Control and Cybernetics

vol. **38** (2009) No. 3

A primal-infeasible interior point algorithm for linearly constrained convex programming*

by

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Abstract: In this paper a primal-infeasible interior point algorithm is proposed for linearly constrained convex programming. A positive primal-infeasible dual-feasible point can be taken as the starting point of this algorithm in a large region. At each iterates it requires to solve approximately a nonlinear system. The polynomial complexity of the algorithm is obtained. It is shown that, after finite iterations a sufficiently good approximation to the optimal point is found, or there is no optimal point in a large region.

Keywords: linearly constrained convex programming; primal-infeasible interior point algorithm; polynomial complexity

1. Introduction

In this paper we consider the linearly constrained convex programming problem:

$$\begin{aligned} &\min \ \ f(x) \\ &\text{subject to} \ \ Ax = b, \ x \geq 0 \end{aligned} \tag{1}$$

where $x \in R^n$, $A \in R^{m\times n}$, $b \in R^m$, $m \leq n$, and $f : R^n \to R$ is a sufficiently smooth convex function.

The dual problem for the problem (1) can be put in the form

$$\begin{aligned} &\max \ \ b^T y - (x^T \nabla f(x) - f(x)) \\ &\text{subject to} \ \ \nabla f(x) - (A^T y + s) = 0, \ s \geq 0. \end{aligned} \tag{2}$$

*Submitted: December 2006; Accepted: February 2009.

†Corresponding author. This author was partly supported by NSAF(10676005).

We let Ω denote the feasible point set of primal-dual problem (1) and (2)

$$\Omega = \{(x, y, s) : \ Ax = b, \nabla f(x) - A^T y - s = 0, (x, s) \geq 0\}.$$

It is well known that $(x, y, s) \in R^n \times R^m \times R^n$ is a primal-dual optimal solution of the problems (1) and (2) if and only if the point (x, y, s) satisfies the following first-order optimality conditions (called KKT-Condition) for the problems (1) and (2):

$$\begin{aligned} A^T y + s &= \nabla f(x), \\ Ax &= b, \\ (x, s) &\geq 0, \\ x^T s &= 0. \end{aligned}$$

If we relax the fourth condition of KKT-condition as follows

$$Xs - \beta_1 \mu e = 0,$$

where $X = diag(x)$, $\mu \geq 0$ and $\beta_1 \in [0, 1]$ are two constants, then KKT-condition is said to be perturbed. Clearly, as $\mu \to 0$, then the point (x, y, s), which satisfies the perturbed KKT-condition, will converge to an optimal point of problems (1) and (2).

In recent years various feasible interior point methods have been developed for solving various optimization problem based on the idea of either reducing the primal-dual complementarity gap $x^T s$ or reducing the value of some primal-dual potential functions, we refer here to publications by Hertog (1994), Nesterov and Nemirovski (1994), Wright (1997) and Ye (1997), and the references therein. The interior point method is also considered to solve convex programming (see, e.g., Güler, 1997; Kortanek, Potra and Ye, 1991; Potra and Wright, 2000; and Shi, 2002) and nonlinear complementarity problems (see, e.g., Ralph and Wright, 1997, 2000, and Potra and Ye 1996). Most of these algorithms have the property of global linear convergence with polynomial complexity. The infeasible-interior-point algorithms for linear programming (see, e.g., Kojima, Megiddo and Mizuo, 1993; Kojima, Noma and Yoshise, 1994, and Mizuno, 1995) have been extended to solve convex programming (see, e.g., Monteiro and Zhou, 1997, and Yamashita et al., 2001), nonlinear complementarity problems (see, e.g., Tseung, 1997, and Wright and Ralph, 1996), and these algorithms have property of global convergence. However, as far as the authors know, there is no result of polynomial complexity of infeasible interior point algorithms for convex programming. This motivates us to further study an infeasible interior point algorithm for convex programming.

In this paper we are interested in a polynomial infeasible interior point method for convex programming. The infeasible interior point method for problems (1) and (2) is more intricate than the analysis of the infeasible interior point algorithm for linear programming because of the nonlinear term of the convex object function for problems (1) and (2). Thus, in order to avoid the nonlinear term, here we present a primal-infeasible dual-feasible interior point algorithm based on the interior point algorithms (Kortanek, Potra and Ye, 1991). The starting point of this algorithm requires a positive primal-infeasible dual-feasible point in a large region. It needs to solve an approximate solution of a nonlinear system in our algorithm. In this paper, under some assumptions, we analyze the complexity of our algorithm, we also prove that after finite number of iterations we get an approximate optimal point, or show that there is no optimal point in a large region.

Throughout the paper, the following notations are used. All vectors are column vectors. We frequently use (x, y) as shorthand for the vector $(x^T, y^T)^T$. R^m denotes the m-dimensional Euclidean space. The set of all $m \times n$ matrices with real entries is denoted by $R^{m\times n}$. The diagonal matrix corresponding to a vector x is denoted by X, i.e., $X = \mathrm{diag}(x)$, and $e = (1, 1, \cdots, 1)^T \in R^n$ We also denote by $\|x\|_1$, $\|x\|_2$ and $\|x\|_\infty$ the 1-, 2- and ∞-norm of x, that is to say, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, $\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$ and $\|x\|_\infty = \max_{1\le i\le n} |x_i|$. For simplicity, we use $\|x\|$ to replace $\|x\|_2$. The superscript ^{T} denotes transpose.

The rest of the paper is organized as follows. In Section 2, we describe the primal-infeasible dual-feasible interior point algorithm for convex programming; in Section 3, we analyze the polynomial convergence of our algorithm.

2. Algorithm

Throughout the present paper, we always make the following assumptions:

Assumption 1 *Without loss of generality we let* $Rank(A) = m$.

This assumption is quite standard for convex programming.

Assumption 2 $f(x)$ *is continuously differentiable and convex.* $\nabla f(x)$ *satisfies the Lipschitz condition with the Lipschitz index* L, *i.e.,*

$$\|\nabla f(x') - \nabla f(x'')\| \le L\|x' - x''\|,$$

where $L > 0$ *is a constant.*

Under the next assumption, Lipschitz condition above is weaker than twice continuous differentiability.

Assumption 3 *If there exists an optimal point* (x^*, y^*, s^*) *of convex programming (1), then we assume that* $(x^*, y^*, s^*) \in \Phi$, *where*

$$\Phi = \{(x, y, s) \in R^n \times R^m \times R^n : (x, s) \ge 0, \|(x, s)\|_\infty \le \rho\}$$

with $\rho > 0$.

Note that Assumption 3 is a frequently used assumption for infeasible interior point methods to obtain polynomial complexity bounds.

