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Exact penalization of pointwise constraints for optimal control problems*

by

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Abstract: In this paper we consider a control problem governed by a semilinear elliptic equation with pointwise control and state constraints. We analyze the existence of an exact penalization of the state constraints. In particular, we prove that the first and second order optimality conditions imply the existence of such a penalization. Finally, we prove some extra regularity of the strict local minima of the control problem, assuming the existence of an exact penalization for them.

Keywords: optimal control, semilinear elliptic equations, pointwise state constraints, exact penalization, first and second order optimality conditions.

1. Introduction

This paper deals with some optimal control problems governed by semilinear elliptic equations, the control being distributed in the domain Ω. Pointwise control and state constraints are considered. It is known that under a stability assumption of the infimum of the control problem with respect to small perturbations of the set of feasible states, there exists an exact penalization of the state constraints. This property was first used by Clarke (1976a) under the name of *calm*. F. Bonnans and E. Casas (1995) used this property to derive the maximum principle of state constrained optimal control problems. This notion has been also used in an abstract framework for optimization problems; see Burke (1991), Clarke (1976b) and Bonnans and Shapiro (2000).

An important property of this stability concept is that almost all problems are stable. Also the existence of an exact penalization for some nonlinear programming problems is known, under the assumption that the first and sufficient

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second order optimality conditions are satisfied. However, as far as we know, there is no analogous result for control problems of PDE with pointwise state constraints. We will prove it in this paper.

The plan of the paper is as follows. After introducing the control problem and studying the existence of solutions, we define the stability concept and establish the first properties. Since our problem is not convex, we will consider local minima and the stability definition will be given in the neighborhoods of the local solutions. Then, by using the sufficient second order optimality conditions recently derived by Casas, De los Reyes and Tröltzsch (2008), we prove that the first and second order conditions imply that the control problem is stable and, consequently, there exists an exact penalization of the state constraints. We finish the paper by showing that the strict local solutions of the control problems are elements of the Sobolev space $H^1(\Omega)$ provided the stability assumption is fulfilled. The $W^{1,s}(\Omega)$-regularity, with $s < n/(n-1)$, for the optimal controls, which follows from the first order optimality conditions, has been known for quite some time. Recently, Casas and Tröltzsch (2008) obtained the $H^1(\Omega)$-regularity under the Slater assumption. Here, we replace the Slater hypothesis by the stability assumption to achieve the same result.

2. The control problem

Let Ω be an open bounded subset of $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz boundary Γ. Let us consider in Ω the following boundary value problem

$$\left\{ \begin{array}{rcll} Ay + a_0(x,y) & = & u & \text{in } \Omega, \\ y & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{2.1}$$

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j}\left(a_{ij}(x)\partial_{x_i} y(x)\right), \quad a_{ij} \in L^\infty(\Omega),\ 1 \le i,j \le n,$$

$$\exists \Lambda_A > 0 \text{ such that } \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \Lambda_A|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \text{ and for a.e. } x \in \Omega.$$

Let us make the following assumptions on a_0.

(A1) The mapping $a_0 : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable and there exists a real number $p > n/2$ such that $a_0(\cdot,0) \in L^p(\Omega)$, $(\partial a_0/\partial y)(x,y) \ge 0$ for almost all $x \in \Omega$. Moreover, for all $M > 0$ there exists a constant $C_{a_0,M} > 0$ such that

$$\left|\frac{\partial a_0}{\partial y}(x,y)\right| + \left|\frac{\partial^2 a_0}{\partial y^2}(x,y)\right| \le C_{a_0,M} \text{ for a.e. } x \in \Omega \text{ and } |y| \le M,$$

$$\left|\frac{\partial^2 a_0}{\partial y^2}(x,y_2) - \frac{\partial^2 a_0}{\partial y^2}(x,y_1)\right| \le C_{a_0,M}|y_2 - y_1| \text{ for a.e. } x \in \Omega,\ |y_1|,|y_2| \le M.$$

The previous assumptions are not very restrictive, except for the one requiring the monotonicity of a_0 with respect to y, which is necessary for the existence and uniqueness of the solution of (2.1). Indeed, for any $u \in L^p(\Omega)$ it is well known that (2.1) has a unique solution $y_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$, depending continuously on the control u; see Grisvard (1985) for the regularity results. Examples of functions a_0 fulfilling the above assumptions are $a_0(x,y) = \theta_1(x) + \theta_2(x) \exp y$ or $a_0(x,y) = \theta_1(x) + \theta_2(x)|y|^\nu y$, with $\nu \geq 1$, $\theta_1 \in L^p(\Omega)$, $0 \leq \theta_2(x)$ and $\theta_2 \in L^\infty(\Omega)$.

Associated with the equation (2.1) we consider the following control problem

$$(\mathrm{P}_\delta) \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))\, dx \\ \alpha(x) \leq u(x) \leq \beta(x) \quad \text{for a.e. } x \in \Omega \quad \text{and} \quad |y_u(x)| \leq \delta \ \forall x \in K. \end{cases}$$

We will make the following assumptions on the data of the control problem.

(A2) The functions α, β are given in $L^\infty(\Omega)$, with $\alpha \leq \beta$ a.e. in Ω. We will denote

$$\mathcal{U}_{\alpha,\beta} = \{u \in L^\infty(\Omega) : \alpha(x) \leq u(x) \leq \beta(x) \quad \text{for a.e. in } \Omega\}$$

$$\mathcal{Y}_\delta = \{y \in C(K) : |y(x)| \leq \delta \quad \forall x \in K\} \quad \text{and} \quad \mathcal{U}_\delta = \{u \in \mathcal{U}_{\alpha,\beta} : y_u \in \mathcal{Y}_\delta\},$$

where K is a nonempty compact subset of $\bar{\Omega}$ and $\delta \geq 0$ is given.

(A3) $L : \Omega \times (\mathbb{R} \times \mathbb{R}) \longrightarrow \mathbb{R}$ is a Carathéodory function, of class C^2 with respect to the last two variables, $L(\cdot, 0, 0) \in L^1(\Omega)$, and for all $M > 0$ there exist a constant $C_{L,M} > 0$ and functions $\psi_{u,M} \in L^2(\Omega)$ and $\psi_{y,M} \in L^1(\Omega)$, such that

$$\left|\frac{\partial L}{\partial u}(x,y,u)\right| \leq \psi_{u,M}(x), \ \left|\frac{\partial L}{\partial y}(x,y,u)\right| \leq \psi_{y,M}(x), \ \|D^2_{(y,u)} L(x,y,u)\| \leq C_{L,M},$$

$$\|D^2_{(y,u)} L(x, y_2, u_2) - D^2_{(y,u)} L(x, y_1, u_1)\| \leq C_{L,M}(|y_2 - y_1| + |u_2 - u_1|),$$

for a.e. $x \in \Omega$ and $|y|, |y_i|, |u|, |u_i| \leq M$, $i = 1, 2$, where $D^2_{(y,u)} L$ denotes the second derivative of L with respect to (y, u), i.e. the associated Hessian matrix.

As a consequence of *(A3)* it follows that for any $M > 0$ there exists a function $\psi_M \in L^1(\Omega)$ such that

$$|L(x,y,u)| \leq \psi_M(x) \quad \text{for a.e. } x \in \Omega \quad \text{and} \quad |y|, |u| \leq M. \tag{2.2}$$

We have the following theorem concerning the existence of a solution of problem (P_δ).

