb{R}$ such that $|h| < dist\,(\text{supp}\{v\}, \partial\mathcal{R}_k)$ we define the translate of v by

$$\mathcal{T}_h v(s,\delta) := v(s+h,\delta)$$

and, if $h \neq 0$, the difference quotient of v by

$$D_h v := (\mathcal{T}_h v - v)/h.$$

We give some elementary property of the operator D_h which will be useful later:

LEMMA 7.12 *One has that*

(1) $\frac{\partial}{\partial\delta} D_h = D_h \frac{\partial}{\partial\delta}$, $\quad \frac{\partial}{\partial s} D_h = D_h \frac{\partial}{\partial s}$;

(2) *all pairs of functions f, g on $\mathcal{R}_k^+$ which vanish in a neighborhood of $\partial\mathcal{R}_k$ satisfy*

$$D_h(fg) = \mathcal{T}_h f D_h g + g D_h f.$$

Moreover, if f, $g \in L^2(\mathcal{R}_k^+)$, one has

$$< D_h f, g >_{L^2(\mathcal{R}_k^+)} = - < f, D_{-h} g >_{L^2(\mathcal{R}_k^+)};$$

(3) *if $v \in \mathcal{H}_1^0(\mathcal{R}_k^+)$ then $D_h v \to \frac{\partial v}{\partial s}$ on $\mathcal{D}'(\mathcal{R}_k^+)$ as $h \to 0$;*

(4) *if $v \in \mathcal{H}_1^0(\mathcal{R}_k^+)$ then $D_h v \in \mathcal{H}_1^0(\mathcal{R}_k^+)$;*

(5) *if $v \in \mathcal{H}_1^0(\mathcal{R}_k^+)$ then $\delta^\theta v \in \mathcal{H}_1^0(\mathcal{R}_k^+)$ for all $\theta \geq \min\{0, (2-\kappa_2)/2\}$;*

(6) *let $\theta \geq \kappa_1$ and $v \in \mathcal{H}_1^0(\mathcal{R}_k^+)$, one has*

$$\int_{\mathcal{R}_k^+} \delta^\theta |D_h v(s,\delta)|^2 \, ds d\delta \leq \int_{\mathcal{R}_k^+} \delta^\theta \left|\frac{\partial v}{\partial s}(s,\delta)\right|^2 ds d\delta.$$

Proof of Lemma 7.12. The assertions (1), ..., (5) are only trivial remarks. Hence it remains to prove (6). We can write

$$v(s+h,\delta) - v(s,\delta) = \int_0^1 h \frac{\partial v}{\partial s}(s+th,\delta)\, dt,$$

hence

$$|v(s+h,\delta) - v(s,\delta)|^2 \leq h^2 \int_0^1 \left|\frac{\partial v}{\partial s}(s+th,\delta)\right|^2 dt. \tag{7.37}$$

Thus, multiplying both members of (7.37) by δ^θ, taking the integral for $(s,\delta) \in \mathcal{R}_k^+$ and using Fubini's Theorem, we deduce

$$\int_{\mathcal{R}_k^+} \delta^\theta |D_h v(s,\delta)|^2 \, dsd\delta \le \int_0^1 \left(\int_{\mathcal{R}_k^+} \delta^\theta \left| \frac{\partial v}{\partial s}(s+th,\delta) \right|^2 dsd\delta \right) dt$$
$$= \int_{\mathcal{R}_k^+} \delta^\theta \left| \frac{\partial v}{\partial s}(s,\delta) \right|^2 dsd\delta. \qquad \blacksquare$$

We are going to prove the main technical result of this section. Preliminarily we give some notations and definitions. Let $\theta \ge 0$, first we introduce the following pre-Hilbert space

$$\mathcal{H}_{1,\theta}(\mathcal{R}_k^+) := \left\{ \delta^{\theta/2} v \in L^2(\mathcal{R}_k^+) \;\middle|\; \delta^{(\kappa_1+\theta)/2} \frac{\partial v}{\partial s}\,,\, \delta^{(\kappa_2+\theta)/2} \frac{\partial v}{\partial \delta} \in L^2(\mathcal{R}_k^+) \right\},$$

endowed by the norm

$$\begin{aligned} \|v\|^2_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)} &:= \int_{\mathcal{R}_k^+} \delta^\theta |v|^2 dsd\delta \;+\; |\, v\,|^2_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)} \\ &:= \int_{\mathcal{R}_k^+} \delta^\theta |v|^2 dsd\delta \;+ \int_{\mathcal{R}_k^+} \delta^{\kappa_1+\theta} \left|\frac{\partial v}{\partial s}\right|^2 + \delta^{\kappa_2+\theta} \left|\frac{\partial v}{\partial \delta}\right|^2 dsd\delta\,. \end{aligned} \tag{7.38}$$

Lemma 7.13 *$\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)$ is a Hilbert space.*

Proof of Lemma 7.13. As first step we define the family

$$L_p^2(\mathcal{R}_k^+) := \{ w \in L^1_{loc}(\mathcal{R}_k^+) \mid \delta^{p/2} w \in L^2(\mathcal{R}_k^+) \}, \quad p \ge 0.$$