ASSUMPTION 4 *Let* $\Omega_1 = \{(x, y, s) : (x, s) > 0$ and $s = \nabla f(x) - A^T y\}$. *Here we suppose that* $\Omega_1 \bigcap \Phi \neq \emptyset$.

Obviously, this assumption is also weaker than one, according to which the feasible interior point set of problems (1) and (2) is not empty.

The central path of problems (1) and (2) is defined as follows

$$S = \left\{(x, y, s) : x > 0, s > 0, Ax - b = 0, A^T y + s - \nabla f(x) = 0, Xs = \frac{x^T s}{n} e\right\}$$

in primal-dual form.

Without considering feasibility, we let the neighborhood set of the central path be as follows:

$$\mathcal{N} = \left\{(x, y, s) : \; x > 0, s > 0, \|Xs - \frac{x^T s}{n} e\| \leq \sigma \frac{x^T s}{n}, \sigma \in (0, 1)\right\}.$$

We note that for an infeasible interior point algorithm it always defines a neighborhood set of the central path just like the set above.

From the definition above, we can easily get the following lemma.

LEMMA 2.1 *For* $(x, y, s) \in \mathcal{N}$,

$$\begin{aligned} \max\{x_i s_i\} &\leq (1+\sigma)\frac{x^T s}{n}, \\ \min\{x_i s_i\} &\geq (1-\sigma)\frac{x^T s}{n}. \end{aligned}$$

The primal-dual affine scaling search direction $(\Delta x, \Delta y, \Delta s)$ at a given infeasible interior point $(x^k, y^k, s^k) \in R^n \times R^m \times R^n$ is computed by applying one step of Newton's method to the perturbed KKT-condition as

$$\left\{ \begin{array}{rcl} A\Delta x &=& -(Ax^k - b) \\ \nabla^2 f(x^k)\Delta x - A^T \Delta y - \Delta s &=& -(\nabla f(x^k) - A^T y^k - s^k) \\ X^k \Delta s + S^k \Delta x &=& -(X^k s^k - \beta_1 \frac{(x^k)^T s^k}{n}) \end{array} \right. , \tag{3}$$

where $\beta_1 > 0$.

According to the idea of the infeasible interior point algorithm (Kojima, Megiddo and Mizuo, 1993; Kojma, Noma and Yoshise, 1994; Mizuno, 1995), the search direction of an infeasible interior point algorithm is the solution of the system above. It is difficult to analyze the complexity of the primal-dual infeasible interior point algorithm for convex problem (1) and (2), because of the nonlinearity of $f(x)$ which appears in the second equation of (3). In order to avoid this difficulty, here we will consider the dual-feasible point, thus we can cancel solving the second equation of the system (3). Since it is dual-feasible point, we choose the next iterate s^{k+1} such that

$$\begin{aligned} s^{k+1} &= s^k + \Delta s(\alpha^k) \\ &= \nabla f(x^k + \alpha^k \Delta x) - A^T(y^k + \alpha^k \Delta y). \end{aligned}$$

That leads to the following: $\Delta s(\alpha^k) = \nabla f(x^k + \alpha^k \Delta x) - \nabla f(x^k) - \alpha^k A^T \Delta y$.

By substituting $\Delta s(\alpha^k)$ in (3), we get

$$\begin{cases} X^k[\nabla f(x^k+\alpha^k\Delta x)-\nabla f(x^k)-\alpha^k A^T\Delta y]+\alpha^k S^k\Delta x = -\alpha^k(X^k s^k-\beta_1\mu e) \\ A\Delta x = -(Ax^k - b) \end{cases} \tag{4}$$

where α^k and μ are some positive real numbers.

We know that it is difficult to find the exact solution of the previous nonlinear system, especially the first equation. In these situations it is reasonable to approximately solve the first equation of (4). Then the system (4) is rewritten in the following form

$$\begin{cases} X^k[\nabla f(x^k+\alpha^k\Delta x)-\nabla f(x^k)-\alpha^k A^T\Delta y]+\alpha^k S^k\Delta x = -\alpha^k(X^k s^k-\beta_1\mu e) \\ \qquad\qquad + \alpha^k r^k \\ A\Delta x = -(Ax^k - b) \end{cases} \tag{5}$$

where r^k satisfies $\|r^k\|_1 \le \nu\mu$ and ν is some small positive constant. Here we can use

Remark 2.1 *Note that, if we also introduce the inexact technique to the linear equation in (5), then the property $A(x^k+\alpha^k\Delta x^k)-b = (1-\alpha^k)(Ax^k-b)$ cannot be valid at each iteration. And then the analysis of the polynomial complexity of our algorithm in Section 3 cannot work. So, above we just consider solving the first equation of (4) inexactly.*

Providing a polynomial complexity for the case of solving both equations of (4) inexactly will be one of our future works. Here we refer to Korzak (2000) and Mizuno and Jarre (1999) for linear programming with inexact computation.

In addition, noting that the equations (5) use the step size α^k, we have to determine α^k before solving the equations (5) in our algorithm.

Now we are in a position to state our primal-infeasible dual-feasible interior point algorithm:

Algorithm 2.1

Step 0. Let (x^0, y^0, s^0) in $\Omega_1 \cap \Phi \cap \mathcal{N}$. Choose ν, β_1, β_2 and σ with the properties $0 < \nu \le 1/2 \le \beta_1 < \beta_2 \le 1$, $0 < \sigma < 1$, $\sigma\beta_1 > 2\nu$, $\beta_2 > \beta_1 + \nu$, and $\rho \ge \rho^0 > 0$. Set $k = 0$ and $\theta^0 = 1$.

Step 1. If the point (x^k, y^k, s^k) satisfies

$$\|Ax^k - b\| \le \epsilon_p \text{ and } (x^k)^T s^k \le \epsilon, \tag{6}$$

then the algorithm is terminated.

Step 2. Compute α^k as follows

$$\alpha^k = \left(\frac{\rho_0}{\rho}\right)^2 \frac{\min\{\sigma\beta_1 - 2\nu, \beta_2 - \beta_1 - \nu/n\} \min_i(x_i^k s_i^k)}{2n(2+\chi+\tau)^2 \left(1 + (1+L)n^{\frac{1}{2}}\right)^2 (x^k)^T s^k}, \tag{7}$$

where

$$\begin{aligned} \chi &= \frac{Ln\rho^2}{\rho_0^2}, \\ \tau &= \frac{\rho\|(s^0, x^0)\|_1}{(x^0)^T s^0}, \\ \rho_0 &= \min\{x_1^0, x_2^0, \cdots, x_n^0, s_1^0, s_2^0, \cdots, s_n^0\}. \end{aligned}$$

Step 3. Let $\mu = \dfrac{(x^k)^T s^k}{n}$, then find $(\Delta x, \Delta y) \in R^n \times R^m$ satisfying the equation (5).

