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Generalized approximate midconvexity*

by

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Abstract: Let X be a normed space and $V \subset X$ a convex set with nonempty interior. Let $\alpha : [0,\infty) \to [0,\infty)$ be a given nondecreasing function. A function $f : V \to \mathbb{R}$ is $\alpha(\cdot)$-*midconvex* if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \alpha(\|x-y\|) \quad \text{for } x, y \in V.$$

In this paper we study $\alpha(\cdot)$-midconvex functions. Using a version of Bernstein-Doetsch theorem we prove that if f is $\alpha(\cdot)$-midconvex and locally bounded from above at every point then

$$f(rx+(1-r)y) \leq rf(x) + (1-r)f(y) + \mathcal{P}_\alpha(r, \|x-y\|)$$

for $x, y \in V$ and $r \in [0,1]$, where $\mathcal{P}_\alpha : [0,1] \times [0,\infty) \to [0,\infty)$ is a specific function dependent on α. We obtain three different estimations of $\mathcal{P}_\alpha$.

This enables us to generalize some results concerning paraconvex and semiconcave functions.

Keywords: approximately midconvex function, convexity, paraconvexity, semiconcavity.

1. Introduction

The existing various notions of generalized convexity are very useful, in particular in optimal control theory (Cannarsa and Sinestrari, 2004) and optimization (for more information and references see Rolewicz, 2005). Therefore, convenient conditions which guarantee generalized convexity are very useful. As we know from the classical theory of convex functions, midconvexity and local upper boundedness guarantee convexity. Not surprisingly, a similar type of behaviour

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can be expected for generalized (mid)convexity. Since our considerations are motivated by the results concerning this problem, let us recall some of them.

Let X be a normed space and V a convex subset of X. A function $f : V \to \mathbb{R}$ is *convex* if

$$f(rx + (1-r)y) \leq rf(x) + (1-r)f(y) \quad \text{for } x, y \in V, r \in [0,1].$$

If the above inequality holds for $r = 1/2$ that is if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad \text{for } x, y \in V$$

then f is called *midconvex* (or Jensen convex). From the very beginning the relation between convexity and midconvexity attracted a lot of attention. In this context one should mention the celebrated Bernstein-Doetsch Theorem.

Bernstein-Doetsch Theorem (Bernstein and Doetsch, 1915; Kuczma, 1985). *Let V be an open convex subset of X and let $f : V \to \mathbb{R}$ be midconvex. If f is locally bounded above at a point then f is convex.*

The notion of approximate convexity was introduced by D.H. Hyers and S.M. Ulam (Hyers and Ulam, 1952; Hyers, Isac and Rassias, 1998). Let $\delta \geq 0$. A function $f : V \to \mathbb{R}$ is called *δ-convex* if

$$f(rx + (1-r)y) \leq rf(x) + (1-r)f(y) + \delta \quad \text{for } x, y \in V, r \in [0,1].$$

Analogously, f is called *δ-midconvex* if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \delta \quad \text{for } x, y \in V.$$

An analogue of Bernstein-Doetsch Theorem adapted to the above definition was obtained by C.T. Ng and K. Nikodem (1993). They proved that if V is convex open and $f : V \to \mathbb{R}$ is δ-midconvex and locally bounded at a point then f is 2δ-convex (a further improvement was recently obtained in Mrowiec, Tabor and Tabor, 2009).

Another type of approximate convexity was introduced by S. Rolewicz (1979 a,b): a function $f : I \to \mathbb{R}$, where I is a subinterval of $\mathbb{R}$ is *γ-paraconvex* if for a certain $\varepsilon > 0$

$$f(rx + (1-r)y) \leq rf(x) + (1-r)f(y) + \varepsilon|x-y|^\gamma \quad \text{for } x, y \in I, r \in [0,1].$$

S. Rolewicz proved that if $\gamma > 2$ then every γ-paraconvex function is convex. In the series of proceeding papers S. Rolewicz has developed a general theory of paraconvex functions, see Rolewicz (2006) for more information and references.

Let $\alpha : [0,\infty) \to [0,\infty)$ be a given nondecreasing function such that

$$\limsup_{r \to 0^+} \alpha(r)/r < \infty\,.$$

A function $f : V \to \mathbb{R}$ is $\alpha(\cdot)$-*paraconvex* if there exists $C > 0$ such that

$$f(rx+(1-r)y) \le rf(x)+(1-r)f(y)+C\alpha(\|x-y\|) \quad \text{for } x, y \in V, r \in [0,1],$$

and *strongly* $\alpha(\cdot)$-*paraconvex* if there exists $C > 0$ such that

$$f(rx + (1-r)y) \le rf(x) + (1-r)f(y) + C\min(r, 1-r)\alpha(\|x-y\|)$$

for $x, y \in V$, $r \in [0,1]$.

There is also an analogous definition of *semiconvexity* (Cannarsa and Sinestrari, 2004), which is especially adapted to the optimal control theory, where the function $\min(r, 1-r)$ in the last definition is replaced by $r(1-r)$, see also Zajíček (2008) (another modification was also studied in Páles, 2003).

