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The nonsmooth maximum principle*

by

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Abstract: We present a brief survey of the nonsmooth maximum principle of optimal control, focusing, in particular, upon the alternative forms of the adjoint equation. We obtain a new version of the theorem that asserts for the first time the full Weierstrass condition together with the Euler form of the adjoint equation, thereby extending a result of de Pinho and Vinter. The new theorem also features stratified hypotheses and conclusions. Two examples illustrate its use.

Keywords: optimal control, Pontryagin maximum principle, nonsmooth analysis.

1. Introduction

The original version

The Pontryagin maximum principle (see Pontryagin et al., 1962) plays a central role in optimal control. The first versions of this celebrated theorem for data that are nonsmooth appeared in Clarke (1973, 1974, 1975), where the adjoint equation is replaced by an inclusion in terms of the *generalized gradient* introduced by the author. Later work treated the case of full endpoint constraints, and since then a number of related or parallel results were developed by other authors; see, for example, Vinter (2000) and Milyutin and Osmolovskii (1998). We also refer the reader to Clarke (2005) for a detailed survey of the nonsmooth analysis approach.

The *nonsmooth maximum principle* can now be considered a well-known result; we proceed to state it, in essentially its original form (see Clarke 1975, 1976b), for the standard optimal control problem in its Mayer formulation. Consider the problem (P) that consists of minimizing the cost functional

$$\ell(x(a), x(b))$$

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subject to the boundary conditions

$$(x(a), x(b)) \in S$$

and the dynamics

$$x'(t) \;=\; f(t, x(t), u(t)) \text{ a.e. } [a, b],$$

where the (measurable) control function $u(\cdot)$ is constrained by

$$u(t) \;\in\; U(t) \text{ a.e. } [a, b].$$

Here $x(t)$ lies in $\Re^n$, and $U(t)$ is a subset of $\Re^m$. It is assumed throughout the article (without further mention) that $f(t, x, u)$ is $\mathcal{L}\times\mathcal{B}$-measurable with respect to t and (x, u),[1] that the multifunction $U(\cdot)$ has $\mathcal{L}\times\mathcal{B}$-measurable graph, that f is locally Lipschitz with respect to x, that the set S is closed, and that ℓ is locally Lipschitz.

Definition 1 We say that a given (admissible) trajectory/control pair (x_*, u_*) is a **strong local minimum** for the problem (P) if, for some $\epsilon > 0$, the process (x_*, u_*) is optimal relative to the other admissible processes (x, u) satisfying

$$\|x - x_*\|_\infty \;:=\; \max_{t\in[a,b]} |x(t) - x_*(t)| \;<\; \epsilon.$$

Definition 2 We say that f is **Lipschitz in** x near $x_*(\cdot)$ if there exist $\epsilon > 0$ and a function k ($\mathcal{L}\times\mathcal{B}$-measurable) such that, for almost every $t \in [a, b]$, for every $u \in U(t)$, we have:

$$|f(t, x_2, u) \;-\; f(t, x_1, u)| \;\le\; k(t, u)|x_2 - x_1| \;\; \forall\, x_1, x_2 \in B(x_*(t), \epsilon).$$

Theorem 1 (Nonsmooth Maximum Principle) *(Clarke, 1975) If (x_*, u_*) is a strong local minimum for the problem (P), if f is Lipschitz in x near x_*, and if $t \mapsto k(t, u_*(t))$ is summable, then there exist an absolutely continuous function $p(\cdot)$ on $[a, b]$ together with a scalar λ_0 equal to 0 or 1, satisfying the nontriviality condition $[NT]$:*

$$\lambda_0 \;+\; |p(t)| \;\ne\; 0, \quad t \in [a, b],$$

the transversality condition $[T]$:

$$(p(a), -p(b)) \;\in\; \lambda_0 \partial_L \ell(x_*(a), x_*(b)) + N_S^L(x_*(a), x_*(b)),$$

the adjoint equation $[A]$:

$$-p'(t) \;\in\; \partial_C \left\langle p(t), f(t, \cdot, u_*(t))\right\rangle (x_*(t)) \quad \text{a.e. } t \in [a, b],$$

[1] This refers to the smallest σ-field containing the products of Lebesgue measurable subsets of $[a, b]$ and Borel measurable subsets of $\Re^n \times \Re^m$. See for example Clarke et al. (1998) for the basic theory of measurable multifunctions.

and the Weierstrass condition $[W]$:

$$\max_{u\in U(t)} \langle p(t), f(t,x_*(t),u)\rangle \text{ at } u=u_*(t) \quad \text{a.e. } t\in[a,b].$$

In the formulation of the theorem, ∂_C denotes the generalized gradient (with respect to the x variable), ∂_L is the limiting subdifferential, and N_S^L the limiting normal cone to S. We refer to Clarke (2005) for a brief summary of these constructs of nonsmooth analysis, or to Clarke et al. (1998) for a detailed presentation. In terms of the (pseudo) Hamiltonian $H(t,x,p,u) := \langle p, f(t,x,u)\rangle$, the adjoint equation and Weierstrass condition may be expressed in the equivalent forms

$$-p'(t) \in \partial_C H(t,\cdot,p(t),u_*(t))(x_*(t)) \text{ a.e.}$$
$$\max_{u\in U(t)} H(t,x_*(t),p(t),u) \text{ at } u=u_*(t) \text{ a.e.}$$

The Hamiltonian is useful in expressing the *Erdmann condition*, which refers to extra information that can be obtained, notably when the problem is autonomous (see Clarke, 2005, or Vinter, 2000, for example). We do not discuss this issue here.