THEOREM 1 *There exists $\delta_0 \geq 0$ such that (P_δ) has at least one feasible control for every $\delta \geq \delta_0$, and there is no feasible control for $\delta < \delta_0$. Moreover, if L is convex with respect to the third variable, then (P_δ) has at least one solution for every $\delta \geq \delta_0$.*

Proof. Using the boundedness of $\mathcal{U}_{\alpha,\beta}$ in $L^\infty(\Omega)$, we deduce the existence of a constant $M_{\alpha,\beta} > 0$ such that

$$\|y_u\|_{C(K)} \leq M_{\alpha,\beta} \quad \forall u \in \mathcal{U}_{\alpha,\beta}. \tag{2.3}$$

It is obvious that (P_δ) has no feasible control for $\delta < 0$ and, at the same time, all elements of $\mathcal{U}_{\alpha,\beta}$ are feasible for $\delta \geq M_{\alpha,\beta}$. Let δ_0 be the infimum of the values δ for which (P_δ) has feasible controls. Then $0 \leq \delta_0 \leq M_{\alpha,\beta}$ and (P_δ) has no feasible control for $\delta < \delta_0$. Let us prove that there exists at least one feasible control for (P_{δ_0}). Let $\{\delta_j\}_{j=1}^\infty$ be a decreasing sequence converging to δ_0 and $\{u_j\} \subset \mathcal{U}_{\alpha,\beta}$ a sequence of controls such that u_j is feasible for (P_{δ_j}). Since $\mathcal{U}_{\alpha,\beta}$ is bounded, we can extract a subsequence, denoted in the same way, converging *weakly in $L^\infty(\Omega)$ towards an element $u_0 \in \mathcal{U}_{\alpha,\beta}$. This implies the uniform convergence of $\{y_{u_j}\}$ to y_{u_0} and therefore

$$|y_{u_0}(x)| = \lim_{j\to\infty} |y_{u_j}(x)| \leq \lim_{j\to\infty} \delta_j = \delta_0 \quad \forall x \in K,$$

which proves that u_0 is a feasible control for (P_{δ_0}). To conclude the proof we must establish the existence of an optimal control of (P_δ) for all $\delta \geq \delta_0$, but this follows by classical arguments. ∎

3. Strongly stable problems

Since problem (P_δ) is not convex, we are interested in local solutions. Let us fix $\bar{u} \in \mathcal{U}_\delta$ and let $\bar{y}$ be its associated state. The control $\bar{u}$ is said to be a local solution of (P_δ) in the sense of the $L^q(\Omega)$ topology, $1 \leq q \leq \infty$, if there exists $r > 0$ such that $\bar{u}$ is the solution of the problem

$$(\mathrm{P}_\delta^{r,q}) \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))\,dx \\ u \in \mathcal{U}_{\alpha,\beta},\ \|u - \bar{u}\|_{L^q(\Omega)} \leq r,\ |y_u(x)| \leq \delta\ \forall x \in K. \end{cases}$$

By taking $r = +\infty$ we identify (P_δ) and $(\mathrm{P}_\delta^{\infty,q})$.

Since $\mathcal{U}_{\alpha,\beta}$ is bounded in $L^\infty(\Omega)$, it is easy to check that $\bar{u}$ is a local solution in the sense of $L^q(\Omega)$, for some $1 \leq q < +\infty$, if and only if it is also a local solution in the sense of $L^t(\Omega)$ for every $1 \leq t < +\infty$. Moreover, if $\bar{u}$ is a local solution in the sense of $L^q(\Omega)$, then it is a local solution in the sense of $L^\infty(\Omega)$ too.

The following concept was introduced by Clarke (1976a) under the denomination of *calm*. We prefer to use the term *stable* instead of *calm*; see Bonnans and Casas (1995).

Definition 1 *We will say that problem* $(\mathrm{P}_\delta^{r,q})$ *is strongly stable (on the right) if there exist* $C_\delta > 0$ *and* $\varepsilon_\delta > 0$ *such that*

$$\inf(\mathrm{P}_\delta^{r,q}) - \inf(\mathrm{P}_{\delta'}^{r,q}) \leq C_\delta(\delta' - \delta) \quad \forall \delta' \in [\delta, \delta + \varepsilon_\delta]. \tag{3.1}$$

The next proposition states that almost all problems are stable.

Proposition 1 *Let δ_0 be as in Theorem 1 and $M_{\alpha,\beta}$ be given by (2.3). Then for every $\delta \geq \delta_0$, except at most a set of zero Lebesgue measure contained in $[\delta_0, M_{\alpha,\beta})$, the problem $(\mathrm{P}_\delta^{r,q})$ is strongly stable.*

Proof. First of all let us remark that $\mathcal{U}_{\alpha,\beta} = \mathcal{U}_\delta$ for every $\delta \geq M_{\alpha,\beta}$, which obviously implies strong stability for every $\delta \geq M_{\alpha,\beta}$ and for every $r > 0$.

On the other hand, if we define $\phi : [\delta_0, +\infty) \longrightarrow \mathbb{R}$ by $\phi(\delta) = \inf(\mathrm{P}_\delta^{r,q})$, then ϕ is a non increasing monotone function and therefore ϕ is differentiable at each point except at most a set of zero Lebesgue measure. Finally, it is obvious that differentiability of ϕ at δ implies strong stability of $(\mathrm{P}_\delta^{r,q})$. ∎

The following proposition justifies the introduction of the stability concept.

Proposition 2 *Let us assume that $\bar{u}$ is a solution of $(\mathrm{P}_\delta^{r,q})$ and this problem is strongly stable. Then the following statements hold.*
1. $(\mathrm{P}_\delta^{r',q})$ is strongly stable for every $0 < r' < r$, with the same numbers $C_\delta > 0$ and $\varepsilon_\delta > 0$.
2. If $q < +\infty$ then $(\mathrm{P}_\delta^{r_t,t})$ is strongly stable, with the same numbers $C_\delta > 0$ and $\varepsilon_\delta > 0$, for every $1 \leq t \leq +\infty$ and r_t given by

$$r_t = \begin{cases} \dfrac{r^{q/t}}{\|\beta - \alpha\|_{L^\infty(\Omega)}^{(q-t)/t}} & \text{if } 1 \leq t < q \\ |\Omega|^{\frac{q-t}{tq}} r & \text{if } q < t < +\infty \\ |\Omega|^{-\frac{1}{q}} r & \text{if } t = +\infty, \end{cases}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.
3. There exists $\rho_0 > 0$ such that for every $\rho \geq \rho_0$ $\bar{u}$ is also a solution of the problem

$$(\mathrm{P}_{\delta,\rho}^{r,q}) \begin{cases} \min J_\rho(u) = J(u) + \rho(\|y_u\|_{C(K)} - \delta)^+ \\ \alpha(x) \leq u(x) \leq \beta(x) \text{ for a.e. } x \in \Omega,\ \|u - \bar{u}\|_{L^q(\Omega)} \leq r. \end{cases}$$

Furthermore, if $\bar{u}$ is a strict solution of $(\mathrm{P}_\delta^{r,q})$, it is also a strict solution of $(\mathrm{P}_{\delta,\rho}^{r,q})$ if $\rho > \rho_0$.

Proof. Let us prove the first part. Given $\delta' \in [\delta, \delta + \varepsilon_\delta]$ and $0 < r' < r$, it is obvious that the set of feasible controls for $(\mathrm{P}_{\delta'}^{r',q})$ is a subset of the corresponding set for $(\mathrm{P}_{\delta'}^{r,q})$, therefore $\inf(\mathrm{P}_{\delta'}^{r',q}) \geq \inf(\mathrm{P}_{\delta'}^{r,q})$. On the other hand, $\bar{u}$ is a solution of $(\mathrm{P}_\delta^{r',q})$, so that $\inf(\mathrm{P}_\delta^{r,q}) = J(\bar{u}) = \inf(\mathrm{P}_\delta^{r',q})$ for every $0 < r' \leq r$. Consequently, (3.1) leads to

$$\inf(\mathrm{P}_\delta^{r',q}) - \inf(\mathrm{P}_{\delta'}^{r',q}) \leq \inf(\mathrm{P}_\delta^{r,q}) - \inf(\mathrm{P}_{\delta'}^{r,q}) \leq C_\delta(\delta' - \delta) \quad \forall \delta' \in [\delta, \delta + \varepsilon_\delta].$$

Let us prove the second part. From the choice of r_t we deduce that $\|u - \bar{u}\|_{L^t(\Omega)} \leq r_t$ implies $\|u - \bar{u}\|_{L^q(\Omega)} \leq r$. Therefore, any feasible point for $(\mathrm{P}_{\delta'}^{r_t,t})$ is also feasible for $(\mathrm{P}_{\delta'}^{r,q})$ for any $\delta' \in [\delta, \delta + \varepsilon_\delta]$. Hence, we have $\inf(\mathrm{P}_{\delta'}^{r,q}) \leq \inf(\mathrm{P}_{\delta'}^{r_t,t})$ and also $\inf(\mathrm{P}_{\delta}^{r,q}) = J(\bar{u}) = \inf(\mathrm{P}_{\delta}^{r_t,t})$. From these properties and (3.1) we deduce

$$\inf(\mathrm{P}_{\delta}^{r_t,t}) - \inf(\mathrm{P}_{\delta'}^{r_t,t}) \leq \inf(\mathrm{P}_{\delta}^{r,q}) - \inf(\mathrm{P}_{\delta'}^{r,q}) \leq C_\delta(\delta' - \delta) \quad \forall \delta' \in [\delta, \delta + \varepsilon_\delta].$$