In a similar manner δ-midconvexity was generalized by A. Házy and Zs. Páles in Házy and Páles (2004, 2005) and Házy (2005). The idea consisted in exchanging the "midconvexity bound" δ by a specific function of $\|x-y\|$: in Házy and Páles (2004) it was replaced by $\varepsilon\|x-y\|+\delta$, in Házy (2005) it was $\varepsilon\|x-y\|^p+\delta$ and in Házy and Páles (2005) $\sum_i \varepsilon_i\|x-y\|^{p_i}$.

We are going to generalize the above definitions in the spirit of paraconvex functions introduced by S. Rolewicz. A very similar definition to the one we propose was used by Cannarsa and Sinestrari (2004, Theorem 2.1.10). However, we would like to underline that contrary to the just mentioned authors, we do not assume that $\limsup_{r\to 0^+} \alpha(r)/r < \infty$.

Definition 1.1 *Let V be a convex subset of a normed space X and let a nondecreasing function $\alpha : [0,\infty) \to [0,\infty)$ be given. A function $f : V \to \mathbb{R}$ is called* $\alpha(\cdot)$-midconvex *if*

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + \alpha(\|x-y\|) \quad \textit{for } x, y \in V.$$

The above definition is also motivated by the notion of $\alpha(\cdot)$-lipschitz functions introduced by R. Mauldin and S. Williams (1986) ($\alpha(\cdot)$-lipschitz function is simply a function which is simultaneously $\alpha(\cdot)$-midconvex and $\alpha(\cdot)$-midconcave).

Now we are ready to present the main problem we investigate in this paper.

Let a nondecreasing function $\alpha : [0,\infty) \to [0,\infty)$ be given. Find a function (if possible—optimal) $\mathcal{P}_\alpha : [0,1] \times [0,\infty) \to [0,\infty)$ such that for every locally bounded above at a point $\alpha(\cdot)$-midconvex function f we have

$$f(rx + (1-r)y) \le rf(x) + (1-r)f(y) + \mathcal{P}_\alpha(r, \|x-y\|)$$

for all x, y in the domain of f and $r \in [0,1]$.

We briefly describe the contents of the article.

In the next section we generalize the estimations obtained in the previously mentioned papers of A. Házy and Zs. Páles. As shown by numerical simulations, this estimation is sharp for the case when $\alpha(x) = x^p$ for all $p \in [0,1]$ (up till now the proof exists only for the cases $p = 0$, Mrowiec, Tabor and Tabor, 2009, and $p = 1$, Boros, 2008).

The third section is motivated by the results of S. Rolewicz (1979 a,b) which showed that a γ-paraconvex function with $\gamma \in (1,2]$ is strongly γ-paraconvex. We obtain a result of this type for $\alpha(\cdot)$-midconvex functions. We would like to add, see Tabor and Tabor (2009), that our result is optimal in the case when $\alpha(x) = x^p$, $p \in [1,2]$.

Fourth section deals with a generalization of Cannarsa and Sinestrari (2004, Theorem 2.10) concerning the comparison of (mid)-semiconcave and semiconcave functions. In particular we show that our technique of proof gives a better estimation then that obtained in Cannarsa and Sinestrari (2004, Theorem 2.10), which consequently gives a partial answer to the question stated in Cannarsa and Sinestrari (2004, Remark 2.1.11).

The last section is inspired by the results of S. Rolewicz (1979b, 2000) and that of K. Nikodem and Zs. Páles (2003/4). We prove that if

$$\liminf_{r \to 0^+} \alpha(r)/r^2 = 0$$

then a locally bounded above $\alpha(\cdot)$-midconvex function is convex. This, in particular, gives a positive answer to the question formulated by S. Rolewicz directly after the proof of Proposition 1 in Rolewicz (2000).

At the end of the introduction we establish some notation. By $\mathbb{N}$ we denote the set of positive integers and by $\mathbb{N}_0$ the set of nonnegative integers. By V we denote a convex subset of a normed space X. We will need the function $d : \mathbb{R} \to \mathbb{R}$ defined by

$$d(r) := 2\mathrm{dist}(r; \mathbb{Z}).$$

In the whole paper we assume that $\alpha : [0, \infty) \to [0, \infty)$ is a given nondecreasing function.

2. Házy and Páles type estimation

Let us first briefly discuss the results of A. Házy and Zs. Páles from Házy and Páles (2004, 2005) and Házy (2005). In Házy and Páles (2004) they proved the following theorem:

THEOREM HP (Házy and Páles, 2004, Theorem 4). *Let V be an open convex subset of a Banach space X and let $f : V \to \mathbb{R}$ be (δ, ε)-midconvex, i.e.*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \varepsilon\|x-y\| + \delta \quad \textit{for } x, y \in V,$$

where $\varepsilon, \delta \geq 0$, and locally bounded above at a point, then it satisfies the inequality

$$f(rx+(1-r)y) \leq rf(x)+(1-r)f(y)+2\varepsilon\varphi(r)\|x-y\|+2\delta \quad \textit{for } x, y \in V, r \in [0,1],$$

where $\varphi : [0,1] \to \mathbb{R}$ *is the Takagi function* $\varphi(r) = \sum_{k=0}^{\infty} \frac{1}{2^k} \mathrm{dist}(2^k r; \mathbb{Z})$ *for* $r \in [0,1]$.

