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Terminal control problem for processes represented by nonlinear multi parameter binary dynamic system*

by

Yakup H. Hacı and Kemal Özen

Department of Mathematics, Faculty of Arts and Sciences,
Çanakkale Onsekiz Mart University
Terzioğlu Campus, 17020, Çanakkale, Turkey
e-mail: yhaciyev@comu.edu.tr, kemalozen@comu.edu.tr

Abstract: In this paper, nonlinear multi parameter binary difference equation system (MPBDS) and optimal piecewise process are analyzed. Since such difference equation system is over-determined, a theorem similar to Frobenius's theorem is proved on Galois field. An illustrative example, which can be solved by applying terminal control problem is given. Then, terminal control problem is examined and it is shown that the principle of optimality is satisfied.

Keywords: Boolean function, difference equation, translation operator, Galois field, piecewise curve.

1. Introduction

The use of nonlinear multi parameter binary difference equation system in sequential machines and coding theory, the command of technical processes with the help of computer, modeling of objects and the imitation processes, designs of contemporary evaluation systems, makes out of it an attractive subject for research.

2. The unique solution condition

One of the domains of finite system theory where very little is known is the nonlinear multi parameter binary difference equation system theory. In general, such difference equation system is defined as follows (Gayshun, 1983):

$$\xi_v s(c) = F_v(c, s(c), x(c)) \quad v = 1, 2, ..., k \tag{1}$$

$$s(c^0) = s^0$$

where $c = (c_1, c_2, ..., c_k) \in G_d = \{c \mid c \in Z^k, c_1^0 \le c_1 \le c_1^{L_1}, ..., c_k^0 \le c_k \le c_k^{L_k}, c_i \in Z\}$ is point in Z^k, determining position; $L_i,\ i = 1, 2, ..., k$, where k

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is a positive integer, is the duration of the stage i of this process. Here, Z is the set of integers, for $s(c) \in S,\ x(c) \in X;\ S = [GF(2)]^m, X = [GF(2)]^r$ are state and input index (alphabet) respectively; $s(c)$ and $x(c)$ are defined over the set Z^k as an m and r dimensional state and input vectors at the point c. $c^0 = (c_1^0, c_2^0, ..., c_k^0)$ is the initial position vector of the system and s^0 is the initial state vector of the system. $c^{L_i} = (c_1^{L_i}, c_2^{L_i}, ..., c_k^{L_i})$ is the point to which the system moves after the stage $i-1$. $\xi_v s(c)$ is translation operator defined as follows (Gayshun, 1983; Burden, Faires, 2001):

$$\xi_v s(c) = s(c + e_v); e_v = (0, ..., 0, \overset{v}{1}, 0, ..., 0)\ ,\ v = 1, 2, ..., k\ .$$

Characteristic Boolean vector functions, $F_v(\cdot) = \{F_{v_1}(\cdot), F_{v_2}(\cdot), ..., F_{v_m}(\cdot)\}$, where $(\cdot)$ denotes $(c, s(c), x(c))$, are defined over the set $Z^k \times [GF(2)]^m \times [GF(2)]^r$ where $GF(2)$ is a Galois field (Anderson, 2004).

If the system (1) defines a nonlinear MPBDS then optimal piecewise process represented by this system is characterized by the pseudo Boolean functional (Musayev, Alp, 2000; Yablonsky, 1989) given by:

$$J(x) = \varphi(s(c^L)) \tag{2}$$

which we use as an objective functional in our problem. Here $L = L_1 + L_2 + ... + L_k$ is the time duration of this process.

Now we show that the system of the translating functions is an over-determined system so we define a piecewise curve as follows.

Let $c^0 = (c_1^0, c_2^0, ..., c_k^0),\ c^1 = (c_1^1, c_2^1, ..., c_k^1), ...,\ c^L = (c_1^L, c_2^L, ..., c_k^L)$ be points in Z^k.

