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From optimal control to non-cooperative differential games: a homotopy approach*

by

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Abstract: We propose a new approach to the study of Nash equilibrium solutions to non-cooperative differential games. The original problem is embedded in a one-parameter family of differential games, where the parameter $\theta \in [0, 1]$ accounts for the strength of the second player. When $\theta = 0$, the second player adopts a myopic strategy and the game reduces to an optimal control problem for the first player. As θ becomes strictly positive, Nash equilibrium solutions can be obtained by studying a bifurcation problem for the corresponding system of Hamilton-Jacobi equations.

Keywords: optimal control, non-cooperative differential games, Nash equilibrium solution, optimal feedback control.

1. Introduction

In this paper we study a class of differential games for two players in infinite time horizon, with dynamics

$$\dot{x} = G(x, u_1, u_2). \tag{1.1}$$

Here $x \in \mathbb{R}^n$ is the state of the system, $u_1, u_2 \in \mathbb{R}^m$ are the controls implemented by the two players, while the upper dot denotes a derivative w.r.t. time. The goal of each player is to maximize his own payoff functional, exponentially discounted in time:

$$J_i \doteq \int_0^\infty e^{-\rho t} L_i(x(t), u_1(t), u_2(t))\, dt \qquad i = 1, 2. \tag{1.2}$$

To simplify the analysis, we assume that the inputs of the two players can be decoupled, namely

$$\begin{aligned} &G(x, u_1, u_2) = G_1(x, u_1) + G_2(x, u_2), \\ &L_i(x, u_1, u_2) = L_{i1}(x, u_1) + L_{i2}(x, u_2). \end{aligned} \tag{1.3}$$

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In the special case of zero-sum games with $L_1 \equiv -L_2$, an extensive mathematical theory is now available (Bardi and Capuzzo-Dolcetta, 1997; Friedman, 1971; Isaacs, 1965). For these problems, much insight can be gained by the analysis of the *value function*, characterized as the unique viscosity solution to a scalar Hamilton-Jacobi equation.

The theory of non-zero-sum games, on the other hand, is far less developed. We remark that, in this case, not even the concept of solution is straightforward. Various possibilities arise, depending on the degree of cooperation and on the information available to the players. Motivated by the classical work of Nash (1951) on non-cooperative equilibrium solutions, we adopt here a similar concept:

DEFINITION 1 *A pair of feedback controls* $x \mapsto (u_1^*(x), u_2^*(x))$ *provides a* non-cooperative equilibrium solution *to the differential game (1.1)-(1.2) if the following holds:*

(i) The control $u_1^*(\cdot)$ *is an optimal feedback, in connection with the optimization problem for the first player:*

$$\textit{maximize} \quad \int_0^\infty e^{-\rho t} L_1(x, u_1, u_2^*(x))\, dt\,, \qquad \textit{subject to} \quad \dot{x} = G(x, u_1, u_2^*(x))\,.$$

(ii) The control $u_2^*(\cdot)$ *is an optimal feedback, in connection with the optimization problem for the second player:*

$$\textit{maximize} \quad \int_0^\infty e^{-\rho t} L_2(x, u_1^*(x), u_2)\, dt\,, \qquad \textit{subject to} \quad \dot{x} = G(x, u_1^*(x), u_2)\,.$$

As shown in the last chapter of Friedman (1971), under suitable regularity conditions the corresponding value functions for the two players satisfy a system of Hamilton-Jacobi equations:

$$\rho V_i = H^{(i)}(x, \nabla V_1, \nabla V_2)\,. \tag{1.4}$$

Unfortunately, systems of this form are hard to study, because they are highly non-linear; moreover they are given in implicit form, i.e. solved for the functions V_i rather than for their derivatives.

The goal of the present paper is to develop a new approach to the analysis of these systems, based on a homotopy method. The original problem (1.1)-(1.2) will be embedded in a family of problems, depending on a parameter $\theta \in [0, 1]$. More precisely, we consider games, whose dynamics is described by

$$\dot{x} = G_1(x, u_1, \theta) + \theta G_2(x, u_2, \theta)\,. \tag{1.5}$$

The payoff functionals are also allowed to depend on θ:

$$J_i = \int_0^\infty e^{-\rho t} \Big(L_{i1}(x, u_1, \theta) + L_{i2}(x, u_2, \theta) \Big)\, dt \qquad i = 1, 2\,. \tag{1.6}$$

Here θ is regarded as the *strength* of the second player. When $\theta = 0$, this player cannot influence in any way the evolution of the system. His optimal strategy is thus the myopic one:

$$u_2 = u_2^{(0)}(x) \;=\; \operatorname*{argmax}_{\omega} L_{22}(x,\omega,0)\,.$$

In this case, the non-cooperative game reduces to an optimal control problem for the first player:

$$\text{maximize} \qquad \int_0^\infty e^{-\rho t}\Big(L_{11}(x,u_1,0) + L_{12}(x,u_2^{(0)}(x(t)),0)\Big)\,dt\,, \tag{1.7}$$

for a system with dynamics

$$\dot x \;=\; G_1(x,u_1,0). \tag{1.8}$$

As soon as the optimal feedback control $u_1^{(0)}(x)$ for the first player is found, this determines in turn the trajectories of the system, and hence the value function for the second player.

When the parameter θ becomes strictly positive, we have a genuine differential game. Our main interest is to understand how the solution of the optimal control problem for $\theta = 0$ can provide useful information about the solutions to the differential game for $\theta > 0$. In rather general terms, we ask the following

Questions: *Let $\Omega \subset I\!\!R^n$ be a compact domain with smooth boundary. Assume that, when $\theta = 0$, the optimal control problem (1.7)-(1.8) for the first player admits an optimal feedback $x \mapsto u_1^{(0)}(x)$, which makes Ω positively invariant for the flow*

$$\dot x = G_1(x,u_1^{(0)}(x),0)\,. \tag{1.9}$$

Call $V_1^{(0)}, V_2^{(0)}$ the corresponding value functions for the two players.

In the above setting, does the differential game (1.5)-(1.6) admit a Nash equilibrium solution in feedback form: $u_1^{(\theta)}, u_2^{(\theta)} : \Omega \mapsto I\!\!R^m$, for $\theta > 0$ sufficiently small ? Is this solution unique ? Do the corresponding value functions $V_1^{(\theta)}, V_2^{(\theta)}$ converge to $V_1^{(0)}, V_2^{(0)}$ as $\theta \to 0$?

We expect that the existence question should have a positive answer, if the optimal feedbacks $u_1^{(0)}, u_2^{(0)}$ are both continuous and if the flow (1.9) is strictly inward pointing at each boundary point $x \in \partial\Omega$.

In the present paper we examine a basic one-dimensional case. Namely, we fix a compact interval $I = [a,b] \subset I\!\!R$ and assume that, when $\theta = 0$, the optimal solution to the control problem (1.7)-(1.8) determines a smooth dynamics

$$\dot x = G_1(x,u_1^*(x),0)$$

having a unique, asymptotically stable equilibrium point $\bar x$ in the interior of I. For $\theta > 0$ small, our analysis indicates that the differential game (1.5)-(1.6) can have one, or else infinitely many Nash equilibrium solutions, all with stable dynamics. To determine which one of these two cases actually occurs, it suffices to check the sign of a specific function, computed at the equilibrium point $\bar{x}$, for $\theta = 0$.

The remainder of the paper is organized as follows. Section 2 contains a review of the optimal control problem in infinite time horizon, and examines the singularity of the H-J equation at a stationary point $\bar{x}$, where $\dot{x} = G(x, u^*(x)) = 0$. In Section 3 we consider the system of equations corresponding to a differential game, and observe that it can be equivalently written as a Pfaffian system. The graph of a solution can be constructed as the concatenation of trajectories of a particular vector field in $\mathbb{R}^3$. After this preliminary material, in Section 4 we consider a parameter-dependent game, and set up the framework for the bifurcation problem, as the parameter θ increases from zero to strictly positive values. In Section 5 we explain the two possible bifurcations that can generically occur, as the parameter θ becomes positive. The discussion is here kept at an informal level. A rigorous mathematical analysis of the bifurcation problem can be found in the companion paper Bressan (2009).

The last two sections contain two examples. The first is a simple linear-quadratic game, where the value functions and the optimal controls can be computed explicitly. In this case, one can directly check the existence and multiplicity of solutions to the differential game for $\theta > 0$. We remark that, when multiple solutions exist, one of these solutions corresponds to affine feedback controls and quadratic value functions. All the other ones have fully non-linear structure.

