

**Incompressible Maxwell-Boussinesq approximation:
Existence, uniqueness and shape sensitivity***

by

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Abstract: We prove the existence and uniqueness of weak solutions to the variational formulation of the Maxwell-Boussinesq approximation problem. Some further regularity in $W^{1,2+\delta}$, $\delta > 0$, is obtained for the weak solutions. The shape sensitivity analysis by the boundary variations technique is performed for the weak solutions. As a result, the existence of the strong material derivatives for the weak solutions of the problem is shown. The result can be used to establish the shape differentiability for a broad class of shape functionals for the models of Fourier-Navier-Stokes flows under the electromagnetic field.

Keywords: magnetohydrodynamic flows, shape sensitivity.

1. Introduction

The problem of magnetohydrodynamic flows has been studied by several authors (see Druet, 2009; Duvaut and Lions, 1972; Gerbeau and Le Bris, 1997, 1999; Hömberg and Sokolowski, 2003; Meir, 1994; Meir and Schmidt, 2001; Sermange and Temam, 1983) and it goes back to the work of Ladyzhenskaya and Solonnikov. At that time the coupled system did not include thermal effects. The full complete problem including the heat transfer seems to be more realistic and not many authors dealt with it. The full Navier-Stokes-Fourier-Maxwell problem was only partially studied in the works of Alekseev (2006), Meir (1994), Meir and Schmidt (2001), where the principal coefficients are assumed constant. On the other hand, in Druet (2009) the coefficients are only temperature dependent,

*Submitted: November 2008; Accepted: September 2009.

and the force term is either globally bounded (truncated) or the thermal expansion coefficient is sufficiently small. Concerning the shape sensitivity analysis we can mention work of Zolésio and his collaborators (see Boisgérault and Zolésio, 2000, 2001; Delfour and Zolésio, 2001; Dziri and Zolésio, 1996, 1997; Dziri, Moubachir and Zolésio, 2004; Murat and Simon, 1976; Pironneau, 1984; Plotnikov and Sokolowski, 2006; Plotnikov, Ruban and Sokolowski, 2008), where the case of Navier-Stokes problem was investigated and also the coupled problem with heat transfer. We refer to Consiglieri, Nečasová and Sokolowski (2008) where an uncoupled complete problem is studied (see Remark 5). It is only in this paper that the problem under study has the principal coefficients varying with the temperature as well as the space variable.

Let Ω be an open bounded subset of \mathbb{R}^3 with the boundary $\partial\Omega \in C^{1,1}$ which is splitted into two parts $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, where Γ_D is an open nonempty subset of $\partial\Omega$ and $\Gamma_N = \partial\Omega \setminus \bar{\Gamma}_D$. The thermoelectromagnetoflow problem reads in Ω :

$$-\nabla \cdot (\nu(T)D\mathbf{u}) + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mu \operatorname{rot} \mathbf{H} \times \mathbf{H} + \rho \mathbf{f} - \rho \mathbf{G}(T)T; \quad (1)$$

$$\nabla \times (\sigma^{-1}(T)\nabla \times \mathbf{H}) = \nabla \times (\sigma^{-1}(T)\mathbf{J}_0 + \mu\mathbf{u} \times \mathbf{H}); \quad (2)$$

$$\operatorname{div} \mathbf{u} = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = \operatorname{div} \mathbf{H} = 0; \quad (3)$$

$$-\nabla \cdot (k(T)\nabla T) + \rho\mathbf{u} \cdot \nabla T = f. \quad (4)$$

Here \mathbf{u} is the fluid velocity vector, T is the temperature, $D\mathbf{u} = (D_{ij}) = (\partial_i u_j + \partial_j u_i)/2$ ($i, j = 1, 2, 3$) is the symmetrized gradient of the velocity, μ the magnetic permeability, p denotes the pressure, ρ is density, \mathbf{f} and f denote the external forces and heat sources, respectively. The coefficients ν, σ, k are temperature dependent functions representing the viscosity of the fluid, the electric and the heat conductivities, respectively. Indeed in order for the setting to be more realistic these coefficients are not only functions of the temperature but also of the space variable. The density ρ is assumed to be constant, we set $\rho = 1$.

The buoyancy force, as in the Boussinesq approximation, is described by $\mathbf{G}(T) = \beta(T)(0, 0, g)^\top$, where β denotes the coefficient of thermal dilatation and g is the constant of gravity. The existence of two body forces in the fluid, the Lorentz force $\mathbf{J} \times \mathbf{B} = (\nabla \times \mathbf{H}) \times (\mu\mathbf{H})$ and the buoyancy force, results from the presence of the magnetic field \mathbf{H} . Moreover (2) results if we take the rotational in the second equation of the steady-state Maxwell equations:

$$\nabla \times \mathbf{E} = 0; \quad \mathbf{J} = \nabla \times \mathbf{H},$$

where \mathbf{E} is the electric intensity field and \mathbf{J} is the current density given by the Ohm's law

$$\mathbf{J} = \mathbf{J}_0 + \sigma(T)(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

where \mathbf{J}_0 denotes a given applied current (see Alekseev, 2006).

Finally, the thermoelectromagnetoflow problem under study has the following boundary conditions

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega; \tag{5}$$

$$T = 0 \text{ on } \Gamma_D, \quad k(T) \frac{\partial T}{\partial \mathbf{n}} + \alpha T = h \quad \text{on } \Gamma_N, \tag{6}$$

where α represents the convective heat transfer coefficient. Here α is a function only depending on the space variable. We refer to Consiglieri (2006, 2008), Druet (2009), Meir and Schmidt (2001), whereby it can be extended to a function also depending on the temperature and even to include radiation effects. To simplify the presentation we assume a homogeneous Dirichlet condition for the velocity of the fluid (see Remark 2).

The outline of the paper is as follows. In Section 2, new existence results are stated under diverse assumptions for the system of strongly coupled elliptic equations governing temperature dependent electromagnetic stationary flow. Fluid velocity, magnetic field intensity and fluid temperature are the state variables. Section 3 is devoted to the proof of existence of a weak solution to the nonlinear coupling of electromagnetics, heat and fluid device, using a fixed point argument. Some well posedness auxiliary existence results are established as well as results on the regularity of solutions. In Section 4, additional regularity of a weak solution to the considered electromagnetic flow problem is proved, assuming more regular external forces and applied current intensities. Assuming Lipschitz type continuity of function parameters with respect to temperature, this solution is shown to be unique for suitable small data. In Section 5, assuming that all coefficients of the elliptic system are constant and the velocity field is divergence free, sensitivity analysis of the unique solution to the considered elliptic system with respect to perturbation of the boundary of the domain occupied by the fluid is performed using the material derivative approach. The existence of strong material derivative of the weak solution to the elliptic system is shown. The elliptic system characterizing this derivative is provided.

2. Assumptions and main existence results

We need some assumptions on the model, which are listed below.

Let us assume that

(H1) $\nu, \sigma, k : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are Caratheodory functions such that

$$\exists \nu^\#, \nu_\# > 0 : \quad \nu_\# \leq \nu(\cdot, \xi) \leq \nu^\#, \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}; \tag{7}$$

$$\exists \sigma^\#, \sigma_\# > 0 : \quad \sigma_\# \leq \sigma(\cdot, \xi) \leq \sigma^\#, \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}; \tag{8}$$

$$\exists k^\#, k_\# > 0 : \quad k_\# \leq k(\cdot, \xi) \leq k^\#, \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}; \tag{9}$$

(H2) $\mathbf{G} = (0, 0, G)$ where $G = g\beta$ with β a real, continuous, and bounded function and we denote by $G^\#$ the upper bound for the function G ;

(H3) $\alpha \in L^q_+(\Gamma_N) = \{\alpha \in L^q(\Gamma_N) : \alpha \geq 0\}$ for q such that $q > 3/2$, which means that its conjugate $q' = q/(q-1)$ verifies $q' < 3$;

(H4) and $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$, $f \in L^2(\Omega)$ and $h \in L^2(\Gamma_N)$.

In the framework of function spaces of the Lebesgue and Sobolev type, the norms are denoted by the symbols $\|\cdot\|$, $\|\cdot\|_1$, $\|\cdot\|_{\Gamma_N}$ in spaces $L^2(\Omega)$, $H^1(\Omega)$, $L^2(\Gamma_N)$, respectively, and the scalar and vector function spaces are not distinguished in our notations. Providing that the meaning remains clear, the canonical norm in $L^p(\Omega)$ for $p \neq 1, 2$ is denoted by $\|\cdot\|_p$. We introduce the Hilbert spaces

$$\begin{aligned} \mathbf{V} &= \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}, \\ \mathbf{V}(\operatorname{rot}) &= \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{rot} \mathbf{v} \in \mathbf{L}^2(\Omega), \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ Z &= \{\xi \in H^1(\Omega) : \xi = 0 \text{ on } \Gamma_D\}, \end{aligned}$$

equipped with their standard scalar products. We recall that the norms $\|\cdot\|_{\mathbf{V}(\operatorname{rot})}$ and $\|\cdot\|_Z$ are equivalent to the usual seminorms $\|\nabla \times \cdot\|$ and $\|\nabla \cdot \cdot\|$ and also to the norms $\|\cdot\|_1$ on spaces $\mathbf{H}^1(\Omega)$ and $H^1(\Omega)$, respectively (see Duvaut and Lions, 1972).

We state the main existence results of the paper.

THEOREM 1 *Under the above assumptions (H1)-(H4), and, in addition, under the following assumptions*

$$\begin{aligned} b > 0 \quad \text{and} \quad \mu a^2 < b^3, & \quad (10) \\ a &= \frac{\nu_{\#}}{\mu \sigma_{\#}} \|\mathbf{J}_0\|, \\ b &= \frac{\nu_{\#}}{\mu \sigma_{\#}} - \left(\|\mathbf{f}\| + \frac{G_{\#}}{k_{\#}} (\|f\| + \|h\|_{\Gamma_N}) \right), \end{aligned}$$

the problem (1)-(6) has a weak solution in the following sense:

The triplet $(\mathbf{u}, \mathbf{H}, T) \in \mathbf{V} \times \mathbf{V}(\operatorname{rot}) \times Z$ satisfies the following integral identities

$$\begin{aligned} & \int_{\Omega} \nu(T) D\mathbf{u} : D\mathbf{v} dx + \int_{\Omega} (\mathbf{v} \otimes \mathbf{u}) : \nabla \mathbf{u} dx = \\ & = \int_{\Omega} \left(\mu (\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T \right) \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{V}; \end{aligned} \quad (11)$$

$$\begin{aligned} & \int_{\Omega} \frac{1}{\sigma(T)} (\nabla \times \mathbf{H}) \cdot (\nabla \times \mathbf{v}) dx = \mu \int_{\Omega} (\mathbf{u} \times \mathbf{H}) \cdot (\nabla \times \mathbf{v}) dx + \\ & + \int_{\Omega} \frac{1}{\sigma(T)} \mathbf{J}_0 \cdot (\nabla \times \mathbf{v}) dx, \quad \forall \mathbf{v} \in \mathbf{V}(\operatorname{rot}); \end{aligned} \quad (12)$$

$$\begin{aligned} & \int_{\Omega} k(T) \nabla T \cdot \nabla \eta dx + \int_{\Omega} \mathbf{u} \cdot \nabla T \eta dx + \int_{\Gamma_N} \alpha T \eta ds = \\ & = \int_{\Omega} f \eta dx + \int_{\Gamma_N} h \eta ds, \quad \forall \eta \in Z. \end{aligned} \quad (13)$$

Moreover, the pair (\mathbf{H}, T) enjoys the additional regularity, actually, it belongs to $\mathbf{W}^{1,2+\epsilon}(\Omega) \times W^{1,2+\epsilon}(\Omega)$ for some $\epsilon, \epsilon > 0$.