If the following conditions are satisfied

a) $c_v^{i+1} \geq c_v^i\ ,\ v = 1, 2, ..., k;\ i = 0, 1, ..., L-1$

b) $\sum_{v=1}^k (c_v^{i+1} - c_v^i) = 1,\ i = 0, 1, ..., L-1$

then we say the curve from c^0 to c^L is a piecewise curve (Gayshun, 1983).

We analyze the behaviour of the system of translating functions in detail on two-dimensional space for simplicity. Consider the case where $L = 5$ and Fig. 1. According to the definition of the set G_d, the piecewise path, on which the system moves from the initial point c^0 to the point c^{L_5} is composed of a right move or an upper move to the point where the system is in any stage. So, there are a lot of paths starting at point c^0 and ending at point c^{L_5}.

Now we determine the state of the system at point c^{L_1}. Since $k = 2$, e_v is either $e_1 = (1, 0)$ or $e_2 = (0, 1)$. From the definition of $\xi_v s(c)$,

$$s(c^{L_1}) = \xi_1 s(c^0) = s(c^0 + e_1) = s((c_1^0, c_2^0) + (1, 0)) = F_1(c^0, s(c^0), x(c^0)).$$

Similarly, the state of the system at point c^{L_2} is determined by

$$s(c^{L_2}) = \xi_2 s(c^{L_1}) = s(c^{L_1} + e_2) = s((c_1^{L_1}, c_2^{L_1}) + (0, 1)) = F_2(c^{L_1}, s(c^{L_1}), x(c^{L_1})),$$

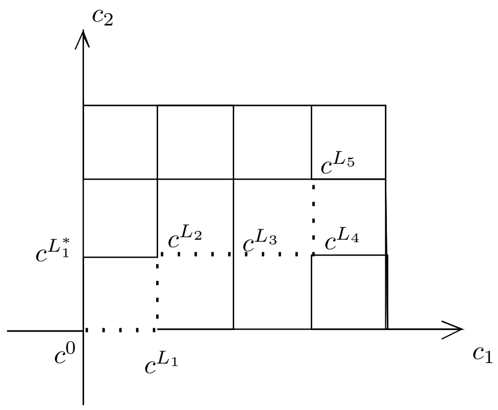

Figure 1. A representative path on G_d for system (1) (dotted line)

the state at point c^{L_3} is determined by

$$s(c^{L_3}) = \xi_1 s(c^{L_2}) = s(c^{L_2}+e_1) = s((c_1^{L_2}, c_2^{L_2})+(1,0)) = F_1(c^{L_2}, s(c^{L_2}), x(c^{L_2})),$$

the state at point c^{L_4} is determined by

$$s(c^{L_4}) = \xi_1 s(c^{L_3}) = s(c^{L_3}+e_1) = s((c_1^{L_3}, c_2^{L_3})+(1,0)) = F_1(c^{L_3}, s(c^{L_3}), x(c^{L_3})),$$

and finally the state at point c^{L_5} is determined by

$$s(c^{L_5}) = \xi_2 s(c^{L_4}) = s(c^{L_4}+e_2) = s((c_1^{L_4}, c_2^{L_4})+(0,1)) = F_2(c^{L_4}, s(c^{L_4}), x(c^{L_4})).$$

As it can be seen from Fig. 1, after any point there is a number of equations for the subsequent state of the system instead of a unique equation. In other words, the subsequent state of the system at point c^0 may be either $s(c^{L_1})$ (to the right of the point c^0) or $s(c^{L_1^*})$ (up from the point c^0). So, the system of equations contains both of them as follows:

$$s(c^{L_1}) = \xi_1 s(c^0) = F_1(c^0, s(c^0), x(c^0))$$
$$s(c^{L_1^*}) = \xi_2 s(c^0) = F_2(c^0, s(c^0), x(c^0)).$$

Many equations for the subsequent state on every stage cause the existence of many solutions of the system (1). Therefore, the sequence of the states of the system is not unique from c^0 to c^{L_5}. So, for every control $x(c)$, system (1) is an over-determined system (Scheid, 1988). Then, for existence of a solution of the problem with system (1) the conditions for a unique solution have to be established.

