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Empirical justification of the uncertain equivalence method*

by

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Abstract: The uncertain equivalence method (UE) is a newly proposed technique for elicitation of 1-D utilities in the case of monotonic preferences. Previous publications argue that the rationale behind the introduction of this technique is that UE estimates are not influenced by certainty effect, UE elicits points that well describe the curvature of the utility function, which is somewhat closer to the true function than the one of the lottery equivalence method (LE), and there is no increase in the width of the elicited UE uncertainty intervals compared to those of the certainty equivalence method (CE). This paper analyzes these assumptions quantitatively on the basis of empirical data from 104 volunteers who constructed their utility functions over monetary prizes using CE, LE and UE. The data was analyzed with the help of four one-tail statistical tests for paired samples. Results showed that: 1) UE results are not influenced by the certainty effect, unlike CE; 2) the UE utility function is more curved than that of LE, but that might be associated with the better selection of approximation nodes and not with the certainty effect; 3) the length of the UE uncertainty intervals is greater than that of the CE intervals, perhaps because of higher complexity of the method, but the increase is only by 30%.

Keywords: one-dimensional utility, monotonic preferences, elicitation methods, statistical tests.

1. Introduction

A typical tool for decisions under risk is utility theory (von Neumann and Morgenstern, 1947), which applies to problems from management, finance and industry (French, 1993; Chalev, Pavlov and Vassiliev, 1990). Central to utility theory is the expected utility rule to choose between alternatives. It is also the basis of most subjective elicitation techniques. Expected utility has been criticized as a decision tool due to empirical deviations, biased elicitation results,

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and contradictions, e.g. Ellsberg's ambiguity aversion (Ellsberg, 1961), inconformity of elicitation results from different methods (McCord and de Neufville, 1983), axiomatic violations (Machina, 1982, 1983), etc. Expected utility is also related to the way people should perceive utilities, and not the way they actually do, thus lacking descriptiveness (Starmer, 2000). That is why other theories for rational choice exist, e.g. regret theory (Loomes and Sugden, 1982), rank-dependent utility (Quiggin, 1981), disappointment aversion (Gul, 1991), as well as descriptive theories like cumulative prospect theory (Tversky and Kahneman, 1992), gambling effect models (Bleichrodt and Schmidt, 2002; Diecidue, Schmidt and Wakker, 2004), etc.

Utility theory models each risky alternative as a lottery – a set of disjoint events associated with a consequence (a prize). The alternatives to choose from are organized in a lottery set L, the prizes forming the set X. Ordinary lotteries apply when L and X are finite. They generate *holistic* prizes x_r (for $r = 1, 2, \ldots,\ t$). Ordinary lotteries are denoted $l = < p_1, x_1; p_2, x_2; \ldots; p_t, x_t >$, where $p_r \geq 0$ is the probability to receive the prize x_r, and $\sum_{r=1}^{t} p_r = 1$. Generalized lotteries of I, II and III type (Pratt, Raiffa and Schlaifer, 1995; Tenekedjiev, 2004) and semi-generalized lotteries of I type (Nikolova, Dimitriakiev and Tenekedjiev, 2006) apply when either X or L (or both) are infinite. However, in all cases it is necessary to construct a utility function over a continuous 1-D set X. Once this is done, lotteries may be ranked in descending order of expected utility.

Many methods are proposed to elicit 1-D utility. Usually they require a dialogue with the decision maker (DM). The scheme of elicitation requires a choice between two gambles. A parameter in one of the gambles changes on an iterative basis, usually according to the bisection method, until the DM is indifferent between the two options. At that point the expected utility rule yields a point estimate of the required utility value (or utility quantile). The procedure is repeated for several points on the curve, and the utility function is constructed by interpolation or approximation based on the elicited points.

If the classical elicitation assumption holds (i.e. that the DM makes choices according to expected utility, Bleichrodt et al., 2001), then all methods should produce the same result. Unfortunately, the elicited values deviate from the true utilities of the DM due to biases, which differ according to method. Several requirements may be imposed to the elicitation techniques to ensure that they provide adequate assessment of the true preferences of the DM (Nikolova, 2006): **R1:** elicited nodes are minimally influenced by biases; **R2:** nodes are chosen so that the interpolated/approximated curve is maximally close to the true one; **R3:** the number of elicited nodes is minimal; **R4:** the number of elicited nodes is preliminarily defined; **R5:** the method is clear enough to the DM; **R6:** the method allows simple elicitation of the uncertainty interval; **R7:** the method can be applied to nominal prizes; **R8:** the method may be applied to elicit scaling constants (constants in a multi-dimensional utility function); **R9:** the method may be applied separately to elicit non-monotonic preferences.

Classical elicitation techniques are the probability equivalence (PE), certainty equivalence (CE), and lottery equivalence (LE) methods. PE compares a certain prize and a reference lottery with varying probability. The reference lottery $< x_{best}(p)x_{worst} >$ gives two prizes – the most preferred x_{best} with probability p and the least preferred – x_{worst} with probability $1-p$. CE compares a fixed-probability reference lottery and a changing prize, that is why this method applies for continuous X. LE proposes to compare two lotteries, which eliminates the illusion of riskless prospects that exists in PE and CE.

Tenekedjiev, Nikolova and Pfliegl (2006) proposed the uncertain equivalence method (UE), which compares two lotteries. The rationale behind UE is that it combines the positive psychological and theoretical elicitation effects of CE and LE, since it overcomes the certainty effect, and guarantees better approximation of the utility function. Initial discussion on the UE method as well as qualitative comparison with other methods is proposed in Tenekedjiev, Nikolova and Pfliegl (2006) and Nikolova (2006). Another modern technique is the trade-off method (TO) (Wakker and Deneffe, 1996), which is least susceptible to biases, such as probability transformation and/or loss aversion (Bleichrodt, Pinto and Wakker, 2001).

This paper focuses on quantitative comparison of UE with LE and CE. The main objective is to justify the advantages of the UE method compared to LE and CE. Previous publications have suggested that UE estimates are not influenced by the certainty effect, better describe the curvature of the utility function than LE and although the method is rather complex it does not lead to increase in the width of the uncertainty intervals as in LE. The results of an empirical study are employed to justify these assumptions with the help of one-tail statistical tests for paired samples. In what follows, Section 2 shortly introduces the essence of the expected-utility based techniques for elicitation of utility. Section 3 focuses on how the real DM elicits utilities with the help of UE and on what the form of the resulting utility estimates is. Section 4 discusses analytical approaches to construct the utility function and their relation to risk attitude of the DM and the ways to measure it. Qualitative comparison of the elicitation methods is proposed in Section 5. Section 6 presents quantitative comparison of UE, LE and CE on the basis of empirical data and statistical tests.