We pick a Cauchy sequence $(v_n)_n$ in $\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)$, there exist u, $g_1, g_2 \in L^2(\mathcal{R}_k^+)$ such that

(1) $v_n \to v := \delta^{-\theta/2} u$ in $L^2_\theta(\mathcal{R}_k^+)$,

(2) $\frac{\partial v_n}{\partial s} \to \delta^{-(\kappa_1+\theta)/2} g_1$ in $L^2_{(\kappa_1+\theta)}(\mathcal{R}_k^+)$,

(3) $\frac{\partial v_n}{\partial \delta} \longrightarrow \delta^{-(\kappa_2+\theta)/2} g_2$ in $L^2_{(\kappa_2+\theta)}(\mathcal{R}_k^+)$.

By (1) we obtain that

$$\frac{\partial v_n}{\partial s} \to \frac{\partial v}{\partial s} \text{ in } \mathcal{D}'(\mathcal{R}_k^+), \quad \frac{\partial v_n}{\partial \delta} \to \frac{\partial v}{\partial \delta} \text{ in } \mathcal{D}'(\mathcal{R}_k^+).$$

So, by (2),(3) and the uniqueness of the distributional limit, the conclusion follows. $\blacksquare$

Next, let $\psi_k \in \mathcal{D}\big(\Omega_k; [0,1]\big)$ with $k = 1, \ldots, m$ be cutoff functions such that $\psi_k = 1$ on $E_k := \Omega \cap \operatorname{supp}\{\chi_k\}$ and $u \in D(A_1)$ we call

$$\begin{cases} z_k := \psi_k u, \\ \hat{z}_k := \hat{\psi}_k \hat{u}_k = (\psi_k \circ X_k)(u \circ X_k), \\ \mathcal{E}_k := X_k^{-1}(E_k)\,. \end{cases}$$

Notice that, by definition of $\{\chi_k\}_{k=0}^m$, one has

$$\Omega_{\delta_0/2} \subset \cup_{k=1}^m E_k\,. \tag{7.39}$$

LEMMA 7.14 *Choosing*

(1) $\theta \geq 0$, *if* $\kappa_1 = 0$ *and* $\kappa_2 \geq 0$,

(2) $\theta \geq \max\{\kappa_1, \kappa_1 + (2-\kappa_2)\}$, *if* $\kappa_1 > 0$ *and* $\kappa_2 \geq 0$,

for each fixed $k \in \{1, \ldots, m\}$, *the differences* $D_h \hat{z}_k$ *are uniformly bounded in* $\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)$.

Proof of Lemma 7.14. Preliminarily we define a *bilinear form with coefficients* $[g^*, g]$, as

$$\mathcal{Q}[g^*, g](u,v) := \int_{\mathcal{R}_k^+} g^*(s,\delta)\delta^{\kappa_1} \frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + g(s,\delta)\delta^{\kappa_2}\frac{\partial u}{\partial \delta}\frac{\partial v}{\partial \delta}\, ds d\delta,$$

where u, $v \in \mathcal{H}_1(\mathcal{R}_k^+)$, g^* and g are two suitable functions on $\mathcal{R}_k^+$. We observe that using Leibniz rule one has

$$\begin{aligned} \mathcal{Q}[\beta_k^{-1}, \beta_k](\hat{\psi}_k u, v) &= \mathcal{Q}[\hat{\psi}_k\beta_k^{-1}, \hat{\psi}_k\beta_k](u,v) \\ &\quad + \int_{\mathcal{R}_k^+} \left(\frac{\partial \hat{\psi}_k}{\partial s}\beta_k^{-1}\right)\delta^{\kappa_1} u \frac{\partial v}{\partial s} + \left(\frac{\partial \hat{\psi}_k}{\partial \delta}\beta_k\right)\delta^{\kappa_2} u\frac{\partial v}{\partial \delta}\, ds d\delta \\ &= \mathcal{Q}[\beta_k^{-1}, \beta_k](u, \hat{\psi}_k v) \\ &\quad - \int_{\mathcal{R}_k^+} \left(\frac{\partial \hat{\psi}_k}{\partial s}\beta_k^{-1}\right)\delta^{\kappa_1} v \frac{\partial u}{\partial s} + \left(\frac{\partial \hat{\psi}_k}{\partial \delta}\beta_k\right)\delta^{\kappa_2} v\frac{\partial u}{\partial \delta}\, ds d\delta \\ &\quad + \int_{\mathcal{R}_k^+} \left(\frac{\partial \hat{\psi}_k}{\partial s}\beta_k^{-1}\right)\delta^{\kappa_1} u \frac{\partial v}{\partial s} + \left(\frac{\partial \hat{\psi}_k}{\partial \delta}\beta_k\right)\delta^{\kappa_2} u\frac{\partial v}{\partial \delta}\, ds d\delta. \end{aligned} \tag{7.40}$$

We suppose also that h is such that

$$h \neq 0, \quad |h| \leq \min_k \left\{ dist\left(\operatorname{supp}\{\hat{z}_k\}\ ,\ \partial\mathcal{R}_k\right)\right\}. \tag{7.41}$$