We remark that the versions of the theorem prior to 1976 use in the transversality $\partial_C\ell$ and N_S^C rather than the potentially smaller constructs $\partial_L\ell$ and N_S^L, but (as B. Mordukhovich was the first to observe) the original proof actually yields this minor improvement without any modifications.

This article will discuss variants of the above result, with emphasis on the two principal hypotheses (the type of local minimum, and the Lipschitz behavior), as well as on the different versions of the adjoint equation $[A]$ and the Weierstrass condition $[W]$ that can be asserted. The nontriviality and transversality conditions will not change. In the context of the calculus of variations, Ioffe and Rockafellar (1996) have treated some similar issues.

The Euler form of the adjoint equation

We now proceed to discuss a variant of Theorem 1 due to de Pinho and Vinter (1995), which features an 'Euler form' of the adjoint equation, one that arises from the Euler inclusion when considering the control problem as a generalized problem of Bolza (see Clarke, 1976b). It requires that $U(t)$ be a closed set for each t, and that near x_* the function f be **integrably Lipschitz** *jointly* in (x,u) in the following sense: there exist $\epsilon>0$ and a summable function k such that, for almost every $t\in[a,b]$, one has, for every x_1,x_2 in $B(x_*(t),\epsilon)$ and u_1,u_2 in $U(t)$:

$$|f(t,x_2,u_2) - f(t,x_1,u_1)| \le k(t)\{|x_2-x_1|+|u_2-u_1|\}.$$

Theorem 2 *(de Pinho and Vinter, 1995) If (x_*,u_*) is a strong local minimum for the problem (P), and if f is integrably Lipschitz jointly in (x,u) near $x_*(\cdot)$,* *then there exist an absolutely continuous function $p(\cdot)$ on $[a,b]$ together with a scalar λ_0 equal to 0 or 1, satisfying the nontriviality condition, the transversality condition, and, for almost every $t \in [a,b]$, the Euler adjoint equation $[EA]$:*

$$(-p'(t),0) \;\in\; \partial_C \langle p(t), f(t,\cdot,\cdot)\rangle\,(x_*(t),u_*(t)) - \{0\} \times N^C_{U(t)}(u_*(t)).$$

Note that in this result, in contrast to Theorem 1, the generalized gradient appearing in the adjoint equation is taken with respect to *both* x and u jointly. The resulting adjoint equation [EA] does not imply the adjoint equation [A] of Theorem 1 in general (or conversely). To clarify its meaning, we observe that the Weierstrass condition of Theorem 1 asserts that a certain maximum relative to $U(t)$ is attained at $u_*(t)$. A necessary (stationarity) condition for that maximum is given by the relation

$$0 \in \partial_C \langle p(t), f(t,x_*(t),\cdot)\rangle\,(u_*(t)) \;-\; N^C_{U(t)}(u_*(t)). \tag{1}$$

In the light of this, the adjoint equation [EA] can be viewed as an amalgam of the adjoint equation [A] together with a stationary form of the Weierstrass condition. It general, however, [EA] neither implies nor is implied by [A] together with (1), so this interpretation is not a precise one.

It follows that Theorems 1 and 2 give different information in general. Theorem 2 has the drawback of requiring that f be Lipschitz in the control variable u. It possesses the feature, however, that in a fully convex problem the normal form of the conditions of Theorem 2 (that is, for $\lambda_0 = 1$) are *sufficient* for optimality. This is not always the case for Theorem 1, essentially because there can be a disparity between the generalized gradient (or convex subdifferential) taken with respect to one variable and the projection on that variable of the generalized gradient taken jointly; an example along these lines is adduced in de Pinho and Vinter (1995).

We remark that if (x_*, u_*) is assumed to be merely a *weak* local minimum, that is with respect to the further constraint $u \in U(t) \cap B(u_*(t), \epsilon)$ (as is the case in de Pinho and Vinter, 1995), the conclusions of Theorem 2 are unaffected, since one may simply replace $U(t)$ by $U(t) \cap B(u_*(t), \epsilon)$. This reflects the regrettable defect of Theorem 2 of not including the Weierstrass condition, responsible for the very phrase 'maximum principle'. This will be remedied below. First, we need to recall some recent advances on the maximum principle.

The stratified maximum principle

A measurable function $R : [a,b] \to (0, +\infty]$ is called a *radius function.*

Definition 3 Let $R(\cdot)$ be a radius function. The process (x_*, u_*) is a **local $W^{1,1}$ minimum of radius** R for the problem (P) if, for some $\epsilon > 0$, it is optimal for (P) relative to the admissible processes (x, u) satisfying

$$\|x - x_*\|_\infty \;<\; \epsilon, \quad \int_a^b |x'(t) - x_*'(t)|\,dt < \epsilon,$$

as well as

$$|x'(t) - x'_*(t)| \ \le \ R(t), \text{ a.e. } t \in [a,b].$$

Note that when R is identically $+\infty$ (which is allowed), this reduces to what is usually referred to as a local $W^{1,1}$ minimum, which is a weaker assumption than that of a strong local minimum. When R is finite, we obtain a type of minimum that is known in the calculus of variations as a 'weak local minimum'.