2. Utility elicitation techniques over a 1-D set of prizes

Let X be a bounded continuous set of prizes (e.g. monetary profits). In most cases, preferences are strictly increasing on X, i.e. $x_i \succ x_j \Leftrightarrow x_i > x_j$; $x_i, x_j \in X$ ($\succ$ is the binary relation "more preferred than"). Let $x_{best} = \sup(X)$, $x_{worst} = \inf(X)$. Then a 1-D utility function $u(.)$ has to be constructed on the interval $[x_{worst}; x_{best}]$. According to the probability interpretation of utility, $u(x_{best})=1$, $u(x_{worst}) = 0$.

It is impossible to determine the utilities of all prizes in the interval $[x_{worst}; x_{best}]$ and that is why only several nodes of the utility curve are elicited and the utility function is interpolated or approximated on this basis. The coordinates of the elicited nodes are $(x_{u_l}; u_l)$, $l = 1, 2, \ldots, z$, where x_{u_l} and u_l are, respectively, a utility quantile and a utility quantile index. The end nodes are always $(x_{u_1}; u_1) = (x_{worst}; 0)$ and $(x_{u_z}; u_z) = (x_{best}; 1)$. The key aspect is the way to choose the nodes on which to build the utility function.

PE is the most intuitive method of eliciting utilities. It defines z utility quantiles $x_{u_1} = x_{worst}, x_{u_2}, \ldots, x_{u_l}, \ldots, x_{u_z} = x_{best}$ and finds their corresponding quantile indices $\hat{u}_1 = 0, \hat{u}_2, \ldots, \hat{u}_l, \ldots,\ \hat{u}_z = 1$. The DM compares the prize x_{u_i} and a p-probability reference lottery. The value of p changes until indifference is reached, and then it equals $\hat{u}_i$. CE chooses z utility quantile indices $u_1 = 0,\ u_2, \ldots, u_l, \ldots, u_z = 1$ and finds their corresponding utility quantiles $\hat{x}_{u_1} = x_{worst}, \hat{x}_{u_2}, \ldots, \hat{x}_{u_l}, \ldots, \hat{x}_{u_z} = x_{best}$. The DM compares a fixed prize and a reference lottery with probability that equals to u_i. She/he is then asked to find $\hat{x}_{u_l}$ that equals, in preference, the reference lottery.

If the elicited nodes do not describe well the utility function, then the interpolation error could be significant (i.e. the deviation of the interpolated points from their values had they have been directly elicited). This is typical for PE, since the selected points may fail to describe the curvature of the utility function (Clemen, 1996, p. 511), which usually slows down the analysis. CE overcomes this problem as it selects nodes guaranteeing that the utility function will be presented with a maximum number of typical points. Thus CE, generates lower interpolation error than PE. On the other hand, PE is easier to apply under different type of preferences, whereas CE is highly dependent on the way preferences change over X.

Empirical studies proved that the CE utility function had higher curvature than the PE utility, which influences attitude towards risk (Hershey, Kunkreuther and Shoemaker, 1982). This is partially caused by the difficulty of comparing lotteries with varying probabilities, and the overestimation of certain prospects. This is called the certainty effect and it affects the results of PE and CE.

McCord and de Neufville argue that comparison between a lottery and a certain quantity leads to incorrect elicitation, and the setup is by itself biased (McCord and de Neufville, 1986). Again, z utility quantiles $x_{u_1} = x_{worst}, x_{u_2}, \ldots, x_{u_l}, \ldots, x_{u_z} = x_{best}$ are selected and their corresponding quantile indices $\hat{u}_1 = 0,\ \hat{u}_2, \ldots, \hat{u}_l, \ldots, \hat{u}_z = 1$ are elicited. The DM compares a reference lottery with probability P, and a lottery between the analyzed prize x_{u_l} and x_{worst} with probabilities p and $(1-p)$. Indifference is reached by changing P, at which moment $\hat{u}_l = P/p$. The LE utility function is not as curved as that of PE and CE due to the elimination of the certainty effect. As a result, decisions based on the LE utility are less risk-averse when dealing with gains, and less risk-prone when dealing with losses. LE estimates are considered to be closer to the true opinion of the DM, although being more uncertain due to the complexity of the scheme.

UE chooses z utility quantile indices $u_1 = 0, u_2, \ldots, u_l, \ldots, u_z = 1$ and finds their corresponding utility quantiles $\hat{x}_{u_1} = x_{worst}, \hat{x}_{u_2}, \ldots, \hat{x}_{u_l}, \ldots, \hat{x}_{u_z} = x_{best}$. The DM compares two lotteries. The first is a reference lottery with probability p times the value of the selected utility u_l. The other lottery gives either x_{u_l} with probability p or x_{worst} *with probability* $1-p$. The DM balances preferences over the two lotteries by changing x_{u_l} in order to find the utility quantile corresponding to u_l.

TO is one of the best techniques for utility elicitation. It finds an unknown number z of points with utility quantile numbers $u_1 = 0$, $u_2 = 1/(z-1), \ldots, u_l = (l-1)/(z-1), \ldots, u_z = 1$. It initially defines a probability p and two artificial consequences G and g, subject to the conditions $0 < p < 1$ and $G \succ g \succ x_{best}$. The utility quantiles $\hat{x}_{u_1} = x_{worst}$, $\hat{x}_{u_2} = x_{u_2}, \ldots, \hat{x}_{u_l}, \ldots, \hat{x}_{u_z}$ are elicited recursively until condition $\hat{x}_{u_z} \succsim x_{best}$ is satisfied. For $l = 2, 3, \ldots, z$ the DM compares two lotteries – the first gives G with probability p, and $\hat{x}_{u_{l-1}}$ otherwise, whereas the second gives g with probability p, and $\hat{x}_{u_l}$ otherwise. Indifference is reached by changing $\hat{x}_{u_l}$. As a result the method generates a set of nodes, equally spaced in terms of utility.

PE, CE, LE, TO and UE are based on preferences over gambles. Tenekedjiev et al. (2006) present a generic theoretical setup of the expected-utility elicitation techniques, based on the preferential equation

$$l_1 \sim l_2, \quad (1)$$

where $l_1 =< P_1, A_1; (1-P_1), B_1 >$ and $l_2 =< P_2, A_2; (1-P_2), B_2 >$. Five of the parameters are fixed, and the equation is solved according to the sixth during a dialog with the DM and usually with the help of the bisection method (Press et al., 1992). Once indifference is reached

$$P_1 \times u(A_1) + (1-P_1) \times u(B_1) = P_2 \times u(A_2) + (1-P_2) \times u(B_2). \quad (2)$$

Table 1 presents the equations and the required results from each of the discussed methods.

PE, CE, LE, TO and UE use the expected utility rule to generate utility estimates. There are also non-expected utility based techniques for utility elicitation, such as the cross-modality matching (Roelofsma and Schut, 2004), adaptive elicitation approach (Chajewska, Koller and Parr, 2000), etc.