We begin by proving the point (2). Set

$$\theta_0 := \max\{\kappa_1, \kappa_1 + (2-\kappa_2)\}$$

in the following we suppose that $\theta \geq \theta_0$. By points (4),(5) of Lemma 7.12 one has that

$$v_{\theta,h,k} := -\delta^\theta D_{-h} D_h \hat{z}_k \in \mathcal{H}^0_1(\mathcal{R}^+_k).$$

Next, we compute $\mathcal{Q}[\beta_k^{-1}, \beta_k](\hat{z}_k, v_{\theta,h,k})$. Using Lemma 7.12, one has

$$\begin{aligned}
&\mathcal{Q}[\beta_k^{-1}, \beta_k](\hat{z}_k, -\delta^\theta D_{-h} D_h \hat{z}_k)\\
&= -\int_{\mathcal{R}^+_k} \beta_k^{-1}\delta^{\kappa_1} \frac{\partial \hat{z}_k}{\partial s}\, \delta^\theta \frac{\partial}{\partial s} D_{-h} D_h \hat{z}_k + \beta_k \delta^{\kappa_2} \frac{\partial \hat{z}_k}{\partial \delta} \frac{\partial}{\partial \delta}(\delta^\theta D_{-h} D_h \hat{z}_k)\, ds d\delta\\
&= \int_{\mathcal{R}^+_k} D_h \left(\beta_k^{-1}\delta^{\kappa_1} \frac{\partial \hat{z}_k}{\partial s}\right) \delta^\theta \frac{\partial}{\partial s} D_h \hat{z}_k + D_h \left(\beta_k \delta^{\kappa_2} \frac{\partial \hat{z}_k}{\partial \delta}\right) \frac{\partial}{\partial \delta}(\delta^\theta D_h \hat{z}_k)\, ds d\delta\\
&= \int_{\mathcal{R}^+_k} \mathcal{T}_h \beta_k^{-1} \delta^{\kappa_1+\theta} \left|\frac{\partial}{\partial s} D_h \hat{z}_k\right|^2 + \mathcal{T}_h \beta_k\, \delta^{\kappa_2+\theta} \left|\frac{\partial}{\partial \delta} D_h \hat{z}_k\right|^2 ds d\delta\\
&+ \int_{\mathcal{R}^+_k} \mathcal{T}_h \beta_k\, \theta\, \delta^{\kappa_2+\theta-1} \frac{\partial \hat{z}_k}{\partial \delta}\, D_{-h} D_h \hat{z}_k\, ds d\delta\\
&+ \int_{\mathcal{R}^+_k} D_h \beta_k^{-1} \delta^{\kappa_1+\theta} \frac{\partial \hat{z}_k}{\partial s} \frac{\partial}{\partial s} D_h \hat{z}_k + D_h \beta_k \delta^{\kappa_2+\theta} \frac{\partial \hat{z}_k}{\partial \delta} \frac{\partial}{\partial \delta} D_h \hat{z}_k\, ds d\delta\\
&+ \int_{\mathcal{R}^+_k} D_h \beta_k\, \theta\, \delta^{\kappa_2+\theta-1} \frac{\partial \hat{z}_k}{\partial \delta} D_h \hat{z}_k\, ds d\delta.
\end{aligned} \tag{7.42}$$

Using (7.40), (7.42) and observing that $\hat{z}_k = \hat{\psi}_k \hat{u}_k$ one has

$$\begin{aligned}
&\int_{\mathcal{R}^+_k} \mathcal{T}_h \beta_k^{-1} \delta^{\kappa_1+\theta} \left|\frac{\partial}{\partial s} D_h \hat{z}_k\right|^2 + \mathcal{T}_h \beta_k\, \delta^{\kappa_2+\theta} \left|\frac{\partial}{\partial \delta} D_h \hat{z}_k\right|^2 ds d\delta\\
&= -\int_{\mathcal{R}^+_k} \mathcal{T}_h \beta_k\, \theta\, \delta^{\kappa_2+\theta-1} \frac{\partial \hat{z}_k}{\partial \delta}\, D_{-h} D_h \hat{z}_k\, ds d\delta\\
&- \int_{\mathcal{R}^+_k} D_h \beta_k^{-1} \delta^{\kappa_1+\theta} \frac{\partial \hat{z}_k}{\partial s} \frac{\partial}{\partial s} D_h \hat{z}_k + D_h \beta_k \delta^{\kappa_2+\theta} \frac{\partial \hat{z}_k}{\partial \delta} \frac{\partial}{\partial \delta} D_h \hat{z}_k\, ds d\delta\\
&- \int_{\mathcal{R}^+_k} D_h \beta_k\, \theta\, \delta^{\kappa_2+\theta-1} \frac{\partial \hat{z}_k}{\partial \delta} D_h \hat{z}_k\, ds d\delta\\
&+ \mathcal{Q}[\beta_k^{-1}, \beta_k](\hat{u}_k, -\hat{\psi}_k \delta^\theta D_{-h} D_h \hat{z}_k)\\
&+ \int_{\mathcal{R}^+_k} \left(\frac{\partial \hat{\psi}_k}{\partial s} \beta_k^{-1}\right) \delta^{\kappa_1+\theta} D_{-h} D_h \hat{z}_k \frac{\partial \hat{u}_k}{\partial s}\, ds d\delta
\end{aligned}$$