Definition 4 The function f is **pseudo-Lipschitz in x near x_* of radius R** if there exist $\epsilon > 0$ and a summable function $k(\cdot)$ such that, for almost every $t \in [a,b]$, the relations

$$x_1, x_2 \in B(x_*(t), \epsilon), \ u \in U(t), \ |f(t, x_1, u) - x'_*(t)| \ \le \ R(t)$$

imply

$$|f(t, x_2, u) - f(t, x_1, u)| \ \le \ k(t)|x_2 - x_1|.$$

Definition 5 An arc p is said to satisfy the **Weierstrass condition of radius R** for (x_*, u_*) (denoted $[W_R]$) if, for almost every $t \in [a,b]$, for every $u \in U(t)$ satisfying $|f(t, x_*(t), u) - x'_*(t)| \le R(t)$, we have

$$\langle p(t), f(t, x_*(t), u) \rangle \ \le \ \langle p(t), f(t, x_*(t), u_*(t)) \rangle \, .$$

Note that when R is identically $+\infty$, this reduces to the usual (global) Weierstrass condition.

Theorem 3 *(Clarke, 2005) If (x_*, u_*) is a local $W^{1,1}$ minimum of radius R for the problem (P), and if f is pseudo-Lipschitz in x near $x_*(\cdot)$ of radius R, where for some $\eta > 0$ we have $R(t) \ge \eta k(t)$* a.e.*, then there exist an absolutely continuous function $p(\cdot)$ on $[a,b]$ together with a scalar λ_0 equal to* 0 *or* 1 *satisfying the nontriviality condition, the transversality condition, the adjoint equation $[A]$:*

$$-p'(t) \ \in \ \partial_C \langle p(t), f(t, \cdot, u_*(t)) \rangle \, (x_*(t)) \quad \text{a.e. } t \in [a,b],$$

and the Weierstrass condition $[W_R]$ of radius R.

If the above holds for a sequence of radius functions R_i (with the parameters ϵ, k, η possibly depending on i) for which

$$\liminf_{i \to \infty} R_i(t) = +\infty \text{ a.e.},$$

then the conclusions hold for an arc p which satisfies the global Weierstrass condition.

Theorem 3 imposes hypotheses only up to radius R, and asserts conclusions relative to that same radius. This *stratified* structure, which is elaborated upon in Clarke (2005), is particularly useful in obtaining multiplier rules for problems with side constraints (see Clarke and de Pinho, 2009) and unbounded controls, as well as developing solution-independent hypotheses, under which the necessary conditions hold. We remark that the theorem fails in the absence of the hypothesis $R(t) \geq \eta k(t)$ a.e., which may be viewed as a requirement that the class of competing arcs (or the radius of optimality) be large enough.

A new nonsmooth maximum principle

The main purpose of this article may now be apparent. It consists of extending Theorem 2 to the case of the stratified hypotheses of Theorem 3. It turns out, however, that more is possible: we are also able to assert the Weierstrass condition alongside the Euler adjoint equation.

In the following, we suppose that $U(t)$ is a closed set for each t, and that $f(t, x, u)$ is locally Lipschitz in (x, u) for each t.

THEOREM 4 *If (x_*, u_*) is a local $W^{1,1}$ minimum of radius R for the problem (P), and if f is pseudo-Lipschitz in x near $x_*(\cdot)$ of radius R, where for some $\eta > 0$ we have $R(t) \geq \eta k(t)$ a.e., then there exist an absolutely continuous function $p(\cdot)$ on $[a, b]$ together with a scalar λ_0 equal to 0 or 1 satisfying the nontriviality condition, the transversality condition, the Euler adjoint equation $[EA]$:*

$$(-p'(t), 0) \in \partial_C \langle p(t), f(t, \cdot, \cdot) \rangle (x_*(t), u_*(t)) - \{0\} \times N^C_{U(t)}(u_*(t))$$

as well as the Weierstrass condition $[W_R]$ of radius R.

If the above holds for a sequence of radius functions R_i (with the parameters ϵ, k, η possibly depending on i) for which

$$\liminf_{i \to \infty} R_i(t) = +\infty \text{ a.e.},$$

then the conclusions hold for an arc p which satisfies the global Weierstrass condition.

Note that even when R is taken to be identically $+\infty$, this extends Theorem 2 in several ways: the Lipschitz behavior in u is less restrictive, the local minimum need only be of $W^{1,1}$ type, and of course the Weierstrass condition is asserted.

We prove Theorem 4 in the next section.

2. Proof of Theorem 4

A. The crux of the proof lies in an appeal to Theorem 3.1.1 of Clarke (2005), a theorem on differential inclusions, which is a template for deriving necessary conditions in several other contexts (such as the calculus of variations, or generalized control systems). We recall the theorem next.

We are given a multifunction F from $[a,b] \times \Re^n$ to the subsets of $\Re^n$. It is assumed that F is measurable and graph-closed. A *trajectory* of F refers to an absolutely continuous function x on $[a,b]$ satisfying $x'(t) \in F(t,x(t))$ a.e. We consider the problem (Q) of minimizing $\ell(x(a),x(b))$ over the trajectories x of F satisfying the boundary constraints $(x(a),x(b)) \in S$.

Definition 6 F is said to satisfy a **pseudo-Lipschitz condition of radius** R near the arc x_* if there exist $\epsilon > 0$ and a summable function k such that, for almost all $t \in [a,b]$, for every x_1 and x_2 in $B(x_*(t),\epsilon)$, one has

$$F(t,x_1) \cap B(x_*'(t),R(t)) \subset F(t,x_2) + k(t)\,|x_2 - x_1|\,B.$$

When R is identically $+\infty$, the above reduces to a (true) Lipschitz condition.