Let c^0, c be in Z^k, K a piecewise curve connecting c^0 and c, $s_K(c)$ a solution of (1) at the point c along the piecewise curve K. The necessary and sufficient condition for a nonlinear multi parameter finite difference equation system to have a unique solution is that the solution $s_K(c)$ be independent of the piecewise curve K.

We can state this in the following theorem.

Theorem 1 *The necessary and sufficient condition for the existence of a unique solution for the difference equation system defined in (1) with initial value* $s(c^0) = s^0$*, is that for every fixed* $x(c)$ *and* $(c, s) \in Z^k \times [GF(2)]^m$ *the following equality holds*

$$F_v(c+e_\mu, F_\mu(c, s(c), x(c)), x(c+e_\mu)) = F_\mu(c+e_v, F_v(c, s(c), x(c)), x(c+e_v)) \quad (3)$$
$$(v, \mu = 1, 2, ..., k).$$

Proof. **(Necessity)** Since $\xi_v s(c)$ is the translation operator we have

$$\xi_v s(c) = s(c + e_v).$$

Apply the operator ξ_μ to both sides of the above equality and get

$$\xi_\mu \xi_v s(c) = \xi_\mu s(c + e_v) = s(c + e_v + e_\mu). \quad (4)$$

Similarly, we get

$$\xi_v \xi_\mu s(c) = s(c + e_\mu + e_v). \quad (5)$$

Since the sum operation is commutative we obtain

$$s(c + e_v + e_\mu) = s(c + e_\mu + e_v). \quad (6)$$

(4), (5) and (6) imply

$$\xi_\mu \xi_v s(c) = \xi_v \xi_\mu s(c). \quad (7)$$

Because $\xi_v s(c) = F_v(c, s(c), x(c))$ we get

$$\begin{aligned}\xi_\mu \xi_v s(c) &= \xi_v s(c + e_\mu)\\ &= F_v(c + e_\mu, s(c + e_\mu), x(c + e_\mu))\\ &= F_v(c + e_\mu, F_\mu(c, s(c), x(c)), x(c + e_\mu)).\end{aligned}$$

In other words, we obtain

$$\xi_\mu \xi_v s(c) = F_v(c + e_\mu, F_\mu(c, s(c), x(c)), x(c + e_\mu)). \quad (8)$$

By similar computations we get

$$\xi_v \xi_\mu s(c) = F_\mu(c + e_v, F_v(c, s(c), x(c)), x(c + e_v)). \quad (9)$$

Therefore by (7), (8) and (9) we obtain

$$F_v(c+e_\mu, F_\mu(c,s(c),x(c)), x(c+e_\mu)) = F_\mu(c+e_v, F_v(c,s(c),x(c)), x(c+e_v)) \quad (v,\mu = 1,2,...,k).$$

This finishes the proof of necessity.

(Sufficiency) Let $(F_v^0(c,\cdot))(s) = s$, $(F_v^r(c,\cdot))(s) = \underbrace{F_v(c, F_v(c, ..., F_v(c,s,x),...))}_{r \text{ times}}$, $(F_1(c^1,\cdot)\otimes F_2(c^2,\cdot)\otimes ...\otimes F_m(c^m,\cdot))(s) = F_1(c^1, F_2(c^2,...,F_m(c^m,s,x),...))$ and $K(c^1,c^2,...,c^L)$, be the piecewise curve connecting the points $(c^1,c^2,...,c^L)$. Here the symbol $\otimes$ denotes modulo 2 multiplication on Galois field. Over this curve we write:

$$\pi(s) = (\prod_{K(c^1,...,c^L)} \otimes F_1^{\Delta l_1}(l,x(l),\cdot)...F_k^{\Delta l_k}(l,x(l),\cdot))(s)$$
$$= (\prod_{i=1}^{L-1} \otimes F_1^{c_1^{i+1}-c_1^i}(c^i,x(c^i),\cdot)...F_k^{c_k^{i+1}-c_k^i}(c^i,x(c^i),\cdot))(s).$$

We can see that the value of $\pi(s)$ is not only dependent on the initial and terminal point of $K(c^1,c^2,...,c^L)$ but also on the points $c^i(1 < i < L)$.