3. Utility elicitation using UE

Ideal DMs have infinite discriminating abilities between alternatives, and their preferences obey a set of rules for rationality (French and Insua, 2000). As a result, they identify a single root of the preferential equations from Table 1. As discussed in Nikolova et al. (2005), real DMs often do not conform to the rationality criteria, and are called fuzzy-rational DMs. They elicit utilities in

Table 1. Realization of the 1-D utility elicitation methods

Method	*Preferential equation*	*Root*	*Selected parameters*	*Elicited nodes*
PE	$< x_{best}(P_1)x_{worst} >\sim x_{u_l}$ $l = 2, 3, \ldots, z-1$	$P_1 = P*$	z and $x_{u_1} = x_{worst}, x_{u_2}, \ldots,$ $x_{u_l}, \ldots, x_{u_z} = x_{best}$	$\hat{u}_l = u\,(x_{u_l}) = P*$
CE	$< x_{best}(u_l)x_{worst} >\sim A_2$ $l = 2, 3, \ldots, z-1$	$A_2 = A^*$	z and $u_2 < \ldots < u_l < \ldots < u_{z-1}$	$u_i = u(\hat{x}_{u_l}) = u(A^*)$
LE	$< x_{best}(P_1)x_{worst} >\sim$ $< p, x_{u_l}; (1-p), x_{worst} >$ $l = 2, 3, \ldots, z-1$	$P_1 = P*$	z and $x_{u_2} < \ldots < x_{u_l} < \ldots <$ $x_{u_{z-1}}$	$\hat{u}_l = u(x_{u_l}) = P*/p$
UE	$< x_{best}(p \times u_l)x_{worst} >\sim$ $< p, A_2; (1-p), x_{worst} >$ $l = 2, 3, \ldots, z-1$	$A_2 = A^*$	z, probability p (usually $p = 2/3$) and $u_2 < \ldots < u_l < \ldots < u_{z-1}$	$u_l = u(\hat{x}_{u_l}) = u(A^*)$
TO	$< p, G; (1-p), \hat{x}_{u_{l-1}} >\sim$ $< p, g; (1-p), B_2 >$ $l = 2, 3, \ldots, z-1$	$B_2 = B*$	z, probability p, prizes G and g where $G \succ g \succ x_{best}$	$u_l = (l-1)/(z-1) =$ $u(\hat{x}_{u_l}) = u(B*)$

the form of uncertainty intervals. Whereas ideal DMs may solve the preferential equations in Table 1 with the help of the bisection method (Press et al., 1992), fuzzy rational DMs should use the triple bisection method (Tenekedjiev, Nikolova and Dimitriakiev, 2004) to elicit uncertainty intervals. If UE is used to elicit utilities, then the fuzzy rational DM should identify the following (in the case of monotonically increasing preferences): 1) the greatest possible x_{down}, such that $< x_{best}(p \times u_i)x_{worst} >\succ< p, x_{down}; (1-p), x_{worst} >$; 2) the smallest possible x_{up}, such that $< p, x_{up}; (1-p), x_{worst} >\succ< x_{best}(p\times u_i)x_{worst} >$. Then $A^* \in (x_{down}; x_{up})$, whereas a point estimate can be $A^* = (x_{down}+x_{up})/2$. Then the uncertainty interval of the root is

$$\hat{x}_{u_l} \in [x_{down}; x_{up}] \equiv [\hat{x}^d_{u_l}; \hat{x}^u_{u_l}], \quad l = 2, 3, \dots, z-1. \tag{3}$$

It is obvious that for the fuzzy rational DM, the relations $\sim$ (indifference) and $\succsim$ (weak preference) are not transitive, and the relations $\succ$ (strict preference) and $\sim$ are not mutually transitive. Let $x_{up} \geq x_2 > x_1 \geq x_{down}$. Then for the fuzzy rational DM:

a) $< p, x_1; (1-p), x_{worst} >\sim< x_{best}(p\times u_l)x_{worst} >$, $< x_{best}(p\times u_l)x_{worst} >\sim < p, x_2; (1-p), x_{worst} >$ and $< p, x_2; (1-p), x_{worst} >\succ< p, x_1; (1-p), x_{worst} >$, although transitivity of $\sim$ assumes $<p, x_2; (1-p), x_{worst} > \sim <p, x_1; (1-p), x_{worst} >$;

b) $< p, x_1; (1-p), x_{worst} >\succsim< x_{best}(p\times u_l)x_{worst} >$, $< x_{best}(p\times u_l)x_{worst} >\succsim < p, x_2; (1-p), x_{worst} >$ and $< p, x_2; (1-p), x_{worst} >\succ< p, x_1; (1-p), x_{worst} >$, although transitivity of $\succsim$ assumes $< p, x_1; (1-p), x_{worst} >\sim< p, x_2; (1-p), x_{worst} >$;

c) $< p, x_2; (1-p), x_{worst} >\succ< p, x_1; (1-p), x_{worst} >$, $< p, x_1; (1-p), x_{worst} > \sim< x_{best}(p \times u_l)x_{worst} >$ and $< p, x_2; (1-p), x_{worst} >\sim< x_{best}(p \times u_l)x_{worst} >$, although mutual transitivity of $\succ$ and $\sim$ assumes $< p, x_2; (1-p), x_{worst} >\succ < x_{best}(p \times u_l)x_{worst} >$;

d) $< x_{best}(p\times u_l)x_{worst} >\sim< p, x_2; (1-p), x_{worst} >$, $< p, x_2; (1-p), x_{worst} > \succ< p, x_1; (1-p), x_{worst} >$ and $< x_{best}(p \times u_l)x_{worst} >\sim< p, x_1; (1-p), x_{worst} >$, although mutual transitivity of $\sim$ and $\succ$ assumes $< x_{best}(p\times u_l)x_{worst} >\succ < p, x_1; (1-p), x_{worst} >$.

If a fuzzy rational DM has monotonically increasing preferences over X, then the uncertainty interval of the root of the UE equation $< x_{best}(p\times u_l)x_{worst} >\sim < p, A_2; (1-p), x_{worst} >$, based on triple bisection, may be elicited using the following algorithm (Tenekedjiev, Nikolova and Pfliegl, 2006):

1) Initialize $p \in (0; 1)$.
2) Initialize x_{down} such that $< x_{best}(p \times u_l)x_{worst} >\succ< p, x_{down}; (1-p), x_{worst} >$.
3) Initialize x_{up} such that $< p, x_{up}; (1-p), x_{worst} >\succ< x_{best}(p\times u_l)x_{worst} >$.
4) Choose a decision precision $\Delta > 0$, such that Δ is the smallest positive number, for which at an arbitrary $A_2 \in [x_{down}, x_{up} - \Delta]$ the DM is not

indifferent between the lotteries $< p, A_2+\Delta; (1-p), x_{worst} >$ and $< p, A_2; (1-p), x_{worst} >$.