Definition 7 F is said to satisfy the *tempered growth condition* of radius R near x_* if there exist $\epsilon > 0$, $\lambda \in (0,1)$, and a summable function r_0 such that for almost every $t \in [a,b]$ we have $0 < r_0(t) \leq \lambda R(t)$ and

$$|x - x_*(t)| \leq \varepsilon \Longrightarrow F(t,x) \cap B(x_*'(t),r_0(t)) \neq \emptyset.$$

The notion of local $W^{1,1}$ minimum of radius R given by Definition 3 clearly carries over to the problem (Q). Note that by taking the minimum of the three parameters ϵ, we may assume that the ϵ defining the local minimum is the same as that of both the pseudo-Lipschitz and the tempered growth conditions.

In the following, $G(t)$ refers to the graph of the multifunction $F(t,\cdot)$.

Theorem 5 *(Clarke, 2005) If x_* is a local $W^{1,1}$ minimum of radius R for the problem (Q), and if F satisfies the pseudo-Lipschitz and tempered growth conditions near x_* for the radius R, where for some $\eta > 0$ we have $R(t) \geq \eta k(t)$* a.e., *then there exist an absolutely continuous function $p(\cdot)$ on $[a,b]$ together with a scalar λ_0 equal to* 0 *or* 1, *satisfying the nontriviality condition, the transversality condition, the Euler equation $[E]$:*

$$p'(t) \in \text{ co } \left\{\omega : (\omega, p(t)) \in N^L_{G(t)}(x_*(t),\dot{x}_*(t))\right\} \text{ a.e. } t \in [a,b],$$

as well as the Weierstrass condition $[W_R]$ of radius R:

$$\langle p(t), v\rangle \leq \langle p(t), \dot{x}_*(t)\rangle \;\; \forall v \in F(t,x_*(t)) \cap B(\dot{x}_*(t),R(t)), \text{ a.e. } t \in [a,b].$$

If the above holds for a sequence of radius functions R_i (with all parameters $\epsilon, k, \lambda, r_0$ possibly depending on i) for which

$$\liminf_{i\to\infty} R_i(t) = +\infty \text{ a.e.},$$

then the conclusions hold for an arc p which satisfies the global Weierstrass condition.

B. The first step in the proof of Theorem 4 is to recast the problem (P) of the preceding section in a form to which Theorem 5 can be applied. The state of the new problem (Q) that we now define will be denoted $z = (x, y)$, for $x \in \Re^n$ and $y \in \Re^m$. Let $\theta : \Re^m \to \Re^m$ be the smooth function given by

$$\theta(u) := \frac{u}{1+|u|^2}.$$

Note that θ is one-to-one, and that θ and $\nabla\theta$ are bounded, with $|\theta(u)| \leq 1\, \forall u$. For a fixed $\delta \in (0,1)$ we define

$$F(t,z) = F(t,x,y) := \{(f(t,x,u), \gamma_\delta(t)\theta(u)) : u \in U(t)\},$$

where $\gamma_\delta(t) := \delta(1-\delta)\eta k(t)/2$. It follows that F is measurable and has closed graph. We set

$$\tilde{\ell}(x_a, y_a, x_b, y_b) := \ell(x_a, x_b),$$
$$\tilde{S} := \{(x_a, y_a, x_b, y_b) : (x_a, x_b) \in S\},$$

and we further define $z_*(t) := (x_*(t), y_*(t))$ where

$$y_*(t) := \int_a^t \gamma_\delta(\tau)\theta(u_*(\tau))\, d\tau.$$

Note that z_* is admissible for the case of the problem (Q) defined by these data; that is, z_* is a trajectory for F satisfying $(z_*(a), z_*(b)) \in \tilde{S}$.

C. We claim that z_* is a local $W^{1,1}$ minimum of radius R for the problem (Q), for the same ϵ. The proof of the claim is by contradiction: suppose there is an arc $z = (x, y)$ admissible for (Q) that satisfies

$$\|z - z_*\|_\infty < \epsilon, \quad \int_a^b |z'(t) - z_*'(t)|\, dt < \epsilon,$$

as well as

$$|z'(t) - z_*'(t)| \leq R(t), \text{ a.e. } t \in [a,b]$$

and $\tilde{\ell}(z(a), z(b)) < \tilde{\ell}(z_*(a), z_*(b))$. Then, $(x(a), x(b))$ lies in S, and (by a standard measurable selection theorem) there is a measurable function $u(\cdot)$ such that

$$u(t) \in U(t) \text{ a.e.}, \quad x'(t) = f(t, x(t), u(t)) \text{ a.e.}$$

That is, the pair (x, u) is an admissible process for the original problem (P). Further, we have

$$\|x - x_*\|_\infty < \epsilon, \quad \int_a^b |x'(t) - x_*'(t)|\, dt < \epsilon,$$

as well as

$$|x'(t) - x'_*(t)| \ \leq \ R(t), \ \text{a.e.}\, t \in [a, b].$$

The fact that $\ell(x(a), x(b)) < \ell(x_*(a), x_*(b))$ contradicts the hypothesis that x_* is a local $W^{1,1}$ minimum of radius R (with constant ϵ) for the problem (P), and the claim is established.