Our second example is a nonlinear "sticky price" model, involving a producer and a group of consumers. We assume that a fraction $\theta \in [0, 1]$ of all consumers join together and implement a long-term strategy, while the remaining fraction $1 - \theta$ consists of individual consumers, who adopt a myopic strategy. Equivalently, one can assume the presence of one large consumer accounting for a fraction θ of the entire market. The case $\theta = 0$ yields an optimal control problem for the producer. As the parameter θ increases form zero to positive values, a detailed analysis shows that the differential game has infinitely many equilibrium solutions in feedback form, all leading to an asymptotically stable dynamics.

Some classes of non-cooperative games in one space dimension were studied in Bressan and Priuli (2006), Cardaliaguet and Plaskacz (2003) and Priuli (2007). These results, although very particular, provide a glimpse of the complexities of the problem. Non-cooperative games with finite horizon and terminal cost were also studied in Bressan and Shen (2004a). In the one-dimensional case, some conditions were derived for the system to be hyperbolic. If this happens, one can differentiate the basic equations and obtain a system of conservation laws for the spatial derivatives of the value functions. In turn, under suitable assumptions, this system can be uniquely solved using the theory of hyperbolic conservation laws (Bressan, 2000; Serre, 2000). In the general multidimensional case, however, as observed in Bressan and Shen (2004b), the system is generically not hyperbolic. As a consequence, the Cauchy problem is ill posed. In other words, a small change in the terminal payoff function can entail large changes in the value functions and in the strategies adopted by the players. Of course, this renders the mathematical model unsuitable for practical applications. For this specific reason, we consider here only the infinite horizon problem. In this case, the value functions are not obtained by solving a Cauchy problem, and the lack of hyperbolicity is not an a-priori obstruction toward the construction of solutions.

The special case of games with linear dynamics and quadratic cost functions has attracted considerable attention in the literature (Basar and Olsder, 1995; Olsder, 2001; Vaisbord and Zhukovskii, 1989). In this case, the system of PDEs always admits a solution, whose components are quadratic polynomials. However, because of the possible ill-posedness of the underlying PDEs, the practical validity of these results needs a careful justification. For applications of game theory to economic modeling, more in general, we refer to Aubin (1979) and Dockner et al. (2000).

We believe that the present homotopy approach can provide a useful tool in the study of noncooperative differential games, which otherwise remain difficult to analyze. As shown in one of the examples, a model with players of different strength can also have a meaningful economic interpretation.

2. Review of the optimal control problem

We first consider a problem of optimal control, with infinite horizon and an exponential discount factor:

$$\text{maximize:} \qquad J(u) \doteq \int_0^\infty e^{-\rho t} L(x(t), u(t))\, dt\,. \tag{2.1}$$

The state of the system is here described by the one-dimensional variable x, whose evolution satisfies

$$\dot{x} = g(x, u) \tag{2.2}$$

with initial condition

$$x(0) = y\,. \tag{2.3}$$

We assume that the functions L, g are smooth, while the control $u \in \mathbf{L}^\infty\,([0, \infty[)$ can be any bounded measurable function. Call $V(y)$ the value function, i.e. the supremum of all payoffs $J(u)$ in (2.1), which are attainable starting from the initial state y. It is well known that $V(\cdot)$ satisfies the Hamilton-Jacobi equation

$$\rho V(x) = H(x, V'(x))\,, \tag{2.4}$$

where the Hamiltonian function is

$$H(x,\xi) = \max_{\omega} \Big\{ \xi \cdot g(x,\omega) + L(x,\omega) \Big\}, \tag{2.5}$$

see, for example Bardi and Capuzzo-Dolcetta (1997) or Bressan and Piccoli (2007). By differentiating (2.4) w.r.t. x we obtain a nonlinear O.D.E. for the derivative $\xi = V'$, namely

$$\rho\xi = H_x(x,\xi) + H_\xi(x,\xi)\xi' . \tag{2.6}$$

In connection with (2.5), call

$$u^*(x,\xi) = \operatorname*{argmax}_{\omega} \Big\{ \xi \cdot g(x,\omega) + L(x,\omega) \Big\} \tag{2.7}$$

the value of the optimal feedback control. From the identity

$$H(x,\xi) = \xi \cdot g(x, u^*(x,\xi)) + L(x, u^*(x,\xi)),$$

differentiating w.r.t. ξ and recalling (2.7) one obtains

$$H_\xi(x,\xi) \;=\; g(x,u^*(x,\xi)) + \Big(\xi \cdot g_u + L_u\Big)\frac{\partial u^*}{\partial \xi} \;=\; g(x,u^*(x,\xi)) . \tag{2.8}$$

When the derivative of the value function $\xi = V'$ is known, one can use (2.7) to compute the optimal feedback control u^*. In turn, this yields a dynamical system which, with slight abuse of notation, we write as

$$\dot x \;=\; g(x,\xi(x)) \;\doteq\; g\Big(x, u^*(x,\xi(x))\Big). \tag{2.9}$$

Under a suitable regularity condition, all trajectories of (2.9) will be optimal.

Next, we seek a stationary solution $x(t) \equiv \bar x$ of the feedback equation (2.9). Calling $\bar\xi = \xi(\bar x) = V'(\bar x)$, the couple $(\bar x, \bar\xi)$ can be determined by solving a system of two equations. The first is (2.6), the second is $g(x,\xi) = H_\xi(x,\xi) = 0$. This yields the system

$$\begin{cases} 0 &= H_\xi(x,\xi)\,, \\ \rho\xi &= H_x(x,\xi)\,. \end{cases} \tag{2.10}$$

To ensure that this stationary solution is stable, we need to check that at the point $x = \bar x$ the solution $\xi = \xi(x)$ of (2.6) satisfies

$$\frac{d}{dx} g(x,\xi(x)) \;=\; g_x + g_\xi\,\xi' \;=\; H_{x\xi} + H_{\xi\xi}\,\xi' \;<\; 0\,. \tag{2.11}$$

Differentiating (2.6) we find

$$\rho\xi' = H_{xx} + 2H_{x\xi}\,\xi' + H_{\xi\xi}(\xi')^2 + H_\xi \xi'' . \tag{2.12}$$

At the stationary point $H_\xi = 0$, hence

$$H_{\xi\xi}(\xi')^2 + (2H_{x\xi} - \rho)\xi' + H_{xx} \;=\; 0\,. \tag{2.13}$$

Assuming that

$$\Delta \;\doteq\; (2H_{x\xi} - \rho)^2 - 4H_{xx}H_{\xi\xi} \;\geq\; 0\,,$$

we find

$$\xi'_\pm \;=\; \frac{\rho - 2H_{x\xi}}{2H_{\xi\xi}} \pm \frac{\sqrt{\Delta}}{2H_{\xi\xi}}\,. \tag{2.14}$$

Hence

$$H_{x\xi} + H_{\xi\xi}\,\xi' \;=\; \frac{1}{2}\left(\rho \pm \sqrt{\Delta}\right),$$

and a stable solution exists, provided that $\rho^2 < \Delta$. Notice that this condition is equivalent to

$$(H_{x\xi} - \rho)H_{x\xi} - H_{xx}H_{\xi\xi} \;>\; 0\,. \tag{2.15}$$

Denoting with an upper dot the derivative w.r.t. time, from the equations (2.9), (2.6) we obtain

$$\begin{cases} \dot{x} \;=\; H_\xi(x,\xi)\,, \\ \dot{\xi} \;=\; \rho\xi - H_x(x,\xi)\,. \end{cases} \tag{2.16}$$

At the equilibrium point $(\bar{x},\bar{\xi})$, the corresponding matrix of partial derivatives is

$$\begin{pmatrix} H_{x\xi} & H_{\xi\xi} \\ -H_{xx} & \rho - H_{x\xi} \end{pmatrix}. \tag{2.17}$$

The eigenvalues of this matrix are

$$\lambda_\pm \;=\; \frac{\rho}{2} \pm \sqrt{\frac{\rho^2}{4} + (H_{x\xi} - \rho)H_{x\xi} - H_{xx}H_{\xi\xi}}\,. \tag{2.18}$$

Assuming that the stability condition (2.15) holds, from (2.18) we see that these eigenvalues are real and distinct, satisfying

$$\lambda_- < 0 < \lambda_+\,, \qquad\qquad |\lambda_-| < \lambda_+\,. \tag{2.19}$$

The stationary point $(\bar{x},\bar{\xi})$ is thus a hyperbolic (saddle) point for the O.D.E. (2.16).

The corresponding eigenvectors $\mathbf{r}_\pm$ of the matrix in (2.17) can be written as

$$\mathbf{r}_- = \begin{pmatrix} 1 \\ \xi'_- \end{pmatrix}, \qquad \mathbf{r}_+ = \begin{pmatrix} 1 \\ \xi'_+ \end{pmatrix}, \tag{2.20}$$

where the coefficients $\xi'_\pm$ are the roots of the equation (2.13), computed at (2.14).