$$+\int_{\mathcal{R}_k^+}\left(\frac{\partial\hat\psi_k}{\partial\delta}\beta_k\right)\delta^{\kappa_2+\theta}D_{-h}D_h\hat z_k\hat u_k\frac{\partial\hat u_k}{\partial\delta}\,dsd\delta$$
$$+\int_{\mathcal{R}_k^+}\left(\frac{\partial\hat\psi_k}{\partial s}\beta_k^{-1}\right)\delta^{\kappa_1}\hat u_k\frac{\partial}{\partial s}(-\delta^\theta D_{-h}D_h\hat z_k)\,dsd\delta$$
$$+\int_{\mathcal{R}_k^+}\left(\frac{\partial\hat\psi_k}{\partial\delta}\beta_k\right)\delta^{\kappa_2}\hat u_k\frac{\partial}{\partial\delta}(-\delta^\theta D_{-h}D_h\hat z_k)\,dsd\delta\,. \tag{7.43}$$

Next, we want to estimate the terms of the second member of (7.43). Using the definition of θ_0, one has that

$$\left|\int_{\mathcal{R}_k^+}\mathcal{T}_h\beta_k\,\theta\,\delta^{\kappa_2+\theta-1}\frac{\partial\hat z_k}{\partial\delta}\,D_{-h}D_h\hat z_k\,dsd\delta\right|$$
$$\le\ \theta\,\|\beta_k\|_{L^\infty(\mathcal{R}_k^+)}\int_{\mathcal{R}_k^+}\delta^{\kappa_2/2}\left|\frac{\partial\hat z_k}{\partial\delta}\right|\delta^{\kappa_2/2+\theta-1}\left|D_{-h}D_h\hat z_k\right|\,dsd\delta$$
$$\le\ \theta\,\|\beta_k\|_{L^\infty(\mathcal{R}_k^+)}\,\|\hat z_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)}\left(\int_{\mathcal{R}_k^+}\delta^{\kappa_2+2(\theta-1)}\left|D_{-h}D_h\hat z_k\right|^2\,dsd\delta\right)^{1/2}$$
$$\le\ \theta\,\|\beta_k\|_{L^\infty(\mathcal{R}_k^+)}\,\|\hat z_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)}\left(\int_{\mathcal{R}_k^+}\delta^{\kappa_1+\theta}\left|D_{-h}D_h\hat z_k\right|^2\,dsd\delta\right)^{1/2}.$$

Indeed, one has that

$$\kappa_2+2(\theta-1)\ge\kappa_1+\theta\quad\Leftrightarrow\quad\theta\ge\kappa_1+(2-\kappa_2)\,.$$

By point (6) of Lemma 7.12, it follows that

$$\left|\int_{\mathcal{R}_k^+}\mathcal{T}_h\beta_k\,\theta\,\delta^{\kappa_2+\theta-1}\frac{\partial\hat z_k}{\partial\delta}\,D_{-h}D_h\hat z_k\,dsd\delta\right|$$
$$\le\ \theta\,\|\beta_k\|_{L^\infty(\mathcal{R}_k^+)}\,\|\hat z_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)}\left(\int_{\mathcal{R}_k^+}\delta^{\kappa_1+\theta}\left|\frac{\partial}{\partial s}D_h\hat z_k\right|^2\,dsd\delta\right)^{1/2} \tag{7.44}$$
$$\le\ \theta\,\|\beta_k\|_{L^\infty(\mathcal{R}_k^+)}\,\|\hat z_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)}\,|D_h\hat z_k|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)}.$$

Next, set

$$M_k:=\max\left\{\left\|\frac{\partial\beta_k^{-1}}{\partial s}\right\|_{L^\infty(\mathcal{R}_k^+)}\ ,\ \left\|\frac{\partial\beta_k}{\partial s}\right\|_{L^\infty(\mathcal{R}_k^+)}\right\}$$

and using Lagrange's Theorem, we have

$$\begin{aligned}&\left|\int_{\mathcal{R}_k^+} D_h\beta_k^{-1}\delta^{\kappa_1+\theta}\frac{\partial \hat{z}_k}{\partial s}\frac{\partial}{\partial s}D_h\hat{z}_k + D_h\beta_k\delta^{\kappa_2+\theta}\frac{\partial \hat{z}_k}{\partial \delta}\frac{\partial}{\partial \delta}D_h\hat{z}_k\, dsd\delta\right|\\ &\le\ M_k\int_{\mathcal{R}_k^+}\delta^{\kappa_1+\theta}\left|\frac{\partial \hat{z}_k}{\partial s}\right|\left|\frac{\partial}{\partial s}D_h\hat{z}_k\right| + \delta^{\kappa_2+\theta}\left|\frac{\partial \hat{z}_k}{\partial \delta}\right|\left|\frac{\partial}{\partial \delta}D_h\hat{z}_k\right|\, dsd\delta.\end{aligned}$$