Step 4. Set $x^{k+1} := x^k + \alpha^k \Delta x, y^{k+1} := y^k + \alpha^k \Delta y, s^{k+1} := \nabla f(x^k + \alpha^k \Delta x) - A^T(y^k + \alpha^k \Delta y)$, $\theta^{k+1} := \theta^k(1-\alpha^k)$ and $k := k+1$, then go to Step 1.

Remark 2.2 *Note that*

$$\begin{aligned} &0 < \min\{\sigma\beta_1 - 2\nu, \beta_2 - \beta_1 - \nu/n\} < 1 < 2n(2+\chi+\tau)^2 \left(1 + (1+L)n^{\frac{1}{2}}\right)^2, \\ &0 < \min_i(x_i^k s_i^k) < (x^k)^T s^k. \end{aligned}$$

From (7) we obtain

$$0 < \alpha^k < 1.$$

This algorithm guarantees that the step size $\alpha \in (0,1)$ and μ reduces at each iteration, since $(x^{k+1})^T s^{k+1} \leq (1 - \alpha_k(1-\beta_1))(x^k)^T s^k$, see Lemma 3.4.

3. Convergence

In this section we analyze the polynomial convergence of Algorithm 2.1. Before we state the complexity result of Algorithm 2.1, we will claim several lemmas. Throughout this section we denote $D = (X^k)^{\frac{1}{2}}(S^k)^{-\frac{1}{2}}$ for a point $(x^k, s^k) > 0$.

Now we first introduce a lemma to show the relation between $\|(x^k, s^k)\|_1$ and $(x^k)^T s^k$, which will help us estimate the solution of (5).

Lemma 3.1 *Assume that the problems (1) and (2) have an optimal point $(x^*, y^*, s^*) \in \Phi$, and the point (x^j, y^j, s^j) is generated by Algorithm 2.1 with $j = 1, 2, \cdots, k$ for $k \geq 1$ and satisfies*

$$(x^j, s^j) > 0, \tag{8}$$

$$(x^j)^T s^j \geq (1 - \alpha^{(j-1)})(x^{j-1})^T s^{j-1}. \tag{9}$$

Then

$$\|(x^k, s^k)\|_1 \leq \frac{2+\chi+\tau}{\theta^k \rho_0}(x^k)^T s^k \tag{10}$$

for $k \geq 1$, where χ and τ are defined in Algorithm 2.1 depending on the starting point and the large region Φ.

Proof. From Algorithm 2.1, by direct computation we have

$$A(x^k - \theta^k x^0 - (1-\theta^k)x^*) = 0 \tag{11}$$

and by the dual-feasible point (x^k, y^k, s^k) we obtain

$$\begin{aligned}(s^k - \theta^k s^0 - (1-\theta^k)s^*) &= (\nabla f(x^k) - A^T y^k) - \theta^k(\nabla f(x^0) - A^T y^0) \\ &\quad -(1-\theta^k)(\nabla f(x^*) - A^T y^*).\end{aligned}$$

Noting $0 < \theta^k < 1$, from the above equations and the fact that $(\dot{x}-\ddot{x})^T(\nabla f(\dot{x}) - \nabla f(\ddot{x})) \geq 0$ we get

$$\begin{aligned}&(x^k - \theta^k x^0 - (1-\theta^k)x^*)^T(s^k - \theta^k s^0 - (1-\theta^k)s^*) \\ &= (x^k - \theta^k x^0 - (1-\theta^k)x^*)^T(\nabla f(x^k) - \theta^k \nabla f(x^0) - (1-\theta^k)\nabla f(x^*))^T \\ &= [\theta^k(x^k - x^0) + (1-\theta^k)(x^k - x^*)]^T[\theta^k(\nabla f(x^k) - \nabla f(x^0)) \\ &\qquad +(1-\theta^k)(\nabla f(x^k) - \nabla f(x^*))] \\ &\geq \theta^k(1-\theta^k)(x^k - x^0)^T(\nabla f(x^k) - \nabla f(x^*)) \\ &\qquad +\theta^k(1-\theta^k)(x^k - x^*)]^T(\nabla f(x^k) - \nabla f(x^0)) \\ &= \theta^k(1-\theta^k)(x^k - x^* + x^* - x^0)^T(\nabla f(x^k) - \nabla f(x^*)) \\ &\quad + \theta^k(1-\theta^k)(x^k - x^0 + x^0 - x^*)]^T(\nabla f(x^k) - \nabla f(x^0)) \\ &\geq -\theta^k(1-\theta^k)(x^* - x^0)^T(\nabla f(x^*) - \nabla f(x^0)),\end{aligned}$$

which implies that $(\theta^k x^0 + (1-\theta^k)x^*)^T s^k + \left(x^k\right)^T(\theta^k s^0 + (1-\theta^k)s^*) \leq (\theta^k x^0 + (1-\theta^k)x^*)^T(\theta^k s^0 + (1-\theta^k)s^*) + \left(x^k\right)^T s^k + \theta^k(1-\theta^k)(x^* - x^0)^T(\nabla f(x^*) - \nabla f(x^0))$.

Thus, according to the definition of the starting point (x^0, y^0, s^0) in our algorithm, Assumption 3 and (8) we know that

$$\begin{aligned}\theta^k \rho_0 \|(x^k, s^k)\|_1 &\leq \theta^k \left[\left(x^0\right)^T s^k + \left(s^0\right)^T x^k\right] \\ &\leq (\theta^k x^0 + (1-\theta^k)x^*)^T s^k + \left(x^k\right)^T(\theta^k s^0 + (1-\theta^k)s^*) \\ &\leq (\theta^k x^0 + (1-\theta^k)x^*)^T(\theta^k s^0 + (1-\theta^k)s^*) + \left(x^k\right)^T s^k \\ &\leq (\theta^k (x^0)^T s^0 + \theta^k(1-\theta^k)((x^*)^T s^0 + (s^*)^T x^0)) + \left(x^k\right)^T s^k \\ &\quad + \theta^k(x^* - x^0)^T(\nabla f(x^*) - \nabla f(x^0)) \\ &\leq \theta^k(1+\delta+\zeta)(x^0)^T s^0 + \left(x^k\right)^T s^k,\end{aligned}$$

where

$$\delta = \frac{(x^* - x^0)^T(\nabla f(x^*) - \nabla f(x^0))}{(x^0)^T s^0},$$
$$\zeta = \frac{(x^*)^T s^0 + (s^*)^T x^0}{(x^0)^T s^0} > 1.$$

By Assumptions 2, 3 and Cauchy-Schwartz inequality we see that

$$\frac{(x^* - x^0)^T(\nabla f(x^*) - \nabla f(x^0))}{(x^0)^T s^0} \le \frac{L\|x^* - x^0\|^2}{(x^0)^T s^0} \le \frac{Ln\rho^2}{\rho_0^2},$$

where $\rho_0 = \min\{x_1^0, x_2^0, \cdots, x_n^0, s_1^0, s_2^0, \cdots, s_n^0\}$.