We finish by proving the third part of the proposition. Let us take C_δ and ε_δ as in (3.1) and let ρ_0 be defined by

$$\rho_0 = \max\left\{C_\delta, \frac{J(\bar{u}) - m_{\alpha,\beta}^r}{\varepsilon_\delta}\right\}, \tag{3.2}$$

$$m_{\alpha,\beta}^r = \inf\{J(u) : u \in \mathcal{U}_{\alpha,\beta},\ \|u - \bar{u}\|_{L^q(\Omega)} \leq r\}.$$

Now, let us take $\rho \geq \rho_0$, $u \in \mathcal{U}_{\alpha,\beta}$ satisfying $\|u - \bar{u}\|_{L^q(\Omega)} \leq r$ and let us set $\delta' = \|y_u\|_{C(K)}$. If $\delta' \leq \delta$, then it is obvious that $J_\rho(\bar{u}) = J(\bar{u}) \leq J(u) = J_\rho(u)$. If $\bar{u}$ is a strict solution of $(\mathrm{P}_{\delta}^{r,q})$, then the above inequality is strict for $u \neq \bar{u}$. On other hand, if $\delta < \delta' \leq \delta + \varepsilon_\delta$, then (3.1) leads to

$$J_\rho(u) = J(u) + \rho(\delta' - \delta) \geq \inf(\mathrm{P}_{\delta'}^{r,q}) + C_\delta(\delta' - \delta) \geq \inf(\mathrm{P}_{\delta}^{r,q}) = J(\bar{u}) = J_\rho(\bar{u}).$$

The first inequality is strict if $\rho > \rho_0$. Finally let us assume that $\delta' > \delta + \varepsilon_\delta$, then

$$J_\rho(u) = J(u) + \rho(\delta' - \delta) > J(u) + \rho\varepsilon_\delta \geq J(u) + J(\bar{u}) - m_{\alpha,\beta}^r \geq J(\bar{u}) = J_\rho(\bar{u}).$$

The last three inequalities imply that $\bar{u}$ is a solution of $(\mathrm{P}_{\delta,\rho}^{r,q})$, the solution being strict if $\bar{u}$ is also a strict solution of $(\mathrm{P}_{\delta}^{r,q})$ and $\rho > \rho_0$. ■

The previous proposition claims that a (strict) local solution of (P_δ) is also a (strict) local solution of

$$(\mathrm{P}_{\delta,\rho}) \begin{cases} \min J_\rho(u) = J(u) + \rho(\|y_u\|_{C(K)} - \delta)^+ \\ \alpha(x) \leq u(x) \leq \beta(x) \text{ for a.e. } x \in \Omega \end{cases}$$

for every $\rho \geq \rho_0$ ($\rho > \rho_0$), with ρ_0 given by (3.2), assuming that there exist $r > 0$ and q such that $(\mathrm{P}_{\delta}^{r,q})$ is strongly stable and $\bar{u}$ is a solution of this problem. In fact, if we take $r > 0$ small enough, then (3.2) implies that $\rho_0 = C_\delta$. This leads to the following result.

Corollary 1 *Let us assume that $\bar{u}$ is a (strict) solution of $(\mathrm{P}_{\delta}^{r,q})$ and this problem is strongly stable. Then $\bar{u}$ is a (strict) local solution of problem $(\mathrm{P}_{\delta,\rho})$ in the sense of $L^q(\Omega)$ for every $\rho \geq C_\delta$ ($\rho > C_\delta$), where $C_\delta > 0$ is given by (3.1).*

From the previous proposition we can also deduce easily the optimality conditions in a qualified form for strongly stable problems. First, let us introduce some notation. By $M(K)$ we denote the space of real and regular Borel measures in K, which is identified with the dual of the continuous function space $C(K)$. $M(K)$ is a Banach space endowed with the norm

$$\|\mu\|_K = |\mu|(K) = \sup\left\{\int_K z(x)\, d\mu(x) : z \in C(K) \text{ and } \|z\|_{C(K)} \leq 1\right\}, \quad (3.3)$$

where $|\mu|$ is the total variation measure; see Rudin (1970).

THEOREM 2 *Let us assume that $\bar{u}$ is a solution of $(\mathrm{P}_\delta^{r,q})$. If $(\mathrm{P}_\delta^{r,q})$ is strongly stable, then there exist $\bar{y} \in H_0^1(\Omega) \cap C(\bar{\Omega})$, $\bar{\varphi} \in W_0^{1,s}(\Omega)$, for every $1 \leq s < n/(n-1)$, and $\bar{\mu} \in M(K)$ such that*

$$\begin{cases} A\bar{y} + a_0(x,\bar{y}) = \bar{u} & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \Gamma. \end{cases} \quad (3.4)$$

$$\begin{cases} A^*\bar{\varphi} + \dfrac{\partial a_0}{\partial y}(x,\bar{y}(x))\bar{\varphi} = \dfrac{\partial L}{\partial y}(x,\bar{y},\bar{u}) + \bar{\mu} & \text{in } \Omega, \\ \bar{\varphi} = 0 & \text{on } \Gamma, \end{cases} \quad (3.5)$$

$$\int_K (z(x) - \bar{y}(x))d\bar{\mu}(x) \leq 0 \quad \forall z \in \mathcal{Y}_\delta, \quad (3.6)$$

$$\int_\Omega (\bar{\varphi} + \frac{\partial L}{\partial u}(x,\bar{y},\bar{u}))(u - \bar{u})\, dx \geq 0 \quad \forall u \in \mathcal{U}_{\alpha,\beta}. \quad (3.7)$$

Proof. According to our assumptions, we have that the mapping control-state $G : L^p(\Omega) \longrightarrow H_0^1(\Omega) \cap C(\bar{\Omega})$, defined by $G(u) = y_u$, is of class C^1 and $z_v = G'(u)v$ satisfies the equation

$$\begin{cases} Az_v + \dfrac{\partial a_0}{\partial y}(x,y_u(x))z_v = v & \text{in } \Omega, \\ z_v = 0 & \text{on } \Gamma. \end{cases} \quad (3.8)$$

Applying the chain rule we deduce that J is of class C^1 in $L^\infty(\Omega)$ and

$$J'(u) = \phi_u + \frac{\partial L}{\partial u}(x,y_u,u), \quad (3.9)$$

where $\phi_u \in W_0^{1,s}(\Omega)$, $1 \leq s < n/(n-1)$, is the solution of

$$\begin{cases} A^*\phi + \dfrac{\partial a_0}{\partial y}(x,y_u(x))\phi = \dfrac{\partial L}{\partial y}(x,y_u,u) & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma. \end{cases} \quad (3.10)$$

Now, let us consider ρ_0 given by Proposition 2 and $F_\rho : C(K) \longrightarrow \mathbb{R}$, with $\rho \geq \rho_0$, defined by

$$F_\rho(y) = \rho(\|y\|_{C(K)} - \delta)^+. \tag{3.11}$$

F_ρ is convex and Lipschitz, ρ being the Lipschitz constant of F_ρ. From Proposition 2 we have that $\bar{u}$ is a local solution of the problem

$$\begin{array}{l} \min \ J_\rho(u) = J(u) + F_\rho(y_u) \\ u \in \mathcal{U}_{\alpha,\beta}, \end{array}$$

and then we can apply the calculus of generalized gradients, introduced by Clarke to deduce that $0 \in J'(\bar{u}) + \partial(F_\rho \circ G)(\bar{u}) + \partial I_{\mathcal{U}_{\alpha,\beta}}(\bar{u})$, where $I_{\mathcal{U}_{\alpha,\beta}}$ is the indicator of the convex set $\mathcal{U}_{\alpha,\beta}$

$$I_{\mathcal{U}_{\alpha,\beta}}(u) = \begin{cases} 0 & \text{if } u \in \mathcal{U}_{\alpha,\beta} \\ +\infty & \text{otherwise.} \end{cases}$$