D. To justify the application of Theorem 5 to z_*, we verify the pseudo-Lipschitz and tempered growth conditions, for the radius R, and for $\tilde{\epsilon} := \min[\epsilon, \eta/2]$. As regards tempered growth (see Definition 7), let us take $\lambda = 1/2$ and $r_0(t) = \tilde{\epsilon}k(t)$. Almost every t satisfies $R(t) \geq \eta k(t)$, in which case we have

$$r_0(t) = \tilde{\epsilon}k(t) \leq \eta k(t)/2 \leq R(t)/2 = \lambda R(t),$$

as required. Now consider any $x \in B(x_*(t), \tilde{\epsilon})$ and any y. The point

$$(f(t, x, u_*(t)), \gamma_\delta(t)\theta(u_*(t)))$$

belongs to $F(t, x, y)$. To confirm tempered growth, it suffices to show that its distance to the point $(f(t, x_*(t), u_*(t)), \gamma_\delta(t)\theta(u_*(t)))$ is no greater than $r_0(t)$. But the pseudo-Lipschitz hypothesis implies

$$|f(t, x_*(t), u_*(t)) - f(t, x, u_*(t))| \leq k(t)|x_*(t) - x| \leq k(t)\tilde{\epsilon} = r_0(t),$$

and so tempered growth is confirmed.

As regards the pseudo-Lipschitz hypothesis (see Definition 6), let x_1, x_2 belong to $B(x_*(t), \tilde{\epsilon})$, choose any points y_1, y_2 in $\Re^m$, and let $(f(t, x_1, u), \gamma_\delta(t)\theta(u))$ be a point in $F(t, x_1, y_1) \cap B((x'_*(t), y'_*(t)), R(t))$. Then $|f(t, x_1, u) - x'_*(t)| \leq R(t)$, and the pseudo-Lipschitz hypothesis for f (Definition 4) gives

$$|f(t, x_2, u) - f(t, x_1, u)| \leq k(t)|x_2 - x_1|.$$

This implies that the distance of the point $(f(t, x_1, u), \gamma_\delta(t)\theta(u))$ to the set $F(t, x_2, y_2)$ is no greater than $k(t)|(x_2, y_2) - (x_1, y_1)|$, as required.

E. Having verified the hypotheses, we deduce the existence of an arc $\tilde{p} = (p, q)$ and a number λ_0 satisfying the conclusions of Theorem 5.

By definition, a point (x, y, x', y') belongs to $G(t)$ iff $(x', y') \in F(t, x, y)$. But F actually depends only upon (t, x), and not upon y. It follows that any limiting normal vector (ω, ν, p, q) to $G(t)$ has $\nu = 0$ (it suffices to prove this for proximal normal vectors, and for those the result is evident from the definition, as we show below). We conclude from the Euler equation $[E]$ that the arc q has $q'(t) = 0$ a.e.

Similarly, the lack of dependence of $\tilde{\ell}$ and $\tilde{S}$ upon both the second and the fourth of the four components allows us to deduce from the transversality condition that $q(a) = q(b) = 0$. Thus, the arc q is identically 0. It follows that p and λ_0 together satisfy the required nontriviality and transversality conditions of Theorem 4. There remain the Euler adjoint equation $[EA]$ and the Weierstrass condition to obtain.

As a preliminary step in obtaining $[EA]$, fix t such that all relevant equations and inclusions hold, and let (ω, ν, p, q) be any proximal normal to $G(t)$ at a typical point $(\bar{x}, \bar{y}, f(t, \bar{x}, \bar{u}), \gamma_\delta(t)\theta(\bar{u}))$ of $G(t)$, where $\bar{u} \in U(t)$. We may consider only such points in a neighborhood of the base point

$$(x_*(t), y_*(t), f(t, x_*(t), u_*(t)), \gamma_\delta(t)\theta(u_*(t))).$$

Then it follows from the definition of proximal normal that for a certain $\sigma > 0$, the following function of (x, y, u) attains a minimum at $(\bar{x}, \bar{y}, \bar{u})$:

$$-\omega \cdot x - \nu \cdot y - p \cdot f(t, x, u) - q \cdot \gamma_\delta(t)\theta(u) + \sigma|(x - \bar{x}, y - \bar{y}, u - \bar{u})|^2$$

relative to all (x, y) in a neighborhood of $(\bar{x}, \bar{y})$ and $u \in U(t)$ in a neighborhood of $\bar{u}$. (We have used here the fact that f is locally Lipschitz with respect to (x, u).)

Writing the necessary condition for this minimum yields $\nu = 0$, as pointed out above. In addition, the minimum with respect to (x, u) implies the stationarity condition[2]

$$(\omega, q\gamma_\delta(t)\nabla\theta(\bar{u})) \in \partial_L \langle -p, f(t, \cdot, \cdot) \rangle (\bar{x}, \bar{u}) + \{0\} \times N^L_{U(t)}(\bar{u}). \tag{2}$$

Now consider a limiting normal $(\omega, 0, p(t), 0)$ to $G(t)$ at the point

$$(x_*(t), y_*(t), x'_*(t), y'_*(t)) = (x_*(t), y_*(t), f(t, x_*(t), u_*(t)), \gamma_\delta(t)\theta(u_*(t))),$$

as occurs in the Euler equation $[E]$ of Theorem 5. By definition, and in view of the preceding analysis, $(\omega, 0, p(t), 0)$ is the limit of a sequence of vectors $(\omega_i, 0, p_i, q_i)$, each of which satisfies (2):

$$(\omega_i, q_i\gamma_\delta(t)\nabla\theta(u_i)) \in \partial_L \langle -p_i, f(t, \cdot, \cdot) \rangle (x_i, u_i) + \{0\} \times N^L_{U(t)}(u_i), \tag{3}$$

where $(x_i, f(t, x_i, u_i), \gamma_\delta(t)\theta(u_i))$ converges to

$$(x_*(t), f(t, x_*(t), u_*(t)), \gamma_\delta(t)\theta(u_*(t))).$$

It follows that $u_i \to u_*(t)$ (since θ is one-to-one, and since, without loss of generality, we may assume $k(t) > 0$ and hence $\gamma_\delta(t) > 0$). Passing to the limit in (3) yields

$$(\omega, 0) \in \partial_L \langle -p(t), f(t, \cdot, \cdot) \rangle (x_*(t), u_*(t)) + \{0\} \times N^L_{U(t)}(u_*(t)). \tag{4}$$

[2] This requires some nonsmooth calculus: see Section 1.10 of Clarke et al. (1998); the fact that f is Lipschitz with respect to u is used here once more.