We also have by (9)

$$(x^k)^T s^k \ge (1 - \alpha^{k-1})(x^{k-1})^T s^{k-1} \ge \theta^k (x^0)^T s^0.$$

And from Assumption 3 it follows that $\zeta \le \tau$. Thus we have

$$\|(x^k, s^k)\|_1 \le \frac{2 + \chi + \tau}{\theta^k \rho_0}(x^k)^T s^k.$$

Heretofore we have completed the proof of this lemma. ∎

Based on Lemma 3.1, we next give out estimates on the solution of (5).

LEMMA 3.2 *Assume that the problems (1) and (2) have an optimal point $(x^*, y^*, s^*) \in \Phi$. Let the points (x^j, y^j, s^j) with $j = 1, 2, \cdots, k$ be generated by Algorithm 2.1 and satisfy (8) and (9). Then there are the following estimates*

$$\|D^{-1}\Delta x\| \le \left(1 + (2 + \chi + \tau)(1 + L)n^{\frac{1}{2}}\frac{\rho}{\rho_0} + \beta_1 + \nu/n\right)\frac{(x^k)^T s^k}{\min_i (x_i s_i)^{\frac{1}{2}}}, \quad (12)$$

$$\|D\Delta s(\alpha^k)\| \le \alpha^k \left(1 + (2 + \chi + \tau)(1 + L)n^{\frac{1}{2}}\frac{\rho}{\rho_0} + \beta_1 + \nu/n\right)\frac{(x^k)^T s^k}{\min_i (x_i s_i)^{\frac{1}{2}}}, \quad (13)$$

where $\Delta s(\alpha^k) = \nabla f(x^k + \alpha^k \Delta x) - \nabla f(x^k) - \alpha^k A^T \Delta y$.

Proof. In the proof, sometimes we will omit the superscript k of x^k and s^k.

Firstly, we consider the following equation: $(\Delta x', \Delta s')$ satisfies

$$\begin{pmatrix} A & 0 \\ S^k & X^k \end{pmatrix}\begin{pmatrix} \Delta x' \\ \Delta s' \end{pmatrix} = \begin{pmatrix} 0 \\ p \end{pmatrix}, \quad (14)$$

where $\Delta s' = \nabla f(x^k + \Delta x') - \nabla f(x^k) - A^T \Delta y'$. Then, by the convexity of $f(x)$, $(\Delta x')^T \Delta s' \ge 0$ we have

$$D^{-1}\Delta x' + D\Delta s' = (X^k S^k)^{-\frac{1}{2}} p,$$

thus we see that the following estimates are valid

$$\|D^{-1}\Delta x'\| = \|D(\Delta s' - (X^k)^{-1}p)\| \le \|(X^kS^k)^{-\frac{1}{2}}p\|, \quad (15)$$

$$\|D\Delta s'\| = \|D^{-1}(\Delta x' - (S^k)^{-1}p)\| \le \|(X^kS^k)^{-\frac{1}{2}}p\|. \quad (16)$$

And obviously we have

$$\begin{aligned}\|D^{-1}\Delta x' + D\Delta s'\|^2 &= \|D^{-1}\Delta x'\|^2 + \|D\Delta s'\|^2 + 2(\Delta x')^T\Delta s' \\ &\ge \|D^{-1}\Delta x'\|^2 + \|D\Delta s'\|^2.\end{aligned}$$

This leads to

$$\begin{aligned}A\Delta x &= -(Ax^k - b) \\ &= -A(\theta^k x^0 + (1-\theta^k)x^*) + b \\ &= -\theta^k A(x^0 - x^*).\end{aligned} \quad (17)$$

Let $\widehat{\Delta x} = \alpha^k(\Delta x + \theta^k(x^0 - x^*))$ and $\widehat{\Delta s} = \nabla f(x^k + \widehat{\Delta x}) - \nabla f(x^k) - \alpha^k A^T\Delta y$ where Δy is the solution of the system (5) at kth iteration of Algorithm 2.1, then $(\widehat{\Delta x}, \widehat{\Delta s})$ satisfies the system (14) with

$$\begin{aligned}p = &-\alpha^k(X^ks^k - \beta_1\mu^k e) + \alpha^k\theta^k S^k(x^0 - x^*) \\ &+ X^k(\nabla f(x^k + \alpha^k(\Delta x + \theta^k(x^0 - x^*))) - \nabla f(x^k + \alpha^k\Delta x)) + \alpha^k r^k.\end{aligned}$$

Now let $(\widehat{\Delta x_1}, \widehat{\Delta s_1})$,$(\widehat{\Delta x_2}, \widehat{\Delta s_2})$, $(\widehat{\Delta x_3}, \widehat{\Delta s_3})$ and $(\widehat{\Delta x_4}, \widehat{\Delta s_4})$ satisfy the system (14) when p is $-\alpha^k(X^ks^k - \beta_1\mu^k e)$, $\alpha^k\theta^k S^k(x^0 - x^*)$, $X^k(\nabla f(x^k + \alpha^k(\Delta x + \theta^k(x^0 - x^*))) - \nabla f(x^k + \alpha^k\Delta x))$ and $\alpha^k r^k$, respectively. By (14) we see that

$$\sum_{i=1}^{4}\begin{pmatrix} A & 0 \\ S^k & X^k\end{pmatrix}\begin{pmatrix}\widehat{\Delta x_i} \\ \widehat{\Delta s_i}\end{pmatrix} = \begin{pmatrix} A & 0 \\ S^k & X^k\end{pmatrix}\begin{pmatrix}\widehat{\Delta x} \\ \widehat{\Delta s}\end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \left\{\begin{array}{l} -\alpha^k(X^ks^k - \beta_1\mu^k e) \\ +\alpha^k\theta^k S^k(x^0 - x^*) \\ +X^k(\nabla f(x^k + \alpha^k(\Delta x + \theta^k(x^0 - x^*))) \\ -\nabla f(x^k + \alpha^k\Delta x)) + \alpha^k r^k\end{array}\right.\end{pmatrix}.$$