The derivative of the value function $x \mapsto \xi(x) \doteq V'(x)$ thus satisfies

$$\xi(\bar{x}) = \bar{\xi}\,, \qquad \xi'(\bar{x}) = \xi'_-\,.$$

REMARK 1 *For each x, consider the control $\hat{u}(x)$ such that*

$$H_\xi(x, \hat{u}(x)) = g(x, \hat{u}(x)) = 0.$$

Choose the function $x \mapsto \hat{\xi}(x)$ so that $\hat{u}(x) = u^*(x, \hat{\xi}(x))$. Call

$$\widehat{V}(x) = \frac{1}{\rho} L(x, \hat{u}(x)) = \frac{1}{\rho} H(x, \hat{\xi}(x))$$

the corresponding payoff function. Since $V(x) \geq \widehat{V}(x)$ for all x, while at the stationary point $V(\bar{x}) = \widehat{V}(\bar{x})$, it follows that $V'(\bar{x}) = \widehat{V}'(\bar{x})$. Recalling that $H_\xi = g$, at the point $\bar{x}$ we have

$$\bar{\xi} \;=\; \widehat{V}'(\bar{x}) \;=\; \frac{d}{dx}\frac{1}{\rho} H(x, \hat{\xi}(x))\Big|_{x=\bar{x}} \;=\; \frac{1}{\rho} H_x(\bar{x}, \bar{\xi})\,. \tag{2.21}$$

This provides another interpretation of the second equation in (2.10).

3. The differential game

We consider a non-cooperative differential game in one space dimension, with infinite horizon, exponentially discounted in time. Let $x \in I\!R$ describe the one-dimensional state of the system, which evolves according to

$$\dot{x} \;=\; G(x, u_1, u_2) \;=\; G_1(x, u_1) + G_2(x, u_2)\,. \tag{3.1}$$

Here u_1, u_2 are the controls implemented by the two players. Let the payoff functionals for the two players be

$$J_i \doteq \int_0^\infty e^{-\rho t} L_i\big(x(t),\, u_1(t),\, u_2(t)\big)\, dt \qquad i = 1, 2\,. \tag{3.2}$$

We assume that the functions L_i can be decomposed as

$$L_i(x, u_1, u_2) = L_{i1}(x, u_1) + L_{i2}(x, u_2)\,. \tag{3.3}$$

Hence, for $\xi_1 \in I\!R$, the function

$$\begin{aligned} u_1^*(x,\xi_1) &= \operatorname*{argmax}_{\omega} \Big\{ \xi_1 \cdot G(x,\omega,u_2) + L_1(x,\omega,u_2) \Big\} \\ &= \operatorname*{argmax}_{\omega} \Big\{ \xi_1 \cdot G_1(x,\omega) + L_{11}(x,\omega) \Big\} \end{aligned} \tag{3.4}$$

does not depend on u_2. Similarly, for $\xi_2 \in I\!R$ the function

$$\begin{aligned} u_2^*(x,\xi_2) &= \operatorname*{argmax}_{\omega} \Big\{ \xi_2 \cdot G(x,u_1,\omega) + L_2(x,u_1,\omega) \Big\} \\ &= \operatorname*{argmax}_{\omega} \Big\{ \xi_2 \cdot G_2(x,\omega) + L_{22}(x,\omega) \Big\} \end{aligned} \tag{3.5}$$

does not depend on u_1.

Let $V_1(x)$, $V_2(x)$ be the value functions for a Nash equilibrium solution. Assuming sufficient regularity, these will satisfy the system of ODEs

$$\rho V_i \ = \ V_i' \cdot G(x,\, u_1^*(x,V_1'),\, u_2^*(x,V_2')) + L_i(x,\, u_1^*(x,V_1'),\, u_2^*(x,V_2')) \qquad i=1,2\,, \tag{3.6}$$

where the u_i^* are defined in (3.4)-(3.5). We write (3) in the form

$$\rho V_i = H^{(i)}(x, V_1', V_2') \tag{3.7}$$

where

$$H^{(i)}(x,\xi_1,\xi_2) = \xi_i \cdot G(x,\, u_1^*(x,\xi_1),\, u_2^*(x,\xi_2)) + L_i(x,\, u_1^*(x,\xi_1),\, u_2^*(x,\xi_2)). \tag{3.8}$$

By a slight abuse of notation, we also write

$$H^{(i)}(x,\xi_1,\xi_2) = \xi_i \cdot G(x,\, \xi_1,\, \xi_2) + L_i(x,\xi_1,\xi_2). \tag{3.9}$$

Differentiating (3.7) w.r.t. x one obtains

$$\rho\xi_1 = H_x^{(1)} + H_{\xi_1}^{(1)}\,\xi_1' + H_{\xi_2}^{(1)}\,\xi_2'\,, \quad \rho\xi_2 = H_x^{(2)} + H_{\xi_1}^{(2)}\,\xi_1' + H_{\xi_2}^{(2)}\,\xi_2'\,. \tag{3.10}$$

Observing that

$$H_{\xi_1}^{(1)} \ = \ H_{\xi_2}^{(2)} \ = \ G, \tag{3.11}$$

the above system can be written in matrix form as

$$\begin{pmatrix} G & H_{\xi_2}^{(1)} \\ H_{\xi_1}^{(2)} & G \end{pmatrix} \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} \ = \ \begin{pmatrix} \rho\xi_1 - H_x^{(1)} \\ \rho\xi_2 - H_x^{(2)} \end{pmatrix}. \tag{3.12}$$

The system (3.12) can be written in standard form, i.e. solved for the derivatives ξ_1', ξ_2', if and only if the determinant of the matrix of coefficients $\left(H_{\xi_j}^{(i)}\right)$ is non-zero, namely

$$G^2 - H_{\xi_1}^{(1)} H_{\xi_2}^{(2)} \ \neq \ 0. \tag{3.13}$$

In this connection, the sign of the product $H^{(1)}_{\xi_2} \cdot H^{(2)}_{\xi_1}$ plays a crucial role. If this sign is always negative, then the matrix will certainly be invertible. We thus expect to find several global solutions to the system (3.12). On the other hand, if the sign is positive, there will be singular points, where the matrix is not invertible. This poses restrictions on global solutions, which must pass through these points.

An alternative way to write the equations (3.12) is as follows. Consider the two differential forms

$$\begin{aligned} \omega_1 &\doteq \Big(H^{(1)}_x - \rho\xi_1\Big)\, dx + G\, d\xi_1 + H^{(1)}_{\xi_2}\, d\xi_2\,, \\ \omega_2 &\doteq \Big(H^{(2)}_x - \rho\xi_2\Big)\, dx + H^{(2)}_{\xi_1}\, d\xi_1 + G\, d\xi_2\,. \end{aligned} \tag{3.14}$$

A solution to the equations (3.12) is a curve, whose tangent vector lies in the kernel of both forms ω_1 and ω_2, at each point. The graph of this solution can thus be constructed by piecing together trajectories of the ODE on $I\!R^3$

$$\frac{d}{ds}\begin{pmatrix} x \\ \xi_1 \\ \xi_2 \end{pmatrix} = \omega_1 \wedge \omega_2 = \begin{pmatrix} G^2 - H^{(1)}_{\xi_2} H^{(2)}_{\xi_1} \\ \Big(H^{(2)}_x - \rho\xi_2\Big) H^{(1)}_{\xi_2} - \Big(H^{(1)}_x - \rho\xi_1\Big) G \\ \Big(H^{(1)}_x - \rho\xi_1\Big) H^{(2)}_{\xi_1} - \Big(H^{(2)}_x - \rho\xi_2\Big) G \end{pmatrix}. \tag{3.15}$$

When the derivatives of the value function $\xi_i = V_i'$ have been determined, one can use (3.4)-(3.5) to compute the optimal feedback controls u_i^*. In turn, this yields a dynamical system which, with slight abuse of notation, we write as

$$\dot x = G(x, \xi_1(x), \xi_2(x)) \doteq G\Big(x,\, u_1^*(x, \xi_1(x)),\, u_2^*(x, \xi_2(x))\Big). \tag{3.16}$$

4. A parameter-dependent game

In this section we consider a family of non-cooperative differential games, depending on a parameter θ. We assume that the evolution of the system is described by

$$\dot x = G(x, u_1, u_2, \theta) = G_1(x, u_1, \theta) + \theta\, G_2(x, u_2, \theta)\,, \tag{4.1}$$

while the payoff functionals take the form

$$J_i \doteq \int_0^\infty e^{-\rho t} L_i\big(x(t),\, u_1(t),\, u_2(t), \theta\big)\, dt \qquad\qquad i = 1, 2\,. \tag{4.2}$$

As before, we assume that the functions L_i can be split as

$$L_i(x, u_1, u_2, \theta) = L_{i1}(x, u_1, \theta) + L_{i2}(x, u_2, \theta)\,.$$

Notice that in the present case not only the functions G_i, L_i depend on the additional parameter θ but, more importantly, the factor θ multiplies G_2 in (4.1). As a consequence, in the special case $\theta = 0$, the second player cannot affect in any way the evolution of the state x.