Taking into account that $\partial(F_\rho \circ G)(\bar{u}) \subset [G'(\bar{u})]^* \partial F_\rho(\bar{y})$, where $\bar{y} = y_{\bar{u}}$, we deduce the existence of $\bar{\mu} \in \partial F_\rho(\bar{y})$, such that $0 \in J'(\bar{u}) + [G'(\bar{u})]^*\bar{\mu} + \partial I_{\mathcal{U}_{\alpha,\beta}}(\bar{u})$. Now, setting $\bar{\varphi} = \phi_{\bar{u}} + \psi_{\bar{\mu}}$, with $\psi_{\bar{\mu}} \in W_0^{1,s}(\Omega)$ being the solution of

$$\begin{cases} A^*\psi + \dfrac{\partial a_0}{\partial y}(x, \bar{y}(x))\psi = \bar{\mu} & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma, \end{cases} \tag{3.12}$$

we deduce (3.4)–(3.7). ■

Remark 1 *From the fact that $\bar{\mu} \in \partial F_\rho(\bar{y})$ we get that $\|\bar{\mu}\|_{M(K)} \leq \rho$. By taking $\rho = \rho_0$ in the proof of the previous theorem, we have that $\|\bar{\mu}\|_{M(K)} \leq \rho_0$.*

Remark 2 *It is well known that* (3.6) *leads to the following decomposition of $\bar{\mu}$*

$$\bar{\mu} = \bar{\mu}_+ - \bar{\mu}_-, \ \text{supp}\bar{\mu}_+ \subset K_+ \ \ \textit{and} \ \ \text{supp}\bar{\mu}_- \subset K_-, \tag{3.13}$$

where $\bar{\mu}_+$ and $\bar{\mu}_-$ are positive measures and

$$K_- = \{x \in K : \bar{y}(x) = -\delta\} \ \ \textit{and} \ \ K_+ = \{x \in K : \bar{y}(x) = +\delta\}.$$

If we set $K_\delta = K_- \cup K_+$, then the support of $\bar{\mu}$ is included in K_δ. In particular, if the state constraint is active at a finite set of points $K_\delta = \{x_j\}_{j=1}^m$, then

$$\bar{\mu} = \sum_{j=1}^m \bar{\lambda}_j \delta_{x_j}, \quad \textit{with} \ \ \bar{\lambda}_j = \begin{cases} \geq 0 & \textit{if} \ \ \bar{y}(x_j) = +\delta \\ \leq 0 & \textit{if} \ \ \bar{y}(x_j) = -\delta, \end{cases} \tag{3.14}$$

where δ_{x_j} denotes the Dirac measure centered at the point x_j.

We finish this section by proving the reciprocal result of the third claim of Proposition 2.

PROPOSITION 3 *Let us assume that $\bar{u} \in \mathcal{U}_\delta$ is a solution of problem $(\mathrm{P}^{r,q}_{\delta,\rho})$, then $(\mathrm{P}^{r,q}_{\delta})$ is strongly stable and C_δ can be chosen equal to ρ.*

Proof. From the inequality

$$J(u) + \rho(\|y_u\|_{C(K)} - \delta)^+ \le J(u) + \rho(\|y_u\|_{C(K)} - \delta')^+ + \rho(\delta' - \delta)$$

for every $\delta' > \delta$ and all $u \in \mathcal{U}_{\alpha,\beta}$, we deduce that

$$\begin{aligned}
&\inf_{u\in\mathcal{U}_{\alpha,\beta},\|u-\bar{u}\|_{L^q(\Omega)}\le r} J(u) + \rho(\|y_u\|_{C(K)} - \delta)^+ \\
&\le \inf_{u\in\mathcal{U}_{\alpha,\beta},\|u-\bar{u}\|_{L^q(\Omega)}\le r} J(u) + \rho(\|y_u\|_{C(K)} - \delta')^+ + \rho(\delta' - \delta),
\end{aligned}$$

and hence

$$\begin{aligned}
&\inf\,(\mathrm{P}^{r,q}_{\delta}) \le J(\bar{u}) = \inf_{u\in\mathcal{U}_{\alpha,\beta},\|u-\bar{u}\|_{L^q(\Omega)}\le r} J(u) + \rho(\|y_u\|_{C(K)} - \delta)^+ \\
&\le \inf_{u\in\mathcal{U}_{\alpha,\beta},\|u-\bar{u}\|_{L^q(\Omega)}\le r} J(u) + \rho(\|y_u\|_{C(K)} - \delta')^+ + \rho(\delta' - \delta) \\
&\le \inf\,(\mathrm{P}^{r,q}_{\delta'}) + \rho(\delta' - \delta).
\end{aligned}$$

4. Second order optimality conditions

Let us start the section by defining the Lagrangian function associated to the problem (P_δ)

$$\mathcal{L} : L^\infty(\Omega) \times M(K) \longrightarrow \mathbb{R}, \quad \mathcal{L}(u,\mu) = J(u) + \int_K y_u(x)\,d\mu(x).$$

The function $\mathcal{L}$ is of class C^2 and we have the following expressions of its derivatives

$$\frac{\partial \mathcal{L}}{\partial u}(u,\mu)v = \int_\Omega \left[\frac{\partial L}{\partial y}(x,y_u,u)z_v + \frac{\partial L}{\partial u}(x,y_u,u)v\right] dx + \int_K z_v\,d\mu,$$

where $z_v \in H^1_0(\Omega) \cap C(\bar{\Omega})$ satisfies (3.8). By using the adjoint state $\varphi_u \in W^{1,s}_0(\Omega)$, solution of

$$\left\{\begin{array}{rcll} A^*\varphi + \dfrac{\partial a_0}{\partial y}(x,y_u(x))\varphi &=& \dfrac{\partial L}{\partial y}(x,y_u,u) + \mu & \text{in } \Omega,\\ \varphi &=& 0 & \text{on } \Gamma, \end{array}\right. \tag{4.1}$$

and using (3.9) we get

$$\frac{\partial \mathcal{L}}{\partial u}(u,\mu)v = \int_\Omega \left[\varphi_u + \frac{\partial L}{\partial u}(x,y_u,u)\right] v\,dx. \tag{4.2}$$

From this expression it follows that (3.7) can be written in the following way:

$$\frac{\partial \mathcal{L}}{\partial u}(\bar u,\bar\mu)(u-\bar u) \geq 0 \quad \forall u \in \mathcal{U}_{\alpha,\beta}. \tag{4.3}$$

For the second derivative of $\mathcal{L}$ we have

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(u,\mu)v^2 = \int_\Omega \left[\frac{\partial L}{\partial y}(x,y_u,u)w_v + \frac{\partial^2 L}{\partial y^2}(x,y_u,u)z_v^2\right.$$
$$\left. +2\frac{\partial^2 L}{\partial y\partial u}(x,y_u,u)z_v v + \frac{\partial^2 L}{\partial u^2}(x,y_u,u)v^2\right] dx + \int_K w_v\,d\mu,$$

where $w_v = G''(u)v^2 \in H_0^1(\Omega)\cap C(\bar\Omega)$ satisfies

$$\begin{cases} Aw_v + \dfrac{\partial a_0}{\partial y}(x,y_u(x))w_v + \dfrac{\partial^2 a_0}{\partial y^2}(x,y_u(x))z_v^2 = 0 & \text{in } \Omega,\\ w_v = 0 & \text{on } \Gamma. \end{cases} \tag{4.4}$$

Using (4.1) and (4.4) in the expression of the second derivative of the Lagrangian function we get

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(u,\mu)v^2 = \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x,y_u,u)z_v^2 + 2\frac{\partial^2 L}{\partial y\partial u}(x,y_u,u)z_v v\right.$$
$$\left. +\frac{\partial^2 L}{\partial u^2}(x,y_u,u)v^2 - \varphi_u\frac{\partial^2 a_0}{\partial y^2}(x,y_u)z_v^2\right] dx. \tag{4.5}$$

Let us remark that the first and second derivatives of $\mathcal{L}$ can be extended to $L^2(\Omega)$, the integrals being well defined for every $v \in L^2(\Omega)$ and continuous with respect to v in the $L^2(\Omega)$ topology.

In order to write the second order optimality conditions we need to introduce the cone of critical directions. For fixed $\bar u \in \mathcal{U}_\delta$ and $\bar y$ being its associated state, we define

$$C_{\bar u} = \{v \in L^2(\Omega) : v \text{ satisfies (4.6), (4.7) and (4.8)}\},$$

$$v(x) = \begin{cases} \geq 0 \text{ if } \bar u(x) = \alpha(x),\\ \leq 0 \text{ if } \bar u(x) = \beta(x),\\ = 0 \text{ if } \bar\varphi(x) + \dfrac{\partial L}{\partial u}(x,\bar y(x),\bar u(x)) \neq 0, \end{cases} \tag{4.6}$$

$$z_v(x) = \begin{cases} \geq 0 \ \text{ if } \bar y(x) = -\delta\\ \leq 0 \ \text{ if } \bar y(x) = +\delta, \end{cases} \tag{4.7}$$

$$\int_K z_v(x)\,d\bar\mu(x) = 0, \tag{4.8}$$

where $z_v \in H_0^1(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\begin{cases} Az_v + \dfrac{\partial a_0}{\partial y}(x,\bar{y})z_v & = & v & \text{in } \Omega \\ z_v & = & 0 & \text{on } \Gamma. \end{cases}$$

Now we have the following result, whose proof can be found in Casas, De Los Reyes and Tröltzsch (2008).