This result was later improved in Házy (2005) and Házy and Páles (2005) for the "midconvexity bound" of the form $\varepsilon\|x-y\|^p + \delta$ and $\sum_i \varepsilon_i \|x-y\|^{p_i}$, respectively. In Házy (2005) and Házy and Páles (2005) the case of t-convexity was also considered.

In this section we generalize some of the above mentioned results for the case of the general mid-convexity bound (not only of the form $\sum_i \varepsilon_i \|x-y\|^{p_i}$). Moreover, we do not restrict ourselves to the case of locally bounded above functions (in this case one can only expect to obtain estimation for $r \in [0,1] \cap \mathbb{Q}$).

We begin with the version of the Bernstein-Doetsch Theorem adapted to $\alpha(\cdot)$-midconvex functions.

Theorem 2.1 *Let* $f : V \to \mathbb{R}$ *be a locally bounded above at one point* $\alpha(\cdot)$-*midconvex function. Then* f *is locally bounded at each point of* $\mathrm{int}V$.

Moreover, if $\lim\limits_{r \to 0^+} \alpha(r) = 0$ *then* f *is continuous on* $\mathrm{int}V$.

Proof. Clearly it is enough to consider the case when $\mathrm{int}V \neq \emptyset$. Let x_0 be the point at which f is locally bounded from above. We can choose such x_0 from $\mathrm{int}V$. Let $r > 0$ be arbitrarily chosen. Let $V_r = \mathrm{int}V \cap B(x_0, r)$, where $B(x_0, r)$ denotes the open ball centered at x_0 with radius r. Since α is nondecreasing we obtain that f is $\alpha(2r)$ midconvex on V_r. By Theorem of Ng and Nikodem (1993) we obtain that f is locally bounded at each point of V_r. As r was arbitrary, we obtain that f is locally bounded at each point of $\mathrm{int}V$.

The proof of continuity of f on $\mathrm{int}V$ is an obvious modification of the proof of continuity in the Bernstein-Doetsch Theorem (Kuczma, 1985; Házy and Páles, 2004) and therefore we skip it. ∎

Let us observe that the above theorem implies that if the domain of f is open to check if f is locally bounded above at every point of the domain it is sufficient to check if f is locally bounded above at one point.

We are going to obtain a global estimation for locally bounded $\alpha(\cdot)$-midconvex functions. We begin with the following proposition.

Proposition 2.1 *Let* $D \subset [0,1]$ *be such that*

$$x \in [0,1/2] \cap D \Rightarrow 2x \in D,\ x \in [1/2,1] \cap D \Rightarrow 2x-1 \in D.$$

Let $h : D \to \mathbb{R}$ *be an upper bounded function such that*

$$h(x) \le h(2x)/2 + \alpha(2x) \quad \text{for } x \in D \cap [0,1/2],$$
$$h(x) \le h(2x-1)/2 + \alpha(2-2x) \quad \text{for } x \in D \cap [1/2,1].$$

Then

$$h(r) \le \sum_{k=0}^{\infty} \frac{1}{2^k} \alpha(d(2^k r)) \quad \text{for } r \in D.$$

Proof. Let

$$C := \sup_{r\in D}\left(h(r) - \sum_{k=0}^{\infty}\frac{1}{2^k}\alpha(d(2^k r))\right).$$

By the assumptions we obtain that $C < \infty$. We are going to prove that $C \le 0$.

For an indirect proof, suppose that this is not the case. Choose an $r_0 \in D$ such that

$$h(r_0) - \sum_{k=0}^{\infty}\frac{1}{2^k}\alpha(d(2^k r_0)) > C/2.$$

Let us first consider the case when $r_0 \in [0, 1/2]$. Then we obtain

$$h(2r_0) \ge 2h(r_0) - 2\alpha(2r_0) > 2\sum_{k=0}^{\infty}\frac{1}{2^k}\alpha(d(2^k r_0)) + 2C/2 - 2\alpha(d(r_0))$$
$$= \sum_{l=0}^{\infty}\frac{1}{2^l}\alpha(d(2^l(2r_0))) + C.$$

Consequently

$$h(2r_0) - \sum_{l=0}^{\infty}\frac{1}{2^l}\alpha(d(2^l(2r_0))) > C,$$

a contradiction with the definition of C.

The case when $r_0 \in [1/2, 1]$ can be treated similarly (we begin with the inequality $h(2r_0 - 1) \ge 2h(r_0) - 2\alpha(2 - 2r_0)$ and proceed analogously). ■

As a direct corollary we get:

COROLLARY 2.1 *Let $h : [0, 1] \to \mathbb{R}$ be an $\alpha(\cdot)$-midconvex function such that $h(0) = h(1) = 0$. Then*

$$h(r) \le \sum_{k=0}^{\infty}\frac{1}{2^k}\alpha(d(2^k r)) \quad \text{for } r \in [0,1]\cap\mathbb{Q}.$$

Moreover, if h is upper bounded then the above inequality holds for all $r \in [0, 1]$.

Proof. Let $r = k/n \in \mathbb{Q}\cap[0, 1]$, where $k \in \mathbb{Z}$, $n \in \mathbb{N}$. We put $D = \{0, 1/n, \ldots, (n-1)/n, 1\}$. Then D satisfies the assumptions of Proposition 2.1. Since D is finite, $h|_D$ is bounded, and therefore Proposition 2.1 applied to $h|_D$ makes the proof complete.