For $\xi_1, \xi_2 \in \mathbb{R}^n$, call

$$u_1^*(x, \xi_1, \theta) = \operatorname*{argmax}_{\omega} \Big\{ \xi_1 \cdot G_1(x, \omega, \theta) + L_{11}(x, \omega, \theta) \Big\}, \tag{4.3}$$

$$u_2^*(x, \xi_2, \theta) = \operatorname*{argmax}_{\omega} \Big\{ \xi_2 \cdot \theta\, G_2(x, \omega, \theta) + L_{22}(x, \omega, \theta) \Big\}, \tag{4.4}$$

the optimal feedback controls of the two players. When $\theta = 0$, the evolution equation (4.1) reduces to

$$\dot{x} = H^{(1)}_{\xi_1} = H^{(2)}_{\xi_2} = G = G_1(x, u_1, 0). \tag{4.5}$$

Since the second player now cannot affect the evolution of the system, his best choice is the myopic strategy:

$$u^\dagger(x) \doteq u_2^*(x, \xi_2, 0) = \operatorname*{argmax}_{\omega} L_{22}(x, \omega, 0), \tag{4.6}$$

depending only on the current state of the system. In particular, we now have

$$H^{(1)}(x, \xi_1, \xi_2, 0) = \xi_1 \cdot G_1\Big(x,\, u_1^*(x, \xi_1, 0),\, 0\Big) + L_1\Big(x,\, u_1^*(x, \xi_1, 0),\, u_2^\dagger(x), 0\Big), \tag{4.7}$$

$$H^{(2)}(x, \xi_1, \xi_2, 0) = \xi_2 \cdot G_1\Big(x,\, u_1^*(x, \xi_1, 0),\, 0\Big) + L_2\Big(x,\, u_1^*(x, \xi_1, 0),\, u_2^\dagger(x), 0\Big). \tag{4.8}$$

The system (3.10) can thus be solved in two steps. Since $H^{(1)}$ is independent of ξ_2, one can first solve the scalar equation

$$\rho \xi_1 = H^{(1)}_x + G\, \xi_1' \tag{4.9}$$

and determine the derivative $\xi_1 = V_1'$ of the value function for the first player. Then solve

$$\rho \xi_2 = H^{(2)}_x + H^{(2)}_{\xi_1}\, \xi_1' + G\, \xi_2' \,, \tag{4.10}$$

determining the derivative $\xi_2 = V_2'$ of the value function for the second player.

Our main goal is to understand how the picture changes for small $\theta > 0$. The differential forms ω_1, ω_2 in (3.14) and the vector field $\mathbf{v} = \omega_1 \wedge \omega_2$ in (3.15) all depend on the parameter θ. We thus focus on the orbits of the vector field $\mathbf{v}$, describing how they change as θ becomes positive. The problem will be somewhat reformulated in the next section, where we analyze the two possible types of bifurcation, near an equilibrium point.

Here we observe that, if the maxima in (4.3)-(4.4) are always attained at points $\omega = u_i^*$, where the Hessian is a non-degenerate (strictly negative) quadratic form, then $\partial u_2^*/\partial\xi_2 = \mathcal{O}(1)\cdot\theta$. Here and in the sequel, the Landau symbol $\mathcal{O}(1)$ denotes a quantity which remains uniformly bounded as the variables range in bounded sets. This implies

$$\frac{\partial G}{\partial \xi_2} \;=\; \frac{\partial}{\partial \xi_2}(G_1+\theta\, G_2) \;=\; \theta\,\frac{\partial G}{\partial u_2}\,\frac{\partial u_2^*}{\partial \xi_2} \;=\; \mathcal{O}(1)\cdot\theta^2. \tag{4.11}$$

For the applications we have in mind, the payoff function L_{12} has the form

$$L_{12}(x,\xi_2,\theta) \;=\; (1-\theta)\,L_{12}(x,u_2^\dagger(x))+\theta\,L_{12}(x,u_2^*(x,\xi_2,\theta))\,. \tag{4.12}$$

An economic model leading to this assumption will be discussed in Example 2, in the last section of this paper. In turn, (4.12) implies

$$\frac{\partial L_{12}}{\partial \xi_2} \;=\; \mathcal{O}(1)\cdot\theta^2\,. \tag{4.13}$$

5. Bifurcation analysis

Assume that, when $\theta = 0$, the optimal control problem for the first player

$$\text{maximize:} \qquad \int_0^\infty e^{-\rho t} L_1(x,u_1,u_2^\dagger(x),0)\,dt\,, \tag{5.1}$$

subject to

$$\dot x \;=\; G_1(x,u_1), \tag{5.2}$$

admits a smooth optimal feedback $x \mapsto u_1^*(x)$. Moreover, assume that the corresponding dynamics

$$\dot x = G_1(x,u_1^*(x)) \tag{5.3}$$

has a unique asymptotically stable equilibrium point. Namely, there exists $\bar x \in I\!\!R$ such that

$$G(\bar x,u_1^*(\bar x)) = 0, \qquad\qquad \frac{d}{dx}\,G(\bar x,u_1^*(\bar x)) \;<\; 0\,. \tag{5.4}$$

We shall study the existence and uniqueness of Nash equilibrium solutions in feedback form, for the differential game (4.1)-(4.2) with $\theta > 0$ small.

Recall that, in the parameter dependent case, the hamiltonian functions are

$$H^{(i)}(x,\xi_1,\xi_2,\theta) \;=\; \xi_i\cdot[G_1(x,\xi_1,\theta)+\theta\,G_2(x,\xi_2,\theta)]+L_{i1}(x,\xi_1,\theta)+L_{12}(x,\xi_2,\theta)\,. \tag{5.5}$$

We begin by writing the parameter-dependent system

$$\begin{pmatrix} G & H^{(1)}_{\xi_2} \\ H^{(2)}_{\xi_1} & G \end{pmatrix} \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} \rho\xi_1 - H^{(1)}_x \\ \rho\xi_2 - H^{(2)}_x \end{pmatrix} \tag{5.6}$$

in a more convenient form. Renaming the variables $(\xi_1, \xi_2) = (y, z)$, we can write (5.6) as

$$\begin{pmatrix} G & \theta^2\alpha \\ \beta & G \end{pmatrix} \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{5.7}$$

Here all functions $G, \alpha, \beta, \phi, \psi$ are smooth functions of x, y, z, θ, while $\theta \geq 0$. If the determinant of the matrix does not vanish, the system (5.7) is equivalent to

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \frac{1}{G^2 - \theta^2\alpha\beta} \begin{pmatrix} G & -\theta\alpha \\ -\beta & G \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{5.8}$$

Notice that for $\theta = 0$ the functions G, ϕ do not depend on the variable z. According to (4.11), (4.13), their partial derivatives w.r.t. z satisfy

(A1) *As $\theta \to 0$, one has*

$$\|G_z(x, y, z, \theta)\|_{\mathcal{C}^1} = \mathcal{O}(1) \cdot \theta^2, \qquad \|\phi_z(x, y, z, \theta)\|_{\mathcal{C}^1} = \mathcal{O}(1) \cdot \theta^2. \tag{5.9}$$

Recalling the analysis in Section 2, by the assumption (5.4), for $\theta = 0$ the ODE

$$\begin{cases} \dot{x} = G(x, y) \\ \dot{y} = \phi(x, y) \end{cases} \tag{5.10}$$

has an equilibrium point of saddle type, at some point $\overline{P} \doteq (\bar{x}, \bar{y})$. At this point, the corresponding Jacobian matrix

$$\begin{pmatrix} G_x & G_y \\ \phi_x & \phi_y \end{pmatrix} \tag{5.11}$$

has two real distinct eigenvalues: $\lambda_- < 0 < \lambda_+$, with $|\lambda_-| < \lambda_+$. An explicit computation yields

$$\lambda_\pm = \frac{\phi_y + G_x \pm \sqrt{(\phi_y - G_x)^2 + 4\phi_x G_y}}{2}. \tag{5.12}$$