REMARK 1 *If $\epsilon > 2/5$ we can deduce the additional regularity on \mathbf{u} as in Consiglieri, Nečasová and Sokolowski (2008). Otherwise, since the operators in the above elliptic equations of the second order have discontinuous coefficients, we can obtain Hölder continuity on $\bar{\Omega}$ of the weak solution T due to the De Giorgi-Nash Theorem if $f, h \in L^q(\Omega)$ for $q > 3$. If σ is taken as a continuous function, then the main operator in (12) has continuous coefficient and the regularity theory can be applied to the weak solution \mathbf{H} . Or, simply, if we suppose that the electric conductivity σ is constant, it will be sufficient to our purposes in the study of the shape sensivity. However, in the sequel the data assumptions are kept as general as possible.*

THEOREM 2 *Let $\epsilon_0 < \epsilon < 1$ and $2 < q < 3$ be such that*

$$\frac{3q}{3 - q} = \frac{(2 + \epsilon_0)(2 + \epsilon)}{\epsilon - \epsilon_0}. \tag{14}$$

If $\mathbf{J}_0 \in \mathbf{L}^q(\Omega)$, then $\mathbf{H} \in \mathbf{L}^{3q/(3-q)}(\Omega)$. Under the assumption $\mathbf{f} \in \mathbf{L}^{2+\delta_1}(\Omega)$, where $\delta_1 > 0$, the weak solution \mathbf{u} given by Theorem 1 enjoys the additional regularity, actually, it belongs to $\mathbf{W}^{1,2+\delta}(\Omega)$ for some $\delta > 0$. Furthermore, under the following Lipschitz-type continuity assumption on the temperature dependent function parameters of the model

$$\exists \bar{\nu} > 0 : \quad |\nu(T^2) - \nu(T^1)| \leq \bar{\nu} |T^2 - T^1|^{3\delta/(2+\delta)}, \tag{15}$$

$$\exists \bar{G} > 0 : \quad g|\beta(T^2) - \beta(T^1)| \leq \bar{G} |T^2 - T^1|, \tag{16}$$

$$\exists \bar{\sigma} > 0 : \quad |\sigma(T^2) - \sigma(T^1)| \leq \bar{\sigma} |T^2 - T^1|^{3\epsilon/(2+\epsilon)}, \tag{17}$$

$$\exists \bar{k} > 0 : \quad |k(T^2) - k(T^1)| \leq \bar{k} |T^2 - T^1|^{3\epsilon/(2+\epsilon)}, \quad \forall T^2, T^1 \in \mathbb{R}, \tag{18}$$

the weak solution $(\mathbf{u}, \mathbf{H}, T)$ is unique for small data.

The existence of the pressure p in the space of distributions follows from the well-known results by using the divergence-free test functions $\mathbf{v} \in \mathbf{C}_0^\infty(\Omega)$ in (11). Moreover, the pressure is unique up to a constant.

3. Proof of Theorem 1

First, we recall the Tychonoff extension to weak topologies of the Schauder fixed point theorem (see Dunford and Schwartz, 1958, pp. 453-456 and 470).

THEOREM 3 *Let K be a nonempty weakly sequentially compact convex subset of a locally convex linear topological vector space V . Let $\mathcal{L} : K \rightarrow K$ be a weakly sequentially continuous operator. Then \mathcal{L} has at least one fixed point.*

Let \mathcal{L} be the mapping of the form

$$\mathcal{L} : (\mathbf{w}, \mathbf{h}, \xi) \in \mathbf{V} \times \mathbf{V}(\text{rot}) \times Z \mapsto (\mathbf{H}, T) \mapsto (\mathbf{u}, \mathbf{H}, T),$$

where the functions \mathbf{u} , \mathbf{H} and T are the solutions for the elliptic boundary value problems (23), (21) and (19), respectively. Indeed, the fixed point argument starts by prescribing $(\mathbf{w}, \mathbf{h}, \xi)$ from $\mathbf{V} \times \mathbf{V}(\text{rot}) \times Z$ next by finding the temperature and the magnetic field and finally the velocity of the fluid. The proofs of such existence results are the straightforward application of the classical existence theory, hence are omitted here.

PROPOSITION 1 *Let $(\mathbf{w}, \xi) \in \mathbf{V} \times Z$ and assume that conditions (9), (H3)-(H4) are fulfilled. Then there exists a unique $T \in Z$ such that*

$$\begin{aligned} & \int_{\Omega} k(\xi) \nabla T \cdot \nabla \eta dx + \int_{\Omega} \mathbf{w} \cdot \nabla T \eta dx + \int_{\Gamma_N} \alpha T \eta ds = \\ & = \int_{\Omega} f \eta dx + \int_{\Gamma_N} h \eta ds, \quad \forall \eta \in Z. \end{aligned} \quad (19)$$

Moreover, the energy estimate holds

$$k_{\#} \|T\|_1 \leq \|f\| + \|h\|_{\Gamma_N}. \quad (20)$$

PROPOSITION 2 *Let $(\mathbf{w}, \mathbf{h}, \xi) \in \mathbf{V} \times \mathbf{V}(\text{rot}) \times Z$ and assume that conditions (8) and (H4) are fulfilled. Then there exists a unique $\mathbf{H} \in \mathbf{V}(\text{rot})$ such that*

$$\begin{aligned} & \int_{\Omega} \frac{1}{\sigma(\xi)} (\nabla \times \mathbf{H}) \cdot (\nabla \times \mathbf{v}) dx = -\mu \int_{\Omega} (\mathbf{h} \times \mathbf{w}) \cdot (\nabla \times \mathbf{v}) dx + \\ & + \int_{\Omega} \frac{1}{\sigma(\xi)} \mathbf{J}_0 \cdot (\nabla \times \mathbf{v}) dx, \quad \forall \mathbf{v} \in \mathbf{V}(\text{rot}). \end{aligned} \quad (21)$$

Moreover, the energy estimate holds

$$\frac{1}{\sigma_{\#}} \|\mathbf{H}\|_1 \leq \mu \|\mathbf{h} \times \mathbf{w}\| + \frac{1}{\sigma_{\#}} \|\mathbf{J}_0\|. \quad (22)$$

PROPOSITION 3 *Let $(\mathbf{w}, \xi) \in \mathbf{V} \times Z$, and T and \mathbf{H} the solutions in accordance with Propositions 1 and 2, respectively, and assume that conditions (7), (H2) and (H4) are fulfilled. Then there exists a unique $\mathbf{u} \in \mathbf{V}$ such that*

$$\begin{aligned} & \int_{\Omega} \nu(\xi) D\mathbf{u} : D\mathbf{v} dx + \int_{\Omega} (\mathbf{v} \otimes \mathbf{w}) : \nabla \mathbf{u} dx = \\ & = \int_{\Omega} \left(\mu (\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T \right) \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{V}. \end{aligned} \quad (23)$$

Moreover, the energy estimate holds

$$\nu_{\#} \|\mathbf{u}\|_1 \leq \mu \|\nabla \times \mathbf{H}\| \|\mathbf{H}\|_{L^3} + \|\mathbf{f}\| + G_{\#} \|T\|_{L^{6/5}}. \quad (24)$$

REMARK 2 For given $\mathbf{g} \in \mathbf{H}^{-1/2}(\partial\Omega)$, there exists a lifting $\mathbf{u}_{\mathbf{g}} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{u}_{\mathbf{g}} = \mathbf{g}$ on $\partial\Omega$ and $\mathbf{u}_{\mathbf{g}}$ verifies

$$-\nabla \cdot (\nu(\xi)D\mathbf{u}_{\mathbf{g}}) + (\mathbf{w} \cdot \nabla)\mathbf{u}_{\mathbf{g}} = -\nabla p_{\mathbf{g}}; \quad \nabla \cdot \mathbf{u}_{\mathbf{g}} = 0 \text{ in } \Omega.$$

If the element $\mathbf{U} = \mathbf{u} - \mathbf{u}_{\mathbf{g}} \in \mathbf{V}$ is determined by a solution to the problem

$$-\nabla \cdot (\nu(\xi)D\mathbf{U}) + (\mathbf{w} \cdot \nabla)\mathbf{U} = -\nabla p_{\mathbf{U}} + \mu(\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T \text{ in } \Omega,$$

then $\mathbf{u} = \mathbf{U} + \mathbf{u}_{\mathbf{g}}$ is the solution to the problem

$$\begin{aligned} -\nabla \cdot (\nu(\xi)D\mathbf{u}) + (\mathbf{w} \cdot \nabla)\mathbf{u} &= -\nabla p + \mu(\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega, \quad \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega. \end{aligned}$$

Therefore, it is assumed that $\mathbf{g} = \mathbf{0}$, observing that in the inhomogeneous case a smallness assumption for the velocity at the boundary will be also needed in order to prove the main results.

In view of Propositions 1, 2 and 3, the operator \mathcal{L} is well defined. Moreover, \mathcal{L} maps the ball

$$\begin{aligned} K = \{(\mathbf{w}, \mathbf{h}, \xi) \in \mathbf{V} \times \mathbf{V}(\text{rot}) \times Z : & \|\mathbf{w}\|_1 \leq R_1, \|\mathbf{h}\|_1 \leq R_2, \\ & \|\xi\|_1 \leq \frac{1}{k_{\#}}(\|f\| + \|h\|_{\Gamma_N})\} \end{aligned}$$

into itself, since by (20), (22) and (24) it follows that

$$\|\mathbf{H}\|_1 \leq \sigma_{\#} \left(\mu R_1 R_2 + \frac{1}{\sigma_{\#}} \|\mathbf{J}_0\| \right) \leq R_2, \tag{25}$$

$$\|\mathbf{u}\|_1 \leq \frac{1}{\nu_{\#}} \left(\mu R_2^2 + \|\mathbf{f}\| + \frac{G_{\#}}{k_{\#}} (\|f\| + \|h\|_{\Gamma_N}) \right) = R_1, \tag{26}$$

where $R_2 > 0$ is such that

$$\frac{\mu\sigma_{\#}}{\nu_{\#}} R_2 \left(\mu R_2^2 + \|\mathbf{f}\| + \frac{G_{\#}}{k_{\#}} (\|f\| + \|h\|_{\Gamma_N}) \right) + \frac{\sigma_{\#}}{\sigma_{\#}} \|\mathbf{J}_0\| \leq R_2$$

or equivalently

$$a \leq R_2(b - \mu R_2^2),$$

if $b > 0$ and $(a/b)^2 < b/\mu$, which is assured by (10).