The case when h is upper bounded follows directly from Proposition 2.1. ■

Now we are ready to present the main result of this section.

THEOREM 2.2 *Let $f : V \to \mathbb{R}$ be an $\alpha(\cdot)$-midconvex function. Then*

$$f(rx + (1-r)y) \le rf(x) + (1-r)f(y) + \sum_{k=0}^{\infty} \frac{1}{2^k} \alpha(d(2^k r)\|x-y\|) \qquad (1)$$

for $x, y \in V$, $r \in [0,1] \cap \mathbb{Q}$.

Moreover, if f is locally bounded above at every point of V then (1) *holds for all $x, y \in V$, $r \in [0,1]$.*

Proof. As α is nondecreasing and d is bounded, we obtain that the right hand side of (1) is well-defined.

Let us fix $x, y \in V$. We define the function $h : [0,1] \to \mathbb{R}$ by the formula

$$h(r) = f(rx + (1-r)y) - rf(x) - (1-r)f(y) \quad \text{for } r \in [0,1].$$

Then $h(0) = h(1) = 0$. By the $\alpha(\cdot)$-midconvexity of f we obtain that

$$h\left(\frac{r+s}{2}\right) \le \frac{h(r)+h(s)}{2} + \alpha(|r-s|\|x-y\|) \quad \text{for } r, s \in [0,1].$$

That is, h is $\theta(\cdot)$-midconvex, with $\theta(w) := \alpha(w\|x-y\|)$. By Corollary 2.1 we obtain that

$$h(q) \le \sum_{k=0}^{\infty} \frac{1}{2^k} \theta(d(2^k q)) = \sum_{k=0}^{\infty} \frac{1}{2^k} \alpha(d(2^k q)\|x-y\|) \quad \text{for } q \in [0,1] \cap \mathbb{Q},$$

that is, for $r \in [0,1] \cap \mathbb{Q}$

$$f(rx + (1-r)y) - rf(x) - (1-r)f(y) \le \sum_{k=0}^{\infty} \frac{1}{2^k} \alpha(d(2^k r)\|x-y\|).$$

Let us discuss now the case when f is locally upper bounded. Then h has the same property. Since the interval $[0,1]$ is compact we obtain that h is upper bounded. Corollary 2.1 completes the proof. ∎

One can easily notice that for $\alpha(r) := \varepsilon r + \delta$ we obtain Theorem HP (in fact we obtain its "upgrade" since we deal also with the case when f is not locally bounded).

3. Rolewicz type estimation

In this section we obtain an estimation, which is to some extent dual to that from the previous section. It occurs that it is optimal for $p \in [1,2]$, see Tabor and Tabor (2009). This means that the next Theorem 3.1 should give a better estimation than Theorem 2.2 in the case when $\lim_{r\to 0^+} \alpha(r)/r = 0$.

By $\mathcal{D}$ we denote the set of dyadic numbers in $\mathbb{Q}$, that is – elements of the form $l/2^n$, where $l \in \mathbb{Z}$ and $n \in \mathbb{N}$. By the degree $\deg(q)$ of a dyadic number q we understand the smallest nonnegative integer n such that $2^n q \in \mathbb{Z}$.

Proposition 3.1 *Let $h : [0,1]\cap\mathcal{D} \to \mathbb{R}$, $h(0) = h(1) = 0$ be an $\alpha(\cdot)$-midconvex function. Then*

$$h(q) \le \sum_{k=0}^{\infty} \alpha(1/2^k) d(2^k q) \quad \text{for } q \in [0,1]\cap\mathcal{D}. \tag{2}$$

Proof. We are going to prove (2) by induction over the degree of q. Obviously (2) is valid if $\deg(q) = 0$, that is if $q = 0$ or $q = 1$. Assume that for a certain $n \in \mathbb{N}_0$ (2) holds true for all $q \in [0,1]\cap\mathcal{D}$ such that $\deg(q) \le n$.

Let $q \in [0,1]\cap\mathcal{D}$, $\deg(q) = n+1$, that is $q = (2l+1)/2^{n+1}$ for a certain $l \in \{0,\dots,2^n - 1\}$. Then $q = (l/2^n + (l+1)/2^n)/2$, and by the inductive assumption

$$h(q) \le [h(l/2^n) + h((l+1)/2^n)]/2 + \alpha(1/2^n)$$

$$\le \alpha(1/2^n) + \sum_{k=0}^{\infty} \alpha(1/2^k) \frac{d(l/2^{n-k}) + d((l+1)/2^{n-k})}{2}. \tag{3}$$

Obviously $l/2^{n-k}, (l+1)/2^{n-k} \in \mathbb{Z}$ for all $k \ge n$. Let us observe that for every $k \in \{0,\dots,n-1\}$ there exists an $m_k \in \mathbb{Z}$ such that

$$l/2^{n-k}, (l+1)/2^{n-k} \in [m_k/2, (m_k+1)/2]. \tag{4}$$

Suppose, for an indirect proof, that this is not the case for a certain $k \in \{0,\dots,n-1\}$. Since

$$(l+1)/2^{n-k} - l/2^{n-k} = 1/2^{n-k} \le 1/2,$$

there would exist an $m \in \mathbb{Z}$ such that $l/2^{n-k} \in [m - 1/2, m)$, $(l+1)/2^{n-k} \in (m, m+1/2]$. This implies that $(l - 2^{n-k}m)/2^{n-k} < 0 < (l+1-2^{n-k}m)/2^{n-k}$, and consequently that $l - 2^{n-k}m < 0 < l + 1 - 2^{n-k}m$, a contradiction.