To make the value of $\pi(s)$ dependent only on the initial and terminal point of the piecewise curve $K(c^1,c^2,...,c^L)$ it suffices to have (3). For $\pi(s)$ we write:

$$\pi(s) = \prod_{(c^1,c^L)} \otimes F_1^{\Delta l_1}(l,x(l),\cdot)...F_k^{\Delta l_k}(l,x(l),\cdot))(s).$$

Let $(c^0,s^0) \in Z^k\times\ [GF(2)]^m$ and consider the function $s(c) = s(c,c^0,s^0,x(c))$. We have

$$s(c,c^0,s^0,x(c)) = \prod_{(c^0,c)} \otimes F_1^{\Delta c_1}(c,x(c),\cdot)...F_k^{\Delta c_k}(c,x(c),\cdot))(s^0).$$

If we evaluate $\xi_v s(c,c^0,s^0,x(c))$ then we get

$$\xi_v s(c,c^0,s^0,x(c)) = (F_v(c,\cdot)\otimes s(c,c^0,\cdot))(s^0) = F_v(c,s(c,c^0,s^0,x(c)))$$

which means that $s(c)$ is the unique solution for the equation system (1). ∎

Now, we can give a suitable illustrative example for optimal control problem which will be analyzed a little later in this paper.

Example 1 (*MPBDS-aided modeling of behaviour of multi dimensional adder*)

The following expression is often effectively calculated for the analysis of discrete description processes by the aid of finite objects which are over-determined and the solution of the some practical problems of the theory of encoders, latticed-sequential machines and automatons:

$$J(c_1^L,c_2^L,...,c_k^L) = \sum_{v_1=0}^{c_1^L-1} ... \sum_{v_k=0}^{c_k^L-1} \sum_{i=1}^{k} x_i(v_1,v_2,...,v_k). \quad (10)$$

Here the components of the vector $x = (x_1, x_2, ..., x_k)$ are defined over the set $\{0, 1\}$. All calculations on the right-hand side of (10) are made on GF(2).

One of the methods for the effective calculation of the sums in (10) is the MPBDS-aided model. For this purpose, we formulate the elements of the set $X = \{x = (x_1, x_2, ..., x_k)|x_i \in GF(2)\}$ with the aid of binary coding as follows:

$$x(c^0) = (0, 0, ..., 0)$$
$$...$$
$$x(c^L) = (1, 1, ..., 1).$$

Considering the concept of the piecewise curve, we associate any vector $x = (x_1, x_2, ..., x_k)$ with any point of k-dimensional lattice. Then we get $x(c) = (x_1(c), x_2(c), ..., x_k(c))$ on the k-dimensional lattice.

Now, let

$$s(c_1, c_2, ..., c_k) = \sum_{v_1=0}^{c_1-1} ... \sum_{v_k=0}^{c_k-1} \sum_{i=1}^{k} x_i(v_1, v_2, ..., v_k). \tag{11}$$

Consequently, as it can be seen from (11), we get the following difference equations on the field GF(2):

$$s(c_1, c_2, ..., c_{v-1}, c_v + 1, c_{v+1}, ..., c_k) =$$
$$= s(c_1, c_2, ..., c_v, ..., c_k) \oplus x_v(c_1, c_2, ..., c_v, ..., c_k), \quad v = 1, 2, ..., k, \tag{12}$$
$$s(0, 0, ..., 0) = 0. \tag{13}$$

Thus, the vector $s(c_1, c_2, ..., c_k)$ in the expressions (12) and (13) is the generalized state of MPBDS, which models the behaviour of k-dimensional adder and the vector $x(c) = (x_1(c), x_2(c), ..., x_k(c)) \in [GF(2)]^k$ is also control.

Application of the criteria of optimality and putting the restrictions for the control $x(c)$ is needed for getting a desired behaviour of a given system.