In the light of (4), the Euler equation therefore implies that $(p'(t), 0)$ lies in the convex hull of the set

$$\partial_L \langle -p(t), f(t, \cdot, \cdot) \rangle (x_*(t), u_*(t)) + \{0\} \times N^L_{U(t)}(u_*(t)),$$

a convex hull, which, in turn, is contained in the corresponding set in which ∂_L and N^L are replaced by ∂_C and N^C. This yields the Euler adjoint equation $[EA]$ of Theorem 4.

F. We turn next to the Weierstrass condition (see Definition 5). Let $u \in U(t)$ satisfy

$$|f(t, x_*(t), u) - x_*'(t)| \le R_\delta(t) \ := \ [1 - \delta(1-\delta)]R(t).$$

Then we have, for almost every t,

$$\begin{aligned} &|(f(t, x_*(t), u), \gamma_\delta(t)\theta(u)) - (x_*'(t), \gamma_\delta(t)\theta(u_*(t)))| \ \le \ R_\delta(t) + 2\gamma_\delta(t) \\ &= \ R_\delta(t) + \delta(1-\delta)\eta k(t) \ \le \ R(t), \end{aligned}$$

so that the Weierstrass condition of radius R (for F) can be invoked to yield (bearing in mind that $q \equiv 0$):

$$\langle p(t), f(t, x_*(t), u) \rangle \ \le \ \langle p(t), f(t, x_*(t), u_*(t)) \rangle \, .$$

This shows that the Weierstrass condition asserted in Theorem 4 holds for the smaller radius R_δ, rather than the radius R we require.

A familiar sequential compactness argument will allow us to obtain the required conclusion. It hinges upon the fact that (see Clarke, 2005) the set of couples (p, λ_0) satisfying the transversality condition, the adjoint equation, and the condition $\|p\|_\infty + \lambda_0 = 1$ is closed relative to uniform convergence of p, weak L^1 convergence of p', and convergence of λ_0 in $\Re$.

We take a sequence δ_i decreasing to 0. As we have seen, we can obtain for each i an arc p_i and a scalar λ_{0_i} satisfying all the conclusions above: nontriviality, transversality, the Euler and adjoint equations, the Weierstrass condition of radius R_{δ_i}. We normalize by setting $\|p_i\|_\infty + \lambda_{0_i} = 1$. This means dividing both p_i and λ_{0_i} by $\|p_i\|_\infty + \lambda_{0_i}$, and no longer imposing that λ_{0_i} must equal 0 or 1; the normalized data continue to satisfy transversality as well as the Euler and adjoint equations.

The Euler equation (or the adjoint equation) provides an estimate of the form $|p_i'(t)| \ \le \ k(t)|p_i(t)|$, which, by a standard weak compactness criterion[3] in L^1, allows us to extract a weakly convergent sequence from $\{p_i'\}$ and then (by Ascoli's Theorem) a further subsequence (we do not relabel) such that $\{p_i\}$ converges uniformly to an arc p, and $\{\lambda_{0_i}\}$ to a scalar λ_0. The limiting couple

[3]See for example Dunford and Schwartz (1967), Theorem IV.8.9.

(p, λ_0) then satisfies the transversality condition, the Euler adjoint equation, and $\|p\|_\infty + \lambda_0 = 1$. It is easy to see that the limiting p satisfies the Weierstrass condition of full radius R. If $\lambda_0 > 0$, we normalize: replace (p, λ_0) by $(p/\lambda_0, 1)$ to conclude.

Finally, the proof of the limiting case of the theorem follows the same lines as the last step above, by considering the couple (p_i, λ_{0_i}) that is obtained for each radius R_i, and passing to the limit along an appropriate subsequence.

3. Two examples

The first example illustrates the advantages of the stratified hypotheses. We consider (for $n = 1$) the problem (P_1) of minimizing the integral cost

$$\int_0^1 \left\{x(t) + |u(t)|^2\right\} \, dt$$

over the processes (x, u) satisfying

$$x'(t) = g(x(t))\theta(u(t))/t \text{ a.e.}, \;\; x(0) = \alpha, \;\; u(t) \in B_m,$$

where B_m is the unit ball in $\Re^m$. We take $\theta : \Re^m \to \Re$ to be continuous, with $\theta(0) = 0$, while $g : \Re \to \Re$ is a smooth function with global Lipschitz constant $K_g > 0$ and values in $[1, 2]$. We also suppose that there are values of u arbitrarily near 0 for which $\theta(u) < 0$.

This can be put into Mayer form by the standard device of introducing another state variable y together with the additional dynamic equation

$$y'(t) = x(t) + |u(t)|^2 \text{ a.e.}$$

Then the problem becomes the one that has the form (P) discussed in §1 : to minimize $y(1)$ subject to

$$(x'(t), y'(t)) = f(t, x(t), y(t), u(t)) := (g(x(t))\theta(u(t))/t, x(t) + |u(t)|^2) \text{ a.e.}$$

and the boundary conditions $(x(0), y(0)) = (\alpha, 0)$, with $U(t) = B_m$.