Thus we obtain that

$$\begin{aligned}\|\alpha^k D^{-1}\Delta x\| &= \|D^{-1}(\widehat{\Delta x_1} + \widehat{\Delta x_2} + \widehat{\Delta x_3} + \widehat{\Delta x_4} - \alpha^k\theta^k(x^0 - x^*))\| \\ &\le \|D^{-1}\widehat{\Delta x_1}\| + \|D^{-1}\widehat{\Delta x_3}\| + \|D^{-1}\widehat{\Delta x_4}\| + \|D\widehat{\Delta s_2}\| \\ &\le \alpha^k\|(XS)^{-\frac{1}{2}}(Xs - \beta_1\mu e)\| + \alpha^k\|(XS)^{-\frac{1}{2}}r^k\| \\ &\quad + \alpha^k\theta^k\|D(s^0 - s^*)\| \\ &\quad + \|D^{-1}[\nabla f(x + \alpha^k(\Delta x + \theta^k(x^0 - x^*))) - \nabla f(x + \alpha^k\Delta x)]\| \\ &\le \alpha^k\|(XS)^{-\frac{1}{2}}(Xs - \beta_1\mu e)\| + \alpha^k\|(XS)^{-\frac{1}{2}}r^k\| \\ &+ \alpha^k\theta^k\|D(s^0 - s^*)\| + \alpha^k\theta^k L\|D^{-1}\|\cdot\|x^* - x^0\|,\end{aligned}$$

where the first inequality uses the fact that $(S^k)^{-1}p_2 = \alpha^k\theta^k(x^0 - x^*)$, where p_2 is the corresponding p with $(\widehat{\Delta x_2}, \widehat{\Delta s_2})$.

Using Assumptions 2, 3 and (8) we have

$$
\begin{aligned}
\|D^{-1}\Delta x\| &\leq \|(XS)^{-\frac{1}{2}}\|(1+\beta_1)x^T s + \nu\|(XS)^{-\frac{1}{2}}\|(x^k)^T s^k/n \\
&\quad + \theta^k\|(XS)^{-\frac{1}{2}}\| \cdot \|X\| \cdot \|(s^0 - s^*)\| \\
&\quad + \theta^k L\|(XS)^{-\frac{1}{2}}\| \cdot \|S\| \cdot \|(x^0 - x^*)\| \\
&\leq \frac{1}{\min\limits_i (x_i s_i)^{\frac{1}{2}}}(1+\beta_1+\nu/n)x^T s \\
&\quad + (1+L)\frac{1}{\min\limits_i (x_i s_i)^{\frac{1}{2}}}\theta^k \frac{(2+\chi+\tau)(x^k)^T s^k}{\theta^k \rho_0} n^{\frac{1}{2}}\rho \\
&\leq \left(1 + (2+\chi+\tau)(1+L)n^{\frac{1}{2}}\frac{\rho}{\rho_0} + \beta_1 + \nu/n\right)\frac{(x^k)^T s^k}{\min\limits_i (x_i s_i)^{\frac{1}{2}}},
\end{aligned}
$$

where the second inequality follows from Lemma 3.1.

Similarly, we can also get

$$
\begin{aligned}
\|D\Delta s(\alpha^k)\| &= \|D(\widehat{\Delta s_1} + \widehat{\Delta s_2} + \widehat{\Delta s_3} + \widehat{\Delta s_4} \\
&\qquad -[\nabla f(x + \alpha^k(\Delta x + \theta^k(x - x^0))) - \nabla f(x + \alpha^k \Delta x)])\| \\
&\leq \alpha^k \left(1 + (2+\chi+\tau)(1+L)n^{\frac{1}{2}}\frac{\rho}{\rho_0} + \beta_1 + \nu/n\right)\frac{(x^k)^T s^k}{\min\limits_i (x_i s_i)^{\frac{1}{2}}}. \qquad \blacksquare
\end{aligned}
$$

Let (x^k, y^k, s^k) be generated by Algorithm 2.1. Under appropriate assumptions, we next show that $(x^k)^T s^k$ does not reduce too much at every iteration. Thus, in our algorithm it does not happen that α^k gets close to 1.

LEMMA 3.3 *Assume that problems (1) and (2) have an optimal point (x^*, y^*, s^*) satisfying $\|(x^*, s^*)\|_\infty \leq \rho$, $(x^{k+1}, y^{k+1}, s^{k+1})$ is generated at the $(k+1)$ st iteration of algorithm 2.1. Then we have*

$$(x^{k+1}, s^{k+1}) > 0, \tag{18}$$

$$(x^{k+1})^T s^{k+1} \geq (1-\alpha^k)(x^k)^T s^k \tag{19}$$

for $k \geq 0$, where α^k is defined as (7).

Before starting the proof, firstly we introduce two auxiliary functions as follows

$$
\begin{aligned}
\varphi_1(\alpha^k) &= \sum_{i=1}^n \left|\frac{\alpha^k \Delta x_i}{x_i^k}\right|^2, \\
\varphi_2(\alpha^k) &= \sum_{i=1}^n \left|\frac{\Delta s_i(\alpha^k)}{s_i^k}\right|^2,
\end{aligned}
$$

which will play an important role in the proof. Obviously, under the condition $(x^k, s^k) > 0$, if $\varphi_1(\alpha^k) < 1$ and $\varphi_2(\alpha^k) < 1$ are valid, then we see that $(x^{k+1}, s^{k+1}) = (x^k + \alpha^k \Delta x, s^k + \Delta s(\alpha^k)) > 0$ must be satisfied. Moreover, under the same assumption, $(x^k, s^k) > 0$, we have

$$\begin{aligned}\varphi_1(\alpha^k) &= \sum_{i=1}^{n} \left|\frac{\alpha^k \Delta x_i}{x_i^k}\right|^2 = \sum_{i=1}^{n} (\alpha^k)^2 \left|\frac{D_{ii}}{x_i^k}\right|^2 (D_{ii}^{-1}\Delta x_i)^2 \\ &\le (\alpha^k)^2 \frac{1}{\min\limits_i \{x_i^k s_i^k\}} \|D^{-1}\Delta x\|^2. \end{aligned} \tag{20}$$

And, similarly, we have

$$\varphi_2(\alpha^k) \le (\alpha^k)^2 \frac{1}{\min\limits_i \{x_i^k s_i^k\}} \|D \Delta s(\alpha^k)\|^2. \tag{21}$$

Proof of Lemma 3.3. The proof uses an induction argument. We first consider the case: $k = 0$.