In order to apply Theorem 3 it remains to prove the weak continuity of \mathcal{L} . Since we have the compact embeddings

$$\mathbf{V}, \mathbf{V}(\text{rot}) \hookrightarrow \hookrightarrow \{\mathbf{w} \in \mathbf{L}^4(\Omega) : \nabla \cdot \mathbf{w} = 0 \text{ in } \Omega, \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \tag{27}$$

$$Z \hookrightarrow \hookrightarrow L^1(\Omega), \tag{28}$$

let $\{(\mathbf{w}_m, \mathbf{h}_m, \xi_m)\}$ be a sequence such that

$$\mathbf{w}_m \rightarrow \mathbf{w}, \quad \mathbf{h}_m \rightarrow \mathbf{h} \quad \text{in } \mathbf{L}^4(\Omega); \quad \xi_m \rightarrow \xi \text{ in } L^1(\Omega). \quad (29)$$

Let $(\mathbf{u}_m, \mathbf{H}_m, T_m)$ be the corresponding weak solutions given by Propositions 1, 2 and 3, for each $m \in \mathbb{N}$. From the estimates (24), (22) and (20), the sequence $\{(\mathbf{u}_m, \mathbf{H}_m, T_m)\}$ is bounded in $\mathbf{V} \times \mathbf{V}(\text{rot}) \times Z$. Then there exists the weak limit $(\mathbf{u}, \mathbf{H}, T) \in \mathbf{V} \times \mathbf{V}(\text{rot}) \times Z$ such that

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \text{ in } \mathbf{V}; \quad \mathbf{H}_m \rightharpoonup \mathbf{H} \text{ in } \mathbf{V}(\text{rot}); \quad T_m \rightharpoonup T \text{ in } Z, \quad (30)$$

possibly for a subsequence, still denoted by $(\mathbf{u}_m, \mathbf{H}_m, T_m)$. Applying (27)-(28) we obtain

$$\mathbf{u}_m \rightarrow \mathbf{u}, \quad \mathbf{H}_m \rightarrow \mathbf{H} \quad \text{in } \mathbf{L}^4(\Omega); \quad T_m \rightarrow T \text{ in } L^1(\Omega). \quad (31)$$

We pass to the limit as $m \rightarrow +\infty$ in the integral identities (23), (21) and (19), replacing $\mathbf{w}, \mathbf{h}, \xi, \mathbf{u}, \mathbf{H}$ and T by the sequences $\mathbf{w}_m, \mathbf{h}_m, \xi_m, \mathbf{u}_m, \mathbf{H}_m$ and T_m , respectively, using (29)-(31) and the continuity properties of the Niemytskii operators in the coefficients combined with (7)-(9). Therefore, we conclude that the limit $(\mathbf{u}, \mathbf{H}, T)$ is a solution corresponding to $(\mathbf{w}, \mathbf{h}, \xi)$ of the required problem (23), (21) and (19).

Then, Theorem 3 guarantees the existence of at least one fixed point which is the required weak solution.

The regularity $(\mathbf{H}, T) \in \mathbf{W}^{1,2+\epsilon}(\Omega) \times W^{1,2+\epsilon}(\Omega)$ for some $\epsilon, \epsilon > 0$ is a consequence of the following regularity results.

PROPOSITION 4 *If $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$ then there exists a constant $\epsilon > 0$ such that the weak solution $\mathbf{H} \in \mathbf{V}(\text{rot})$ of (12) belongs to $\mathbf{W}^{1,2+\epsilon}(\Omega)$, i.e.*

$$\|\nabla \mathbf{H}\|_{2+\epsilon} \leq K_1,$$

with a constant $K_1 > 0$ only dependent on the data.

Proof. Adapting the regularity theory for elliptic equations of the second order (see Groger, 1989), we obtain $\mathbf{H} \in \mathbf{W}^{1,2+\epsilon}(\Omega)$ with $2 + \epsilon < 6$ since

$$\mathbf{J}_0 - \sigma(T)\mu\mathbf{H} \times \mathbf{u} \in \mathbf{L}^2(\Omega) \hookrightarrow (\mathbf{W}^{1,6/5}(\Omega))'.$$

The following result is a consequence of the regularity of solutions to the mixed boundary value problems for elliptic equations (see Groger, 1989).

PROPOSITION 5 *If $f \in L^2(\Omega)$ and $h \in L^2(\Gamma_N)$ then there exists a constant $\epsilon > 0$ such that the weak solution $T \in Z$ of (13) belongs to $W^{1,2+\epsilon}(\Omega)$, i.e.*

$$\|\nabla T\|_{2+\epsilon} \leq K_2,$$

with a constant $K_2 > 0$ only dependent on the data.

Proof. According to Groger (1989) we obtain $T \in W^{1,2+\epsilon}(\Omega)$ with $2 + \epsilon < 3$ since $f, h \in (W^{1,3/2}(\Omega))'$. ■

4. Proof of Theorem 2

The regularity $\mathbf{u} \in \mathbf{W}^{1,2+\delta}(\Omega)$ for some $\delta > 0$ is a consequence of the following regularity results.

PROPOSITION 6 *For every $2 < q < 3$, if $\mathbf{J}_0 \in \mathbf{L}^q(\Omega)$ then the weak solution $\mathbf{H} \in \mathbf{V}(\text{rot})$ of (12) belongs to $\mathbf{L}^{3q/(3-q)}(\Omega)$.*

Proof. Adapting the regularity theory for elliptic equations of the second order (see Groger, 1989), the desired result is obtained, provided by $\mathbf{J}_0 - \sigma(T)\mu\mathbf{H} \times \mathbf{u} \in \mathbf{L}^q(\Omega)$. ■

PROPOSITION 7 *If q is given as in (14) and $\mathbf{f} \in \mathbf{L}^{2+\delta_1}(\Omega)$ for some $\delta_1 > 0$, then there exists a constant $\delta > 0$ such that the weak solution $\mathbf{u} \in \mathbf{V}$ of (11) belongs to $\mathbf{W}^{1,2+\delta}(\Omega)$, i.e.*

$$\|\nabla \mathbf{u}\|_{2+\delta} \leq K_3,$$

with a constant $K_3 > 0$ only dependent on the data.

Proof. For every $x_0 \in \bar{\Omega}$, $0 < r < R$ small enough, $\Omega(x_0, R) := \Omega \cap B(x_0, R)$, $\theta \in]0, 1[$ and some positive constants B_1, B_2 , independent of \mathbf{u}, \mathbf{H} and T , we have the following reverse estimate (see Consiglieri and Shilkin, 2000, Lemma 3.2)

$$\begin{aligned} \left(\int_{\Omega(x_0, r)} |\nabla \mathbf{u}|^2 dx \right)^{1/2} &\leq \theta \left(\int_{\Omega(x_0, R)} |\nabla \mathbf{u}|^2 dx \right)^{1/2} \\ &\quad + \frac{B_1}{R-r} \left(\int_{\Omega(x_0, R)} |\nabla \mathbf{u}|^{6/5} dx \right)^{5/6} \\ &\quad + \frac{B_2}{R-r} \left(\int_{\Omega(x_0, R)} (|\mathbf{u} \otimes \mathbf{u}|^2 + |\mathbf{F}|^2 + |\mathbf{f}|^2 + 1) dx \right)^{1/2} \end{aligned}$$

where $\mathbf{F} = \mu \text{rot} \mathbf{H} \times \mathbf{H} - \mathbf{G}(T)T$. By Propositions 4 and 6, we have $\mathbf{H} \in \mathbf{W}^{1,2+\epsilon}(\Omega) \cap \mathbf{L}^{(2+\epsilon_0)(2+\epsilon)/(\epsilon-\epsilon_0)}(\Omega)$. Thus, it follows that $\text{rot} \mathbf{H} \times \mathbf{H} \in \mathbf{L}^{2+\epsilon_0}(\Omega)$ and $\mathbf{F} \in \mathbf{L}^{2+\epsilon_0}(\Omega)$. Since $\mathbf{u} \otimes \mathbf{u} \in \mathbf{L}^3(\Omega)$ then the Gehring inequality (cf. Gehring, 1973) guarantees the higher integrability $\mathbf{u} \in \mathbf{W}^{1,2+\delta}(\Omega)$ for some $0 < \delta < \min\{\epsilon_0, \delta_1\}$. ■

Now, we prove the uniqueness. To this end, let $(\mathbf{u}^1, \mathbf{H}^1, T^1)$ and $(\mathbf{u}^2, \mathbf{H}^2, T^2)$ be two weak solutions to problem (11), (13), and (12). Arguing as in Consiglieri, Nečasová and Sokolowski (2008), the respective differences $\bar{\mathbf{u}} = \mathbf{u}^1 - \mathbf{u}^2$, $\bar{\mathbf{H}} =$