Furthermore, terminal control problem is derived for the k-dimensional binary adder as follows:

$$J(x) = \varphi(s(c^L)) \rightarrow min, \tag{14}$$
$$s(c_1, c_2, ..., c_{v-1}, c_v + 1, c_{v+1}, ..., c_k) =$$
$$= s(c_1, c_2, ..., c_v, ..., c_k) \oplus x_v(c_1, c_2, ..., c_v, ..., c_k), \quad v = 1, 2, ..., k, \tag{15}$$
$$s(0, 0, ..., 0) = 0, \tag{16}$$
$$x(c) \in \widehat{X}. \tag{17}$$

3. Principle of optimality

Definition 1 *If for every control $x(c)$ the system (1) has a unique solution, then we say that the control $x(c)$ is an admissible control.*

Denote the set of admissible control vector functions by $\widehat{X}$, acting from $\widehat{G_d} = G_d \backslash \{c^L\}$ into $[GF(2)]^r$ where i.e. $x(c) = (x_1(c), x_2(c), ..., x_r(c))$.

Now for nonlinear MPBDS we can analyze the following terminal control problem.

In order for a given nonlinear MPBDS to go from s^0 to $s^*(c^L)$ in L steps a control $x(c) \in \widehat{X}$ must exist such that the functional in (2) has a minimal value:

$$\xi_v s(c) = F_v(c, s(c), x(c)) \ , \ c \in G_d, \ v = 1, 2, ..., k$$
$$s(c^0) = s^0$$
$$x(c) \in \widehat{X}, \ c \in \widehat{G_d}$$
$$J(x) = \varphi(s(c^L)) \to \min .$$

Here the translating functions are Boolean and the function, which characterizes the process is pseudo Boolean. Therefore, the pseudo Boolean expressions of the translating functions must be obtained. These expressions can be obtained by the operations given in Hacıyev (Hacı) (2007). After this step, the problem becomes

$$\xi_v s(c) = \widehat{F_v}(c, s(c), x(c)) \ , \ c \in G_d, \ v = 1, 2, ..., k \tag{18}$$
$$s(c^0) = s^0$$
$$x(c) \in \widehat{X}, \ c \in \widehat{G_d} \tag{19}$$
$$J(x) = \varphi(s(c^L)) \to \min . \tag{20}$$

Here $\widehat{F_v}(\cdot)$ $(v = 1, 2, ...k)$ denotes the pseudo Boolean expression of the Boolean vector function $F_v(\cdot)(v = 1, 2, ...k)$.

Now we show that the principle of optimality (Boltyanskii, 1978) is satisfied for the terminal control problem, which we have considered. Therefore, we formulate the problem (18)-(20) as an optimal problem:

$$\xi_v s(c) = \widehat{F_v}(c, s(c), x(c)) \ , c \in G_d(\sigma), \ v = 1, 2, ..., k \tag{21}$$
$$s(\sigma) = \chi$$
$$x(c) \in \widehat{X}, \ c \in G_d(\sigma) \tag{22}$$
$$J(x) = \varphi(s(c^L)) \to \min \tag{23}$$

where $\chi \in S = [GF(2)]^m$, $\sigma \in G_d$, $G_d(\sigma) = \{c \mid \sigma_1 \le c_1 \le c_1^{L_1}, ..., \sigma_k \le c_k \le c_k^{L_k}\}$. If we substitute $\sigma = c^0$ and $\chi = s^0$ into the problem (21)-(23), we obtain the first problem we stated above. If the conditions for the existence of a unique solution are satisfied, then for the given initial condition $s(\sigma) = \chi$ and given $x(c)(c \in G_d(\sigma))$ we find a unique $s(c)$. In other words, the functional (23) is the function of the parameters χ and $x(c)(c \in G_d(\sigma))$:

$$J(x) = J(\chi, x(G_d(\sigma))). \tag{24}$$

Here, $x(G_d(\sigma))$ where $c \in G_d(\sigma)$ denotes the range of the control $x(c)$:

$$x(G_d(\sigma)) = \{x(c) \mid c \in G_d(\sigma)\}.$$

From the exact solution conditions of the system (18) we find that the control process in (18)-(20) can be analyzed in the set $G_d(\sigma)$ and moreover in

$$G_{d_1}(\sigma) = \{c \mid c_1^0 \le c_1 \le \sigma_1, ..., c_k^0 \le c_k < \sigma_k\}. \quad (25)$$