Hence, by Holder's inequality one obtains

$$\begin{aligned}&\left|\int_{\mathcal{R}_k^+} D_h\beta_k^{-1}\delta^{\kappa_1+\theta}\frac{\partial \hat{z}_k}{\partial s}\frac{\partial}{\partial s}D_h\hat{z}_k + D_h\beta_k\delta^{\kappa_2+\theta}\frac{\partial \hat{z}_k}{\partial \delta}\frac{\partial}{\partial \delta}D_h\hat{z}_k\, dsd\delta\right|\\ &\le\ M_k\left\|\hat{z}_k\right\|_{\mathcal{H}_1(\mathcal{R}_k^+)}\left|D_h\hat{z}_k\right|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)}.\end{aligned} \tag{7.45}$$

Next, one has

$$\begin{aligned}&\left|\int_{\mathcal{R}_k^+} D_h\beta_k\,\theta\,\delta^{\kappa_2+\theta-1}\frac{\partial \hat{z}_k}{\partial \delta}D_h\hat{z}_k\, dsd\delta\right|\\ &\le\ \theta M_k\int_{\mathcal{R}_k^+}\delta^{\kappa_2/2}\left|\frac{\partial \hat{z}_k}{\partial \delta}\right|\delta^{\kappa_2/2+\theta-1}\left|D_h\hat{z}_k\right|\, dsd\delta\\ &\le\ \theta M_k\left\|\hat{z}_k\right\|_{\mathcal{H}_1(\mathcal{R}_k^+)}\left(\int_{\mathcal{R}_k^+}\delta^{\kappa_1+\theta}\left|D_h\hat{z}_k\right|^2\, dsd\delta\right)^{1/2}\\ &\le\ \theta M_k\left\|\hat{z}_k\right\|_{\mathcal{H}_1(\mathcal{R}_k^+)}\left|D_h\hat{z}_k\right|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)}.\end{aligned} \tag{7.46}$$

Furthermore, since

$$\mathcal{Q}[\beta_k^{-1},\beta_k](\hat{u}_k,-\hat{\psi}_k\delta^\theta D_{-h}D_h\hat{z}_k) = -\int_{\mathcal{R}_k^+}(\hat{f}_k-\lambda\hat{u}_k)\hat{\psi}_k\delta^\theta D_{-h}D_h\hat{z}_k\,\beta_k\, dsd\delta$$

and $\theta \ge \theta_0 \ge \kappa_1$, one has

$$\begin{aligned}&\left|\mathcal{Q}[\beta_k^{-1},\beta_k](\hat{u}_k,-\hat{\psi}_k\delta^\theta D_{-h}D_h\hat{z}_k)\right| \le \left|\int_{\mathcal{R}_k^+}(\hat{f}_k-\lambda\hat{u}_k)D_{-h}D_h\hat{z}_k\,\delta^\theta\beta_k\, dsd\delta\right|\\ &\le\left(\int_{\mathcal{R}_k^+}\left|\beta_k(\hat{f}_k-\lambda\hat{u}_k)\right|^2\delta^{\theta-\kappa_1}\, dsd\delta\right)^{1/2}\left(\int_{\mathcal{R}_k^+}\left|\frac{\partial}{\partial s}D_h\hat{z}_k\right|^2\delta^{\theta+\kappa_1}\, dsd\delta\right)^{1/2}\\ &\le\left\|\beta_k(\hat{f}_k-\lambda\hat{u}_k)\right\|_{L^2(\mathcal{R}_k^+)}\left|D_h\hat{z}_k\right|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)}.\end{aligned} \tag{7.47}$$

Next, one has

$$\begin{aligned}
&\left|\int_{\mathcal{R}_k^+} \frac{\partial\hat{\psi}_k}{\partial s}\beta_k^{-1}\delta^{\kappa_1+\theta}D_{-h}D_h\hat{z}_k\frac{\partial\hat{u}_k}{\partial s}\,dsd\delta\right| \\
&\leq \left\|\frac{\partial\hat{\psi}_k}{\partial s}\beta_k^{-1}\right\|_{L^\infty(\mathcal{R}_k^+)}\left(\int_{\mathcal{R}_k^+}\delta^{\kappa_1+\theta}|D_{-h}D_h\hat{z}_k|^2\,dsd\delta\right)^{1/2}\left(\int_{\mathcal{R}_k^+}\delta^{\kappa_1+\theta}\left|\frac{\partial\hat{u}_k}{\partial s}\right|^2\,dsd\delta\right)^{1/2} \\
&\leq \left\|\frac{\partial\hat{\psi}_k}{\partial s}\beta_k^{-1}\right\|_{L^\infty(\mathcal{R}_k^+)}\left(\int_{\mathcal{R}_k^+}\delta^{\kappa_1+\theta}\left|\frac{\partial}{\partial s}D_h\hat{z}_k\right|^2\,dsd\delta\right)^{1/2}\left(\int_{\mathcal{R}_k^+}\delta^{\kappa_1+\theta}\left|\frac{\partial\hat{u}_k}{\partial s}\right|^2\,dsd\delta\right)^{1/2} \\
&\leq \left\|\frac{\partial\hat{\psi}_k}{\partial s}\beta_k^{-1}\right\|_{L^\infty(\mathcal{R}_k^+)}|D_h\hat{z}_k|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)}\,\|\hat{u}_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)}\,.
\end{aligned} \tag{7.48}$$