$\mathbf{H}^1 - \mathbf{H}^2$ and $\bar{T} = T^1 - T^2$ satisfy

$$\begin{aligned}
\frac{\nu_{\#}}{2} \|D\bar{\mathbf{u}}\|^2 &\leq \frac{\bar{\nu}}{\nu_{\#}} \|\bar{T}\|_6^{6\delta/(2+\delta)} \|D\mathbf{u}^2\|_{2+\delta}^2 + C_2^2 \|D\bar{\mathbf{u}}\|^2 \|\nabla \mathbf{u}^2\| + \\
&+ \frac{C_1}{\nu_{\#}} \left(\mu \|(\nabla \times \mathbf{H}^1) \times \mathbf{H}^1 - (\nabla \times \mathbf{H}^2) \times \mathbf{H}^2\|_{6/5} + \right. \\
&+ \left. G^{\#} \|\bar{T}\|_{6/5} + \bar{G} \|\bar{T}\|_6 \|T^2\|_{3/2} \right)^2; \\
\frac{1}{4\sigma^{\#}} \|\nabla \times \bar{\mathbf{H}}\|^2 &\leq \sigma^{\#} \left\| \left(\frac{1}{\sigma(T^2)} - \frac{1}{\sigma(T^1)} \right) \nabla \times \mathbf{H}^2 \right\|^2 + \\
&+ \sigma^{\#} \mu \|\mathbf{u}^1 \times \mathbf{H}^1 - \mathbf{u}^2 \times \mathbf{H}^2\|^2 + \sigma^{\#} \left\| \left(\frac{1}{\sigma(T^1)} - \frac{1}{\sigma(T^2)} \right) \mathbf{J}_0 \right\|^2; \\
\frac{k_{\#}}{2} \|\nabla \bar{T}\|^2 &\leq \frac{\bar{k}}{k_{\#}} \|\bar{T}\|_6^{6\epsilon/(2+\epsilon)} \|\nabla T^2\|_{2+\epsilon}^2 + \frac{C_1}{k_{\#}} \|\bar{\mathbf{u}}\|_6^2 \|\nabla T^2\|_{3/2}^2, \tag{32}
\end{aligned}$$

where C_1, C_2 are the Sobolev constants of the embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $H^1(\Omega) \hookrightarrow L^4(\Omega)$, respectively. Using the Lipschitz continuity assumptions (15)-(18), and applying Hölder and Young inequalities leads to

$$\begin{aligned}
\frac{\nu_{\#}}{2} \|D\bar{\mathbf{u}}\|^2 &\leq \frac{\bar{\nu}}{\nu_{\#}} \|\bar{T}\|_6^{6\delta/(2+\delta)} \|D\mathbf{u}^2\|_{2+\delta}^2 + C_2^2 \|D\bar{\mathbf{u}}\|^2 \|\nabla \mathbf{u}^2\| + \\
&+ \frac{C_1}{\nu_{\#}} \left(\mu \|\nabla \times \bar{\mathbf{H}}\| \|\mathbf{H}^1\|_3 + \mu \|\nabla \times \mathbf{H}^2\| \|\bar{\mathbf{H}}\|_3 + G^{\#} \|\bar{T}\|_{6/5} + \bar{G} \|\bar{T}\|_6 \|T^2\|_{3/2} \right)^2; \\
\frac{1}{4(\sigma^{\#})^2} \|\nabla \times \bar{\mathbf{H}}\|^2 &\leq \frac{\bar{\sigma}}{(\sigma^{\#})^2} \|\bar{T}\|_6^{6\epsilon/(2+\epsilon)} (\|\nabla \mathbf{H}^2\|_{2+\epsilon}^2 + \|\mathbf{J}_0\|_{2+\epsilon}^2) + \\
&+ \mu (\|\bar{\mathbf{u}}\|_4^2 \|\mathbf{H}^1\|_4^2 + \|\mathbf{u}^2\|_4^2 \|\bar{\mathbf{H}}\|_4^2).
\end{aligned}$$

Let K_1, K_2 and K_3 be the upper bounds derived in Propositions 4, 5 and 7, respectively, and K_4 stand for the upper bound in estimate (20), namely,

$$K_4 = \frac{1}{k_{\#}} (\|f\| + \|h\|_{\Gamma_N}).$$

Next, in view of (25)-(26), we set

$$R_1 = \frac{1}{\nu_{\#}} (\mu R_2^2 + \|\mathbf{f}\| + G^{\#} K_4),$$

where R_2 is chosen such that

$$\left(1 - \frac{\mu\sigma^{\#}}{\nu_{\#}} (\mu R_2^2 + \|\mathbf{f}\| + G^{\#} K_4) \right) R_2 = \frac{\sigma^{\#}}{\sigma^{\#}} \|\mathbf{J}_0\|,$$

we have

$$\begin{aligned} \frac{\nu_{\#}}{2} \|D\bar{\mathbf{u}}\|^2 &\leq \frac{\bar{\nu}}{\nu_{\#}} \|\bar{T}\|_6^{6\delta/(2+\delta)} K_3^2 + C_2^2 \|D\bar{\mathbf{u}}\|^2 R_1 + \\ &+ \frac{C_1}{\nu_{\#}} \left(\mu R_2 (\|\nabla \times \bar{\mathbf{H}}\| + \|\bar{\mathbf{H}}\|_3) + G^{\#} \|\bar{T}\|_{6/5} + \bar{G} \|\bar{T}\|_6 K_4 \right)^2; \\ \frac{1}{4(\sigma_{\#})^2} \|\nabla \times \bar{\mathbf{H}}\|^2 &\leq \frac{\bar{\sigma}}{(\sigma_{\#})^2} \|\bar{T}\|_6^{6\epsilon/(2+\epsilon)} (K_1^2 + \|\mathbf{J}_0\|_{2+\epsilon}^2) + \\ &+ \mu (\|\bar{\mathbf{u}}\|_4^2 R_2^2 + R_1^2 \|\bar{\mathbf{H}}\|_4^2). \end{aligned}$$

Now, sum the above two inequalities with (32) rewritten as follows as

$$\frac{k_{\#}}{2} \|\nabla \bar{T}\|^2 \leq \frac{\bar{k}}{k_{\#}} \|\bar{T}\|_6^{6\epsilon/(2+\epsilon)} K_2^2 + \frac{C_1}{k_{\#}} \|\bar{\mathbf{u}}\|_6^2 K_4^2.$$

As a result,

$$\begin{aligned} &\left(\frac{\nu_{\#}}{2} - C^2 R_1 - C\mu R_2^2 - \frac{C}{k_{\#}} K_4^2 \right) \|D\bar{\mathbf{u}}\|^2 + \\ &+ \left(\frac{1}{4(\sigma_{\#})^2} - \frac{2C\mu^2}{\nu_{\#}} R_2^2 - C\mu R_1^2 \right) \|\nabla \bar{\mathbf{H}}\|^2 + \\ &+ \left(\frac{k_{\#}}{2} - \frac{C}{\nu_{\#}} (\bar{\nu} K_3^2 + (G^{\#} + \bar{G} K_4)^2) - \right. \\ &\left. - \frac{C\bar{\sigma}}{(\sigma_{\#})^2} (K_1^2 + \|\mathbf{J}_0\|_{2+\epsilon}^2) - \frac{C\bar{k}}{k_{\#}} K_2^2 \right) \|\nabla \bar{T}\|^2 \leq 0, \end{aligned}$$

with C standing for different Sobolev constants, and the uniqueness of solution holds under smallness assumption on the data.

5. Shape sensivity analysis

In this section we deal with the shape sensivity analysis to the model correspondent to Theorem 1, when the coefficients ν , k , σ and α are assumed constants. First, a family of mappings $\mathcal{T}_{\tau} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ associated with a given velocity field $V(\tau, x)$ is constructed. The evolution of geometrical domains, if the vector field V is chosen, is governed by the real parameter τ . So we denote by $\Omega_{\tau} = \mathcal{T}_{\tau}(\Omega)$ the variable domain depending on *two parameters*, a vector field V and the real variable τ . We call the family of perturbations Ω_{τ} of a given initial configuration Ω , and the variable τ has the meaning of time in our setting. Indeed, the variable domains Ω_{τ} are defined by the images of the mapping, which is given by the system of differential equations

$$\frac{d}{d\tau} x(\tau) = V(\tau, x(\tau)), \quad x(0) = X,$$

with the solution denoted by $x(\tau) = x(\tau, X), \tau \in (0, \tau_1), X \in \mathbb{R}^3$, for some positive constant τ_1 . Therefore, each specific family parametrized by τ is defined in the direction of a given vector field V , and it is denoted by $\Omega_\tau = \{x \in \mathbb{R}^3 \mid x = x(\tau, X), X \in \Omega\}$.

In our setting all equations defined in variable domain Ω_τ can be transported to the reference domain, which is also called the fixed domain Ω , using the inverse transformation $\mathcal{T}_\tau^{-1} : \Omega_\tau \rightarrow \Omega$.

Let us assume the following additional hypotheses:

(H5) The field V is compactly supported with respect to the spatial variable x , i.e.,

$$V \in C(0, \tau_1; \mathcal{D}^2(\Omega; \mathbb{R}^3)), \text{ supp}V \subset \Omega,$$

and it is divergence free.

(H6) In the variable domain setting, the elements

$$\mathbf{f}^\tau \in \mathbf{L}^2(\Omega_\tau), \quad \mathbf{J}_0^\tau \in \mathbf{L}^2(\Omega_\tau), \quad f^\tau \in L^2(\Omega_\tau) \quad \text{and} \quad h^\tau \in L^2(\Gamma_N^\tau), \quad (33)$$

stand for the data in boundary value problems in Ω_τ , are simply given by restrictions to Ω_τ of some functions

$$\mathbf{f} \in \mathbf{H}^1(\mathbb{R}^3), \quad \mathbf{J}_0 \in \mathbf{H}^1(\mathbb{R}^3), \quad f \in H^1(\mathbb{R}^3) \quad \text{and} \quad h \in H^1(\mathbb{R}^3) \quad (34)$$

defined in the whole space. In this way the shape derivatives of all the data vanish, except for h , and the material derivatives are just given by the scalar products of the gradients of the data with respect to spatial variables with the velocity vector field, e.g., $\dot{f} = \nabla f \cdot V$, provided that all data are given in the Sobolev spaces $H^1(\mathbb{R}^3)$.

Notice that (H5) implies the additional constraint, $|\Omega| = \text{constant}$, this means that for our shapes the admissible domains have the given volume.

5.1. Perturbed problem

We consider in (H5) that the velocity field $V(\tau, x)$ is divergence free, which implies that also our \mathbf{u} and \mathbf{H} conserve the divergencelessness. This simplifies the situation and we do not need to apply Bogovskii operator, since for pressure we use the standard Rham theorem.

DEFINITION 1 *We call a perturbed problem to the model (1)-(6) in a perturbed domain the following system of equations in Ω_τ*

$$-\nabla \cdot (\nu D\mathbf{u}^\tau) + (\mathbf{u}^\tau \cdot \nabla)\mathbf{u}^\tau + \nabla p^\tau = \mu \text{rot}\mathbf{H}^\tau \times \mathbf{H}^\tau + \mathbf{f}^\tau - \mathbf{G}(T^\tau)T^\tau; \quad (35)$$

$$\nabla \times (\nabla \times \mathbf{H}^\tau) = \nabla \times (\mathbf{J}_0^\tau + \sigma \mu \mathbf{u}^\tau \times \mathbf{H}^\tau); \quad (36)$$

$$\text{div}\mathbf{u}^\tau = \text{div}\mathbf{H}^\tau = 0; \quad (37)$$

$$-\nabla \cdot (k \nabla T^\tau) + \mathbf{u}^\tau \cdot \nabla T^\tau = f^\tau; \quad (38)$$

with the boundary conditions:

$$\mathbf{u}^\tau = \mathbf{0}, \quad \mathbf{H}^\tau \cdot \mathbf{n}^\tau = 0 \quad \text{on } \partial\Omega_\tau; \tag{39}$$

$$T^\tau = 0 \quad \text{on } \Gamma_D^\tau; \quad k \frac{\partial T^\tau}{\partial \mathbf{n}^\tau} + \alpha T^\tau = h^\tau \quad \text{on } \Gamma_N^\tau. \tag{40}$$

We introduce the Hilbert spaces

$$\begin{aligned} \mathbf{V}^\tau &= \{ \mathbf{v} \in \mathbf{H}_0^1(\Omega_\tau) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_\tau \} \\ \mathbf{V}^\tau(\operatorname{rot}) &= \{ \mathbf{v} \in \mathbf{L}^2(\Omega_\tau) : \operatorname{rot} \mathbf{v} \in \mathbf{L}^2(\Omega_\tau), \\ &\quad \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_\tau, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_\tau \} \\ Z^\tau &= \{ \xi \in H^1(\Omega_\tau) : \xi = 0 \text{ on } \Gamma_D^\tau \} \end{aligned}$$

equipped with their standard inner products.