DEFINITION 2 *We say that the control $x(c)(c \in G_d(\sigma))$ which minimizes the functional (23) in the problem (21)-(23) is optimal control with respect to the initial pair (σ, χ) on the region $G_d(\sigma)$.*

THEOREM 2 ***(Principle of Optimality)*** *Assume that $x^0(c)$ is an optimal control with respect to the initial pair (c^0, s^0) on the region G_d and $s^0(c)$ is appropriate optimal trajectory. Then $x^0(c)$ is optimal with respect to the initial pair $(\sigma, s^0(\sigma))$ on the region $G_d(\sigma)$ for every $\sigma \in G_d$.*

Proof. Assume the contrary. Then there exist $x(c)(c \in G_d(\sigma))$ such that we have

$$J(\chi, x(G_d(\sigma))) < J(\chi, x^0(G_d(\sigma))). \quad (26)$$

We choose a new control process $\widetilde{x}(c)(c \in G_d)$ as follows:

$$\widetilde{x}(c) = \begin{cases} x^0(c) & \text{, for } c \in G_{d_1}(\sigma) \\ x(c) & \text{, for } c \in G_d(\sigma). \end{cases} \quad (27)$$

As it can be seen, (27) is an admissible control process such that

$$J(s^0, \widetilde{x}(G_d)) = J(s^0, \widetilde{x}(G_{d_1}(\sigma) \cup G_d(\sigma))). \quad (28)$$

According to the condition, $s^0(\sigma) = \chi$. Thus we have

$$J(s^0, \widetilde{x}(G_{d_1}(\sigma) \cup G_d(\sigma))) = J(s^0(\sigma), \widetilde{x}(G_d(\sigma))) = J(\chi, x(G_d(\sigma)))$$
$$< J(\chi, x^0(G_d(\sigma))) = J(s^0(\sigma), x^0(G_d(\sigma))) = J(s^0, x^0(G_d)), \quad (29)$$

and by using (28) and (29) we can obtain

$$J(s^0, \widetilde{x}(G_d)) < J(s^0, x^0(G_d)). \quad (30)$$

The inequality (30) is contradicting the hypothesis that the control $x^0(c)(c \in G_d)$ is optimal. This finishes the proof of the theorem. ■

4. Conclusion

It is shown that nonlinear multi parameter binary difference equation system is over-determined. A theorem, which ensures existence of a solution for nonlinear multi parameter binary difference equation system is proved. This theorem similar to Frobenius's theorem is needed for existence of the optimal control problem solution. It is shown that the principle of optimality is provided for terminal control problem.

References

ANDERSON, J.A. (2004) *Discrete Mathematics with Combinatorics*, Second edition. Prentice Hall.

BOLTYANSKII, V.G. (1978) *Optimal Control of Discrete Systems.* John Wiley, New York.

BURDEN, R.L. and FAIRES, J.D. (2001) *Numerical Analysis*, Seventh edition. Brooks/Cole, Pacific Grove, Australia.

GAYSHUN, I.V. (1983) *Completely Solvable Multidimensional Differential Equations.* Science and Technique, Minsk (in Russian).

HACIYEV (HACI), Y.H. (2007) Pseudo Boolean Expressions of Multi Parameter Boolean Vector Translating Functions. II. *Turkish World Mathematics Symposium, Sakarya* (in Turkish).

MUSAYEV, B. and ALP, M. (2000) *Functional Analysis.* Balcı Publications, Kütahya (in Turkish).

SCHEID, F. (1988) *Schaum's Outline of Theory and Problems of Numerical Analysis*, Second edition. Schaum's Outline Series. McGraw-Hill.

YABLONSKY, S.V. (1989) *Introduction to Discrete Mathematics.* Mir Publishers, Moscow.