Now let us notice that $d(m) = 0$ and $d(m + \frac{1}{2}) = 1$ for all $m \in \mathbb{Z}$ and that d is piecewise affine, more precisely,

$$d(\frac{x+y}{2}) = (d(x) + d(y))/2 \quad \text{for } x, y \in [m/2, (m+1)/2], m \in \mathbb{Z}.$$

Concluding, by applying (3) and (4) we obtain that

$$h(q) \le \alpha(1/2^n) + \sum_{k=0}^{n-1} \alpha(1/2^k) d((2l+1)/2^{n+1-k}) =$$

$$= \sum_{k=0}^{n} \alpha(1/2^k) d((2l+1)/2^{n+1-k}) = \sum_{k=0}^{\infty} \alpha(1/2^k) d(2^k q). \qquad \blacksquare$$

Now we are ready to present the main result of this section.

THEOREM 3.1 *Let $f : V \to \mathbb{R}$ be an $\alpha(\cdot)$-midconvex function. Then for all $x, y \in V$ and $r \in [0,1] \cap \mathcal{D}$ we have*

$$f(rx + (1-r)y) - rf(x) - (1-r)f(y) \leq \sum_{k=0}^{\infty} \alpha(\|x-y\|/2^k) \cdot d(2^k r). \quad (5)$$

Moreover, if f is locally upper bounded and

$$\sum_{k=0}^{\infty} \alpha(1/2^k) < \infty \quad (6)$$

then f is continuous on int$V$ *and* (5) *holds for all $x, y \in V$ and $r \in [0,1]$.*

Before proceeding to the proof let us make some comments on condition (6). From this condition we obtain that $\lim_{k\to\infty} \alpha(1/2^k) = 0$, and consequently, by the fact that α is nondecreasing, that $\lim_{r\to 0^+} \alpha(r) = 0$. Moreover, it is equivalent to the fact that the right hand side of (5) is finite for all $x, y \in V$, $r \in [0,1]$ (observe that the value of the right hand side of (5) at the point $r = 1/3$ is equal to $\frac{2}{3}\sum_{k=0}^{\infty} \alpha(\|x-y\|/2^k)$).

Proof of Theorem 3.1 Let us fix $x, y \in V$ and define the function $h : [0,1] \to \mathbb{R}$ by the formula

$$h(r) = f(rx + (1-r)y) - rf(x) - (1-r)f(y) \quad \text{for } r \in [0,1].$$

Clearly $h(0) = h(1) = 0$, and by the $\alpha(\cdot)$-midconvexity of f we obtain that

$$h\left(\frac{s+t}{2}\right) \leq \frac{h(s)+h(t)}{2} + \alpha(|s-t|\|x-y\|) \quad \text{for } s, t \in [0,1].$$

It means that h is θ-midconvex with $\theta(w) := \alpha(w\|x-y\|)$. By Proposition 3.1

$$h(r) \leq \sum_{k=0}^{\infty} \theta(1/2^k) d(2^k r) = \sum_{k=0}^{\infty} \alpha(\|x-y\|/2^k) d(2^k r) \quad \text{for } r \in [0,1] \cap \mathcal{D}, \quad (7)$$

which consequently yields that for $r \in [0,1] \cap \mathcal{D}$

$$f(rx + (1-r)y) - rf(x) - (1-r)f(y) \leq \sum_{k=0}^{\infty} \alpha(\|x-y\|/2^k) \cdot d(2^k r). \quad (8)$$

Now let us assume that f is locally bounded above. Since α is nondecreasing, it follows from (6) that $r \mapsto \sum_{k=0}^{\infty} \alpha(\|x-y\|/2^k)d(2^k r)$ is a well-defined function for every $x, y \in V$. Furthermore, due to the continuity of d it is continuous.

By Theorem 2.1 f is continuous on int$V$. Applying Theorem 2.1 to h we obtain that h is continuous on $(0,1)$. Since $r \mapsto \sum_{k=0}^{\infty} \alpha(\|x-y\|/2^k)d(2^k r)$ is continuous and h is continuous on $(0,1)$ the formula (7) holds for all $r \in (0,1)$. Since it trivially holds for $r = 0, 1$ we obtain that (8) holds for all $r \in [0,1]$. ■

4. Semiconcavity

We will show that Theorem 3.1 can be applied to obtain a generalization of Cannarsa and Sinestrari (2004, Theorem 2.1.10). More precisely, we prove that under relatively weak assumptions a (mid)-semiconcave function is semiconcave. For more information on this subject we refer the reader to Cannarsa and Sinestrari (2004, Chapter 2). For the convenience of the reader we state the result obtained in the proof of Cannarsa and Sinestrari (2004, Theorem 2.1.10) in the form slightly adapted to our settings.