THEOREM 3 *Let $\bar{u}$ be a feasible control of problem* (P_δ)*, $\bar{y}$ the associated state and $(\bar{\varphi},\bar{\mu}) \in W_0^{1,s}(\Omega) \times M(K)$, for all $1 \le s < n/(n-1)$, satisfying* (3.5)-(3.7)*. Assume further that*

$$\frac{\partial^2 L}{\partial u^2}(x,\bar{y}(x),\bar{u}(x)) \geq \omega \quad \text{if} \quad |\bar{\varphi}(x) + \frac{\partial L}{\partial u}(x,\bar{y}(x),\bar{u}(x))| \le \tau, \ \text{a.e.}, \tag{4.9}$$

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u},\bar{\mu})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\}. \tag{4.10}$$

Then there exist $r_{\bar{u}} > 0$ and $\alpha > 0$ such that the following inequality holds

$$J(\bar{u}) + \frac{\alpha}{2}\|u-\bar{u}\|^2_{L^2(\Omega)} \le J(u) \quad \text{if} \quad \|u-\bar{u}\|_{L^\infty(\Omega)} \le r_{\bar{u}} \text{ and } u \in \mathcal{U}_\delta. \tag{4.11}$$

The next theorem provides a sufficient condition for the problem ($P_\delta^{r_{\bar{u}}}$) to be strongly stable.

THEOREM 4 *Let us assume that $\bar{u} \in \mathcal{U}_\delta$ and the first order necessary optimality conditions* (3.4)-(3.7) *and the second order sufficient condition* (4.9)-(4.10) *are fulfilled. Then there exists $r > 0$ such that the problem*

$$(P_\delta^{r,\infty}) \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x,y_u(x),u(x))\,dx \\ u \in \mathcal{U}_{\alpha,\beta}, \ \|u-\bar{u}\|_{L^\infty(\Omega)} \le r, \ |y_u(x)| \le \delta \ \forall x \in K \end{cases}$$

is strongly stable.

Proof. Let us argue by contradiction. If the statement of the theorem is not true, then ($P_\delta^{1/k,\infty}$) is not strongly stable for any $k \geq 1$. Therefore, there exists $\delta_k \in (\delta, \delta + 1/k)$ such that

$$\inf (P_\delta^{1/k,\infty}) - \inf (P_{\delta_k}^{1/k,\infty}) > k(\delta_k - \delta). \tag{4.12}$$

Let u_k be a feasible control for problem ($P_{\delta_k}^{1/k,\infty}$) such that $J(\bar{u}) - J(u_k) > k(\delta_k - \delta)$. Then we have that $\|u_k - \bar{u}\|_{L^\infty(\Omega)} \le 1/k$. Let us take

$$\rho_k = \|u_k - \bar{u}\|_{L^2(\Omega)} \quad \text{and} \quad v_k = \frac{1}{\rho_k}(u_k - \bar{u}). \tag{4.13}$$

By taking a subsequence, denoted in the same way, we can assume

$$\lim_{k\to\infty} \rho_k = 0, \quad \|v_k\|_{L^2(\Omega)} = 1 \ \forall k \ \text{ and } \ v_k \rightharpoonup v \ \text{ weakly in } L^2(\Omega). \tag{4.14}$$

The rest of the proof is divided into three steps.

Step 1. $v \in C_{\bar u}$. From the definition of u_k we have that

$$J(u_k) - J(\bar u) + k(\delta_k - \delta) < 0. \tag{4.15}$$

Dividing the above expression by ρ_k and passing to the limit we get

$$J'(\bar u)v + \limsup_{k\to\infty} \frac{k(\delta_k - \delta)}{\rho_k} = \lim_{k\to\infty} \frac{J(\bar u + \rho_k v_k) - J(\bar u)}{\rho_k} + \limsup_{k\to\infty} \frac{k(\delta_k - \delta)}{\rho_k} \le 0.$$

Since $\delta_k > \delta$ for every k, the above inequality implies

$$J'(\bar u)v \le 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\delta_k - \delta}{\rho_k} = 0. \tag{4.16}$$

On the other hand, it is obvious that

$$v_k(x) = \begin{cases} \ge 0 & \text{if } \bar u(x) = \alpha(x) \\ \le 0 & \text{if } \bar u(x) = \beta(x). \end{cases}$$

Since the set of functions of $L^2(\Omega)$ satisfying the previous sign condition is closed and convex, it is also weakly closed, therefore v satisfies this sign condition, too. Then, (3.7) implies

$$\text{if } \bar\varphi(x) + \frac{\partial L}{\partial u}(x, \bar y(x), \bar u(x)) > 0 \Rightarrow \bar u(x) = \alpha(x) \Rightarrow v(x) \ge 0,$$

analogously

$$\text{if } \bar\varphi(x) + \frac{\partial L}{\partial u}(x, \bar y(x), \bar u(x)) < 0 \Rightarrow \bar u(x) = \beta(x) \Rightarrow v(x) \le 0.$$

These properties lead to

$$\frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu)v = \int_\Omega \left[\bar\varphi + \frac{\partial L}{\partial u}(x, \bar y, \bar u)\right] v \, dx = \int_\Omega \left|\bar\varphi + \frac{\partial L}{\partial u}(x, \bar y, \bar u)\right| |v| \, dx. \tag{4.17}$$

Using (3.13) we obtain for every $k \ge \|\bar\mu\|_{M(K)}$

$$\int_K (y_{u_k}(x) - \bar y(x)) \, d\bar\mu(x) = \int_{K_+} (y_{u_k}(x) - \delta) \, d\bar\mu_+(x) - \int_{K_-} (y_{u_k}(x) + \delta) \, d\bar\mu_-(x)$$

$$\le (\delta_k - \delta)\bar\mu_+(K_+) - (-\delta_k + \delta)\bar\mu_-(K_-) = \|\bar\mu\|_{M(K)}(\delta_k - \delta) \le k(\delta_k - \delta). \tag{4.18}$$

From (4.15) and (4.18) it follows that

$$\mathcal{L}(u_k,\bar\mu) - \mathcal{L}(\bar u,\bar\mu) = J(u_k) - J(\bar u) + \int_K (y_{u_k} - \bar y)\, d\bar\mu < 0. \tag{4.19}$$

Dividing this expression by ρ_k and passing to the limit we deduce

$$\int_\Omega \left[\bar\varphi + \frac{\partial L}{\partial u}(x,\bar y,\bar u)\right] v\, dx = \frac{\partial \mathcal{L}}{\partial u}(\bar u,\bar\mu)v \le 0.$$

This inequality, along with (4.17), implies that

$$\frac{\partial \mathcal{L}}{\partial u}(\bar u,\bar\mu)v = \int_\Omega \left|\bar\varphi + \frac{\partial L}{\partial u}(x,\bar y,\bar u)\right| |v|\, dx = 0. \tag{4.20}$$

This identity and the sign condition satisfied by v implies that (4.6) holds. Let us prove (4.7) and (4.8). Since

$$\frac{\partial \mathcal{L}}{\partial u}(\bar u,\bar\mu)v = J'(\bar u)v + \int_K z_v\, d\bar\mu,$$

(4.16) and (4.20) lead to

$$\int_K z_v\, d\bar\mu \ge 0. \tag{4.21}$$

On the other hand, using (4.16) we get

$$\begin{aligned} x\in K_+ &\Rightarrow z_v(x) = \lim_{k\to\infty}\frac{y_{u_k}(x)-\bar y(x)}{\rho_k} \le \lim_{k\to\infty}\frac{\delta_k-\delta}{\rho_k} = 0,\\ x\in K_- &\Rightarrow z_v(x) = \lim_{k\to\infty}\frac{y_{u_k}(x)-\bar y(x)}{\rho_k} \ge \lim_{k\to\infty}\frac{-\delta_k+\delta}{\rho_k} = 0.\end{aligned}$$

These inequalities and (4.21) imply (4.7) and (4.8), which concludes the proof of $v \in C_{\bar u}$.