5) If $(x_{up} - x_{down}) \leq \Delta$, then go to step 21.
6) Calculate $A_2 = (x_{down} + x_{up})/2$.
7) Ask the DM for her preferences over $< x_{best}(p \times u_l)x_{worst} >$ and $< p, A_2; (1-p), x_{worst} >$.
8) If $< x_{best}(p \times u_l)x_{worst} > \succ < p, A_2; (1-p), x_{worst} >$, put $x_{down} = A_2$, go to step 5.
9) If $< p, A_2; (1-p), x_{worst} > \succ < x_{best}(p \times u_l)x_{worst} >$, put $x_{up} = A_2$, go to step 5.
10) If $< x_{best}(p \times u_l)x_{worst} > \sim < p, A_2; (1-p), x_{worst} >$, put $x_{down,ind} = A_2$ and $x_{up,ind} = A_2$.
11) If $(x_{down,ind} - x_{down}) \leq \Delta$, then go to step 16.
12) Calculate $A_2 = (x_{down,ind} + x_{down})/2$.
13) Ask the DM for her preferences over $< x_{best}(p \times u_l)x_{worst} >$ and $< p, A_2; (1-p), x_{worst} >$.
14) If $< x_{best}(p \times u_l)x_{worst} > \succ < p, A_2; (1-p), x_{worst} >$, put $x_{down} = A_2$, go to step 11.
15) If $< x_{best}(p \times u_l)x_{worst} > \sim < p, A_2; (1-p), x_{worst} >$, put $x_{down,ind} = A_2$, go to step 11.
16) If $(x_{up} - x_{up,ind}) \leq \Delta$, then go to step 21.
17) Calculate $A_2 = (x_{up,ind} + x_{up})/2$.
18) Ask the DM for her preferences over $< x_{best}(p \times u_l)x_{worst} >$ and $< p, A_2; (1-p), x_{worst} >$.
19) If $< p, A_2; (1-p), x_{worst} > \succ < x_{best}(p \times u_l)x_{worst} >$, put $x_{up} = A_2$, go to step 16.
20) If $< x_{best}(p \times u_l)x_{worst} > \sim < p, A_2; (1-p), x_{worst} >$, put $x_{up,ind} = A_2$, go to step 16.
21) The uncertainty interval of the root is $A^* \in [x_{down}; x_{up}]$, whereas the point estimate of A^* is $\hat{x}_{u_i} = (x_{down} + x_{up})/2$. The end.

Tenekedjiev (2007a) presents a modification of this algorithm for the case of strictly decreasing preferences.

4. Analytical approximation and risk attitude in the utility function

The result of applying an elicitation method is a set of elicited nodes. UE, in particular, results in z nodes with the following characteristics:

$$\left| \begin{array}{l} \{(\hat{x}^d_{u_l};\hat{x}^u_{u_l};u_l)/l=1,2,\ldots,z\},\ \text{where} \\ \hat{x}^d_{u_1}\le \hat{x}^d_{u_2}\le \ldots \le \hat{x}^d_{u_z}, \\ \hat{x}^u_{u_1}\le \hat{x}^u_{u_2}\le \ldots \le \hat{x}^u_{u_z}, \\ \hat{x}^d_{u_l}<\hat{x}^u_{u_l},\ \text{for}\ l=2,3,\ldots,z-1, \\ \hat{x}^d_{u_1}=\hat{x}^u_{u_1}, \hat{x}^d_{u_z}=\hat{x}^u_{u_z}, \\ 0=u_1<u_2<\ldots<u_z=1. \end{array} \right. \tag{4}$$

The utility function might be linearly interpolated or approximated (using an analytical form) over these nodes. The objective is to build a function that fits well the data and reflects the true risk attitude of the DM. A proper measure of risk attitude is the local risk aversion function $r(x)$, proposed in Pratt (1964)

$$r(x) = -u''(x)/u'(x). \tag{5}$$

This function not only compares risk attitude of different DMs, but also the risk attitude of a single DM depending on the set of prizes. As discussed in French (1993): 1) most individuals are risk averse over gains and small losses, and their risk averseness decreases with the increase of prizes; 2) most individuals are risk prone over losses, and their risk proneness decreases with the increase of losses. This defines the typical form of the utility function and its corresponding risk aversion function, depicted in Fig. 1.

If the utility function is linearly interpolated, then the left and right bound of $r(x)$ for each inner interpolation node may be calculated using approximation of the derivatives with finite differences:

$$r(x_l^-) = -2\left(\frac{u(x_{l+1})-u(x_l)}{x_{l+1}-x_l} - \frac{u(x_l)-u(x_{l-1})}{x_l-x_{l-1}}\right) \times \frac{x_l-x_{l-1}}{(x_{l+1}-x_{l-1})[u(x_l)-u(x_{l-1})]},\quad \text{for } l=2,3,\ldots,z-1, \tag{6}$$

$$r(x_l^+) = -2\left(\frac{u(x_{l+1})-u(x_l)}{x_{l+1}-x_l} - \frac{u(x_l)-u(x_{l-1})}{x_l-x_{l-1}}\right) \times \frac{x_{l+1}-x_l}{(x_{l+1}-x_{l-1})[u(x_{l+1})-u(x_l)]},\quad \text{for } l=2,3,\ldots,z-1. \tag{7}$$

Here, x_l is the midpoint of the uncertainty interval $[\hat{x}^d_{u_l};\hat{x}^u_{u_l}]$. Since x_l^- and x_l^+ are equal, then the graphics of $r(x)$ would contain vertical sections. If those are replaced by their midpoints, then

$$r(x_l) = \frac{r(x_l^-)+r(x_l^+)}{2},\quad \text{for } l=2,3,\ldots,z-1. \tag{8}$$

The second derivatives in the first and last node may be calculated as left and right second derivatives:

$$r(x_1) = r(x_2^-), \tag{9}$$

$$r(x_z) = r(x_{z-1}^+). \tag{10}$$

Different analytical forms are also proposed to approximate the utility function depending on risk attitude (Clemen, 1996; Keeney and Raiffa, 1993). Nikolova (2007) introduced the following form:

$$u(x) = \frac{\operatorname{arctg}\left[a(x - x_0)\right] - \operatorname{arctg}\left[a(x_1 - x_0)\right]}{\operatorname{arctg}\left[a(x_z - x_0)\right] - \operatorname{arctg}\left[a(x_1 - x_0)\right]}. \tag{11}$$

where $a \in (0, \infty)$, $x_0 \in (-\infty, \infty)$. The risk aversion function for (11) is

$$r(x) = -\frac{u''(x)}{u'(x)} = \frac{2a^2(x - x_0)}{1 + a^2(x - x_0)^2}. \tag{12}$$

The arctg-approximated utility (11) has an analytical inverse function:

$$x(u) = \frac{\operatorname{tg}\left\{u \times \operatorname{arctg}\left[a(x_z - x_o)\right] + (1 - u)\operatorname{arctg}\left[a(x_1 - x_0)\right]\right\}}{a} + x_0. \tag{13}$$

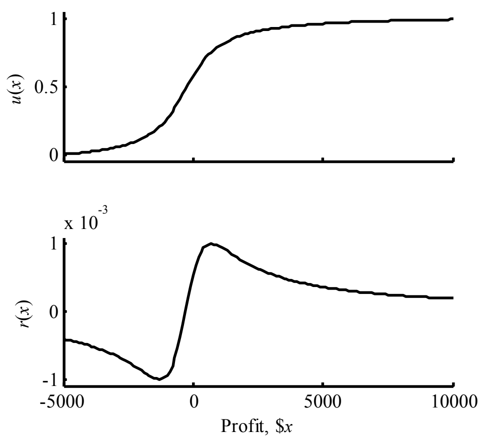


Figure 1. Typical utility function and its corresponding local risk aversion function.