To find the slopes of the stable and unstable manifolds through $\overline{P}$ we differentiate the equation $Gy' = \phi$, obtaining

$$G_x y' + G_y (y')^2 + G y'' = \phi_x + \phi_y y'. \tag{5.13}$$

Solving for y', since $G = 0$, one finds

$$y'_\pm \;=\; \frac{\phi_y - G_x \pm \sqrt{(\phi_y - G_x)^2 + 4G_y\phi_x}}{2G_y}\,. \tag{5.14}$$

This could also be derived by observing that the eigenvectors r_-, r_+ of the Jacobian matrix (5.11) satisfy

$$r_\pm \doteq \begin{pmatrix} 1 \\ y'_\pm \end{pmatrix}, \qquad\qquad y'_\pm = \frac{\lambda_\pm - G_x}{G_y}\,.$$

The eigenvalue $\lambda_- < 0$ corresponds to the stable branch:

$$\frac{d}{dx} G(x, y(x)) \;=\; G_x + G_y y'_- \;=\; \lambda_- \;<\; 0. \tag{5.15}$$

Differentiating (5.13) once again, we obtain

$$G_{xx}y' + 2G_{xy}(y')^2 + 2G_x y'' + G_{yy}(y')^2 + 3G_y y' y'' + Gy''' = \phi_{xx} + 2\phi_{xy}y' + \phi_{yy}(y')^2 + \phi_y y''. \tag{5.16}$$

At the point $\overline{P} = (\bar x, \bar y)$ where $G = 0$, we thus have

$$\Big[2G_x + 3G_y y' - \phi_y\Big] y'' \;=\; \phi_{xx} + 2\phi_{xy}y' + \phi_{yy}(y')^2 - G_{xx}y' - 2G_{xy}(y')^2 - G_{yy}(y')^2. \tag{5.17}$$

Since $y' = y'_-$ is already known, the equation (5.17) can be solved for second derivative y''. Indeed, recalling (5.15), the coefficient of y'' can be written as

$$3(G_x + G_y y') - (\phi_y + G_x) \;=\; \frac{\phi_y + G_x - 3\sqrt{(\phi_y - G_x)^2 + 4G_y\phi_x}}{2} \;<\; 0.$$

Always in the case $\theta = 0$, let

$$y = y(x) \tag{5.18}$$

be the equation of the smooth stable manifold for the equation (5.10). The corresponding equation for the component z in (5.7) takes the form

$$G(x, y(x))z' = \psi(x, y(x), z) - \beta(x, y(x), z)y'(x)\,. \tag{5.19}$$

We assume that, always for $\theta = 0$, the value function V_2 for the second player is also smooth. To have a bounded solution of (5.19) through the point $x = \bar x$, where $G = 0$, the right hand side must also vanish at this point. In other words, there should be a value $\bar z = V_2'(\bar x)$ such that, calling y'_- the lower root in (5.14), at the point $(x,y,z,\theta) = (\bar x, \bar y, \bar z, 0)$, the following equations hold,

$$\begin{cases} G &=& 0, \\ \phi &=& 0, \\ \beta y'_- - \psi &=& 0. \end{cases} \tag{5.20}$$

Recalling that $H^{(1)}_{\xi_1} = H^{(2)}_{\xi_2} = G$, by (5.4) we deduce

$$\beta_z y'_- - \psi_z \;=\; H^{(2)}_{\xi_1\xi_2}\frac{d}{dx}\xi_1(x) - \Big(\rho\xi_2 - H^{(2)}_x\Big)_{\xi_2} \;=\; \frac{\partial G_1}{\partial \xi_1}\frac{d\xi_1}{dx} - \rho + \frac{\partial G_1}{\partial x} \;<\; 0\,. \tag{5.21}$$

Differentiating (5.19) one more time, we obtain

$$(G_x + G_y y')z' + Gz'' \;=\; \psi_x + \psi_y y' + \psi_z z' - (\beta_x + \beta_y y' + \beta_z z')y' - \beta y''. \tag{5.22}$$

At the singular point one has $G(\bar x, \bar y) = 0$. Since the value $y''(\bar x)$ is already known from (5.17), we can now solve (5.22) for the derivative $z'(\bar x)$,

$$(G_x + G_y y'_- + \beta_z y'_- - \psi_z)z' \;=\; \psi_x + \psi_y y' - (\beta_x + \beta_y y')y' - \beta y''. \tag{5.23}$$

By (5.15) and (5.21), the factor on the left hand side of (5.23) is strictly negative. Therefore, when $\theta = 0$, the implicit O.D.E. (5.19) has a regular solution passing through the point $(\bar x, \bar y, \bar z)$, with slope $y'(\bar x) = y'_-$ given at (5.14) and $z'(\bar x)$ computed by (5.23).

The previous analysis motivates the following assumptions.

(A2) *There exists a point* $\overline{P} \doteq (\bar x, \bar y, \bar z)$ *such that, when* $\theta = 0$,

$$G(\bar x, \bar y) \;=\; \phi(\bar x, \bar y) \;=\; 0\,, \qquad G_y(\bar x, \bar y) \;\neq\; 0\,, \tag{5.24}$$

and at $\overline{P}$ *the matrix*

$$\begin{pmatrix} G_x & G_y \\ \phi_x & \phi_y \end{pmatrix} \tag{5.25}$$

has two real distinct eigenvalues: $\lambda_- < 0 < \lambda_+$ *with* $|\lambda_-| < \lambda_+$. *Moreover, calling* $\Sigma_- \doteq \{(x,y)\,;\quad y = y'_- x\}$ *the stable eigenspace corresponding to the eigenvalue* λ_-, *at* $\overline{P}$ *one has*

$$\beta y'_- - \psi \;=\; 0, \tag{5.26}$$

together with

$$\beta_z \xi'_- - \psi_z < 0\,. \tag{5.27}$$

Notice that the shorter notation used in (5.24) is meaningful, because when $\theta = 0$, the functions G, ϕ do not depend on η. It is important to observe that the assumption (5.27) is always satisfied by the system (5.6), because of (5.21).

Finally, we make an assumption of global nature:

(A3) *When $\theta = 0$, the stable solution $x \mapsto W^0(x) \doteq (y^0(x), z^0(x))$ passing through $\overline{P} = (\bar{x}, \bar{y}, \bar{z})$ is well defined on a neighborhood of the compact interval $I \doteq [a, b]$ containing $\bar{x}$ in the interior, and*

$$\begin{aligned} &G(x, y^0(x)) > 0 \qquad \text{if} \quad a \le x < \bar{x}\,,\\ &G(x, y^0(x)) < 0 \qquad \text{if} \quad \bar{x} < x \le b\,. \end{aligned} \tag{5.28}$$

The structure of solutions to the system (5.7) for $\theta > 0$ depends in a crucial way on the sign of the quantities α, β. Indeed, we have the following alternatives.

THEOREM 1 *Let the implicit ODE (5.7) satisfy the assumptions (A1)–(A3).*

(i) When $\theta = 0$, assume that at the point $\overline{P} = (\bar{x}, \bar{y}, \bar{z})$ one has $\alpha\beta < 0$. Then for each $\theta > 0$ small enough the system (5.7) has infinitely many smooth solutions defined on the interval $I = [a, b]$, close to the reference solution (y^0, z^0).

(ii) On the other hand, if $\alpha\beta > 0$, then, under a generic transversality condition, for each $\theta > 0$ small enough the system (5.7) has a unique smooth solution $(y^\theta, z^\theta) : I \mapsto \mathbb{R}^2$. As $\theta \to 0$, this solution approaches (y^0, z^0), uniformly on I.

All of the above solutions determine a stable dynamics.

A proof of the above results, including a precise statement of the transversality condition, can be found in Bressan (2009). As a motivation, here we observe that, if $\alpha\beta < 0$, then for $\theta > 0$ the implicit ODE (5.7) can be rewritten in the standard form (5.8). Given a solution $x \mapsto (y(x), z(x))$ defined on the interval I, one can thus construct infinitely many nearby solutions, by slightly changing the initial data at $x = \bar{x}$.