THEOREM 4 *Assuming (H2), (H5), (33) and (10) with the constants a and b under the perturbed data, i.e.*

$$\begin{aligned} a &= \frac{\nu}{\mu\sigma} \|\mathbf{J}_0^\tau\| \\ b &= \frac{\nu}{\mu\sigma} - \left(\|\mathbf{f}^\tau\| + \frac{G^\#}{k} (\|f^\tau\| + \|h^\tau\|_{\Gamma_N}) \right), \end{aligned}$$

the problem (35)-(40) has a weak solution in the following sense:

The triple $(\mathbf{u}^\tau, \mathbf{H}^\tau, T^\tau) \in \mathbf{V}^\tau \times \mathbf{V}^\tau(\operatorname{rot}) \times Z^\tau$ and it satisfies

$$\begin{aligned} &\nu \int_{\Omega_\tau} D\mathbf{u}^\tau : D\mathbf{v}^\tau dx_\tau + \int_{\Omega_\tau} D\mathbf{u}^\tau : (\mathbf{u}^\tau \otimes \mathbf{v}^\tau) dx_\tau = \\ &= \int_{\Omega_\tau} \left(\mu(\nabla \times \mathbf{H}^\tau) \times \mathbf{H}^\tau + \mathbf{f}^\tau - \mathbf{G}(T^\tau)T^\tau \right) \cdot \mathbf{v}^\tau dx_\tau, \quad \forall \mathbf{v}^\tau \in \mathbf{V}^\tau; \\ &\int_{\Omega_\tau} (\nabla \times \mathbf{H}^\tau) \cdot (\nabla \times \mathbf{w}^\tau) dx_\tau = \sigma\mu \int_{\Omega_\tau} (\mathbf{u}^\tau \times \mathbf{H}^\tau) \cdot (\nabla \times \mathbf{w}^\tau) dx_\tau + \\ &+ \int_{\Omega_\tau} \mathbf{J}_0^\tau \cdot (\nabla \times \mathbf{w}^\tau) dx_\tau, \quad \forall \mathbf{w}^\tau \in \mathbf{V}^\tau(\operatorname{rot}); \\ &k \int_{\Omega_\tau} \nabla T^\tau \cdot \nabla \eta^\tau dx_\tau + \int_{\Omega_\tau} \mathbf{u}^\tau \cdot \nabla T^\tau \eta^\tau dx_\tau + \alpha \int_{\Gamma_N^\tau} T^\tau \eta^\tau ds_\tau = \\ &= \int_{\Omega_\tau} f^\tau \eta^\tau dx_\tau + \int_{\Gamma_N^\tau} h^\tau \eta^\tau ds_\tau, \quad \forall \eta^\tau \in Z^\tau. \end{aligned}$$

Proof. See the proof of Theorem 1. ■

THEOREM 5 *If the assumptions of Theorem 4 are fulfilled, the solution $(\mathbf{u}^\tau, \mathbf{H}^\tau, T^\tau)$ in accordance to Theorem 4 is such that $(\mathbf{H}^\tau, T^\tau)$ belongs to $\mathbf{W}^{1,2+\epsilon}(\Omega_\tau) \times W^{1,2+\epsilon}(\Omega_\tau)$ for some $\epsilon, \epsilon > 0$. Moreover, if we assume $\mathbf{f}^\tau \in \mathbf{L}^{2+\delta_1}(\Omega_\tau)$ for some $\delta_1 > 0$ and $\mathbf{J}_0^\tau \in \mathbf{L}^q(\Omega_\tau)$ with q given as in (14), then $\mathbf{u}^\tau \in \mathbf{W}^{1,2+\delta}(\Omega_\tau)$ for some $\delta > 0$, and $(\mathbf{u}^\tau, \mathbf{H}^\tau, T^\tau)$ is unique under small data.*

Proof. See the proof of Theorems 1 and 2. ■

5.2. Transported problem

The transported solution to the fixed domain is denoted by $\mathbf{u}_\tau = \mathbf{u}^\tau \circ \mathcal{T}_\tau$, $\mathbf{H}_\tau = \mathbf{H}^\tau \circ \mathcal{T}_\tau$, $T_\tau = T^\tau \circ \mathcal{T}_\tau$ with data $\mathbf{f}_\tau = \mathbf{f}^\tau \circ \mathcal{T}_\tau$, $\mathbf{G}_\tau = \mathbf{G}^\tau \circ \mathcal{T}_\tau$, $\mathbf{J}_{0\tau} = \mathbf{J}_0^\tau \circ \mathcal{T}_\tau$, $f_\tau = f^\tau \circ \mathcal{T}_\tau$ and $h_\tau = h^\tau \circ \mathcal{T}_\tau$.

We begin by recalling the following result (see Sokolowski and Zolésio, 1992).

PROPOSITION 8 *The unit normal vector field on Γ_τ is given by*

$$\mathbf{n}_\tau(\mathcal{T}_\tau(X)) = (\|\nabla \mathcal{T}_\tau^{-1} \cdot \mathbf{n}\|_{\mathbb{R}^3}^{-1} (\nabla \mathcal{T}_\tau^{-1} \cdot \mathbf{n})(X))$$

for $X \in \Gamma$. For any $f \in L^1(\Gamma_\tau)$,

$$\int_{\Gamma_\tau} f ds_\tau = \int_\Gamma f \circ \mathcal{T}_\tau \|M(\mathcal{T}_\tau) \cdot \mathbf{n}\|_{\mathbb{R}^3} ds,$$

where $M(\mathcal{T}_\tau) = \det(\nabla \mathcal{T}_\tau) \nabla \mathcal{T}_\tau^{-1}$ is the cofactor matrix of the Jacobian matrix $J\mathcal{T}_\tau$.

We recall the following important results, which give us the answer to the question, what happens with grad, div or curl after applying the transformation of domain.

PROPOSITION 9 *Denote by $J\mathcal{T}_\tau$ the Jacobian of \mathcal{T}_τ i.e. $J\mathcal{T}_\tau = \partial_{x_i}(\mathcal{T}_\tau)_j$, and for any matrix B the transposed matrix is denoted by B^\top . The following relations hold, with $J\mathcal{T}_\tau^\top = \nabla \mathcal{T}_\tau$,*

- (i) $(\text{grad } w) \circ \mathcal{T}_\tau = (J\mathcal{T}_\tau^{-\top} \nabla)(w \circ \mathcal{T}_\tau)$ for all $w \in H^1(\Omega)$;
- (ii) $(\text{div } \mathbf{w}) \circ \mathcal{T}_\tau = \zeta(\tau)^{-1} (\zeta(\tau) J\mathcal{T}_\tau^{-1} \nabla) \cdot (\mathbf{w} \circ \mathcal{T}_\tau)$ for all $\mathbf{w} \in \mathbf{H}^1(\Omega)$;
- (iii) $(\text{curl } \mathbf{w}) \circ \mathcal{T}_\tau = (J\mathcal{T}_\tau^{-\top} \nabla) \times (\mathbf{w} \circ \mathcal{T}_\tau)$ for all $\mathbf{w} \in \mathbf{H}^1(\Omega)$.

REMARK 3 *From Proposition 9, it follows that functions, which are divergence free on Ω_τ , generally lose this property when they are transported to the fixed domain. For more details see Remark 6.2 in Hömberg and Sokolowski (2003). This is the reason, why we assume, additionally, for simplicity, that $\text{div } V = 0$.*

We introduce the following notations

$$\begin{aligned} \zeta(\tau) &= \det(J\mathcal{T}_\tau), \\ \varrho(\tau) &= \nabla \mathcal{T}_\tau^{-1}, \\ A(\tau) &= \zeta(\tau) \varrho(\tau)^\top \varrho(\tau), \\ B(\tau) &= \zeta(\tau) \varrho(\tau), \\ \omega(\tau) &= \|M(\mathcal{T}_\tau) \cdot \mathbf{n}\|_{\mathbb{R}^3}. \end{aligned}$$

DEFINITION 2 We call the weak variational formulation of the transported problem to the model (1)-(6) in the fixed domain (relative to the perturbed problem in accordance with Definition 1) the following system of equations

$$\begin{aligned} & \nu \int_{\Omega} A(\tau) : (D\mathbf{u}_\tau D\mathbf{v}) dx + \int_{\Omega} B(\tau) \nabla \mathbf{u}_\tau : (\mathbf{v} \otimes \mathbf{u}_\tau) dx = \\ & = \int_{\Omega} \zeta(\tau) \left(\mu ((\varrho(\tau) \nabla) \times \mathbf{H}_\tau) \times \mathbf{H}_\tau + \mathbf{f}_\tau - \mathbf{G}(T_\tau) T_\tau \right) \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{V}; \\ & \int_{\Omega} ((\varrho(\tau) \nabla) \times \mathbf{H}_\tau) \cdot ((\varrho(\tau) \nabla) \times \mathbf{w}) dx = \\ & = \int_{\Omega} \zeta(\tau) (\sigma \mu \mathbf{u}_\tau \times \mathbf{H}_\tau + \mathbf{J}_{0\tau}) \cdot ((\varrho(\tau) \nabla) \times \mathbf{w}) dx, \quad \forall \mathbf{w} \in \mathbf{V}(\text{rot}); \\ & k \int_{\Omega} A(\tau) : (\nabla T_\tau \otimes \nabla \eta) dx + \int_{\Omega} B(\tau) : (\mathbf{u}_\tau \otimes \nabla T_\tau) \eta dx + \\ & + \alpha \int_{\Gamma_N} T_\tau \eta \omega(\tau) ds = \int_{\Omega} f_\tau \eta \zeta(\tau) dx + \int_{\Gamma_N} h_\tau \eta \omega(\tau) ds, \quad \forall \eta \in Z. \end{aligned}$$

In particular, with $\mathbf{v}_\tau = \mathbf{v}^\tau \circ \mathcal{T}_\tau$, $\mathbf{w}_\tau = \mathbf{w}^\tau \circ \mathcal{T}_\tau$ and $\eta_\tau = \eta^\tau \circ \mathcal{T}_\tau$, for all $\mathbf{v}^\tau \in \mathbf{V}^\tau$, $\mathbf{w}^\tau \in \mathbf{V}^\tau(\text{rot})$, and $\eta^\tau \in Z^\tau$.