It is obvious that $(x^0, s^0) > 0$. Then we have by (17)

$$A(\Delta x + (x^0 - x^*)) = 0.$$

Under the assumption that the problems (1) and (2) have an optimal point (x^*, y^*, s^*) satisfying $\|(x^*, s^*)\|_\infty \le \rho$.
Let $(\widehat{\Delta x}, \widehat{\Delta s}) = (\alpha^0(\Delta x + (x^0 - x^*)), \nabla f(x^k + \widehat{\Delta x}) - \nabla f(x^k) - \alpha^k A^T \Delta y)$ satisfy (14) with

$$\begin{aligned} p = & -\alpha^0(X^0 s^0 - \beta_1 \mu^0 e) + \alpha^0 \theta^0 S^0 (x^0 - x^*) \\ & + X^0(\nabla f(x^0 + \alpha^0(\Delta x + \theta^0(x^0 - x^*))) - \nabla f(x^0 + \alpha^0 \Delta x)) + \alpha^0 r^0. \end{aligned}$$

Now let $(\widehat{\Delta x_1}, \widehat{\Delta s_1})$,$(\widehat{\Delta x_2}, \widehat{\Delta s_2})$, $(\widehat{\Delta x_3}, \widehat{\Delta s_3})$ and $(\widehat{\Delta x_4}, \widehat{\Delta s_4})$ satisfy the system (14) when p is $-\alpha^0(X^0 s^0 - \beta_1 \mu^0 e)$, $\alpha^0 \theta^0 S^0 (x^0 - x^*)$, $X^0(\nabla f(x^0 + \alpha^0(\Delta x + \theta^0(x^0 - x^*))) - \nabla f(x^0 + \alpha^0 \Delta x))$ and $\alpha^0 r^0$, respectively. Thus, as in Lemma 3.2 we also have

$$\begin{aligned} \|D^{-1}\Delta x\| \le & \|(X^0 S^0)^{-\frac{1}{2}}\|(1+\beta_1)(x^0)^T s^0 + \|(X^0 S^0)^{-\frac{1}{2}}\|\nu (x^0)^T s^0/n \\ & + \theta^0 \|(X^0 S^0)^{-\frac{1}{2}}\| \cdot \|X^0\| \cdot \|(s^0 - s^*)\| \\ & + \theta^0 L \|(X^0 S^0)^{-\frac{1}{2}}\| \cdot \|S^0\| \cdot \|(x^0 - x^*)\| \\ \le & (1 + \beta_1 + \nu/n) \frac{(x^0)^T s^0}{\min\limits_i (x_i^0 s_i^0)^{1/2}} + (1+L) \frac{1}{\min\limits_i (x_i^0 s_i^0)^{1/2}} \frac{(x^0)^T s^0}{\rho_0} \rho n^{1/2} \\ \le & \left(1 + \beta_1 + \nu/n + \frac{\rho}{\rho_0}(1+L) n^{1/2}\right) \frac{(x^0)^T s^0}{\min\limits_i (x_i^0 s_i^0)^{1/2}}. \end{aligned}$$

And similarly,

$$\|D\Delta s(\alpha^0)\| \leq \alpha^0 \left(1+\beta_1+\nu/n+\frac{\rho}{\rho_0}(1+L)n^{1/2}\right)\frac{(x^0)^T s^0}{\min\limits_i (x_i^0 s_i^0)^{1/2}}.$$

Thus, by the facts that $\tau > 1$, $\beta_1+\nu/n < \beta_2 < 1$, $\nu < 1$, (7) with $k = 0$, (20) and (21) we easily see that $\varphi_1(\alpha^0) < 1$ and $\varphi_2(\alpha^0) < 1$ hold. That is to say,

$$(x^1, s^1) = (x^0 + \alpha^0\Delta x, s^0 + \Delta s(\alpha^0)) > 0.$$

Furthermore, we have

$$\begin{aligned}
(x^1)^T s^1 &= (x^0+\alpha^0\Delta x)^T(s^0+\Delta s(\alpha^0))\\
&= (x^0)^T s^0 + (\alpha^0(s^0)^T\Delta x + (x^0)^T\Delta s(\alpha^0)) + \alpha^0(\Delta x)^T\Delta s(\alpha^0)\\
&= (x^0)^T s^0 - \alpha^0(1-\beta_1)(x^0)^T s^0 + \alpha^k(\sum_{i=1}^n r_i^0) + \alpha^0(\Delta x)^T\Delta s(\alpha^0)\\
&= (1-\alpha^0)(x^0)^T s^0 + \alpha^0\beta_1(x^0)^T s^0 + \alpha^k(\sum_{i=1}^n r_i^0) + \alpha^0(\Delta x)^T\Delta s(\alpha^0)\\
&\geq (1-\alpha^0)(x^0)^T s^0 + \alpha^0(\beta_1-\nu/n)(x^0)^T s^0 - \alpha^0\|D^{-1}\Delta x\|\cdot\|D\Delta s(\alpha^0)\|.
\end{aligned}$$

Then, by the definition of α^0, the estimates of $\|D^{-1}\Delta x\|$ and $\|D\Delta s(\alpha^0)\|$ above, $0 < \beta_1 < 1$, $0 < \sigma < 1$ and $\sigma\beta_1 - 2\nu < \beta_1 - \nu/n < \beta_2 < 1$ we see that for α^0 with the form of (7) the following is satisfied

$$\alpha^0\beta_1(x^0)^T s^0 - \alpha^0\|D^{-1}\Delta x\|\cdot\|D\Delta s(\alpha^0)\| \geq 0.$$

Thus, we have completed the proof of the case $k = 0$: $(x^1, s^1) > 0$ and $(x^1)^T s^1 \geq (1-\alpha^0)(x^0)^T s^0$.

We can now assume that it is valid that

$$(x^k, s^k) > 0, \tag{22}$$

$$(x^k)^T s^k \geq (1-\alpha^{(k-1)})(x^{k-1})^T s^{k-1} \tag{23}$$

for any integer $k \geq 0$.

We next consider the case of $k+1$. By the hypothesis of this lemma, (22) and (23) we see that Lemmas 3.1 and 3.2 hold. Thus, by Lemma 3.2, (7), (20) and (21) we easily obtain that $\varphi_1(\alpha^k) < 1$ and $\varphi_2(\alpha^k) < 1$ hold, which implies that