To analyze the case where $\alpha\beta > 0$, we write (5.7) as a Pfaffian system

$$\begin{cases} \omega_1 \;\doteq\; -\,\phi\,dx + G\,dy + \theta^2\alpha\,dz \;=\; 0\,,\\ \omega_2 \;\doteq\; -\,\psi\,dx + \beta\,dy + G\,dz \;=\; 0\,. \end{cases} \tag{5.29}$$

The graph of a solution to (5.7) can then be obtained by suitably concatenating trajectories of the vector field

$$\mathbf{v} \;=\; \omega_1 \wedge \omega_2 \;=\; \begin{pmatrix} G^2 - \theta^2\alpha\beta \\ G\phi - \theta^2\alpha\psi \\ G\psi - \beta\phi \end{pmatrix}. \tag{5.30}$$

It $\alpha\beta > 0$, then the first component of $\mathbf{v}$ vanishes along the two surfaces

$$\Sigma^{\pm}_\theta \;\doteq\; \Big\{(x, y, z)\,;\quad G = \pm\theta\sqrt{\alpha\beta}\Big\}.$$

Hence, either $\mathbf{v} = 0 \in I\!R^3$, or else $\mathbf{v}$ is vertical. The only way to connect trajectories of the vector field $\mathbf{v}$ forming the graph of a smooth function $x \mapsto W^\theta(x) = (y^\theta(x), z^\theta(x))$ is to cross the surfaces $\Sigma^\pm_\theta$ somewhere along the two curves where $\mathbf{v}$ vanishes, namely

$$\gamma^\pm_\theta \doteq \left\{(x,y,z)\,; \qquad G = \pm\,\theta\sqrt{\alpha\beta}\,, \qquad \phi = \pm\,\theta\sqrt{\frac{\alpha}{\beta}}\,\psi\right\}. \tag{5.31}$$

One thus needs to study the stable and unstable manifolds through points on the curves $\gamma^\pm_\theta$. Under a generic transversality assumption, one obtains the existence of a unique heteroclinic orbit connecting a point $p^- \in \gamma^-_\theta$ to a point $p^+ \in \gamma^+_\theta$ (see Fig. 1).

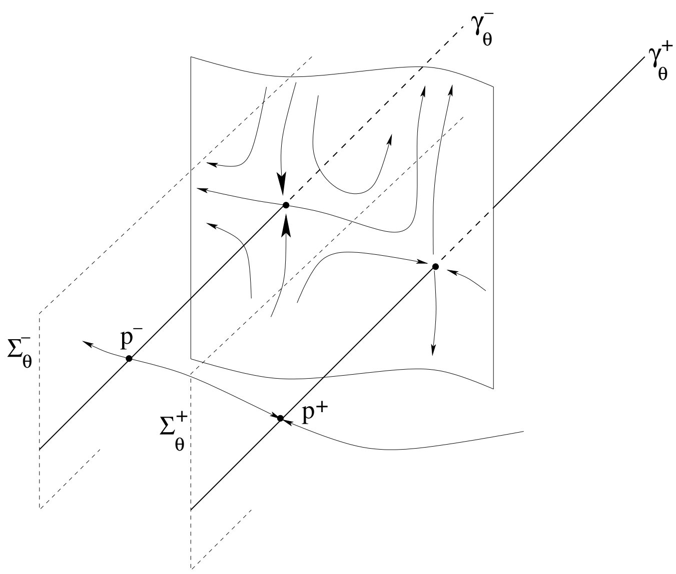

Figure 1. A unique heteroclinic orbit connects a pair of points p^-, p^+, on the lines $\gamma^\pm_\theta$

In a second step, one needs to check that this local solution can be extended to the entire interval I, remaining close to the reference solution W^0. This is achieved by showing that this extension remains close to the singular solution $\widetilde{W}^\theta(x) = (\tilde y^\theta(x), \tilde z^\theta(x))$ of the intermediate problem

$$\left\{\begin{aligned} G(x,y,\bar z,\theta)y' &= \phi(x,y,\bar z,\theta)\,,\\ \beta(x,y,z,\theta)y' + G(x,y,\bar z,\theta)z' &= \psi(x,y,z,\theta)\,, \end{aligned}\right. \tag{5.32}$$

passing through the point $(\bar x^\theta, \bar\xi^\theta, \bar\eta^\theta)$, where the corresponding equations (5.20) hold. Notice that in (5.32) we set $z = \bar z$ in the argument of G, ϕ. Hence the first equation can be solved independently of the second one. As $\theta \to 0$, one then shows that the solution $(\tilde{y}^\theta, \tilde{z}^\theta)$ of (5.32) approaches the solution (y^0, z^0) of (5.7), corresponding to $\theta = 0$.

In connection with the original system (5.6), the above results show that for $\theta > 0$ the differential game (4.1)-(4.2) has one or infinitely many Nash equilibrium solutions in feedback form, depending on the sign of the product $H^{(1)}_{\xi_2} \cdot H^{(2)}_{\xi_1}$, at the point $(x, \xi, \eta, \theta) = (\bar{x}, \bar{\xi}, \bar{\eta}, 0)$. This will be illustrated by two examples, in the next two sections.

6. A linear-quadratic game

Consider the parameter dependent, linear-quadratic game

$$\dot{x} \;=\; G(x, u, v, \theta) \;\doteq\; -x + u + \theta v. \tag{6.1}$$

The payoff functions are

$$J^u \doteq \int_0^\infty e^{-\rho t}\Big[ax - \frac{u^2}{2}\Big]\, dt\,, \tag{6.2}$$

$$J^v \doteq \int_0^\infty e^{-\rho t}\Big[bx - \frac{v^2}{2}\Big]\, dt\,. \tag{6.3}$$

We assume $\rho > 0$, while $a, b \neq 0$. Call $U(x)$, $V(x)$ the value functions for the two players, and set $\xi = U'$, $\eta = V'$. The optimal controls are then computed as

$$u^*(x, \xi) \;=\; \operatorname*{argmax}_\omega \Big(\xi \cdot u - \frac{u^2}{2}\Big) \;=\; \xi\,, \tag{6.4}$$

$$v^*(x, \eta, \theta) \;=\; \operatorname*{argmax}_\omega \Big(\theta\eta \cdot v - \frac{v^2}{2}\Big) \;=\; \theta\eta\,. \tag{6.5}$$

The value functions U, V can be found by solving a system of Hamilton-Jacobi equations, depending on the parameter θ,

$$\begin{cases} \rho U &= H(x, U', V', \theta)\,, \\ \rho V &= K(x, U', V', \theta)\,. \end{cases} \tag{6.6}$$

Here

$$\begin{aligned} H(x, \xi, \eta) &= \xi \cdot \Big(-x + u^*(x, \xi) + \theta\, v^*(x, \eta, \theta)\Big) + \Big(ax - \frac{(u^*(x, \xi))^2}{2}\Big) \\ &= \frac{\xi^2}{2} + (a - \xi)x + \theta^2 \xi\eta\,, \end{aligned} \tag{6.7}$$

$$\begin{aligned} K(x, \xi, \eta) &= \eta \cdot \Big(-x + u^*(x, \xi) + \theta\, v^*(x, \eta, \theta)\Big) + \Big(bx - \frac{(v^*(x, \eta, \theta))^2}{2}\Big) \\ &= \xi\eta + (b - \eta)x + \frac{\theta^2\eta^2}{2}\,. \end{aligned} \tag{6.8}$$

Notice that

$$H_\xi \;=\; K_\eta \;=\; -x+\xi+\theta^2\eta \;\doteq\; G(x,\xi,\eta,\theta)\,. \tag{6.9}$$

Differentiating (6.6) we obtain the system

$$\begin{cases} \rho\xi &= H_x + H_\xi\,\xi' + H_\eta\,\eta'\,,\\ \rho\eta &= K_x + K_\xi\,\xi' + K_\eta\,\eta'\,. \end{cases} \tag{6.10}$$

More explicitly,

$$\begin{pmatrix} \xi+\theta^2\eta - x & \theta^2\xi \\ \eta & \xi+\theta^2\eta-x \end{pmatrix}\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} (1+\rho)\xi - a \\ (1+\rho)\eta - b \end{pmatrix}. \tag{6.11}$$

We consider first the case of $\theta = 0$, so that (6.11) reduces to

$$\begin{cases} \rho\xi &= (a-\xi) + (\xi - x)\,\xi'\,,\\ \rho\eta &= (b-\eta) + \eta\,\xi' + (\xi-x)\;\eta'\,. \end{cases} \tag{6.12}$$

The stationary solution with $\dot x = \xi - x = 0$ is found to be

$$\bar x = \xi(\bar x) = \frac{a}{1+\rho}\,. \tag{6.13}$$

Substituting in (6.12) the linear functions

$$\xi = Ax+B \qquad\qquad \eta = Cx + D\,,$$

we obtain

$$\begin{cases} \rho(Ax+B) &= (a-Ax-B) + (Ax+B-x)\,A\,,\\ \rho(Cx+D) &= (b-Cx-D) + (Cx+D)\,A + (Ax+B-x)\;C\,, \end{cases} \tag{6.14}$$

$$\begin{aligned} &A^2 - (2+\rho)A = 0\,, && B(1+\rho-A) = a\,,\\ &(2+\rho-2A)C = 0\,, && (1+\rho-A)D = b + BC\,. \end{aligned}$$