Theorem CS (Cannarsa and Sinestrari, 2004, Theorem 2.1.10). *Let $\tilde{\omega} : [0,\infty) \to [0,\infty)$ be a nondecreasing upper-semicontinuous function such that $\lim_{\rho \to 0^+} \tilde{\omega}(\rho) = 0$. Let $S \subset X$ be arbitrary and let $u : S \to \mathbb{R}$ be a function such that*

$$u(x) + u(y) - 2u(\frac{x+y}{2}) \leq \frac{\|x-y\|}{2}\tilde{\omega}(\|x-y\|)$$

for any $x, y \in S$ such that the segment $[x,y]$ is contained in S.

Then for every $x, y \in S$ such that $[x,y] \subset S$ we have

$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \leq \lambda(1-\lambda)\|x-y\|\omega(\|x-y\|) \quad (9)$$

for $\lambda \in [0,1] \cap \mathcal{D}$, where

$$\omega(z) = \sum_{k=0}^{\infty} \tilde{\omega}(\frac{z}{2^k}) \quad \text{for } z \in [0,\infty).$$

If $\tilde{w}$ is linear, then we can take $\omega = \tilde{\omega}$.

Moreover, if u is continuous then (9) holds for all $\lambda \in [0,1]$.

Let us begin with the following proposition.

Proposition 4.1 *Let $\tilde{\omega} : [0,\infty) \to [0,\infty)$ be a nondecreasing function. Let $S \subset X$ be arbitrary and let $u : S \to \mathbb{R}$ be such that*

$$u(x) + u(y) - 2u(\frac{x+y}{2}) \leq \frac{\|x-y\|}{2}\tilde{\omega}(\|x-y\|)$$

for any $x, y \in S$ such that the segment $[x,y]$ is contained in S.

Then for every $x, y \in S$ *such that* $[x, y] \subset S$ *we have*

$$\lambda u(x)+(1-\lambda)u(y)-u(\lambda x+(1-\lambda)y) \leq \frac{1}{4}\sum_{k=0}^{\infty}\frac{\|x-y\|}{2^k}\tilde{\omega}(\frac{\|x-y\|}{2^k})d(2^k\lambda) \quad (10)$$

for $\lambda \in [0,1] \cap \mathcal{D}$.

Moreover, if u *is locally lower bounded then* (10) *holds for all* $\lambda \in [0,1]$.

Proof. We fix $x, y \in S$ such that $[x, y] \in S$. Take $V = [x, y]$, $f = -u|_V$, $\alpha(r) = r\tilde{w}(r)/4$ for $r \in [0, \infty)$ and apply Theorem 3.1. ∎

To show that above proposition improves Theorem CS we need to observe that the right hand side of (10) can be estimated from above by the right hand side of (9):

Observation 4.1 *Let* $\tilde{\omega} : [0,\infty) \to [0,\infty)$ *be a nondecreasing function and let*

$$\omega(z) := \sum_{k=0}^{\infty}\tilde{\omega}(\frac{z}{2^k}) \quad \text{for } z \in [0,\infty).$$

Then

$$\frac{1}{4}\sum_{k=0}^{\infty}\frac{z}{2^k}\tilde{\omega}(\frac{z}{2^k})d(2^k\lambda) \leq \lambda(1-\lambda)z\omega(z) \quad \text{for } z \in [0,\infty), \lambda \in [0,1]. \quad (11)$$

Moreover, if $\tilde{\omega}$ *is linear then the analogue of* (11) *holds with* $\tilde{\omega}$ *in place of* ω.

Proof. To prove (11) in the general case it is enough to check that for every $k \in \mathbb{N}$

$$\frac{1}{4}\frac{z}{2^k}\tilde{\omega}(z/2^k)d(2^k\lambda) \leq \lambda(1-\lambda)z\tilde{\omega}(z/2^k),$$

or equivalently that

$$d(2^k\lambda)/2^k \leq 4\lambda(1-\lambda). \quad (12)$$

Let $f_k(\lambda) := d(2^k\lambda)/2^k$. Clearly, f_k is Lipschitz with $\mathrm{lip}(f) = 2$ and satisfies $f_k(0) = f_k(1) = 0$. This implies that

$$|f_k(\lambda)| = |(1-\lambda)(f_k(\lambda) - f_k(0)) + \lambda(f_k(\lambda) - f_k(1))|$$
$$\leq |1-\lambda|\mathrm{lip}(f_k)|\lambda - 0| + |\lambda|\mathrm{lip}(f_k)|\lambda - 1| = 4\lambda(1-\lambda),$$

which consequently proves (12).

Now let us consider the case when $\tilde{\omega}$ is linear. We have to prove the following inequality

$$\sum_{k=0}^{\infty} d(2^k\lambda)/4^k \leq 4\lambda(1-\lambda) \quad \text{for } \lambda \in [0,1].$$

However, as one can easily show even the equality holds true (Yamaguti and Hata, 1983). ∎

Jointly the two above results form a generalization of Theorem CS. They also give a partial answer to the question posed in Cannarsa and Sinestrari (2004, Remark 2.1.1) – the technique we use is an improvement to that used in the proof of Theorem CS (observe that in fact the proof of Theorem CS is divided into two: the case of linear and nonlinear $\tilde{\omega}$; while in our Theorem 3.1 we use a unified approach).