5. Qualitative comparison of the elicitation methods

A comparison of elicitation methods against the nine requirements mentioned before is proposed in Nikolova (2006). Table 2 summarizes the results:

Table 2. Comparison of PE, CE, LE, UE and TO against nine requirements

Requirements	**PE**	**CE**	**LE**	**UE**	**TO**
$R1$	*	*	**	***	****
$R2$	*	**	*	**	**
$R3$	**	***	**	***	*
$R4$	**	**	**	**	*
$R5$	***	***	*	*	*
$R6$	**	**	**	**	*
$R7$	**	*	**	*	*
$R8$	**	*	**	*	*
$R9$	**	*	**	*	*

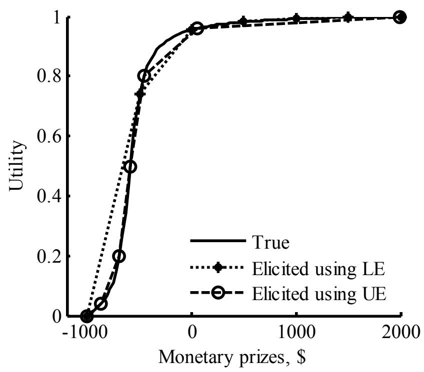


Figure 2. True utility function and its approximation using LE and UE

R1: PE and CE perform the worst (marked by *), since they are influenced by the certainty effect, whereas TO performs the best as it overcomes the distortion of probabilities. LE is worse than UE, since the error in the elicited nodes is $1/p$ times larger than the error in the elicited root. **R2**: the best methods are CE, UE and TO (i.e. the quantile methods that change prize to reach indifference), and not PE and LE (i.e. the index methods that change probability in a lottery to reach indifference). Fig. 2 presents an approximation of the true utility function by UE and LE and shows that both methods deviate from the true curve, but the quantile methods find more appropriate nodes for elicitation. **R3**: the best methods are UE and CE, where the elicited nodes are informative for the curvature of the function; the worst method is TO, as the number of elicited nodes is unknown. In the case of PE or LE, some of the elicited nodes may turn out useless. **R4**: only TO does not fulfil this criterion as the number of elicited nodes is initially unknown. **R5**: the best methods are PE and CE (which compare prize and lottery), and the worst are LE, UE and TO, which compare two lotteries. **R6**: the recurrent methods (TO) do not allow eliciting the uncertainty interval of the root. **R7**: only index methods apply over nominal prizes. **R8**: only index methods allow eliciting scaling constants, since it is necessary to define the utility of a prize. **R9**: non-monotonic preferences do not influence the index methods, whereas quantile methods need to be modified and their results rescaled using an index method.

Empirical research has proven that there is no method that is appropriate for all cases and all DMs. Selection of a method depends both on the type of prizes and the knowledge of the DM in quantitative analysis (Nikolova, 2006). The UE method fits best the case when a DM, well acquainted with utility elicitation analyses, has monotonic preferences over continuous set of prizes, and the same holds for TO. These two methods together can also analyze non-monotonic preferences. CE may be used by beginner-DMs to analyze their monotonic preferences over a continuous set of prizes. LE is suitable for elicitation of monotonic preferences of intermediate and advanced DMs over discrete sets of prizes. It can also be used in combination with UE and TO for elicitation of non-monotonic preferences.

If it is not clear whether UE or TO is the right approach, then the characteristics of the situation must be taken into account. Applying TO requires the following: 1) g and G values must be meaningful; 2) a small number of nodes should be elicited; 3) it is not necessary to elicit uncertainty intervals; 4) the DM and the decision analyst must be well acquainted with the TO since often there is a need to select new G and g and start the analysis all over again. If any of these conditions does not hold, then UE is the better approach.

6. Quantitative comparison of the elicitation methods

The introduction of UE is justified by the expectation that the method generates estimates less influenced by biases and allows for better construction of the utility function. The qualitative comparisons in the previous section, as well as discussions in Tenekedjiev (2007b) suggest that the following three assumptions hold and are the main rationale behind the UE method:

Assumption 1: Decisions based on the CE utility are more risk averse than those made based on the UE utility due to the certainty effect.

Assumption 2: The curvature of the LE utility function is greater than that of the UE utility, since UE is less influenced by biases.

Assumption 3: The UE uncertainty intervals do not differ in length from the ones of CE.

An empirical study was conducted to prove quantitatively these assumptions. A total of 104 volunteers constructed their utility functions over monetary prizes in the interval [–\$10000; \$30000] using CE, LE (at $p = 1/2$) and

UE (at $p = 2/3$). The participants were Bachelor and Master students in Industrial Management at Technical University – Varna (Bulgaria), who are well acquainted with quantitative decision analysis techniques. When CE and UE were used, the participants elicited nine utility quantiles with indices 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 and 0.9. When LE was used, the participants elicited the utilities of nine inner prizes: -\$6000, -\$2000, \$2000, \$6000, \$10000, \$14000, \$18000, \$22000 and \$26000. Estimates from each DM were collected in two two-hour sessions, with a minimum of 48 hours interval between each session. The order of methods used and nodes elicited for each DM was random in order to avoid the anchoring effect (Tversky and Kahneman, 1974). In that way, each DM elicited a total of 27 nodes. Table 3 presents one data sample collected from DM No. 69. Fig. 3 presents the linearly interpolated CE, UE and LE utility functions and their corresponding linearly interpolated local risk aversion functions for DM No. 69, whereas Fig. 4 presents the arctg-approximated utility functions and their corresponding local risk aversions.

Local risk aversion shall be used as a basis to demonstrate the first two assumptions. It shall be calculated both empirically (using (8)) and analytically (using (12)). In both cases, two different measures shall be calculated for each data sample: 1) the mean value of the module of the local risk aversion in all nodes; 2) the maximum of the module of the local risk aversion in all nodes. If empirical local risk aversion is used, then analysis shall only consider values at the inner nodes. The third assumption shall be proven on the basis of the length of the elicited uncertainty intervals.