$$(x^{k+1}, s^{k+1}) > 0. \tag{24}$$

In addition, we have

$$
\begin{aligned}
(x^{k+1})^T s^{k+1} &= (x^k + \alpha^k \Delta x)^T (s^k + \Delta s(\alpha^k)) \\
&= (x^k)^T s^k + (\alpha^k (s^k)^T \Delta x + (x^k)^T \Delta s(\alpha^k)) + \alpha^k (\Delta x)^T \Delta s(\alpha^k) \\
&= (x^k)^T s^k - \alpha^k (1-\beta_1)(x^k)^T s^k + \alpha^k (\sum_{i=1}^n r_i^k) + \alpha^k (\Delta x)^T \Delta s(\alpha^k) \\
&= (1-\alpha^k)(x^k)^T s^k + \alpha^k \beta_1 (x^k)^T s^k + \alpha^k (\sum_{i=1}^n r_i^k) + \alpha^k (\Delta x)^T \Delta s(\alpha^k) \\
&\geq (1-\alpha^k)(x^k)^T s^k + \alpha^k (\beta_1 - \nu/n)(x^k)^T s^k - \alpha^k \nu (x^k)^T s^k / n \\
&\quad + \alpha^k \|D^{-1}\Delta x\| \cdot \|D \Delta s(\alpha^k)\| \\
&\geq (1-\alpha^k)(x^k)^T s^k + \alpha^k (\beta_1 - \frac{\nu}{n})(x^k)^T s^k \\
&\quad - (\alpha^k)^2 \left(1 + (2+\chi+\tau)(1+L) n^{\frac{1}{2}} \frac{\rho}{\rho_0} + \beta_1 + \frac{\nu}{n} \right)^2 \frac{((x^k)^T s^k)^2}{\min_i (x_i s_i)},
\end{aligned}
$$

where the third equality comes from the equation (5), and the last inequality uses Lemma 3.2.

Thus, by the definition (7) of α^k and $\sigma\beta_1 - 2\nu < \beta_1 - \nu/n$, we see that

$$
(\beta_1 - \frac{\nu}{n})(x^k)^T s^k - \alpha^k \left(1 + (2+\chi+\tau)(1+L) n^{\frac{1}{2}} \frac{\rho}{\rho_0} + \beta_1 + \frac{\mu}{n} \right)^2 \frac{((x^k)^T s^k)^2}{\min_i (x_i s_i)} > 0,
$$

which implies that it holds that $(x^{k+1})^T s^{k+1} \geq (1-\alpha^k)(x^k)^T s^k$.

Hereunto, we have completed the proof of Lemma 3.3. ∎

From the proof above we note that, for $k = 0$, $(\Delta x, \Delta s(\alpha^k))$ also satisfies (12) and (13).

In the next lemma we show that the complementarity gap of the problems (1) and (2) will decrease at each iteration of Algorithm 2.1, and it is also shown that the sequence $\{(x^k, y^k, s^k)\}$ remains in the neighborhood $\mathcal{N}$ of the central path.

LEMMA 3.4 *Suppose that $(x^{k+1}, y^{k+1}, s^{k+1})$ is generated by Algorithm 2.1, $(x^k, s^k) \in \mathcal{N}$, and the assumptions of Lemmas 3.1, 3.2, 3.3 hold. Then the following statements are true*

$$
(x^{k+1})^T s^{k+1} \leq (1 - \alpha^k (1-\beta_2))(x^k)^T s^k, \tag{25}
$$

$$
(x^{k+1}, s^{k+1}) \in \mathcal{N}, \tag{26}
$$

where α^k is defined as in (7).

Proof. First, we have by (5)

$$\begin{aligned} X^{k+1}s^{k+1} &= (X^k + \alpha^k \Delta X)(s^k + \Delta s(\alpha^k)) \\ &= X^k s^k + (X^k \Delta s(\alpha^k) + \alpha^k S^k \Delta x) + \alpha^k \Delta X \Delta s(\alpha^k) \\ &= X^k s^k - \alpha^k (X^k s^k - \beta_1 \frac{(x^k)^T s^k}{n} e) + \alpha^k r^k + \alpha^k \Delta X \Delta s(\alpha^k), \end{aligned} \tag{27}$$

and

$$(x^{k+1})^T s^{k+1} = (x^k)^T s^k - \alpha^k (1-\beta_1)(x^k)^T s^k + \alpha^k (\sum_{i=1}^n r_i^k) + \alpha^k \Delta x^T \Delta s(\alpha^k). \tag{28}$$

We next seek to prove the inequality (25).

$$\begin{aligned} &(1-\alpha^k(1-\beta_2))(x^k)^T s^k - (x^{k+1})^T s^{k+1} \\ &= \alpha^k(\beta_2-\beta_1)(x^k)^T s^k - \alpha^k \Delta x^T \Delta s(\alpha^k) - \alpha^k (\sum_{i=1}^n r_i^k) \\ &\geq \alpha^k(\beta_2-\beta_1)(x^k)^T s^k - \alpha^k \|r^k\|_1 - \alpha^k \|D^{-1}\Delta x\| \cdot \|D \Delta s(\alpha^k)\| \\ &\geq \alpha^k(\beta_2-\beta_1-\nu/n)(x^k)^T s^k \\ &\quad -(\alpha^k)^2 \left[\left(1+\beta_1+\nu/n+(2+\chi+\tau)(1+L)n^{\frac{1}{2}} \frac{\rho}{\rho_0} \right)^2 \frac{((x^k)^T s^k)^2}{\min_i x_i^k s_i^k} \right]. \end{aligned}$$

If the right-hand side of the last inequality above is not less than zero, i.e.,

$$\alpha^k \leq \frac{(\beta_2-\beta_1-\nu/n)\min_i x_i^k s_i^k}{\left(1+\beta_1+\nu/n+(2+\chi+\tau)(1+L)n^{\frac{1}{2}}\frac{\rho}{\rho_0}\right)^2 (x^k)^T s^k}, \tag{29}$$

then the inequality (25) must be valid.

By (7) we easily see that the inequality (29) is satisfied. The proof of (25) is terminated.

For (26), from Lemma 3.3, we just need to prove the following inequality

$$\|X^{k+1}s^{k+1} - \frac{(x^{k+1})^T s^{k+1}}{n} e\| \leq \sigma \frac{(x^{k+1})^T s^{k+1}}{n}.$$

Then, by (27) and (28) we obtain

$$\begin{aligned} \|X^{k+1}s^{k+1} - \frac{(x^{k+1})^T s^{k+1}}{n} e\| &\leq (1-\alpha^k)\|X^k s^k - \frac{(x^k)^T s^k}{n} e\| \\ &\quad + \alpha^k \|r^k - \frac{\sum_{i=1}^n r_i^k}{n} e\| + \alpha^k \|\Delta X \Delta s(\alpha^k) - \frac{\Delta x^T \Delta s(\alpha^k)}{n} e\|. \end{aligned}$$

By the fact that $\dfrac{\Delta x^T \Delta s(\alpha^k)}{n} e$ and $r^k - \left(\sum_{i=1}^{n} r_i^k / n \right) e$ are the orthogonal projections of $\Delta X \Delta s(\alpha^k)$ and r^k on the one dimensional subspace by e respectively, we obtain that

$$\|X^{k+1} s^{k+1} - \frac{(x^{k+1})^T s^{k+1}}{n} e\| \leq (1-\alpha^k)\sigma \frac{(x^k)^T s^k}{n} + \alpha^k \|r^k\| + \alpha^k \|\Delta X \Delta s(\alpha^k)\|.$$