We thus have two solutions. The first one is

$$A=0,\qquad B=\frac{a}{1+\rho}\,,\qquad C=0,\qquad D=\frac{b}{1+\rho}\,,$$

corresponding to the stable dynamics

$$\dot x \;=\; \bar x - x\,.$$

The second one is

$$A = 2+\rho\,,\qquad B=-a\,,\qquad C=0\,,\qquad D=-b,$$

corresponding to the unstable dynamics

$$\dot{x} \;=\; (1+\rho)(x-\bar{x})\,.$$

Next, we study the bifurcation occurring for $\theta > 0$, in a neighborhood of the singular point

$$(\bar{x}, \bar{\xi}, \bar{\eta}) \;=\; \left(\frac{a}{1+\rho}\,,\;\frac{a}{1+\rho}\,,\;\frac{b}{1+\rho}\right).$$

Comparing (6.11) with (5.7) we see that

$$\begin{aligned} &G = \xi + \theta^2\eta - x\,, \qquad \alpha = \xi\,, \qquad \beta = \eta\,,\\ &\phi = (1+\rho)\xi - a\,, \qquad \psi = (1+\rho)\eta - b\,. \end{aligned}$$

One can easily check that the assumption (5.9) holds, and that the conditions (5.15) and (5.27) are satisfied, with $\xi'_- = 0$. To see which case in Theorem actually occurs, for $\theta = 0$ at the singular point $(\bar{x}, \bar{\xi}, \bar{\eta})$, we compute

$$\alpha\cdot\beta \;=\; \xi\eta \;=\; \frac{ab}{(1+\rho)^2}\,, \qquad\qquad G_x + \frac{\psi}{\beta}G_\xi \;=\; \rho - \frac{b}{\eta} \;=\; -1\,.$$

We thus have two cases.

CASE 1: If $ab > 0$, then for each $\theta > 0$ small there exists a unique solution $(\xi^\theta,\, \eta^\theta)$ of the implicit ODE (6.11) close to the stable solution $(\xi^0,\, \eta^0) \equiv \left(\frac{a}{1+\rho}\,,\,\frac{b}{1+\rho}\right)$.

CASE 2: If $ab < 0$, then for each $\theta > 0$ small, there exist infinitely many solutions $x \mapsto (\xi^\theta,\, \eta^\theta)$ close to the stable solution $(\xi^0,\, \eta^0)$.

In both of the above cases, the couple of constant functions

$$\xi^\theta(x) \;\equiv\; \bar{\xi} \;=\; \frac{a}{1+\rho}\,, \qquad\qquad \eta^\theta(x) \;\equiv\; \bar{\eta} \;=\; \frac{b}{1+\rho}\,, \tag{6.15}$$

provides a solution to (6.11) also for $\theta > 0$. The corresponding dynamics

$$\dot{x} \;=\; \frac{a+\theta^2 b}{1+\rho} - x$$

admits the point $\bar{x}^\theta = \frac{a+\theta^2 b}{1+\rho}$ as the unique globally stable equilibrium.

However, a major difference must be pointed out. Let $I \subset I\!\!R$ be a compact interval containing the point $\bar{x} = \frac{\alpha}{1+\rho}$ in its interior. Then, as in (5.30), the graph of any solution of (6.11) defined on the entire interval I must be a concatenation of trajectories for the vector field

$$\mathbf{v}^\theta = \begin{pmatrix} (\xi+\theta^2\eta-x)^2-\theta^2\xi\eta \\ [b-(1+\rho)\eta]\,\theta^2\xi-[a-(1+\rho)\xi]\,(\xi+\theta^2\eta-x) \\ [a-(1+\rho)\xi]\,\eta-[b-(1+\rho)\eta]\,(\xi+\theta^2\eta-x) \end{pmatrix}. \tag{6.16}$$

We observe that, along the surfaces

$$\Gamma_\pm = \Big\{(x,\xi,\eta)|,;\quad x=(\xi+\theta^2\eta)\pm\theta\sqrt{\xi\eta}\Big\},$$

the first component of $\mathbf{v}^\theta$ vanishes. Therefore, the graph of any regular solution of (6.11) globally defined on I and close to the solution (ξ^0,η^0) must cross the surfaces $\Gamma_\pm$ at points $P_\pm$, where the vector field $\mathbf{v}^\theta$ vanishes. This can happen only at the points

$$P_\pm = (\bar{x}_\pm,\bar{\xi},\bar{\eta})\,, \qquad\qquad \bar{x}_\pm = \frac{a+\theta^2 b}{1+\rho}\pm\frac{\theta\sqrt{ab}}{1+\rho}\,.$$

The only trajectory of $\mathbf{v}^\theta$ connecting these points P_-,P_+ corresponds to the constant solution (6.15).

On the other hand, if $ab<0$, then for $\theta>0$ the determinant of the coefficient matrix in (6.11) is uniformly positive definite as (ξ,η) range in a neighborhood of $(\frac{a}{1+\rho}\,,\ \frac{b}{1+\rho})$. Hence we can rewrite the implicit ODE (6.11) in the standard form

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \frac{1}{(\xi+\theta^2\eta-x)^2-\theta^2\xi\eta}\begin{pmatrix} \xi+\theta^2\eta-x & -\theta^2\xi \\ -\eta & \xi+\theta^2\eta-x \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{6.17}$$

Since all coefficients are locally smooth, by slightly changing the initial data, say at $x=\bar{x}$, we can construct infinitely many other solutions, globally defined on the interval I.

7. A nonlinear sticky price game

Let $p(t)$ denote the price of a good at time t. We assume that this good can be produced by one of the players, at rate $a(t)$, and consumed by the other player at rate $b(t)$. In a very simplified model, the variation of the price in time can be described by the differential equation

$$\dot{p} = (b-a)p\,. \tag{7.1}$$

Here the functions $t\mapsto a(t)$ and $t\mapsto b(t)$ represent the *controls* implemented by the two players. According to (7.1), the price increases when consumption is larger than production, and decreases otherwise.

In addition, we consider the exponentially discounted payoffs

$$J^{prod} = \int_0^\infty e^{-\beta t}\big[p(t)\cdot b(t) - c\big(a(t)\big)\big]\,dt\,, \tag{7.2}$$

$$J^{cons} = \int_0^\infty e^{-\beta t}\big[\phi(b(t)) - p(t)b(t)\big]\,dt\,. \tag{7.3}$$

The payoff for the producer is given by the profit generated by sales, minus the cost $c(a)$ of producing the good at rate a. The payoff for the consumer is measured by a utility function $\phi(b)$, minus the price paid to buy the good. For the sake of definiteness, throughout the following we choose

$$c(a) = \frac{a^2}{2}\,, \qquad\qquad \phi(b) = 2\sqrt{b}\,. \tag{7.4}$$

As a preliminary we observe that, if the consumer adopts the myopic strategy:

$$b^\dagger(p) \ \doteq\ \operatorname*{argmax}_b \ \Big\{\phi(b) - p\,b\Big\} \ =\ \frac{1}{p^2}\,, \tag{7.5}$$

then the differential game would reduce to an optimal control problem for the producer. Namely:

$$\text{maximize:} \qquad J^{\rm prod} \ \doteq\ \int_0^\infty e^{-\rho t}\left[\frac{1}{p(t)} - \frac{a^2(t)}{2}\right] dt\,, \tag{7.6}$$

where the evolution of the price is governed by the ODE

$$\dot p \ =\ \frac{1}{p} - p\,a\,. \tag{7.7}$$

We now want to study an intermediate situation, where a fraction $\theta \in [0,1]$ of all consumers join forces and play strategically, while the remaining ones still behave myopically. Equivalently, we may also think of one large consumer, whose actions are motivated by a long-term strategy, and several small consumers who simply maximize their instantaneous payoff.