Four one-tail statistical tests for paired samples shall be used to analyze the statistical significance of the differences, where Δ is a random variable with realizations – the differences between the sample with greater values and the sample with smaller values.

1) Bootstrap mean test to check, whether the mean value of Δ is zero. The null hypothesis is $H_0 : E_\Delta = 0$, whereas the alternative hypothesis is $H_1 : E_\Delta > 0$. Due to the symmetry of the distribution of Δ around the mean at H_0, each synthetic sample of 104 realizations of Δ may be doubled to a synthetic sample of 208 realizations by adding measures symmetrical to the initial ones. After generating N synthetic samples, N synthetic estimates of E_Δ may be calculated, which are doubled to $2N$ synthetic estimates of E_Δ by adding estimates symmetric to the initial ones. The required value of $p_{value,1}$ of the test equals the number of synthetic estimates of E_Δ that exceed the calculated mean value, divided by $2N$ (Efron and Tibshirani, 1993).

2) Bootstrap median test to check, whether the median of Δ is zero. The null hypothesis is $H_0 : \Delta_{0.5} = 0$, and the alternative hypothesis is $H_1 : \Delta_{0.5} > 0$. Similar doubled samples may be formed for the estimates of $\Delta_{0.5}$ as with the estimates of E_Δ in the Bootstrap mean test. The required value $p_{value,2}$ of the test equals the number of synthetic estimates of $\Delta_{0.5}$ that exceed the calculated median, divided by $2N$.

Table 3. Uncertainty intervals of quantiles and quantile indices on the inner nodes of the utility function in the interval [−\$10000; \$30000] for DM No. 69, elicited using CE, UE and LE. The shaded boxes indicate the nodes that have not been elicited by the DM

	u_l	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
CE	$\hat{x}_{u_l}^d$	–10000	–4800	–3400	–3200	–2400	–1400	100	1100	6000	14000	30000
	$\hat{x}_{u_l}^u$	–10000	–2900	–1900	–1400	–1000	200	1800	2600	7800	16000	30000
UE	$\hat{x}_{u_l}^d$	–10000	–7300	–4800	–4000	–2100	–1300	1000	4500	7500	16000	30000
	$\hat{x}_{u_l}^u$	–10000	–5900	–3200	–2200	–100	800	3600	6300	9500	18200	30000

	x_l	**-10000**	**-6000**	**-2000**	**2000**	**6000**	**10000**	**14000**	**18000**	**22000**	**26000**	**30000**
LE	$\hat{u}_l^d$	0	0.09	0.27	0.49	0.63	0.72	0.78	0.80	0.85	0.91	1
	$\hat{u}_l^u$	0	0.15	0.33	0.57	0.73	0.86	0.92	0.96	0.97	0.99	1

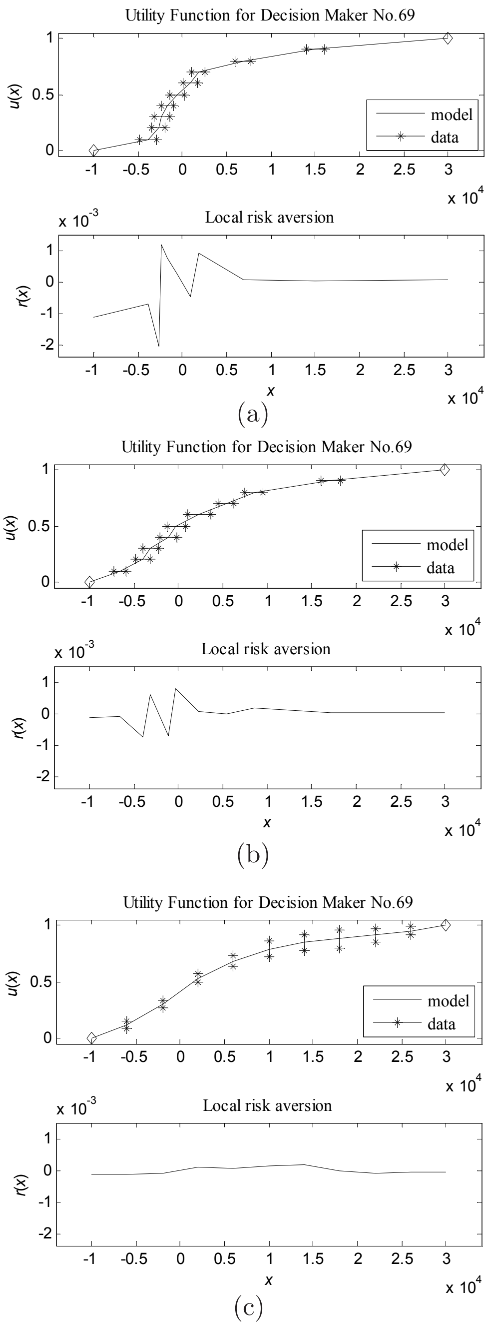
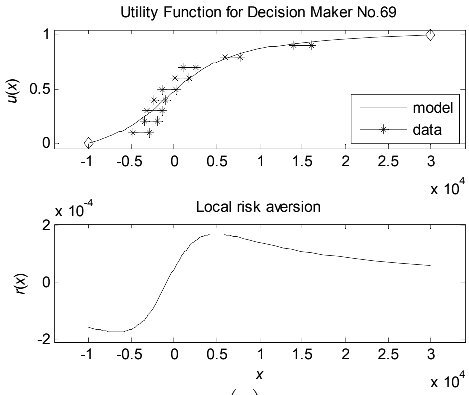
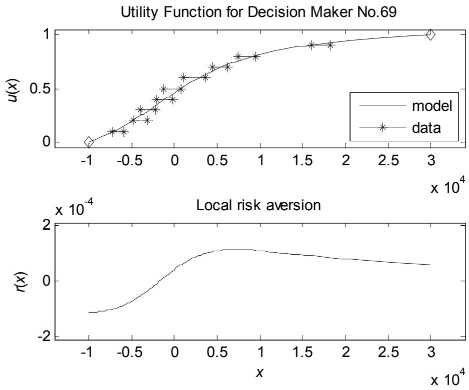


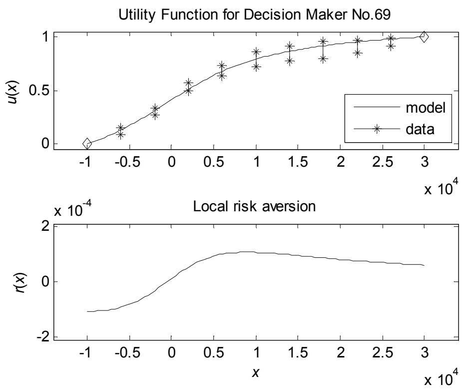
Figure 3. Linearly interpolated CE (a), UE (b) and LE (c) utility functions for DM No. 69, and their corresponding linearly interpolated local risk aversion functions



(a)



(b)



(c)

Figure 4. Arctg-approximated CE (a), UE (b) and LE (c) utility functions for DM No. 69, and their corresponding local risk aversion functions

3) Sign test to check, whether the median of Δ is zero. The null hypothesis is $H_0 : \Delta_{0.5} = 0$, whereas the alternative hypothesis is $H_1 : \Delta_{0.5} > 0$. The required $p_{value,3}$ of the test is calculated by a modification of the function *signtest* of MATLAB Statistical Toolbox (The MathWorks, 2006).