Likewise, one can prove

$$\begin{aligned}
&\left|\int_{\mathcal{R}_k^+}\left(\frac{\partial\hat{\psi}_k}{\partial\delta}\beta_k\right)\delta^{\kappa_2+\theta}D_{-h}D_h\hat{z}_k\hat{u}_k\frac{\partial\hat{u}_k}{\partial\delta}\,dsd\delta\right| \\
&\leq \left\|\frac{\partial\hat{\psi}_k}{\partial s}\beta_k^{-1}\right\|_{L^\infty(\mathcal{R}_k^+)}|D_h\hat{z}_k|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)}\,\|\hat{u}_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)}\,.
\end{aligned} \tag{7.49}$$

Finally, we estimate the last two terms of (7.43). One has

$$\begin{aligned}
&\left|\int_{\mathcal{R}_k^+}\left(\frac{\partial\hat{\psi}_k}{\partial s}\beta_k^{-1}\right)\delta^{\kappa_1}\hat{u}_k\frac{\partial}{\partial s}(-\delta^\theta D_{-h}D_h\hat{z}_k)\,dsd\delta\right| \\
&= \left|\int_{\mathcal{R}_k^+}D_h\left(\frac{\partial\hat{\psi}_k}{\partial s}\beta_k^{-1}\hat{u}_k\right)\delta^{\kappa_1+\theta}\frac{\partial}{\partial s}(D_h\hat{z}_k)\,dsd\delta\right| \\
&\leq c_k\,\|\hat{u}_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)}\,|D_h\hat{z}_k|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)}
\end{aligned} \tag{7.50}$$

for some positive constant c_k. Next, one has

$$\begin{aligned}
&\left|\int_{\mathcal{R}_k^+}\left(\frac{\partial\hat{\psi}_k}{\partial\delta}\beta_k\right)\delta^{\kappa_2}\hat{u}_k\frac{\partial}{\partial\delta}(-\delta^\theta D_{-h}D_h\hat{z}_k)\,dsd\delta\right| \\
&\leq \int_{\mathcal{R}_k^+}\left|\frac{\partial\hat{\psi}_k}{\partial\delta}\beta_k\hat{u}_k\right|\theta\,\delta^{\kappa_2+\theta-1}|D_{-h}D_h\hat{z}_k|\,dsd\delta \\
&+ \int_{\mathcal{R}_k^+}\left|D_h\left(\frac{\partial\hat{\psi}_k}{\partial\delta}\beta_k\hat{u}_k\right)\right|\delta^{\kappa_2+\theta}\left|\frac{\partial}{\partial\delta}(D_h\hat{z}_k)\right|\,dsd\delta\,.
\end{aligned} \tag{7.51}$$

By Holder's inequality, one obtains

$$\int_{\mathcal{R}_k^+} \left| D_h \left(\frac{\partial \hat{\psi}_k}{\partial \delta} \beta_k \hat{u}_k \right) \right| \delta^{\kappa_2+\theta} \left| \frac{\partial}{\partial \delta}(D_h \hat{z}_k) \right| dsd\delta$$

$$\le \left(\int_{\mathcal{R}_k^+} \left| D_h \left(\frac{\partial \hat{\psi}_k}{\partial \delta} \beta_k \hat{u}_k \right) \right|^2 \delta^{\kappa_2+\theta}\, dsd\delta \right)^{1/2} \left(\int_{\mathcal{R}_k^+} \delta^{\kappa_2+\theta} \left| \frac{\partial}{\partial \delta}(D_h \hat{z}_k) \right| dsd\delta \right)^{1/2}$$

$$\le c_k' \, \|\hat{u}_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)} \, |D_h \hat{z}_k|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)} \tag{7.52}$$

for some positive constant c_k'. Moreover, one has

$$\int_{\mathcal{R}_k^+} \delta^{\kappa_2/2} \left| \frac{\partial \hat{\psi}_k}{\partial \delta} \beta_k \hat{u}_k \right| \theta\, \delta^{\kappa_2/2+\theta-1} |D_{-h} D_h \hat{z}_k| \, dsd\delta$$

$$\le \theta \left(\int_{\mathcal{R}_k^+} \delta^{\kappa_2} \left| \frac{\partial \hat{\psi}_k}{\partial \delta} \beta_k \hat{u}_k \right|^2 dsd\delta \right)^{1/2} \left(\int_{\mathcal{R}_k^+} \delta^{\kappa_1+\theta} \left| \frac{\partial}{\partial s} D_h \hat{z}_k \right|^2 dsd\delta \right)^{1/2} \tag{7.53}$$