Thus,

$$\begin{aligned}
&\sigma \frac{(x^{k+1})^T s^{k+1}}{n} - \|X^{k+1} s^{k+1} - \frac{(x^{k+1})^T s^{k+1}}{n} e\| \\
&\geq \sigma \frac{(x^{k+1})^T s^{k+1}}{n} - ((1-\alpha^k)\sigma \frac{(x^k)^T s^k}{n} + \alpha^k \|r^k\| + \alpha^k \|\Delta X \Delta s(\alpha^k)\|) \\
&\geq \alpha^k \sigma \beta_1 \frac{(x^k)^T s^k}{n} \\
&\quad -\alpha^k (\|r^k\| + |(\sum_{i=1}^{n} r_i^k)/n|) - \alpha^k (\|\Delta X \Delta s(\alpha^k)\| + |\frac{\Delta x^T \Delta s(\alpha^k)}{n}|) \\
&\geq \alpha^k (\sigma\beta_1 - 2\nu) \frac{(x^k)^T s^k}{n} - 2\alpha^k \|D^{-1}\Delta x\| \cdot \|D \Delta s(\alpha^k)\| \\
&\geq \alpha^k (\sigma\beta_1 - 2\nu) \frac{(x^k)^T s^k}{n} \\
&\quad -2(\alpha^k)^2 \left(1 + \beta_1 + \nu/n + (2+\chi+\tau)(1+L) n^{\frac{1}{2}} \frac{\rho}{\rho_0}\right)^2 \frac{((x^k)^T s^k)^2}{\min_i x_i^k s_i^k}.
\end{aligned}$$

By (7), we see that the following inequality holds

$$\begin{aligned}
(\sigma\beta_1 - 2\nu) \frac{(x^k)^T s^k}{n} - 2\alpha^k \left(1 + \beta_1 + \nu/n + (2+\chi+\tau)(1+L) n^{\frac{1}{2}} \frac{\rho}{\rho_0}\right)^2 \times \\
\frac{((x^k)^T s^k)^2}{\min_i x_i^k s_i^k} \geq 0.
\end{aligned}$$

Thus, it is true that $(x^{k+1}, y^{k+1}, s^{k+1}) \in \mathcal{N}$.

We have obtained all of our desired results. ∎

Using Lemma 3.4, by $(x^0, y^0, s^0) \in \Phi \cap \Omega_1 \cap \mathcal{N}$ we see that $\{(x^k, y^k, s^k)\}_{k \geq 0} \subseteq \mathcal{N}$. Since $\mu = \frac{(x^k)^T s^k}{n}$, by (25) we see that

$$\mu = \prod_{i=0}^{k} (1 - \alpha^i (1-\beta_2))(x^0)^T s^0 \to 0$$

as $k \to \infty$.

Now we are in a position to provide the complexity result of Algorithm 2.1.

THEOREM 3.1 *Suppose that the problems (1) and (2) have an optimal point in Φ. Then Algorithm 2.1 will terminate in at most*

$$\left\lceil \frac{\Xi}{-\ln(1-\tilde{\alpha}(1-\beta_2))} \right\rceil \tag{30}$$

iterations, where

$$\tilde{\alpha} = \left(\frac{\rho_0}{\rho}\right)^2 \frac{\min\{(\sigma\beta_1 - 2\nu), (\beta_2 - \beta_1 - \nu/n)\}(1-\sigma)}{2(2+\chi+\tau)^2\ n^2 \left(1+(1+L)n^{\frac{1}{2}}\right)^2}, \tag{31}$$

$$\Xi = \max\left\{\ln\frac{(x^0)^T s^0}{\varepsilon}, \ln\frac{\|Ax^0 - b\|}{\varepsilon_p}\right\}, \tag{32}$$

and $\lceil \xi \rceil$ denotes the least integer which is not less than ξ.

Proof. By Lemma 2.1, we see that $\alpha^k \geq \tilde{\alpha}$.

Using Lemma 3.4, we have

$$\begin{aligned}
&(x^{k+1})^T s^{k+1} \leq (1-\alpha^k(1-\beta_2))(x^k)^T s^k \leq (1-\widetilde{\alpha}(1-\beta_2))(x^k)^T s^k,\\
&\|Ax^{k+1} - b\| = (1-\alpha^k)\|Ax^k - b\| \leq (1-\widetilde{\alpha})\|Ax^k - b\|\\
&\quad \leq (1-\widetilde{\alpha}(1-\beta_2))\|Ax^k - b\|.
\end{aligned}$$

Then we easily obtain the result of Theorem 3.1. ∎

From all the proof above, we easily have

THEOREM 3.2 *If we insert "**Step 1 ′ If** (x^k, y^k, s^k) **does not satisfy one of (10), (12) and (13), then Algorithm 2.1 terminates.**" between Step 1 and Step 2, and "**Step 3 ′ If** $(\Delta x^k, \Delta s(\alpha^k))$ **with** α^k **not satisfying either (18) or (19), then Algorithm 2.1 terminates.**" between Step 3 and Step 4, of Algorithm 2.1, then Algorithm 2.1 will terminate in at most*

$$\left\lceil \frac{\Xi}{-\ln(1-\tilde{\alpha}(1-\beta_2))} \right\rceil \tag{33}$$

iterations.

Furthermore, if it terminates by Step 1, (x^k, y^k, s^k) is the ε-optimal solution of the primal-dual problem (1) and (2); or else, if it terminates by Step 1′ or Step 3′, there is no optimal solution (x^*, y^*, s^*) of the problems (1) and (2) satisfying $\|(x^*, s^*)\|_\infty \leq \rho$.

Proof. If the algorithm is terminated by Step 1 at the kth iteration, then the iterative point (x^k, y^k, s^k) satisfies our requirements, so we say that our desired approximate optimal point has been obtained.

On the other hand, if the algorithm is terminated by Step 1′ or Step 3′, we next use a contradiction argument to show that there is no optimal solution

(x^*, y^*, s^*) of the problems (1) and (2) satisfying $\|(x^*, s^*)\|_\infty \leq \rho$. At first, suppose that there exists an optimal point (x^*, y^*, s^*) satisfying $\|(x^*, s^*)\|_\infty \leq \rho$. Then by Lemmas 3.1-3.4 and Theorem 3.1 we see that (10), (12), (13), (18) and (19) must be valid. There is a contradiction. Thus, we see that in this case there is no optimal solution (x^*, y^*, s^*) of the problems (1) and (2) satisfying $\|(x^*, s^*)\|_\infty \leq \rho$. ∎

Acknowledgement

The authors would like to thank the referees for their valuable suggestions and careful reading of the manuscript.

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