4) Signrank test to check, whether the median of Δ is zero. The null hypothesis is $H_0 : \Delta_{0.5} = 0$, whereas the alternative hypothesis is $H_1 : \Delta_{0.5} > 0$. The required $p_{value,4}$ of the test is calculated by a modification of the function *signrank* of MATLAB Statistical Toolbox (The MathWorks, 2006).

A) Justification of Assumption 1

It is expected that the local risk aversion for the CE utility function is higher than that for the UE utility function, which would prove that decisions based on the CE utility are more risk averse than those based on the UE utility. Four samples containing the four different risk aversion measures are formed for all DMs separately for the CE and UE utility functions. The mean values (m), standard deviations (σ), medians ($x_{0.5}$) and the interquartile intervals ($x_{0.75} - x_{0.25}$) on all DMs for the four samples of local risk aversion measures and for their difference (Δ) are given in Tables 4, 5, 6 and 7.

The calculated p_{value} for all four statistical tests for each of the four paired samples are presented in Table 8. At the significance level of α=0.05, all tests reject the null hypothesis H_0 that risk aversion measures are equal for both methods and accept the alternative hypothesis H_1 that CE risk aversion measures are higher than those of UE, and the difference is statistically significant. Such conclusion may be made even at a significance level of α=0.01. This shows that UE is superior to CE as it overcomes the certainty effect and generates utilities that better reflect the true risk attitude of the DM.

Table 4. Comparison of mean value of the module of empirical local risk aversion of the UE and CE utility functions for all DMs

	UE	***CE***	Δ
m	0.1620e-3	0.1975e-3	0.0355e-3
σ	0.1195e-3	0.1372e-3	0.1060e-3
$x_0.5$	0.1307e-3	0.1559e-3	0.0226e-3
$x_{0.75} - x_{0.25}$	0.0889e-3	0.1269e-3	0.0995e-3

Table 5. Comparison of maximum of module of empirical local risk aversion of the UE and CE utility functions for all DMs

	UE	***CE***	Δ
m	0.4180e-3	0.4962e-3	0.0781e-3
σ	0.4178e-3	0.3645e-3	0.3666e-3
$x_0.5$	0.2885e-3	0.3893e-3	0.0545e-3
$x_{0.75} - x_{0.25}$	0.2216e-3	0.3331e-3	0.3002e-3

Table 6. Comparison of mean value of the module of analytical local risk aversion of the UE and CE utility functions for all DMs

	UE	***CE***	Δ
m	0.4382e-4	0.5942e-4	0.1560e-4
σ	0.3729e-4	0.3936e-4	0.2380e-4
$x_0.5$	0.3668e-4	0.5370e-4	0.1283e-4
$x_0.75 - x_0.25$	0.4555e-4	0.5452e-4	0.3133e-4

Table 7. Comparison of maximum of module of analytical local risk aversion of the UE and CE utility functions for all DMs

	UE	***CE***	Δ
m	0.6319e-4	0.8485e-4	0.2166e-4
σ	0.5134e-4	0.5382e-4	0.3237e-4
$x_{0.5}$	0.5290e-4	0.8076e-4	0.1653e-4
$x_{0.75} - x_{0.25}$	0.5669e-4	0.7535e-4	0.3697e-4

Table 8. P_{value} of the statistical tests for each of the four paired samples

	Empirical mean	***Empirical max***	***Approximated mean***	***Approximated max***
$p_{value,1}$	0	1.0000e-3	0	0
$p_{value,2}$	0	0	0	0
$p_{value,3}$	2.9954e-4	0.0012	3.4608e-8	1.1399e-8
$p_{value,4}$	9.7277e-5	5.7190e-4	2.1858e-9	5.1014e-10

B) Justification of Assumption 2

It is expected that the risk aversion for the LE utility function is higher than that for the UE utility function, which could prove that the curvature of the LE utility is greater than that of the UE utility. Four samples containing the four different local risk aversion measures are formed for all DM separately for the LE and UE utility functions. The mean values, standard deviations, medians and interquartile intervals on all DMs for the four samples of local risk aversion measures and for their difference (Δ) are given in Tables 9, 10, 11 and 12.

The calculated p_{value} for all four statistical tests for each of the four paired samples are presented in Table 13. At the significance level of α=0.05, test results are contradicting and do not convincingly confirm the initial assumption. For the first two paired samples, all tests reject the null hypothesis H_0 that risk aversion measures are equal and accept the alternative hypothesis H_1 that UE risk aversion measures are higher than those of LE, and the difference is statistically significant. For the third paired sample, three of the tests fail to reject the null hypothesis, and for the fourth paired sample, one test fails to reject the null hypothesis.

Table 9. Comparison of mean value of the module of empirical local risk aversion of the LE and UE utility functions for all DMs

	LE	***UE***	Δ
m	0.1203e-3	0.1620e-3	0.0417e-3
σ	0.0656e-3	0.1195e-3	0.0991e-3
$x_{0.5}$	0.1119e-3	0.1307e-3	0.0206e-3
$x_{0.75} - x_{0.25}$	0.0604e-3	0.0889e-3	0.0626e-3

Table 10. Comparison of maximum of module of empirical local risk aversion of the LE and UE utility functions for all DMs

	LE	***UE***	Δ
m	0.3052e-3	0.4180e-3	0.1129e-3
σ	0.2125e-3	0.4178e-3	0.3680e-3
$x_{0.5}$	0.2625e-3	0.2885e-3	0.0491e-3
$x_{0.75} - x_{0.25}$	0.1458e-3	0.2216e-3	0.2354e-3

Table 11. Comparison of mean value of the module of analytical local risk aversion of the LE and UE utility functions for all DMs

	LE	***UE***	Δ
m	0.3925e-4	0.4382e-4	0.0457e-4
σ	0.2303e-4	0.3729e-4	0.2758e-4
$x_{0.5}$	0.3321e-4	0.3668e-4	0.0147e-4
$x_{0.75} - x_{0.25}$	0.3391e-4	0.4555e-4	0.2602e-4

Table 12. Comparison of maximum module of approximated risk aversion of the LE and UE utility function for all DMs