$$\le \theta\, c_k'' \, \|\hat{u}_k\|_{L^2(\mathcal{R}_k^+)} \, |D_h \hat{z}_k|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)}$$

for some positive constant c_k''. Thus, using (7.52), (7.53) in (7.51) one has

$$\left| \int_{\mathcal{R}_k^+} \left(\frac{\partial \hat{\psi}_k}{\partial \delta} \beta_k \right) \delta^{\kappa_2} \hat{u}_k \frac{\partial}{\partial \delta} (-\delta^\theta D_{-h} D_h \hat{z}_k) \, dsd\delta \right|$$
$$\le (c_k' \vee \theta\, c_k'') \, \|\hat{u}_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)} \, |D_h \hat{z}_k|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)} \, . \tag{7.54}$$

Lastly, using the estimates (7.44), (7.45), (7.46), (7.47), (7.48), (7.49), (7.50), (7.54) in (7.43) we obtain

$$\int_{\mathcal{R}_k^+} \mathcal{T}_h \beta_k^{-1} \delta^{\kappa_1+\theta} \left| \frac{\partial}{\partial s} D_h \hat{z}_k \right|^2 + \mathcal{T}_h \beta_k\, \delta^{\kappa_2+\theta} \left| \frac{\partial}{\partial \delta} D_h \hat{z}_k \right|^2 dsd\delta$$
$$\le C_k \Big(\|\hat{u}_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)} + \|\hat{f}_k\|_{L^2(\mathcal{R}_k^+)} \Big) |D_h \hat{z}_k|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)}$$

for some positive constant C_k. On the other hand, one has

$$\int_{\mathcal{R}_k^+} \mathcal{T}_h \beta_k^{-1} \delta^{\kappa_1+\theta} \left| \frac{\partial}{\partial s} D_h \hat{z}_k \right|^2 + \mathcal{T}_h \beta_k\, \delta^{\kappa_2+\theta} \left| \frac{\partial}{\partial \delta} D_h \hat{z}_k \right|^2 dsd\delta$$
$$\ge C_k' |D_h \hat{z}_k|^2_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)}$$

for some positive constant C'_k. Thus, we have obtained that

$$|D_h \hat{z}_k|_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)} \le \big(C_k/C'_k\big)\Big(\|\hat{u}_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)} + \|\hat{f}_k\|_{L^2(\mathcal{R}_k^+)}\Big). \tag{7.55}$$

Using (7.55) and point (6) of Lemma 7.12, we got

$$\|D_h \hat{z}_k\|^2_{\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)} \le \int_{\mathcal{R}_k^+} \delta^\theta \left|\frac{\partial \hat{z}_k}{\partial s}\right|^2 dsd\delta + \big(C_k/C'_k\big)^2 \Big(\|\hat{u}_k\|_{\mathcal{H}_1(\mathcal{R}_k^+)} + \|\hat{f}_k\|_{L^2(\mathcal{R}_k^+)}\Big)^2$$

for all $k = 1, \dots, m$, $\theta \ge \theta_0$ and h satisfying (7.41). Finally, by choosing $\theta = \theta_0 = 0$ and $v_{0,h,k} = -D_{-h}D_h\hat{z}_k$, the case (1) can be treated in an analogous way. ■

Using Lemma 7.14, it follows that, for suitable θ, κ_1, κ_2, from the family $D_h\hat{z}_k$ we can extract a weakly convergent subsequence in $\mathcal{H}_{1,\theta}(\mathcal{R}_k^+)$: let $\hat{z}'_k$ be the weak limit. >From point (3) of Lemma 7.12 and the uniqueness of the distributional limit it follows that

$$\frac{\partial \hat{z}_k}{\partial s} = \hat{z}'_k \in \mathcal{H}_{1,\theta}(\mathcal{R}_k^+), \quad k = 1, \dots, m,$$

that is, choosing

$$\theta = \begin{cases} 0 \quad if \quad \kappa_1 = 0 \\ \max\{\kappa_1, \kappa_1 + (2-\kappa_2)\} \quad if \quad \kappa_1 > 0\,, \end{cases}$$

since $\hat{z}_{k|_{\mathcal{E}_k}} = \hat{u}_k$, we have proved that

(1) if $\kappa_1 = 0$ and $\kappa_2 \ge 0$, then

$$\frac{\partial^2 \hat{u}_k}{\partial s^2},\ \delta^{\kappa_2/2}\frac{\partial^2 \hat{u}_k}{\partial\delta\partial s} \ \in L^2(\mathcal{E}_k);$$

(2) if $\kappa_1 > 0$ and $\kappa_2 \ge 0$, then

$$\delta^{(\kappa_1+\theta)/2}\frac{\partial^2 \hat{u}_k}{\partial s^2},\ \delta^{(\kappa_2+\theta)/2}\frac{\partial^2 \hat{u}_k}{\partial\delta\partial s} \ \in L^2(\mathcal{E}_k).$$