We now ask whether the admissible process $u_* \equiv 0$ is a strong local minimum, a question we seek to answer by applying the necessary conditions to the proposed optimal process

$$(x_*(t), y_*(t), u_*(t)) := (\alpha, \alpha t, 0).$$

Notice that the hypothesis of Lipschitz integrability of Theorem 2 fails: the Lipschitz constant of f relative to x is in general at least $K_g\theta(u)/t$, which is not summable when $\theta(u) \neq 0$. Thus, that result cannot be invoked here, much less the classical maximum principle.

However, we now verify the pseudo-Lipschitz condition for any radius function R which is a finite constant. The inequality $|f(t,x_1,y_1,u)-(x_*'(t),y_*'(t))| \le R$ of Definition 4 implies $|\theta(u)|/t \le R$, whence

$$|f(t,x_1,y_1,u)-f(t,x_2,y_2,u)| \le (K_gR+1)|x_2-x_1| = k_R(t)|x_2-x_1|,$$

where $k_R(t) \equiv K_gR+1$. Thus, Theorem 3 applies (with $\eta = R/(K_gR+1)$). In fact, the theorem applies in its stated limiting form, so that (if $u_* = 0$ corresponds to a local minimum) there must exist an arc (p,q) satisfying the global conclusions of the maximum principle as given by Theorem 3.

The adjoint and transversality conditions imply that q is the constant $-\lambda_0$, and that $p(1) = -\lambda_0$. If λ_0 were 0, then p,q would also both vanish at $t=1$, contradicting nontriviality. Thus $\lambda_0 = 1$. Now the adjoint equation [A] gives $p'(t) = 1$, so that $p(t) = t-1$. The Weierstrass condition asserts that (almost everywhere on $[0,1]$) the maximum over $u \in B_m$ of the function

$$u \mapsto (t-1)g(\alpha)\theta(u)/t - |u|^2$$

is attained at $u = 0$ (where the value of the function is 0). Since we have $(t-1)g(\alpha)\theta(u) > 0$ for some values of u, and for t arbitrarily small, this is clearly absurd. Thus $u_* = 0$ is ruled out by Theorem 3 as a strong local minimum (or even as a local $W^{1,1}$ minimum).

We remark that if θ is assumed to be locally Lipschitz, then Theorem 4 also applies, and leads to the same conclusion.

A second example

We now give an example in which the Euler adjoint equation [EA] of Theorem 4 yields more precise information than the usual adjoint equation [A]. The availability of the Weierstrass equation will also be useful in the analysis, as we shall see.

We consider the problem (P_2) (with $m = n = 1$) of minimizing

$$\int_0^1 |x(t)-u(t)|\,dt$$

over the processes (x,u) satisfying

$$x'(t) = g(x(t))u(t),\ x(0) = \alpha,\ \int_0^1 u(t)^2\,dt \ \le\ 1.$$

We assume that g is continuously differentiable, that $|g(x)| \le 1\ \forall\, x$, and that for some constant $K_g > 0$ we have

$$|g(x_2)-g(x_1)| \ \le\ K_g|x_2-x_1| \quad \forall\, x_1,x_2 \in \Re,\ t \in [0,1].$$

We remark that under the stated hypotheses, routine arguments prove that (P_2) admits a solution (x_*, u_*). We turn now to the necessary conditions, which, in contrast to the first example, are employed here to identify the solution.

We use the standard device to express the problem in the Mayer form, by introducing two new state variables, y and z. Thus, we are led to introduce the problem (P) of minimizing the endpoint cost $y(1)$ over the processes (x, y, z, u), satisfying the dynamics

$$\begin{aligned} &x'(t) = g(x(t))u(t) \text{ a.e.} \\ &y'(t) = |x(t) - u(t)| \text{ a.e.} \\ &z'(t) = u(t)^2 \text{ a.e.} \end{aligned}$$

and subject to the boundary conditions

$$x(0) = \alpha,\ y(0) = 0,\ z(0) = 0,\ z(1) \leq 1.$$

This has the form of the problem (P) considered in Section 1 (with $U(t) = \Re$); the solution is denoted (x_*, y_*, z_*, u_*) for the evident choices of the functions y_*, z_*.

With an eye to applying Theorem 4, note that in this example f is locally Lipschitz in (x, u) for each t. We proceed to verify the pseudo-Lipschitz hypothesis. For any positive integer i, let the radius function R_i be defined by

$$R_i(t) := i + u_*(t)^2.$$

Then the inequality $|f(t, x_1, y_1, z_1, u) - (x_*'(t), y_*'(t), z_*'(t))| \leq R_i(t)$ of Definition 4 implies $|u^2 - u_*(t)^2| \leq R_i(t)$, whence

$$|u| \leq \left\{R_i(t) + u_*(t)^2\right\}^{1/2} = \left\{i + 2u_*(t)^2\right\}^{1/2},$$

in the light of which we deduce

$$|f(t, x_1, y_1, z_1, u) - f(t, x_2, y_2, z_2, u)| \leq k_i(t)|x_2 - x_1|,$$

where $k_i(t) := K_g[i + 2u_*(t)^2]^{1/2} + 1 \in L^1$. Note that $R_i(t) \geq k_i(t)/(2K_g)$, so that both Theorems 3 and 4 apply (with $\eta = (2K_g)^{-1}$) in their limiting forms. We deduce the existence of an arc (p, q, r) satisfying the global conclusions of either theorem (but not both simultaneously). We proceed now to compare the resulting necessary conditions, depending upon which adjoint equation is used: [A] or [EA].