Step 2. $v = 0$. Using (4.19) and making a Taylor development we get

$$\begin{aligned} 0 &> \mathcal{L}(u_k,\bar\mu) - \mathcal{L}(\bar u,\bar\mu) = \mathcal{L}(\bar u + \rho_k v_k,\bar\mu) - \mathcal{L}(\bar u,\bar\mu)\\ &= \rho_k \frac{\partial \mathcal{L}}{\partial u}(\bar u,\bar\mu)v_k + \frac{\rho_k^2}{2}\frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k,\bar\mu)v_k^2\\ &= \rho_k \frac{\partial \mathcal{L}}{\partial u}(\bar u,\bar\mu)v_k + \frac{\rho_k^2}{2}\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u,\bar\mu)v_k^2 + \frac{\rho_k^2}{2}[\frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k,\bar\mu) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u,\bar\mu)]v_k^2, \end{aligned} \tag{4.22}$$

with $w_k = \bar u + \theta_k(u_k - \bar u)$ and $0 < \theta_k < 1$. Considering the second derivative of $\mathcal{L}$ given by (4.5), using the weak convergence $v_k \rightharpoonup v$ in $L^2(\Omega)$ and the strong convergence $z_{v_k} \to z_v$ in $C(\bar\Omega)$, it is easy to pass to the limit and get

$$\lim_{k\to\infty} \int_\Omega \left\{ 2\frac{\partial^2 L}{\partial y \partial u}(x,\bar y,\bar u) z_{v_k} v_k + [\frac{\partial^2 L}{\partial y^2}(x,\bar y,\bar u) - \bar\varphi \frac{\partial^2 a_0}{\partial y^2}(x,\bar y)] z_{v_k}^2 \right\} dx$$
$$= \int_\Omega \left\{ 2\frac{\partial^2 L}{\partial y \partial u}(x,\bar y,\bar u) z_v v + [\frac{\partial^2 L}{\partial y^2}(x,\bar y,\bar u) - \bar\varphi \frac{\partial^2 a_0}{\partial y^2}(x,\bar y)] z_v^2 \right\} dx \tag{4.23}$$

and

$$\lim_{k\to\infty} [\frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k,\bar\mu) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u,\bar\mu)] v_k^2 = 0. \tag{4.24}$$

If we prove that

$$\int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar y,\bar u) v^2\, dx \le \liminf_{k\to\infty} \left(\frac{2}{\rho_k}\frac{\partial \mathcal{L}}{\partial u}(\bar u,\bar\mu) v_k + \int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar y,\bar u) v_k^2\, dx \right), \tag{4.25}$$

then we conclude from (4.22)-(4.25) and (4.5) that $\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u,\bar\mu) v^2 \le 0$, which leads to the desired result $v = 0$, owing to (4.10). Let us prove (4.25). First, let us remark that (3.7) implies that

$$(\bar\varphi(x) + \frac{\partial L}{\partial u}(x,\bar y(x),\bar u(x)))(u(x) - \bar u(x)) \ge 0 \quad \text{a.e.} \quad \forall u \in \mathcal{U}_{\alpha,\beta},$$

therefore

$$(\bar\varphi(x) + \frac{\partial L}{\partial u}(x,\bar y(x),\bar u(x))) v_k(x) \ge 0 \quad \text{a.e. } \forall k \ge 1. \tag{4.26}$$

Let us denote

$$\Omega^\tau = \{x \in \Omega : |\bar\varphi(x) + \frac{\partial L}{\partial u}(x,\bar y,\bar u)| \le \tau\}.$$

With the help of (4.26) we get

$$\frac{2}{\rho_k}\frac{\partial \mathcal{L}}{\partial u}(\bar u,\bar\mu) v_k + \int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar y,\bar u) v_k^2\, dx$$
$$\ge \frac{2}{\rho_k}\int_{\Omega\setminus\Omega^\tau} (\bar\varphi + \frac{\partial L}{\partial u}(x,\bar y,\bar u)) v_k + \int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar y,\bar u) v_k^2\, dx$$
$$\ge \int_{\Omega\setminus\Omega^\tau} [\frac{2\tau}{\rho_k}|v_k| + \frac{\partial^2 L}{\partial u^2}(x,\bar y,\bar u) v_k^2]\, dx + \int_{\Omega^\tau} \frac{\partial^2 L}{\partial u^2}(x,\bar y,\bar u) v_k^2\, dx. \tag{4.27}$$

Now, from the definition of u_k, ρ_k and v_k we have

$$k\rho_k |v_k(x)| = k|u_k(x) - \bar u(x)| \le k\|u_k - \bar u\|_{L^\infty(\Omega)} \le 1,$$

which implies $\frac{2\tau}{\rho_k}|v_k(x)| \geq 2k\tau v_k^2(x)$. This inequality and (4.27) lead to

$$\begin{aligned}
&\frac{2}{\rho_k}\frac{\partial \mathcal{L}}{\partial u}(\bar{u},\bar{\mu})v_k + \int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v_k^2\,dx \\
&\geq \int_{\Omega\setminus\Omega^\tau}[2k\tau + \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})]v_k^2\,dx + \int_{\Omega^\tau}\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v_k^2\,dx. \qquad (4.28)
\end{aligned}$$

From *(A3)* there follows the existence of k_0 such that

$$2k\tau + \frac{\partial^2 L}{\partial u^2}(x,\bar{y}(x),\bar{u}(x)) \geq 1 \text{ a.e. } \forall k \geq k_0. \qquad (4.29)$$

Inserting this inequality into (4.28), using (4.9) and taking the lower limit we get

$$\begin{aligned}
&\liminf_{k\to\infty}\left\{\frac{2}{\rho_k}\frac{\partial \mathcal{L}}{\partial u}(\bar{u},\bar{\mu})v_k + \int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v_k^2\,dx\right\} \\
&\geq \liminf_{k\to\infty}\int_{\Omega\setminus\Omega^\tau}[2k_0\tau + \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})]v_k^2\,dx + \liminf_{k\to\infty}\int_{\Omega^\tau}\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v_k^2\,dx \\
&\geq \int_{\Omega\setminus\Omega^\tau}[2k_0\tau + \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})]v^2\,dx + \int_{\Omega^\tau}\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v^2\,dx \\
&\geq \int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v^2\,dx,
\end{aligned}$$

which proves (4.25).

Step 3. Final contradiction. We have proved that $v_k \rightharpoonup 0$ weakly in $L^2(\Omega)$, then $z_{v_k} \to 0$ strongly in $C(\bar{\Omega})$. By using (4.22), (4.23), (4.24), (4.28), (4.29) and (4.9) we obtain

$$\begin{aligned}
&0 < \min\{\omega,1\} = \min\{\omega,1\}\lim_{k\to\infty}\|v_k\|^2_{L^2(\Omega)} \\
&\leq \liminf_{k\to\infty}\left\{\int_{\Omega\setminus\Omega^\tau}[2k\tau + \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})]v_k^2\,dx + \int_{\Omega^\tau}\frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v_k^2\,dx\right\} \\
&\leq \liminf_{k\to\infty}\left\{\frac{2}{\rho_k}\frac{\partial \mathcal{L}}{\partial u}(\bar{u},\bar{\mu})v_k + \int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v_k^2\,dx\right\} \leq 0,
\end{aligned}$$

which is a contradiction. ∎

Remark 3 *If we assume that*

$$\exists \Lambda_L > 0 \text{ such that } \frac{\partial^2 L}{\partial u^2}(x,y,u) \geq \Lambda_L \quad \forall y,u \in \mathbb{R} \text{ and for a.e. } x \in \Omega, \qquad (4.30)$$

then Theorem 4 holds with $r = r_{\bar{u}}$, where $r_{\bar{u}}$ is given by Theorem 3. To prove this we can follow the steps of the proof of Theorem 4 with the following changes.