Denoting by $b(\cdot)$ the control implemented by the single large consumer, and by $J^{\rm cons}$ his expected payoff, we obtain the system

$$\dot p \ =\ \Big[\theta b + (1-\theta)b^\dagger - a\Big]\,p \ =\ \left[\theta b + \frac{1-\theta}{p^2} - a\right]\,p \tag{7.8}$$

with payoff functionals

$$J^{\rm prod} \ =\ \int_0^\infty e^{-\rho t}\left[\theta p\,b + \frac{1-\theta}{p} - \frac{a^2}{2}\right] dt\,, \tag{7.9}$$

$$J^{\rm cons} \;=\; \int_0^\infty e^{-\rho t}\,\Big[2\sqrt{b} - p\,b\Big]\,dt\,. \tag{7.10}$$

To derive the corresponding system of Hamilton-Jacobi equations, we first compute the optimal feedback controls. Assuming $\xi < 0$ and $\theta\eta < 1$, we find

$$a^*(p,\xi,\theta) = \operatorname*{argmax}_a \left\{\xi\cdot\Big(\theta b + \frac{1-\theta}{p^2} - a\Big)p + \Big(\theta p b + \frac{1-\theta}{p} - \frac{a^2}{2}\Big)\right\} = -p\xi\,, \tag{7.11}$$

$$b^*(p,\eta,\theta) = \operatorname*{argmax}_a \left\{\eta\cdot\Big(\theta b + \frac{1-\theta}{p^2} - a\Big)p + \Big(2\sqrt{b} - p\,b\Big)\right\} = \frac{1}{(1-\theta\eta)^2p^2}\,. \tag{7.12}$$

The corresponding Hamiltonian functions for the producer and for the single large consumer are computed as

$$\begin{aligned} H(p,\xi,\eta,\theta) &= p\,\xi\left[\frac{1-\theta}{p^2} + \frac{\theta}{(1-\theta\eta)^2p^2} + p\xi\right] + p\left[\frac{1-\theta}{p^2} + \frac{\theta}{(1-\theta\eta)^2p^2}\right] - \frac{(p\xi)^2}{2} \\ &= \frac{\xi+1}{p}\left[(1-\theta) + \frac{\theta}{(1-\theta\eta)^2}\right] + \frac{p^2\xi^2}{2} \end{aligned} \tag{7.13}$$

$$\begin{aligned} K(p,\xi,\eta,\theta) &= \; p\,\eta\left[\frac{1-\theta}{p^2} + \frac{\theta}{(1-\theta\eta)^2p^2} + p\xi\right] + \frac{2}{(1-\theta\eta)p} - \frac{p}{(1-\theta\eta)^2p^2} \\ &= \; \frac{1}{p}\left[(1-\theta)\eta + \frac{1}{1-\theta\eta}\right] + p^2\xi\eta\,. \end{aligned} \tag{7.14}$$

The Hamilton-Jacobi system of equations takes the form

$$\begin{pmatrix} \xi p^2 + \dfrac{1}{p} + \dfrac{\theta^2\eta^2}{(1-\theta\eta)^2p} & \dfrac{2(\xi+1)\theta^2}{(1-\theta\eta)^3p} \\ \eta p^2 & \xi p^2 + \dfrac{1}{p} + \dfrac{\theta^2\eta^2}{(1-\theta\eta)^2p} \end{pmatrix}\begin{pmatrix}\xi' \\ \eta'\end{pmatrix}$$

$$= \begin{pmatrix} \rho\xi - \xi^2 p + \dfrac{\xi+1}{p^2}\left[(1-\theta) + \dfrac{\theta}{(1-\theta\eta)^2}\right] \\ \rho\eta - 2\xi\eta p + \dfrac{1}{p^2}\left[(1-\theta)\eta + \dfrac{1}{1-\theta\eta}\right]\end{pmatrix}. \tag{7.15}$$

After renaming the variables $x \doteq p,\ y = \xi,\ z = \eta$, the system (7.15) has the same form as (5.7), with

$$G = H_\xi = \xi p^2 + \frac{1}{p} + \frac{\theta^2\eta^2}{(1-\theta\eta)^2p}\,, \qquad \theta^2\alpha = \frac{2(\xi+1)\theta^2}{(1-\theta\eta)^3p}\,, \qquad \beta = \eta p^2\,, \tag{7.16}$$

$$\phi = \rho\xi - \xi^2 p + \frac{\xi+1}{p^2}\left[(1-\theta) + \frac{\theta}{(1-\theta\eta)^2}\right],$$

$$\psi = \rho\eta - 2\xi\eta p + \frac{1}{p^2}\left[(1-\theta)\eta + \frac{1}{1-\theta\eta}\right]. \tag{7.17}$$

When $\theta = 0$, the Hamiltonian function (7) reduces to

$$H(p,\xi) = \frac{\xi+1}{p} + \frac{p^2\xi^2}{2}, \tag{7.18}$$

and the system (2.16) takes the form

$$\begin{cases} \dot{p} &=& H_\xi &=& \dfrac{1}{p} + \xi p^2, \\ \dot{\xi} &=& \rho\xi - H_p &=& -\dfrac{1+\xi}{p^2} + \xi^2 p. \end{cases} \tag{7.19}$$

The coordinates $(\bar{p},\bar{\xi})$ of a stationary point are found by solving

$$\xi = -\frac{1}{p^3}, \qquad p^3 - \rho p^2 = 2. \tag{7.20}$$

To check the stability of the stationary solution, according to (2.11) we need to establish the inequality

$$H_{p\xi} + H_{\xi\xi}\xi' < 0. \tag{7.21}$$

Toward this goal, recalling (7.18), we compute

$$H_{pp} = \xi^2 + \frac{2(1+\xi)}{p^3}, \qquad H_{p\xi} = 2\xi p - \frac{1}{p^2}, \qquad H_{\xi\xi} = p^2. \tag{7.22}$$

By differentiating the equation

$$\xi' = \frac{\rho\xi - H_p}{H_\xi} = \left(\frac{1+\xi}{p^2} + \rho\xi - \xi^2 p\right)\frac{p}{1+p^3\xi} \tag{7.23}$$

we obtain

$$\begin{aligned} \rho\xi' &= H_{pp} + 2H_{p\xi}\xi' + H_{\xi\xi}(\xi')^2 + H_\xi\xi'' \\ &= \xi^2 + \frac{2(1+\xi)}{p^3} + 2\left(2\xi p - \frac{1}{p^2}\right)\xi' + p^2(\xi')^2 + \left(\frac{1}{p} + \xi p^2\right)\xi''. \end{aligned} \tag{7.24}$$

By (7.20) one has

$$\xi = -p^{-3}, \qquad \rho = p - \frac{2}{p^2}. \tag{7.25}$$

Hence

$$p^2(\xi')^2 + 2\left(2\xi p - \frac{1}{p^2} - \frac{\rho}{2}\right)\xi' + \xi^2 + \frac{2(1+\xi)}{p^3} \ = \ 0,$$

$$p^2(\xi')^2 - \left(\frac{6}{p^2} + \rho\right)\xi' + \frac{2}{p^3} - \frac{1}{p^6} \ = \ 0. \tag{7.26}$$

Using (7.25) we compute the discriminant

$$\Delta \ \doteq \ \left(\frac{6}{p^2} + \rho\right)^2 - 4\left(\frac{2}{p} - \frac{1}{p^4}\right) \ = \ \frac{20}{p^4} + p^2 > 0\,.$$

Choosing the smaller root in (7.26), we find

$$\xi'_- \ = \ \frac{\left(\frac{4}{p^2} + p\right) - \sqrt{\frac{20}{p^4} + p^2}}{2p^2}\,. \tag{7.27}$$

Inserting this in (7.21) and using (7.22) we get

$$H_{p\xi} + H_{\xi\xi}\xi' \ = \ -\frac{3}{p^2} + \left(\frac{2}{p^2} + \frac{p}{2}\right) - \frac{1}{2}\sqrt{\frac{20}{p^4} + p^2} \ < \ 0.$$

This establishes the existence of a locally stable solution.

To apply Theorem 1, we first need to compute the value $\bar{\eta}$ such that

$$\beta\xi'_- - \psi = \eta p^2\,\xi'_- - \left(\rho\eta - 2\xi\eta p + \frac{1}{p^2}(\eta + 1)\right) \ = \ 0.$$

Using (7.27) we find

$$\bar{\eta} \ = \ \frac{1}{p^2}\cdot\left(p^2\xi'_- - \rho + 2\xi p - \frac{1}{p^2}\right)^{-1} \ < \ 0\,. \tag{7.28}$$

We now check the sign of the product $\alpha\,\beta$, when $\theta = 0$, at the equilibrium point $(\bar{p}, \bar{\xi}, \bar{\eta})$. Recalling (7.16)-(7.17), we find

$$\alpha\cdot\beta \ = \ \frac{2(\xi+1)}{p}\cdot\eta p^2 \ = \ \frac{2(p^3-1)\eta}{p^2} \ < \ 0\,, \tag{7.29}$$

because $\bar{\eta} < 0$ and $\bar{p} > 2^{1/3}$, by (7.28) and (7.20). Therefore, the first alternative in Theorem 1 applies. For $\theta > 0$ the differential game admits infinitely many Nash equilibrium solutions, all leading to a stable dynamics.

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