THEOREM 6 Suppose that the assumptions (H2), (H5) and (34) are fulfilled and additionally assume that (10) holds for the constants a and b under the transported data:

$$\begin{aligned} a &= \frac{\nu}{\mu\sigma} \|\mathbf{J}_{0\tau}\| \\ b &= \frac{\nu}{\mu\sigma} - \left(\|\mathbf{f}_\tau\| + \frac{G^\#}{k} (\|f_\tau\| + \|h_\tau\|_{\Gamma_N}) \right), \end{aligned}$$

then the triple $(\mathbf{u}_\tau, \mathbf{H}_\tau, T_\tau) \in \mathbf{V} \times \mathbf{V}(\text{rot}) \times Z$ is a weak solution in the sense of Definition 2. Moreover, the solution $(\mathbf{u}_\tau, \mathbf{H}_\tau, T_\tau)$ is such that $(\mathbf{H}_\tau, T_\tau)$ belongs to $\mathbf{W}^{1,2+\epsilon}(\Omega) \times W^{1,2+\epsilon}(\Omega)$ for some $\epsilon, \epsilon > 0$, and if $\mathbf{f}_\tau \in \mathbf{L}^{2+\delta_1}(\Omega)$ for some $\delta_1 > 0$ and $\mathbf{J}_{0\tau} \in \mathbf{L}^q(\Omega)$ with q given as in (14) then $\mathbf{u}_\tau \in \mathbf{W}^{1,2+\delta}(\Omega)$ for some $\delta > 0$. Furthermore $(\mathbf{u}_\tau, \mathbf{H}_\tau, T_\tau)$ is unique under small data.

Proof. See the proof of Theorems 1 and 2. ■

As we introduce the forms

$$(F1) \quad \alpha_0(\tau, \mathbf{u}, \mathbf{v}) = \nu \int_{\Omega} \zeta(\tau) (\varrho(\tau) D\mathbf{u}) : (\varrho(\tau) D\mathbf{v}) dx = \nu \int_{\Omega} A(\tau) : (D\mathbf{u} D\mathbf{v}) dx$$

$$(F2) \quad \alpha_1(\tau, \mathbf{u}, \mathbf{v}) = \int_{\Omega} \zeta(\tau) (\varrho(\tau) \nabla \mathbf{u}) : (\mathbf{v} \otimes \mathbf{u}) dx = \int_{\Omega} B(\tau) \nabla \mathbf{u} : (\mathbf{v} \otimes \mathbf{u}) dx$$

$$(F3) \quad \alpha_2(\tau, \mathbf{H}, \mathbf{v}) = \mu \int_{\Omega} \zeta(\tau) \left(((\varrho(\tau) \nabla) \times \mathbf{H}) \times \mathbf{H} \right) \cdot \mathbf{v} dx$$

$$(F4) \quad \alpha_3(\tau, \mathbf{f}, T, \mathbf{v}) = \int_{\Omega} \zeta(\tau) (\mathbf{f} - \mathbf{G}(T)T) \cdot \mathbf{v} dx$$

$$(F5) \quad \beta_1(\tau, \mathbf{H}, \mathbf{w}) = \int_{\Omega} ((\varrho(\tau)\nabla) \times \mathbf{H}) \cdot ((\varrho(\tau)\nabla) \times \mathbf{w}) dx$$

$$(F6) \quad \beta_2(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w}) = \sigma\mu \int_{\Omega} \zeta(\tau) (\mathbf{u} \times \mathbf{H}) \cdot ((\varrho(\tau)\nabla) \times \mathbf{w}) dx$$

$$(F7) \quad \beta_3(\tau, \mathbf{J}_0, \mathbf{w}) = \int_{\Omega} \zeta(\tau) \mathbf{J}_0 \cdot ((\varrho(\tau)\nabla) \times \mathbf{w}) dx$$

$$(F8) \quad \gamma_1(\tau, T, \eta) = k \int_{\Omega} A(\tau) : (\nabla T \otimes \nabla \eta) dx$$

$$(F9) \quad \gamma_2(\tau, \mathbf{u}, T, \eta) = \int_{\Omega} \zeta(\tau) \mathbf{u} \cdot (\varrho(\tau)\nabla) T \eta dx = \int_{\Omega} B(\tau) : (\mathbf{u} \otimes \nabla T) \eta dx$$

$$(F10) \quad \gamma_3(\tau, T, \eta) = \alpha \int_{\Gamma_N} T \eta \omega(\tau) ds$$

$$(F11) \quad \gamma_4(\tau, f, \eta) = \int_{\Omega} f \eta \zeta(\tau) dx$$

$$(F12) \quad \gamma_5(\tau, h, \eta) = \int_{\Gamma_N} h \eta \omega(\tau) ds$$

we can state the following corollary:

COROLLARY 1 *Assume (H5). Let $|\tau| \leq \tau_1$ and τ_1 be small enough, then there exist real valued functions g_i satisfying $g_i(\tau) = o(\tau)$, $i = 0, \dots, 11$ and forms $\tilde{\alpha}_i(\tau, \dots)$, $i = 0, 1, 2, 3$, $\tilde{\beta}(\tau, \dots)$, $i = 1, 2, 3$, and $\tilde{\gamma}(\tau, \dots)$, $i = 1, \dots, 5$, such that the following statements are valid.*

(B1) For all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$

$$\alpha_0(\tau, \mathbf{u}, \mathbf{v}) = \alpha_0(0, \mathbf{u}, \mathbf{v}) + \tau \alpha_{0,\tau}(0, \mathbf{u}, \mathbf{v}) + \tilde{\alpha}_0(\tau, \mathbf{u}, \mathbf{v})$$

$$\alpha_{0,\tau}(0, \mathbf{u}, \mathbf{v}) = \nu \int_{\Omega} A'(0) : (D\mathbf{u}D\mathbf{v}) dx$$

$$\tilde{\alpha}_0(\tau, \mathbf{u}, \mathbf{v}) \leq g_0(\tau) \|\mathbf{u}\|_1 \|\mathbf{v}\|_1.$$

(B2) For all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$

$$\alpha_1(\tau, \mathbf{u}, \mathbf{v}) = \alpha_1(0, \mathbf{u}, \mathbf{v}) + \tau \alpha_{1,\tau}(0, \mathbf{u}, \mathbf{v}) + \tilde{\alpha}_1(\tau, \mathbf{u}, \mathbf{v})$$

$$\alpha_{1,\tau}(0, \mathbf{u}, \mathbf{v}) = \int_{\Omega} B'(0) \nabla \mathbf{u} : (\mathbf{v} \otimes \mathbf{u}) dx$$

$$\tilde{\alpha}_1(\tau, \mathbf{u}, \mathbf{v}) \leq g_1(\tau) \|\mathbf{u}\|_1^2 \|\mathbf{v}\|_1.$$

(B3) For all $\mathbf{H} \in \mathbf{H}^1(\Omega)$ and $\mathbf{v} \in \mathbf{V}$

$$\alpha_2(\tau, \mathbf{H}, \mathbf{v}) = \alpha_2(0, \mathbf{H}, \mathbf{v}) + \tau \alpha_{2,\tau}(0, \mathbf{H}, \mathbf{v}) + \tilde{\alpha}_2(\tau, \mathbf{H}, \mathbf{v})$$

$$\alpha_{2,\tau}(0, \mathbf{H}, \mathbf{v}) = \mu \int_{\Omega} ((\varrho'(0)\nabla) \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{v} dx$$

$$\tilde{\alpha}_2(\tau, \mathbf{H}, \mathbf{v}) \leq g_2(\tau) \|\nabla \times \mathbf{H}\| \|\mathbf{H}\|_1 \|\mathbf{v}\|_1.$$

(B4) For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $T \in Z$ and $\mathbf{v} \in \mathbf{V}$

$$\alpha_3(\tau, \mathbf{f}, T, \mathbf{v}) = \alpha_3(0, \mathbf{f}, T, \mathbf{v}) + \tilde{\alpha}_3(\tau, \mathbf{f}, T, \mathbf{v})$$

$$\tilde{\alpha}_3(\tau, \mathbf{f}, T, \mathbf{v}) \leq g_3(\tau) (\|\mathbf{f}\| + G^\# \|T\|) \|\mathbf{v}\|.$$

(B5) For all $\mathbf{H}, \mathbf{w} \in \mathbf{V}(\text{rot})$

$$\beta_1(\tau, \mathbf{H}, \mathbf{w}) = \beta_1(0, \mathbf{H}, \mathbf{w}) + \tau \beta_{1,\tau}(0, \mathbf{H}, \mathbf{w}) + \tilde{\beta}_1(\tau, \mathbf{H}, \mathbf{w})$$

$$\beta_{1,\tau}(0, \mathbf{H}, \mathbf{w}) = \int_{\Omega} A'(0) : (\nabla \times \mathbf{H}) \otimes (\nabla \times \mathbf{w}) dx$$

$$\tilde{\beta}_1(\tau, \mathbf{H}, \mathbf{w}) \leq g_4(\tau) \|\mathbf{H}\|_1 \|\mathbf{w}\|_1.$$