If $(\mathrm{P}_\delta^{r_{\bar u},\infty})$ *is not strongly stable, then for any* $k \geq 1$ *there exists* $\delta_k \in (\delta, \delta + 1/k)$ *such that* $\inf\,(\mathrm{P}_\delta^{r_{\bar u},\infty}) - \inf\,(\mathrm{P}_{\delta_k}^{r_{\bar u},\infty}) > k(\delta_k - \delta)$*. Let* u_k *be a solution of problem* $(\mathrm{P}_{\delta_k}^{r_{\bar u},\infty})$*. Since* $\{u_k\}_{k=1}^\infty \subset \mathcal{U}_{\alpha,\beta}$ *we can take a subsequence, denoted in the same way, converging to* $\tilde u \in \mathcal{U}_{\alpha,\beta}$ **weakly in* $L^\infty(\Omega)$*. Since* $y_{u_k} \to y_{\tilde u}$ *in* $C(\bar\Omega)$ *and* $\delta_k \to \delta$ *it is easy to prove that* $\tilde u \in \mathcal{U}_\delta$ *and* $\|\tilde u - \bar u\|_{L^\infty(\Omega)} \leq r_{\bar u}$*, therefore* $\tilde u$ *is a feasible control for* $(\mathrm{P}_\delta^{r_{\bar u},\infty})$*. Hence, using* (4.11) *and that* $\bar u$ *is a feasible control for every problem* $(\mathrm{P}_{\delta_k}^{r_{\bar u},\infty})$ *we get*

$$J(\bar u) \leq J(\tilde u) \leq \liminf_{k\to\infty} J(u_k) \leq \limsup_{k\to\infty} J(u_k) \leq J(\bar u).$$

According to (4.11) *we have that* $\bar u$ *is the unique global solution of* $(\mathrm{P}_\delta^{r_{\bar u},\infty})$*, therefore the above inequalities imply that* $\tilde u = \bar u$ *and then the whole sequence* $\{u_k\}_{k=1}^\infty$ *converges to* $\bar u$ **weakly in* $L^\infty(\Omega)$ *and* $y_{u_k} \to \bar y$ *in* $C(\bar\Omega)$*. Moreover, from the same inequalities we also get that* $J(u_k) \to J(\bar u)$*. From these properties and the assumption* (4.30) *it is easy to deduce that* $u_k \to \bar u$ *strongly in* $L^2(\Omega)$*. Now we can continue as in the proof of Theorem 4, though some simplifications are possible thanks to the assumption* (4.30).

For some usual functions L Theorems 3 and 4 can be improved. Let us consider the following structure assumption on the function L defining the cost functional J.

(A4) $L(x, y, u) = L_0(x, y) + \dfrac{\Lambda}{2}u^2$, where $L_0 : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable and $\Lambda > 0$, $L_0(\cdot, 0) \in L^1(\Omega)$, and for all $M > 0$ there exist a constant $C_{L_0,M} > 0$ and a function $\psi_{L_0,M} \in L^1(\Omega)$ such that for a.e. $x \in \Omega$ and $|y|, |y_i| \leq M$, $i = 1, 2$,

$$\left|\frac{\partial L_0}{\partial y}(x, y)\right| \leq \psi_{L_0,M}(x), \quad \left|\frac{\partial^2 L_0}{\partial y^2}(x, y)\right| \leq C_{L_0,M},$$

$$\left|\frac{\partial^2 L_0}{\partial y^2}(x, y_2) - \frac{\partial^2 L_0}{\partial y^2}(x, y_1)\right| \leq C_{L_0,M}(|y_2 - y_1|).$$

Under the assumption *(A4)*, (3.4)-(3.7) and (4.9), (4.11) can be replaced by

$$J(\bar u) + \frac{\alpha}{2}\|u - \bar u\|^2_{L^2(\Omega)} \leq J(u) \quad \text{if} \quad \|u - \bar u\|_{L^2(\Omega)} \leq r_{\bar u} \text{ and } u \in \mathcal{U}_\delta; \quad (4.31)$$

see Casas, De Los Reyes and Tröltzsch (2008). If we use (4.31) and we argue as in the proof of Theorem 4, then we obtain the following theorem:

THEOREM 5 *Under the assumptions of Theorem 4 and (A4) we have that the problem*

$$(\mathrm{P}_\delta^{r_{\bar u},2}) \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))\,dx \\ u \in \mathcal{U}_{\alpha,\beta},\ \|u - \bar u\|_{L^2(\Omega)} \leq r_{\bar u},\ |y_u(x)| \leq \delta\ \forall x \in K \end{cases}$$

is strongly stable.

As an immediate consequence of (4.31), the previous theorem and Proposition 2 we have the following result:

Corollary 2 *Let us assume that* $\bar{u} \in \mathcal{U}_\delta$ *and the first order necessary optimality conditions* (3.4)-(3.7) *and the second order sufficient condition* (4.9) *are fulfilled. If, moreover, (A4) holds, then* $\bar{u}$ *is a strict local solution of* $(\mathrm{P}_{\delta,\rho})$ *in the sense of* $L^q(\Omega)$, *for any* $1 \le q \le +\infty$ *and for every* $\rho > C_\delta$, *where* $C_\delta > 0$ *is given by* (3.1).

5. Regularity of the local optimal controls

In this section we study the regularity of local solutions of problem $(\mathrm{P}_{\delta,\rho})$, with $\rho > 0$. To carry out this analysis we will assume *(A4)*. The next theorem provides a first regularity result.

Theorem 6 *Let* $\bar{u}$ *be a local solution of problem* $(\mathrm{P}_{\delta,\rho})$. *Let us also assume that (A4) holds,* $(\bar{y}, \bar{u}, \bar{\varphi}, \bar{\mu})$ *satisfies the optimality system* (3.4)-(3.7) *and* $\alpha, \beta \in W^{1,s}(\Omega)$ *for some* $s < n/(n-1)$. *Then,* $\bar{u} \in W^{1,s}(\Omega)$. *Furthermore if* $\alpha, \beta \in C(\bar{\Omega})$ *and the function* $\psi_{L_0,M}$, *introduced in (A4), belongs to* $L^p(\Omega)$, *with* $p > n/2$, *then* $\bar{u} \in C(\bar{\Omega} \setminus K_\delta)$.

Proof. Since $\bar{u}$ is a local solution of $(\mathrm{P}_{\delta,\rho})$, there exist $\bar{y} \in H_0^1(\Omega) \cap C(\bar{\Omega})$, $\bar{\varphi} \in W_0^{1,s}(\Omega)$ for every $1 \le s < n/(n-1)$ and $\bar{\mu} \in M(K)$ such that

$$\left\{ \begin{array}{rcll} A\bar{y} + a_0(x, \bar{y}) & = & \bar{u} & \text{in } \Omega, \\ \bar{y} & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{5.1}$$

$$\left\{ \begin{array}{rcll} A^*\bar{\varphi} + \dfrac{\partial a_0}{\partial y}(x, \bar{y}(x))\bar{\varphi} & = & \dfrac{\partial L_0}{\partial y}(x, \bar{y}) + \bar{\mu} & \text{in } \Omega, \\ \bar{\varphi} & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{5.2}$$

$$\int_K (y - \bar{y})\, d\bar{\mu} + F_\rho(\bar{y}) \le F_\rho(y) \quad \forall y \in C(K), \tag{5.3}$$

$$\int_\Omega (\bar{\varphi} + \Lambda \bar{u})(u - \bar{u})\, dx \ge 0 \quad \forall u \in \mathcal{U}_{\alpha,\beta}, \tag{5.4}$$

where $F_\rho : C(K) \longrightarrow \mathbb{R}$ is defined by $F_\rho(y) = \rho(\|y\|_{C(K)} - \delta)^+$.

From (5.4) and assumption *(A4)* we get

$$\bar{u}(x) = \mathrm{Proj}_{[\alpha(x),\beta(x)]}\left(-\frac{1}{\Lambda}\bar{\varphi}(x)\right) = \max\{\alpha(x), \min\{\beta(x), -\frac{1}{\Lambda}\bar{\varphi}(x)\}\}. \tag{5.5}$$

This identity, along with the fact that $\alpha, \beta, \bar{\varphi} \in W^{1,s}(\Omega)$, leads to the $W^{1,s}(\Omega)$-regularity for $\bar{u}$.