5. Superstability phenomenon

S. Rolewicz (1979b) proved that if a function $f : I \to \mathbb{R}$, where I is a subinterval of $\mathbb{R}$, satisfies with a certain $C \geq 0$, $p > 2$ the inequality

$$f(rx+(1-r)y) \leq rf(x)+(1-r)f(y)+C|x-y|^p \quad \text{for } x, y \in I, r \in [0,1], \tag{13}$$

then it is convex. This statement can be understood as a superstability phenomenon – a respective perturbation of the convexity condition still guarantees convexity.

We are going to generalize this result for $\alpha(\cdot)$-midconvex functions. We will need the following lemma.

LEMMA 5.1 *Let $N \in \mathbb{N}$, $N \geq 2$, $C \in \mathbb{R}$, and let $x_0, \ldots, x_N \in \mathbb{R}$, $x_0 = x_N = 0$, be a given sequence such that*

$$x_k \leq (x_{k-1} + x_{k+1})/2 + C \quad \text{for } k = 1, \ldots, N-1. \tag{14}$$

Then

$$x_k \leq Ck(N-k) \quad \text{for } k = 0, \ldots, N.$$

Proof. The proof goes by induction over N. If $N = 2$ the assertion is trivial. Suppose that the assertion holds for a given $N \geq 2$. We prove that it holds for $N + 1$.

Let $x_0, \ldots, x_{N+1}$, $x_0 = x_{N+1} = 0$ be a given sequence satisfying (14). Consider the sequence y_k, $k = 0, \ldots, N$, defined by

$$y_k = x_k - \frac{k}{N} x_N.$$

Then, y_k satisfies (14) and $y_0 = y_N = 0$. By the inductive assumption we obtain that $y_k \leq Ck(N-k)$, that is

$$x_k \leq \frac{k}{N} x_N + Ck(N-k) \quad \text{for } k = 0, \ldots, N. \tag{15}$$

We estimate x_N. By the assumptions, applying (15) for $k = N-1$, we obtain

$$x_N \leq (x_{N-1} + x_{N+1})/2 + C \leq \frac{N-1}{2N} x_N + C(N-1)/2 + C.$$

Consequently, we obtain that $x_N \leq CN$. Making use of this inequality we obtain from (15)

$$x_k \leq Ck(N+1-k) \quad \text{for } k = 0, \ldots, N.$$

Obviously, the above inequality is valid for $k = N + 1$. ∎

PROPOSITION 5.1 *Let* $h : [0,1] \cap \mathbb{Q} \to \mathbb{R}$ *be an* $\alpha(\cdot)$*-midconvex function such that* $h(0) = h(1) = 0$*. Let* $N \in \mathbb{N}$*. Then*

$$h(k/N) \leq k(N-k) \cdot \alpha(2/N) \quad \textit{for } k = 0, \dots, N.$$

Proof. The assertion is obvious for $N = 1$. So assume that $N \geq 2$ and define the sequence $x_k := h(k/N)$ for $k = 0, \dots, N$, put $C = \alpha(2/N)$ and apply Lemma 5.1. ■

S. Rolewicz (2000) proved (Proposition 1) (see also Nikodem and Páles, 2003/4, and Rolewicz, 1979b) that if f is $\alpha(\cdot)$ paraconvex and absolutely continuous on each line segment contained in the domain of f, where α satisfies the condition

$$\liminf_{r \to 0^+} \alpha(r)/r^2 = 0,$$

then f is convex. In the comment after the proof of Proposition 1 of Rolewicz (2000), he also asked whether the assumption of absolute continuity on the lines can be removed.

The following theorem gives the generalization of the above mentioned result (to observe this just put $A = 0$). It also shows that the answer to S. Rolewicz's question is positive (it is enough to assume that f is locally bounded above).

THEOREM 5.1 *We assume that*

$$A := \liminf_{r \to 0^+} \alpha(r)/r^2 < \infty.$$

Let $f : V \to \mathbb{R}$ *be a given* $\alpha(\cdot)$*-midconvex function locally bounded above at every point. Then*

$$f(rx + (1-r)y) \leq rf(x) + (1-r)f(y) + 4Ar(1-r)\|x-y\|^2 \tag{16}$$

for $x, y \in V, r \in [0,1]$.

Proof. Without loss of generality we may assume that $\alpha(r) > 0$ for all $r > 0$ (in the opposite case we apply the Theorem for $\alpha_\delta(r) := \alpha(r) + \delta r^2$ and go with $\delta \to 0^+$).

Choose arbitrary $x, y \in V$, $x \neq y$ and define the function

$$h(r) := f(rx + (1-r)y) - rf(x) - (1-r)f(y) \quad \text{for } r \in [0,1].$$

Clearly, $\lim_{r \to 0} \alpha(r) = 0$, which by Theorem 2.1 and local upper boundedness of f implies that h is continuous on $(0,1)$. By applying Proposition 5.1 we obtain

$$h(K/N) \leq K(N-K)\alpha(2\|x-y\|/N) \quad \text{for } K \in \mathbb{N}_0, N \in \mathbb{N}, K \leq N. \tag{17}$$

Now, choose a sequence $r_k \to 0$ such that

$$\lim_{k\to\infty} \alpha(r_k)/r_k^2 = A. \tag{18}$$

Let us put

$$N_k := \lceil 2\|x-y\|/r_k \rceil \in \mathbb{N} \quad \text{for } k \in \mathbb{N}.$$