- (B6) For all $\mathbf{u} \in \mathbf{V}$, $\mathbf{H} \in \mathbf{V}(\text{rot})$, $\mathbf{w} \in \mathbf{V}(\text{rot})$
- $$\beta_2(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w}) = \beta_2(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) + \tau\beta_{2,\tau}(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) + \widetilde{\beta}_2(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w})$$
- $$\beta_{2,\tau}(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) = \sigma\mu \int_{\Omega} (\mathbf{u} \times \mathbf{H}) \cdot ((\varrho'(0)\nabla) \times \mathbf{w}) dx$$
- $$\widetilde{\beta}_2(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w}) \leq g_5(\tau) \|\mathbf{u} \times \mathbf{H}\|_2 \|\mathbf{w}\|_{\mathbf{V}(\text{rot})}.$$
- (B7) For all $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$, $\mathbf{w} \in \mathbf{V}(\text{rot})$
- $$\beta_3(\tau, \mathbf{J}_0, \mathbf{w}) = \beta_2(0, \mathbf{J}_0, \mathbf{w}) + \tau\beta_{3,\tau}(0, \mathbf{J}_0, \mathbf{w}) + \widetilde{\beta}_3(\tau, \mathbf{J}_0, \mathbf{w})$$
- $$\beta_{3,\tau}(0, \mathbf{J}_0, \mathbf{w}) = \int_{\Omega} \mathbf{J}_0 \cdot ((\varrho'(0)\nabla) \times \mathbf{w}) dx$$
- $$\widetilde{\beta}_3(\tau, \mathbf{J}_0, \mathbf{w}) \leq g_6(\tau) \|\mathbf{J}_0\| \|\nabla \times \mathbf{w}\|.$$
- (B8) For all $T, \eta \in Z$
- $$\gamma_1(\tau, T, \eta) = \gamma_1(0, T, \eta) + \tau\gamma_{1,\tau}(0, T, \eta) + \widetilde{\gamma}_1(\tau, T, \eta)$$
- $$\gamma_{1,\tau}(0, T, \eta) = k \int_{\Omega} A'(0) : (\nabla T \otimes \nabla \eta) dx$$
- $$\widetilde{\gamma}_1(\tau, T, \eta) \leq g_7(\tau) \|T\|_1 \|\eta\|_1.$$
- (B9) For all $\mathbf{u} \in \mathbf{V}$ and $T, \eta \in Z$
- $$\gamma_2(\tau, \mathbf{u}, T, \eta) = \gamma_2(0, \mathbf{u}, T, \eta) + \tau\gamma_{2,\tau}(0, \mathbf{u}, T, \eta) + \widetilde{\gamma}_2(\tau, \mathbf{u}, T, \eta)$$
- $$\gamma_{2,\tau}(0, \mathbf{u}, T, \eta) = \int_{\Omega} B'(0) : (\mathbf{u} \otimes \nabla T) \eta dx$$
- $$\widetilde{\gamma}_2(\tau, \mathbf{u}, T, \eta) \leq g_8(\tau) \|\mathbf{u}\|_1 \|T\|_1 \|\eta\|_1.$$
- (B10) For all $T \in Z, \eta \in Z$
- $$\gamma_3(\tau, T, \eta) = \gamma_3(0, T, \eta) + \tau\gamma_{3,\tau}(0, T, \eta) + \widetilde{\gamma}_3(\tau, T, \eta)$$
- $$\gamma_{3,\tau}(0, T, \eta) = \alpha \int_{\Gamma_N} T \eta \omega'(0) ds$$
- $$\widetilde{\gamma}_3(\tau, T, \eta) \leq g_9(\tau) \|T\|_1 \|\eta\|_1.$$
- (B11) For all $f \in L^2(\Omega)$ and $\eta \in Z$
- $$\gamma_4(\tau, f, \eta) = \gamma_4(0, f, \eta) + \widetilde{\gamma}_4(\tau, f, \eta)$$
- $$\widetilde{\gamma}_4(\tau, f, \eta) \leq g_{10}(\tau) \|f\| \|\eta\|.$$
- (B12) For all $h \in L^2(\Gamma_N)$ and $\eta \in Z$
- $$\gamma_5(\tau, h, \eta) = \gamma_5(0, h, \eta) + \tau\gamma_{5,\tau}(0, h, \eta) + \widetilde{\gamma}_5(\tau, h, \eta)$$
- $$\gamma_{5,\tau}(0, h, \eta) = \int_{\Gamma_N} h \eta \omega'(0) ds$$
- $$\widetilde{\gamma}_5(\tau, h, \eta) \leq g_{11}(\tau) \|h\|_{\Gamma_N} \|\eta\|_1.$$

Proof. The expressions (B1)-(B12) are consequence of the following derivatives with respect to τ at $\tau = 0$

$$\zeta'(0) = \text{div } V(0) = 0, \quad \varrho'(0) = B'(0) = -\nabla V(0), \quad A'(0) = -2D(V(0)),$$

observing that as in Section 1, $D(V(0))$ denotes the symmetrized part of $\nabla V(0)$, i.e. $D(V(0)) = \frac{1}{2}(\nabla V(0) + (\nabla V)^T(0))$. For the proof see Sokolowski and Zolésio (1992), Section 2.13. ■

Applying Taylor polynomials of degree one we can prove the stability results.

PROPOSITION 10 *Under the assumptions of Theorem 6, if $(\mathbf{u}_\tau, \mathbf{H}_\tau, T_\tau)$ is the transported solution corresponding to $(\mathbf{u}, \mathbf{H}, T)$ and the following assumptions are fulfilled*

- (M1) $\|\mathbf{f}_\tau - \mathbf{f}\| \leq C|\tau|$
 (M2) $\|\mathbf{G}(T_\tau)T_\tau - \mathbf{G}(T)T\|_{6/5} \leq C|\tau|$
 (M3) $\|\mathbf{J}_{0\tau} - \mathbf{J}_0\| \leq C|\tau|$
 (M4) $\|h_\tau - h\|_{\Gamma_N} \leq C|\tau|$
 (M5) $\|f_\tau - f\| \leq C|\tau|$

then we have

$$\|\mathbf{u}_\tau - \mathbf{u}\|_1 \leq C|\tau|; \quad (41)$$

$$\|\mathbf{H}_\tau - \mathbf{H}\|_1 \leq C|\tau|; \quad (42)$$

$$\|T_\tau - T\|_1 \leq C|\tau|, \quad (43)$$

with C denoting different constants.

Proof. For τ small enough and $\xi_i \in [0, \tau]$, $i = 0, \dots, 4$, we can write

$$\begin{aligned} \zeta(\tau) &= 1 + \tau\zeta'(\xi_0), \\ \varrho(\tau) &= I + \tau\varrho'(\xi_1), \\ A(\tau) &= I + \tau A'(\xi_2), \\ B(\tau) &= I + \tau B'(\xi_3), \\ \omega(\tau) &= 1 + \tau\omega'(\xi_4), \end{aligned}$$

where $\zeta(\tau) \geq c_{\tau_1} > 0$ for $|\tau| \leq \tau_1$ and A , B , ϱ are positive definite for $|\tau| \leq \tau_1$.

Observing that α_0 is linear with respect to the second argument, we write

$$\begin{aligned} \alpha_0(\tau, \mathbf{u}_\tau, \mathbf{v}) - \alpha_0(0, \mathbf{u}, \mathbf{v}) &= \alpha_0(0, \mathbf{u}_\tau - \mathbf{u}, \mathbf{v}) + \\ &+ \tau\nu \int_\Omega A'(\xi_2) : (D\mathbf{u}_\tau D\mathbf{v}) dx. \end{aligned} \quad (44)$$

Observe that α_i , ($i = 1, 2, 3$), is no more linear with respect to the secondary argument, thus we write

$$\begin{aligned} \alpha_1(\tau, \mathbf{u}_\tau, \mathbf{v}) - \alpha_1(0, \mathbf{u}, \mathbf{v}) &= \alpha_1(0, \mathbf{u}_\tau, \mathbf{v}) - \alpha_1(0, \mathbf{u}, \mathbf{v}) + \\ &+ \tau \int_\Omega B'(\xi_3) \nabla \mathbf{u}_\tau : (\mathbf{v} \otimes \mathbf{u}_\tau) dx; \end{aligned} \quad (45)$$

$$\begin{aligned} \alpha_2(\tau, \mathbf{H}_\tau, \mathbf{v}) - \alpha_2(0, \mathbf{H}, \mathbf{v}) &= \alpha_2(0, \mathbf{H}_\tau, \mathbf{v}) - \alpha_2(0, \mathbf{H}, \mathbf{v}) + \\ &+ \tau\mu \int_\Omega \left(\zeta'(\xi_0) (\nabla \times \mathbf{H}_\tau) \times \mathbf{H}_\tau + ((\varrho'(\xi_1) \nabla) \times \mathbf{H}_\tau) \times \mathbf{H}_\tau \right) \cdot \mathbf{v} dx; \end{aligned} \quad (46)$$

$$\begin{aligned} \alpha_3(\tau, \mathbf{f}_\tau, T_\tau, \mathbf{v}) - \alpha_3(0, \mathbf{f}, T, \mathbf{v}) &= \alpha_3(0, \mathbf{f}_\tau, T_\tau, \mathbf{v}) - \\ &- \alpha_3(0, \mathbf{f}, T, \mathbf{v}) + \tau \int_\Omega \zeta'(\xi_0) \left(\mathbf{f}_\tau - \mathbf{G}(T_\tau)T_\tau \right) \cdot \mathbf{v} dx. \end{aligned} \quad (47)$$

Considering that \mathbf{u} is the particular case ($\tau = 0$) to the perturbed \mathbf{u}_τ , it follows that

$$\text{RHS of (44)} + \text{RHS of (45)} = \text{RHS of (46)} + \text{RHS of (47)}.$$

If we set $\mathbf{v} = \mathbf{u}_\tau - \mathbf{u}$ and argue as in the proof of Theorem 2, we get

$$\begin{aligned} \left(\frac{\nu}{2} - C_2^2 \|\mathbf{u}\|_1\right) \|\mathbf{u}_\tau - \mathbf{u}\|_1^2 &\leq \frac{C}{\nu} \left(\mu \|(\nabla \times \mathbf{H}_\tau) \times \mathbf{H}_\tau - (\nabla \times \mathbf{H}) \times \mathbf{H}\|_{6/5} \right. \\ &\left. + \|\mathbf{f}_\tau - \mathbf{f}\| + \|\mathbf{G}(T_\tau)T_\tau - \mathbf{G}(T)T\|_{6/5} + C|\tau|\right)^2. \end{aligned} \tag{48}$$

Now, observing that β_1 and β_3 are linear with respect to the second argument we write

$$\begin{aligned} \beta_1(\tau, \mathbf{H}_\tau, \mathbf{w}) - \beta_1(0, \mathbf{H}, \mathbf{w}) &= \beta_1(0, \mathbf{H}_\tau - \mathbf{H}, \mathbf{w}) + \\ &+ \tau \int_\Omega A'(\xi_2) : (\nabla \times \mathbf{H}_\tau \otimes \nabla \times \mathbf{w}) dx; \\ \beta_3(\tau, \mathbf{J}_{0\tau}, \mathbf{w}) - \beta_3(0, \mathbf{J}_0, \mathbf{w}) &= \beta_3(0, \mathbf{J}_{0\tau} - \mathbf{J}_0, \mathbf{w}) + \\ &+ \tau \int_\Omega B'(\xi_3) : (\mathbf{J}_{0\tau} \otimes \nabla \times \mathbf{w}) dx, \end{aligned}$$

while the remaining term reads

$$\begin{aligned} \beta_2(\tau, \mathbf{u}_\tau, \mathbf{H}_\tau, \mathbf{w}) - \beta_2(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) &= \beta_2(0, \mathbf{u}_\tau, \mathbf{H}_\tau, \mathbf{w}) - \\ &- \beta_2(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) + \tau \sigma \mu \int_\Omega B'(\xi_3) : \left((\mathbf{u}_\tau \times \mathbf{H}_\tau) \otimes \nabla \times \mathbf{w}\right) dx. \end{aligned}$$