Either adjoint equation implies that q and r are constants, and transversality yields

$$p(1) = 0,\ q = -\lambda_0,\ \ r \leq 0,\ \ r = 0 \text{ if } \int_0^1 u_*(t)^2\, dt < 1.$$

Suppose that $\lambda_0 = 0$. Then $p(1) = q = 0$, so that $r < 0$ by nontriviality. Consequently, $\int_0^1 u_*(t)^2\,dt = 1$. Also, when $q = 0$, either adjoint equation implies $-p'(t) = g'(x_*(t))u_*(t)p(t)$, whence $p \equiv 0$ by Gronwall's Lemma. But then the Weierstrass condition asserts that (almost everywhere) ru^2 is maximized at $u = u_*(t)$, so that $u_* \equiv 0$. Since this contradicts $\int_0^1 u_*(t)^2\,dt = 1$, we deduce $\lambda_0 = 1$ necessarily.

We therefore take $q = -1$. The Weierstrass condition involves the maximization of the Hamiltonian

$$H = \phi(t)u - |x_*(t) - u| + ru^2$$

over $u \in \Re$, where $\phi(t) := p(t)g(x_*(t))$. Either adjoint equation implies

$$-p'(t) = p(t)g'(x_*(t))u_*(t) - \lambda_1(t),\ \lambda_1(t) \in \partial_C|\cdot - u_*(t)|(x_*(t)), \tag{5}$$

where $|\lambda_1(t)| \leq 1$. It follows from this and from the state equation that the function ϕ has derivative $\lambda_1(t)g(x_*(t))$, which is bounded in absolute value by 1. Since $\phi(1) = 0$, we deduce $|\phi(t)| < 1$ for almost every $t \in [0, 1]$.

If $r = 0$, then the maximization of $\phi(t)u - |x_*(t) - u|$ implies that $u_*(t) = x_*(t)$ a.e., which is clearly optimal when it happens to be admissible. We now exclude this trivial case, so that we may take r strictly negative.

Note that for $r < 0$, the maximum of H is attained at a unique point (by strict concavity). A necessary and sufficient condition for this maximization to occur at $u = u_*(t)$ is:

$$\phi(t) + \lambda_2(t) + 2ru_*(t) = 0,\ \lambda_2(t) \in \partial_C h(t, \cdot)(u_*(t)), \tag{6}$$

where $h(t, u) := -|u - x_*(t)|$.

When $x_* > (pg(x_*) + 1)/(-2r)$ (let us call this zone 1), we find that the choice

$$u_* = (pg(x_*) + 1)/(-2r) < x_* \text{ and } \lambda_2(t) = +1$$

satisfies this condition. When $x_* < (pg(x_*) - 1)/(-2r)$ (zone 2), it holds for

$$u_* = (pg(x_*) - 1)/(-2r) > x_* \text{ and } \lambda_2(t) = -1.$$

In the remaining case (zone 3), where

$$(pg(x_*) - 1)/(-2r) \ \leq\ x_* \leq (pg(x_*) + 1)/(-2r), \tag{7}$$

it follows that $u_*(t) = x_*(t)$, and we find (from (6)):

$$\lambda_2(t) = -pg(x_*) - 2rx_* \in [-1, +1]. \tag{8}$$

It follows that the optimal control is given by a feedback $u = F(x, p)$ in the phase-plane (x, p), defined differently on each of the three zones. The state equation determines $x'(t)$ in each zone. The missing ingredient is the differential equation for $p'(t)$, which must come from the adjoint equation.

Suppose that form $[A]$ of the adjoint equation is used. Then in zone 1, we have $u < x$ and $u = (pg(x)+1)/(-2r)$, so the phase plane dynamics become

$$-p' = pg'(x)(pg(x)+1)/(-2r) - 1, \;\; x' = pg(x)(pg(x)+1)/(-2r). \tag{9}$$

Similarly, we find that in zone 2 we have

$$-p' = pg'(x)(pg(x)-1)/(-2r) + 1, \;\; x' = pg(x)(pg(x)-1)/(-2r). \tag{10}$$

However, the adjoint equation (5) does not specify $p'(t)$ when x_* lies in zone 3; that is, we do not know $\lambda_1(t)$ precisely, since the generalized gradient appearing there is the interval $[-1,+1]$.

If, however, $[EA]$ is used, then the same dynamics as above result in zones 1 and 2, while in zone 3 we have

$$(-p', 0) = (pg'(x)u - \lambda_3, pg(x) + \lambda_3 + 2rx)$$

for some λ_3, since the (joint) generalized gradient of $|x-u|$ at $(0,0)$ is the set

$$\{(\lambda_3, -\lambda_3) : |\lambda_3| \le 1\}.$$

It follows from this and (8) that

$$\lambda_3 = \lambda_2 = -pg(x) - 2rx.$$

Thus the λ_1 of (5) must be λ_3. Accordingly, the phase plane dynamics in zone 3 is given by

$$-p' = pg'(x)x + pg(x) + 2rx, \;\; x' = pg(x)x. \tag{11}$$

The phase-plane system (9)(10)(11) may now be used in the usual way together with the boundary conditions $x(0) = \alpha$, $p(1) = 0$ in order to determine the optimal control u_*, as well as the value of r, which must be taken to respect the saturation condition $\int_0^1 u_*(t)^2\,dt = 1$. The case $g \equiv 1$ can be completed analytically.

We remark that another approach to finding the correct expression for p' in zone 3 (in the absence of $[EA]$) would be to exploit the Erdmann condition ($H =$ constant) for this autonomous problem.

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