	LE	***UE***	Δ
m	0.5416e-4	0.6319e-4	0.0902e-4
σ	0.2974e-4	0.5134e-4	0.3925e-4
$x_{0.5}$	0.4459e-4	0.5290e-4	0.0606e-4
$x_{0.75} - x_{0.25}$	0.4036e-4	0.5669e-4	0.3933e-4

Table 13. P_{value} of the statistical tests for each of the four paired samples

	Empirical mean	***Empirical max***	***Approximated mean***	***Approximated max***
$p_{value,1}$	0	0	0.0120	1.0000e-3
$p_{value,2}$	0	0	0.1540	0.0085
$p_{value,3}$	6.5581e-5	0.0121	0.3120	0.0707
$p_{value,4}$	3.2798e-6	6.2637e-4	0.1661	0.0247

The results lead to the conclusion that the UE utility function is somewhat more curved than that of LE. A possible reason could be that UE chooses elicitation nodes that are more typical for the utility curve than those chosen by the LE method (since it chooses utilities and elicits their corresponding prizes, whereas LE chooses prizes and elicits their utility). Thus the resulting function may better describe the true curvature of the utility function.

C) Justification of assumption 3

As shown in Table 1, the final LE uncertainty intervals are $1/p$ times wider than those resulting directly from the elicitation, which makes LE results $1/p$ times more uncertain that those of PE. Since usually $p = 1/2$, then LE intervals are in most cases twice as wide as those of PE. Even though the UE scheme is more complex than that of CE, Table 1 shows that the final intervals from UE coincide with the elicited ones, i.e. there is no increase in uncertainty. To test this statement quantitatively, two samples are formed containing the length of the uncertainty intervals of all DMs, elicited using CE and UE.

The mean values, standard deviations, medians and the interquartile intervals on all DMs for samples of CE and UE interval lengths, as well as for their difference (Δ) are given in Table 14. The calculated p_{value} of all four statistical tests for the paired sample are presented in the first column of Table 17. At the significance level of α=0.05, all tests reject the null hypothesis H_0 that the length of the CE and UE uncertainty intervals are equal, and accept the alternative hypothesis H_1 that UE intervals are wider than those of CE, and the difference is statistically significant. This contradicts the initial expectations that UE and CE intervals are of the same length. It is likely that the higher complexity of the UE scheme compared to CE makes people identify wider intervals.

However, it is also necessary to identify how much are UE intervals wider than those of CE. That is why two additional samples were constructed, containing the lengths of the CE uncertainty intervals increased by, respectively, 28% and 30%. The mean values, standard deviations, medians and interquartile intervals on all DMs for the two new samples of uncertainty interval lengths are given in Tables 15 and 16.

The calculated p_{value} for all four statistical tests for the two new paired samples are presented in the second and third columns of Table 17. At the significance level of α=0.05, the results show that: 1) when comparing UE intervals with the ones from CE, increased by 28%, only one test fails to reject the null hypothesis and accepts that both lengths are equal (p_{value} is shaded); 2) when comparing UE intervals with the ones from CE, increased by 30%, all tests fail to reject the null hypothesis and prove that both lengths are equal (all p_{value} are shaded).

In can be concluded that despite the initial expectations, UE is more complex than CE and the uncertainty in the scheme is higher than that in CE. That leads to wider uncertainty intervals, but the increase is only by about 30%, unlike the ($1/p \times 100\%$ increase in the LE-PE case (which in the common case is 50%).

Table 14. Comparison of the length of uncertainty intervals elicited using CE and UE

	CE	***UE***	Δ
m	2.2775e+3	2.9702e+3	0.6927e+3
σ	0.3052e+3	0.3232e+3	0.3828e+3
$x_{0.5}$	2.2667e+3	3.0000e+3	0.7139e+3
$x_{0.75} - x_{0.25}$	0.2833e+3	0.4000e+3	0.3694e+3

Table 15. Comparison of the length of uncertainty intervals of UE and those of CE, increased by 28%

	1.28×CE	***UE***	Δ
m	2.9151e+3	2.9702e+3	0.0550e+3
σ	0.3906e+3	0.3232e+3	0.4378e+3
$x_{0.5}$	2.9013e+3	3.0000e+3	0.0711e+3
$x_{0.75} - x_{0.25}$	0.3627e+3	0.4000e+3	0.3924e+3

Table 16. Comparison of the length of uncertainty intervals of UE and those of CE, increased by 30%

	1.30×CE	***UE***	Δ
m	2.9607e+3	2.9702e+3	0.0095e+3
σ	0.3967e+3	0.3232e+3	0.4421e+3
$x_{0.5}$	2.9467e+3	3.0000e+3	0.0261e+3
$x_{0.75} - x_{0.25}$	0.3683e+3	0.4000e+3	0.3894e+3

Table 17. P_{value} of the statistical tests for each of the paired samples

	CE vs UE	***(1.28×CE) vs UE***	***(1.30×CE) vs UE***
$p_{value,1}$	0	0.0345	0.3750
$p_{value,2}$	0	0.065	0.0700
$p_{value,3}$	2.3129e-22	0.0197	0.1404
$p_{value,4}$	1.3699e-17	0.0349	0.2941

7. Conclusions

The paper discussed classical and modern techniques for elicitation of utilities, based on the rule that rational individuals make their choice according to maximum expected utility. As claimed in many works, any elicitation method leads to different distortion of results, which depend on the structure of the elicitation scheme. The paper focused mostly on the UE method and its advantages over CE and LE. The rationale behind the introduction of UE was based on three assumptions, namely that UE results are not influences by the certainty effect, the UE utility curve is quite similar to that of the LE utility, and that the length of the UE uncertainty intervals does not differ from that of the CE intervals. These assumptions were quantitatively analyzed on the basis of data from an empirical experiment with 104 volunteers, who elicited utilities over one and same interval of prizes using three methods – CE, LE and UE. Four one-tail statistical tests for paired samples were employed to test the statistical significance of the expected effects.

Statistical results proved some of the initial assumptions, and rejected others. Analysis of risk aversion proved that CE results are indeed more risk averse than the UE ones because of the certainty effect. A greater curvature of the UE function compared to the LE one turned out to be statistically significant, contrary to the initial assumption. This was explained by the fact that UE selects elicitation nodes which are much more representative (typical) for the function curvature than LE. Analysis of the length of the UE and CE uncertainty intervals proved that contrary to what was expected, UE intervals are wider than those of CE and the difference is statistically significant. However, tests also showed that the increase in length is only by about 30%. This fact was explained by the higher complexity of UE compared to that of CE, which leads to more uncertain results.

Despite the fact that two of the assumptions were not justified on the collected data, a general conclusion may be drawn that UE generates estimates of higher quality than those of CE or LE, due to: 1) elimination of certainty effect; 2) selection of more typical elicitation nodes; 3) small increase in the length of the uncertainty intervals, regardless of the complexity of the scheme.

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