Considering that \mathbf{H} is the particular case ($\tau = 0$) to the perturbed \mathbf{H}_τ , we set $\mathbf{w} = \mathbf{H}_\tau - \mathbf{H}$ to get the following estimate

$$\|\mathbf{H}_\tau - \mathbf{H}\|_1 \leq \|\mathbf{J}_{0\tau} - \mathbf{J}_0\| + \sigma \mu \|\mathbf{u}_\tau \times \mathbf{H}_\tau - \mathbf{u} \times \mathbf{H}\| + C|\tau|. \tag{49}$$

Next

- (N1) $\gamma_1(\tau, T_\tau, \eta) - \gamma_1(0, T, \eta) = \gamma_1(0, T_\tau - T, \eta) + \tau k \int_\Omega A'(\xi_2) : (\nabla T_\tau \otimes \nabla \eta) dx;$
- (N2) $\gamma_2(\tau, \mathbf{u}_\tau, T_\tau, \eta) - \gamma_2(0, \mathbf{u}, T, \eta) = \gamma_2(0, \mathbf{u}_\tau, T_\tau, \eta) - \gamma_2(0, \mathbf{u}, T, \eta) + \tau \int_\Omega B'(\xi_3) : (\mathbf{u}_\tau \otimes \nabla T_\tau) \eta dx;$
- (N3) $\gamma_3(\tau, T_\tau, \eta) - \gamma_3(0, T, \eta) = \gamma_3(0, T_\tau - T, \eta) + \tau \alpha \int_{\Gamma_N} T_\tau \eta \omega'(\xi_4) ds;$
- (N4) $\gamma_4(\tau, f_\tau, \eta) - \gamma_4(0, f, \eta) = \gamma_4(0, f_\tau - f, \eta) + \tau \int_\Omega f_\tau \eta \zeta'(\xi_0) dx;$
- (N5) $\gamma_5(\tau, h_\tau, \eta) - \gamma_5(0, h, \eta) = \gamma_5(0, h_\tau - h, \eta) + \tau \int_{\Gamma_N} h_\tau \eta \omega'(\xi_4) ds.$

We set $\eta = T_\tau - T$ to get

$$\begin{aligned} \frac{k}{2} \|\nabla(T_\tau - T)\|^2 + \alpha \|T_\tau - T\|_{\Gamma_N}^2 &\leq \frac{C}{k} \left(\|(\mathbf{u}_\tau - \mathbf{u}) \cdot \nabla T\|_{6/5} + \right. \\ &\left. + \|f_\tau - f\| + \|h_\tau - h\|_{\Gamma_N} + C|\tau|\right)^2. \end{aligned} \tag{50}$$

Now we add the three inequalities (48),(49),(50) and from assumptions (M1)-(M5), we get (41)-(43). ■

Finally, we are in the position to formulate the existence theorem for the material derivative of our problem.

DEFINITION 3 *The following limit in the function space norm \mathcal{H}*

$$\dot{f} = \lim_{\tau \rightarrow 0} \frac{f(\tau) - f(0)}{\tau}$$

is called the strong material derivative \dot{f} of f in \mathcal{H} .

REMARK 4 *The shape derivative u' of $u(\tau)$ in the direction of the vector field V is defined by the formula $u' = \dot{u} - \nabla u \cdot V$ provided that there exists the material derivative \dot{u} .*

We recall that $A(0) = B(0) = \varrho(0) = I$, $\zeta(0) = \omega(0) = 1$, $\dot{\zeta} = \zeta'(0) = 0$, $\dot{\varrho} = \varrho'(0)$, $\dot{A} = A'(0)$, $\dot{B} = B'(0)$ and $\dot{\omega} = \omega'(0)$, and we state the following result on the existence of material derivatives.

THEOREM 7 *Assume (H2), (H5), $\dot{\mathbf{f}} \in \mathbf{L}^2(\Omega)$, $\dot{\mathbf{J}} \in \mathbf{L}^2(\Omega)$, $\dot{f} \in L^2(\Omega)$, $\dot{h} \in L^2(\Gamma_N)$ and moreover (10) holds with constants a and b given as*

$$a = \frac{\nu}{\mu\sigma} \|\dot{\mathbf{J}}_0\|$$

$$b = \frac{\nu}{\mu\sigma} - \left(\|\dot{\mathbf{f}}\| + \frac{\dot{G}^\#}{k} (\|\dot{f}\| + \|\dot{h}\|_{\Gamma_N}) \right),$$

then the triple $(\dot{\mathbf{u}}, \dot{\mathbf{H}}, \dot{T}) \in \mathbf{V} \times \mathbf{V}(\text{rot}) \times Z$ satisfies

$$\begin{aligned} & \nu \int_{\Omega} (\dot{A}D\mathbf{u} + D\dot{\mathbf{u}}) : D\mathbf{v} dx + \int_{\Omega} (\dot{B}\nabla\mathbf{u} + \nabla\dot{\mathbf{u}}) : (\mathbf{v} \otimes \mathbf{u}) dx + \int_{\Omega} \nabla\mathbf{u} : (\mathbf{v} \otimes \dot{\mathbf{u}}) dx \\ & = \mu \int_{\Omega} \left(((\dot{\varrho}\nabla) \times \mathbf{H}) \times \mathbf{H} + (\nabla \times \dot{\mathbf{H}}) \times \mathbf{H} + (\nabla \times \mathbf{H}) \times \dot{\mathbf{H}} \right) \cdot \mathbf{v} dx + \\ & + \int_{\Omega} \left(\dot{\mathbf{f}} - \dot{\mathbf{G}}(T)T - \mathbf{G}(T)\dot{T} \right) \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{V}; \\ & \int_{\Omega} ((\dot{\varrho}\nabla) \times \mathbf{H} + \nabla \times \dot{\mathbf{H}}) \cdot (\nabla \times \mathbf{w}) dx + \int_{\Omega} (\nabla \times \mathbf{H}) \cdot ((\dot{\varrho}\nabla) \times \mathbf{w}) dx = \\ & = \sigma\mu \int_{\Omega} \left(\dot{\mathbf{u}} \times \mathbf{H} + \mathbf{u} \times \dot{\mathbf{H}} \right) \cdot (\nabla \times \mathbf{w}) dx + \sigma\mu \int_{\Omega} (\mathbf{u} \times \mathbf{H}) \cdot ((\dot{\varrho}\nabla) \times \mathbf{w}) dx + \\ & + \int_{\Omega} \dot{\mathbf{J}}_0 \cdot (\nabla \times \mathbf{w}) dx + \int_{\Omega} \mathbf{J}_0 \cdot (\dot{\varrho}\nabla) \times \mathbf{w} dx, \quad \forall \mathbf{w} \in \mathbf{V}(\text{rot}); \\ & k \int_{\Omega} (\dot{A}\nabla T + \nabla\dot{T}) \cdot \nabla\eta dx + \int_{\Omega} (\dot{B} : \mathbf{u} \otimes \nabla T + \dot{\mathbf{u}} \cdot \nabla T + \mathbf{u} \cdot \nabla\dot{T}) \eta dx \\ & + \alpha \int_{\Gamma_N} (\dot{T} + T\dot{\omega}) \eta ds = \int_{\Omega} \dot{f} \eta dx + \int_{\Gamma_N} (\dot{h} + h\dot{\omega}) \eta ds, \quad \forall \eta \in Z; \end{aligned}$$

and the following estimates

$$\begin{aligned}\|\dot{T}\|_1 &\leq C\left((1 + \|\mathbf{u}\|_1 + \|\dot{\mathbf{u}}\|_1)\|T\|_1 + \|\dot{f}\| + \|\dot{h}\|_{\Gamma_N} + \|h\|_{\Gamma_N}\right); \\ \|\dot{\mathbf{u}}\|_1 &\leq C\left((\|\dot{\mathbf{H}}\|_1 + \|\mathbf{H}\|_1)\|\mathbf{H}\|_1 + \|\dot{\mathbf{f}}\| + \dot{G}^\# \|T\|_1 + G^\# \|\dot{T}\|_1 + \|\mathbf{u}\|_1\right); \\ \|\dot{\mathbf{H}}\|_1 &\leq C\left(\|\mathbf{H}\|_1 + \mu\sigma(\|\dot{\mathbf{u}} \times \mathbf{H}\| + \|\mathbf{u} \times \dot{\mathbf{H}}\| + \|\mathbf{u} \times \mathbf{H}\|) + \|\dot{\mathbf{J}}_0\| + \|\mathbf{J}_0\|\right).\end{aligned}$$

Proof. We subtract the perturbed solution and the transported solution and we pass to the limit with τ tending to 0 (for details, see Consiglieri, Nečasová and Sokolowski, 2008, for analogous proof). ■

6. Concluding remarks

In order to overcome the problem of losing divergence free behavior we can apply the Piola transform, which is given by the following mapping:

$$\begin{aligned}P_I : \mathbf{V} &\rightarrow \mathbf{V}^\tau; \\ \mathbf{v} &\mapsto (J\mathcal{T}_\tau \cdot \mathbf{v}) \circ \mathcal{T}_\tau^{-1}.\end{aligned}$$

Denoting

$$\hat{\mathbf{u}}_\tau := (J\mathcal{T}_\tau)^{-1} \cdot (\mathbf{u}_\tau \circ \mathcal{T}_\tau) \text{ defined on } \Omega$$

and

$$\mathbf{u}_\tau = P_I(\hat{\mathbf{u}}_\tau) \text{ is defined on } \Omega_\tau,$$

the mapping P_I can be applied on velocity field and also on magnetic field to conserve the divergencelessness and that $\mathbf{u} \cdot \mathbf{n} = 0$ and $\mathbf{H} \cdot \mathbf{n} = 0$. By the same method as in Section 5 we get the stability and material derivative for $\hat{\mathbf{u}}$ and then we just apply the inverse mapping to conclude the results in Consiglieri, Nečasová and Sokolowski (2008).

REMARK 5 In Consiglieri, Nečasová and Sokolowski (2008) we get the stability depending not only on the data but also on assumption of behavior of \mathbf{H} , but it is not the case in our present problem.

Acknowledgment

Luisa Consiglieri was partially supported by Fundação para a Ciência e a Tecnologia, Financiamento Base 2008 - ISFL/1/209. The work of Šárka Nečasová was partially supported by project n. 201050005 of GACR and also in the framework of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503. The final version was supported by the project n. 201/08/0012 of GACR. The works of Šárka Nečasová and Jan Sokolowski were partially supported by the project between Czech Academy and CNRS. We would also like to thank to the referees and the editor for their valuable comments and suggestions.

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