Thus, from (7.36) we deduce that

$$\begin{cases} \frac{\partial}{\partial\delta}\left(\delta^{\kappa_2}\frac{\partial \hat{u}_k}{\partial\delta}\right) \ \in L^2(\mathcal{E}_k) \quad \text{in case} \quad (1), \\ \delta^{(\theta-\kappa_1)/2}\frac{\partial}{\partial\delta}\left(\delta^{\kappa_2}\frac{\partial \hat{u}_k}{\partial\delta}\right) \ \in L^2(\mathcal{E}_k) \quad \text{in case} \quad (2). \end{cases}$$

So, recalling (7.33) the conclusion follows. For instance, let us estimate $\partial^2_{\varepsilon_1} u$ when $\kappa_1 = 0$. Using (7.39) one has

$$\int_{\Omega_{\delta_0/2}} \left|\partial^2_{\varepsilon_1} u\right|^2 dx \le \int_{\cup_{k=1}^m E_k} \left|\partial^2_{\varepsilon_1} u\right|^2 dx \le \sum_{k=1}^m \int_{E_k} \left|\partial^2_{\varepsilon_1} u\right|^2 dx$$
$$= \sum_{k=1}^m \int_{\mathcal{E}_k} \left|\beta_k^{-1}\frac{\partial}{\partial s}(\beta_k^{-1}\frac{\partial \hat{u}_k}{\partial s})\right|^2 \beta_k\, dsd\delta \le C\sum_{k=1}^m \int_{\mathcal{E}_k} \left|\frac{\partial^2 \hat{u}_k}{\partial s^2}\right|^2 + \left|\frac{\partial \hat{u}_k}{\partial s}\right|^2 dsd\delta,$$

where C is a suitable positive constant.

References

Adams, R. (1975) *Sobolev Spaces.* Academic Press, New York, San Francisco, London.

Agmon, S. (1965) *Elliptic Boundary Value Problems.* Van Nostrand Company.

Alabau-Boussouira, F., Cannarsa, P. and Fragnelli, G. (2006) Carleman estimates for degenerate parabolic operators with applications to null controllability. *J. Evol. Eqs* **6**, 161-204.

Baiocchi, C. and Capelo, A. (1983) *Variational and Quasivariational Inequalities, Applications to Free Boundary Problems.* Wiley, Chichester.

Bellman, R.E. (1960) *Introduction to Matrix Analysis.* McGraw-Hill.

Bensoussan, A., Da Prato, G., Delfour, M.C. and Mitter, S.K. (1993) *Representation and Control of Infinite Dimensional Systems. Systems and Control: Foundations and applications.* Birkhäuser.

Cannarsa, P., Fragnelli, G. and Rocchetti, D. (2007) Null Controllability of degenerate parabolic operators with drift. *Networks and Heterogeneous Media* **2** (4).

Cannarsa, P., Martinez, P. and Vancostenoble, J. (2004) Persistent regional controllability for a class of degenerate parabolic equations. *Commun. Pure Appl. Anal.* **3**, 607-635.

Cannarsa, P. and Sinestrari, C. (2004) *Semiconcave Functions, Hamilton-Jacobi Equations and Optimal Control.* Birkhäuser, Boston.

Cazenave, T. and Haraux, A. (1998) *An Introduction to Semilinear Evolution Equations. Oxford Lecture Series in Mathematics and its Applications* **13**, Oxford University Press.

Davies, E.B. (1995) *Spectral Theory and Differential Operators.* Cambridge Univ. Press. Cambridge.

Delfour, M.C. and Zolésio, J.P. (1994) Shape analysis via oriented distance functions. *J. Funct. Anal.* **123**(1), 129-201.

Egorov, Y.V. and Shubin, M.A. (1994) *Partial Differential Equations VI.* Springer Berlin, Heidelberg, New York.

Fichera, G. 1956 Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine. *Lincei - Memorie Sc. fisiche, ecc.* **VIII, V** (I, I).

Freidlin, M. (1985) *Functional Integration and Partial Differential Equations.* Princeton University Press.

Gilbarg, D. and Trudinger, N.S. (1983) *Elliptic Partial Differential Equations of Second Order* . Springer Verlag, Berlin, Heidelberg, New York, Tokyo.

Lions, J.L. and Magenes, E. (1972) *Non-homogeneous Boundary Value Problems and Applications.* **I**, Springer, Berlin.

Martinez, P. and Vancostenoble, J. (2006) Carleman estimates for one-dimensional degenerate heat equations. *J. Evol. Eqs* **6**, 325-362.

Necas, J. (1967) *Les méthodes directes en théorie des équations elliptiques.* Masson.

Oleinik, O.A. (1966) Alcuni risultati sulle equazioni lineari e quasi lineari ellittico-paraboliche a derivate parziali del secondo ordine. *Lincei - Rend. Sc. fis. mat. e nat.* **XL**.

Oleinik, O.A. and Radkewitch, E.V. (1973) *Second Order Equations with Non-negative Characteristic Form.* American Math. Soc. and Plenum Press, New York.

Showalter, R.E. (1977) *Hilbert Space Methods for Partial Differential Equations.* Pitman.

Temam, R. (1977) *Navier-Stokes Equations.* North Holland, Amsterdam.