Then, by (18) and the fact that α is nondecreasing we get

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{\alpha(2\|x-y\|/N_k)}{(2\|x-y\|/N_k)^2} \\
&\le \limsup_{k\to\infty} \frac{\alpha(2\|x-y\|/N_k)}{\alpha(r_k)} \cdot \limsup_{k\to\infty} \frac{(r_k)^2}{(2\|x-y\|/N_k)^2} \cdot \limsup_{k\to\infty} \frac{\alpha(r_k)}{(r_k)^2} \\
&\le 1 \cdot 1 \cdot A = A.
\end{aligned}$$

By the definition of A we directly obtain that

$$\lim_{k\to\infty} \frac{\alpha(2\|x-y\|/N_k)}{(2\|x-y\|/N_k)^2} = A.$$

Let us now choose an arbitrary $r \in (0,1)$. Then, there exists a sequence $(K_k)_{k\in\mathbb{N}}$ of positive integers such that $K_k/N_k \to r$. By applying (17) and the above inequality we obtain

$$\begin{aligned}
h(r) &= \lim_{k\to\infty} h(K_k/N_k) \le \limsup_{k\to\infty} K_k(N_k - K_k)\alpha(2\|x-y\|/N_k) \\
&= \limsup_{k\to\infty} K_k(N_k - K_k) \frac{\alpha(2\|x-y\|/N_k)}{(2\|x-y\|/N_k)^2} (2\|x-y\|/N_k)^2 \\
&\le \limsup_{k\to\infty} 4\tfrac{K_k}{N_k}(1 - \tfrac{K_k}{N_k}) \frac{\alpha(2\|x-y\|/N_k)}{(2\|x-y\|/N_k)^2} \|x-y\|^2 = 4r(1-r)A\|x-y\|^2.
\end{aligned}$$

By the definition of h this makes the proof complete. ∎

References

Bernstein, F. and Doetsch, G. (1915) Zur Theorie der konvexen Funktionen. *Math. Annalen* **76**, 514–526.

Boros, Z. (2008) An inequality for the Takagi function. *Math. Inequal. Appl.* **11**, 757–765.

Cannarsa, P. and Sinestrari, C. (2004) Semiconvex functions, Hamilton-Jacobi equations and optimal control. *Progress in Nonlinear Differential Equations and their Applications* **58**, Birkhäuser, Boston.

Házy, A. (2005) On approximate t-convexity. *Mathematical Inequal. Appl.* **8** (3), 389–402.

HÁZY, A. and PÁLES, ZS. (2004) On approximately midconvex functions. *Bulletin London Math. Soc.* **36** (3), 339–350.

HÁZY, A. and PÁLES, ZS. (2005) On approximately t-convex functions. *Publ. Math. Debrecen* **66** (3-4), 489–501.

HYERS, D.H., ISAC, G. and RASSIAS, TH.M. (1998) *Stability of Functional Equations in Several Variables.* Birkhäuser, Basel.

HYERS, D.H. and ULAM, S.M. (1952) Approximately convex functions. *Proc. Amer. Math. Soc.* **3**, 821–828.

KUCZMA, M. (1985) *An Introduction to the Theory of Functional Equations and Inequalities.* PWN – Uniwersytet Śląski, Warszawa - Kraków - Katowice.

MAULDIN, R. and WILLIAMS, S. (1986) On the Hausdorff dimension of some graphs. *Trans. Amer. Math. Soc.* **298** (2), 793–803.

MROWIEC, J., TABOR, JACEK and TABOR, JÓZEF (2009) Approximately midconvex functions. In: C. Bandle, A. Gilányi, L. Losonczi, Z. Páles, M. Plum, eds., Inequalities and Applications. *International Series of Numerical Mathematics* **157**, Birkhäuser, 261–265.

NG, C.T. and NIKODEM, K. (1993) On approximately convex functions. *Proc. Amer. Math. Soc.* **118**, 103–108.

NIKODEM, K. and PÁLES, ZS. (2003/2004) On t-convex functions. *Real Anal. Exchange* **29** (1), 1–16.

PÁLES, ZS. (2003) On approximately convex functions. *Proc. Amer. Math. Soc.* **131** (1), 243–252.

ROLEWICZ, S. (1979) On paraconvex multifunctions. *Oper. Research Verf. (Methods Oper. Res.)* **31**, 540–546.

ROLEWICZ, S. (1979) On γ-paraconvex multifunctions. *Math. Japonica* **24** (3), 293–300.

ROLEWICZ, S. (2000) On $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions. *Control and Cybernetics* **29**, 367–377.

ROLEWICZ, S. (2005) Paraconvex analysis. *Control and Cybernetics* **34**, 951–965.

ROLEWICZ, S. (2006) An extension of Mazur's theorem on Gateaux differentiability to the class of strongly $\gamma(\cdot)$-paraconvex functions. *Studia Math.* **172** (3), 243–248.

TABOR, JACEK and TABOR, JÓZEF (2009) Takagi functions and approximate midconvexity. *Journal of Mathematical Analysis and Applications* **356** (2), 729–737.

YAMAGUTI, M. and HATA, M. (1983) Weierstrasse's function and chaos. *Hokkaido Math. J.* **12**, 333–342.

ZAJÍČEK, L. (2008) Differentiability of Approximately Convex, Semiconcave and Strongly Paraconvex Functions. *Journal of Convex Analysis* **15**, 1–15.