On the other hand, from the assumption on $\psi_{L_0,M}$ and (3.5) we deduce that $\bar{\varphi} \in C(\bar{\Omega} \setminus K_\delta)$. Finally, the identity (5.5) and the continuity of the functions $\bar{\varphi}, \alpha$ and β in $\bar{\Omega} \setminus K_\delta$ imply the continuity of $\bar{u}$ in the same domain. ∎

The previous regularity result on the control $\bar{u}$ can be improved if $\bar{y} \in \mathcal{Y}_\delta$ and there is a finite number of points, where the state constraints are active. More precisely, let us assume that $K_\delta = \{x_j\}_{j=1}^m \subset K$. Then the structure of the Lagrange multiplier $\bar{\mu}$ is given by (3.13). If we denote by $\bar{\varphi}_j$, $1 \leq j \leq m$, and $\bar{\varphi}_0$ the solutions of

$$\left\{ \begin{array}{rcll} A^*\bar{\varphi}_j + \dfrac{\partial a_0}{\partial y}(x, \bar{y}(x))\bar{\varphi}_j &=& \delta_{x_j} & \text{in } \Omega, \\ \bar{\varphi}_j &=& 0 & \text{on } \Gamma, \end{array} \right. \tag{5.6}$$

and

$$\left\{ \begin{array}{rcll} A^*\bar{\varphi}_0 + \dfrac{\partial a_0}{\partial y}(x, \bar{y}(x))\bar{\varphi}_0 &=& \dfrac{\partial L_0}{\partial y}(x, \bar{y}) & \text{in } \Omega, \\ \bar{\varphi}_0 &=& 0 & \text{on } \Gamma, \end{array} \right. \tag{5.7}$$

then the adjoint state associated to $\bar{u}$ is given by

$$\bar{\varphi} = \bar{\varphi}_0 + \sum_{j=1}^m \bar{\lambda}_j \bar{\varphi}_j. \tag{5.8}$$

Now, we have the following regularity result:

THEOREM 7 *Let us assume that $p > n$ in (A1) and (A4) and $\psi_{L_0,M} \in L^p(\Omega)$. Suppose also that $\bar{u}$ is a local solution of problem* $(\mathrm{P}_{\delta,\rho})$, *$\alpha, \beta, a_{ij} \in C^{0,1}(\bar{\Omega})$, for $1 \leq i, j \leq n$, $(\bar{y}, \bar{u}, \bar{\varphi}, \bar{\mu})$ satisfies the optimality system* (5.1)-(5.4) *and Γ is of class $C^{1,1}$. If the active set consists of finitely many points, i.e. $\bar{y} \in \mathcal{Y}_\delta$ and $K_\delta = \{x_j\}_{j=1}^m \subset K$, then $\bar{u}$ belongs to $C^{0,1}(\bar{\Omega})$ and $\bar{y}$ to $W^{2,p}(\Omega)$.*

Because of the properties of a_{ij}, Γ and $p > n$ we get that $\bar{y}, \bar{\varphi}_0 \in W^{2,p}(\Omega) \subset C^1(\bar{\Omega})$. On the other hand, $\bar{\varphi}_j(x) \to +\infty$ when $x \to x_j$, hence $\bar{\varphi}$ has singularities at the points x_j where $\bar{\lambda}_j \neq 0$. Consequently, $\bar{\varphi}$ cannot be Lipschitz.

Surprisingly, this does not lower the regularity of $\bar{u}$. Notice that (5.5) implies that $\bar{u}$ is identically equal to α or β in a neighborhood of x_j, depending on the sign of $\bar{\lambda}_j$. This implies the desired result; see Casas (2007) for the details.

Now the question arises if this Lipschitz property remains also valid for an infinite number of points, where the pointwise state constraints are active. Unfortunately, the answer is negative. In fact, the optimal control can even fail to be continuous if K_δ is an infinite and numerable set. For a counterexample the reader is referred to Casas and Tröltzsch (2008).

In the next theorem we will state the $H^1(\Omega)$-regularity of strict local solutions $\bar{u}$ of $(\mathrm{P}_{\delta,\rho})$. This results implies the $H^1(\Omega)$-regularity of strict local solutions of (P_δ) assuming that (P_δ^r) is strongly stable for every $r > 0$ small enough. Roughly speaking we can say that the stability implies the $H^1(\Omega)$-regularity of strict local solutions of (P_δ). The same result was proved in Casas and Tröltzsch (2008) under the Slater assumption. We have to remark that $\bar{u}$ is assumed to be a local solution in the sense of the $L^q(\Omega)$-topology, with $1 \le q < +\infty$, a posteriori it is also a local solution in the sense of $L^\infty(\Omega)$. Let us remember that if the first and second order optimality conditions are fulfilled, then $\bar u$ is a strict local solution of $(\mathrm{P}_{\delta,\rho})$ in the sense of $L^q(\Omega)$ for all $1 \le q \le +\infty$; see Corollary (2).

THEOREM 8 *Suppose that $\bar u$ is a strict local minimum of $(\mathrm{P}_{\delta,\rho})$ in the sense of the $L^2(\Omega)$ topology. We also assume that assumption (A4) holds, $a_{ij} \in C(\bar\Omega)$ for $1 \le i,j \le n$, $\alpha,\beta \in L^\infty(\Omega) \cap H^1(\Omega)$ and $\psi_{L_0,M} \in L^p(\Omega)$ in (A4), with $p > n/2$. Then $\bar u \in H^1(\Omega)$.*

Proof. Fix $r > 0$ such that $\bar u$ is a strict solution of the problem $(\mathrm{P}^{r,2}_{\delta,\rho})$.

Now we select a sequence $\{x_k\}_{k=1}^\infty$ dense in K and consider the family of control problems

$$(\mathrm{Q}_k)\left\{\begin{array}{l}\min J_{\rho,k}(u) = J(u) + \rho(\max\limits_{1\le j\le k} |y_u(x_j)| - \delta)^+\\ \alpha(x) \le u(x) \le \beta(x) \text{ for a.e. } x \in \Omega,\ \|u-\bar u\|_{L^2(\Omega)} \le r.\end{array}\right.$$

Obviously, $\bar u$ is a feasible control for every problem (Q_k). Therefore, the existence of a global minimum u_k of (Q_k) follows easily by standard arguments. It is also easy to check that $u_k \to \bar u$ in $L^2(\Omega)$, owing to the assumption *(A4)* and $y_{u_k} \to \bar y$ in $H^1_0(\Omega) \cap C(\bar\Omega)$.

Since $\bar u$ is solution of $(\mathrm{P}^{r,2}_{\delta,\rho})$, there exist $\bar y \in H^1_0(\Omega) \cap C(\bar\Omega)$, $\bar\varphi \in W^{1,s}_0(\Omega)$ for every $1 \le s < n/(n-1)$ and $\bar\mu \in M(K)$ such that (5.1)-(5.4) hold.

Analogously, the fact that u_k is a solution of (Q_k) implies the existence of $y_k \in H^1_0(\Omega) \cap C(\bar\Omega)$, $\varphi_k \in W^{1,s}_0(\Omega)$ for every $1 \le s < n/(n-1)$ and $\mu_k \in M(K)$ such that

$$\left\{\begin{array}{rcll} Ay_k + a_0(x,y_k) &=& u_k & \text{in } \Omega,\\ y_k &=& 0 & \text{on } \Gamma.\end{array}\right. \tag{5.9}$$

$$\left\{\begin{array}{rcll} A^*\varphi_k + \dfrac{\partial a_0}{\partial y}(x,y_k(x))\varphi_k &=& \dfrac{\partial L_0}{\partial y}(x,y_k) + \mu_k & \text{in } \Omega,\\ \varphi_k &=& 0 & \text{on } \Gamma,\end{array}\right. \tag{5.10}$$

$$\int_K (y-y_k)d\mu_k + F_{\rho,k}(y_k) \le F_{\rho,k}(y) \quad \forall y \in C(K), \tag{5.11}$$

$$\int_\Omega (\varphi_k + \Lambda u_k)(u-u_k)\,dx \ge 0 \ \ \forall u \in \mathcal{U}_{\alpha,\beta}, \tag{5.12}$$

where the function $F_{\rho,k} : C(K) \longrightarrow \mathbb{R}$ is defined by

$$F_{\rho,k}(y) = \rho(\max_{1\le j\le k} |y(x_j)| - \delta)^+.$$

From (5.11) it follows that

$$\mu_k = \sum_{j=1}^{k} \lambda_{k,j}\delta_{x_j}, \ \sum_{j=1}^{k} |\lambda_{k,j}| \leq \rho \text{ and } \lambda_{k,j} = \begin{cases} \geq 0 & \text{if } y_k(x_j) \geq +\delta \\ \leq 0 & \text{if } y_k(x_j) \leq -\delta \\ 0 & \text{if } |y_k(x_j)| < \delta. \end{cases} \quad (5.13)$$

The rest of the proof follows identically as in Casas and Tröltzsch (2008). ■

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