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Normality of the maximum principle for absolutely continuous solutions to Bolza problems under state constraints*

by

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Dedicated to Richard Vinter

Abstract: It is well known that for a Bolza optimal control problem under state constraints every local minimizer satisfies a constrained maximum principle which may be degenerate. In the recent years several researchers proposed sufficient conditions for its nondegeneracy, e.g. Arutyanov and Assev (1997), Rampazzo and Vinter (1999, 2000), Galbraith and Vinter (2003). In all these papers the most important assumption links dynamics of a control system with tangent cones to constraints. It is the so called inward pointing condition of control theory that is in the same spirit with the well known Slater and Managasarian-Fromowitz conditions of mathematical programming.

We propose here two sufficient conditions for normality when the boundary of constraints is C^1 and the end point is free. The first one applies to *every* nondegenerate maximum principle without any assumptions on the initial state. The second one applies to *every* maximum principle, but involves an additional assumption on the initial conditions.

Keywords: optimal control, maximum principle under state constraints, normal necessary conditions.

1. Introduction

Let $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\mathcal{Z}$ be a complete separable metric space, $f : [0,1] \times \mathbb{R}^n \times \mathcal{Z} \to \mathbb{R}^n$, $\ell : [0,1] \times \mathbb{R}^n \times \mathcal{Z} \to \mathbb{R}$, $C \subset \mathbb{R}^n$ be closed and $\Omega \subset \mathbb{R}^n$ be open with the C^1 boundary $\partial\Omega$. Below, we denote by $N_C(x)$ the Clarke normal cone to C at $x \in C$ and we set $K = \overline{\Omega}$.

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Consider a measurable set-valued map $U : [0,1] \rightsquigarrow \mathcal{Z}$ with nonempty closed values and the Bolza optimal control problem

$$\text{minimize } \varphi(x(0), x(1)) + \int_0^1 \ell(s, x(s), u(s))ds \tag{1.1}$$

over all admissible trajectory/control pairs (x, u) of the following control system under state constraints

$$\begin{cases} x'(s) = f(s, x(s), u(s)), \quad u(s) \in U(s) & \text{a.e. in } [0,1] \\ x(s) \in K \quad \text{for all } \ s \in [0,1] \\ x(0) \in C. \end{cases} \tag{1.2}$$

When we write *a.e.* without making precise with respect to what measure, we always mean the Lebesgue measure.

Recall that a trajectory/control pair (x, u) (with $x(\cdot)$ absolutely continuous and $u(\cdot)$ measurable) is called admissible if it satisfies system (1.2).

If $(z, \bar{u})$ is a strong local minimizer of the above Bolza problem, then under some regularity assumptions, there exist $\lambda \in \{0, 1\}$, a normalized function with bounded total variation $\psi : [0,1] \to \mathbb{R}^n$ and an absolutely continuous function $p(\cdot) : [0,1] \to \mathbb{R}^n$, not vanishing simultaneously, such that $p(\cdot)$ is a solution to the adjoint system

$$-p'(s) = \frac{\partial f}{\partial x}(s, z(s), \bar{u}(s))^*(p(s) + \psi(s)) - \lambda \frac{\partial \ell}{\partial x}(s, z(s), \bar{u}(s)),$$

satisfying the transversality condition

$$(p(0), -p(1) - \psi(1)) \in \lambda \nabla \varphi(z(0), z(1)) + N_C(z(0)) \times \{0\},$$

the maximum principle

$$\begin{aligned} &\langle p(s) + \psi(s), f(s, z(s), \bar{u}(s))\rangle - \lambda \ell(s, z(s), \bar{u}(s)) = \\ &\max\nolimits_{u \in U(s)}(\langle p(s) + \psi(s), f(s, z(s), u)\rangle - \lambda \ell(s, z(s), u)) \end{aligned} \tag{1.3}$$

a.e. in $[0,1]$ and

$$\psi(t) = \int_{[0,t]} \nu(s) d\mu(s) \ \ \forall\, t \in (0,1] \tag{1.4}$$

for a positive (scalar) Radon measure μ on $[0,1]$ and a Borel measurable $\nu(\cdot) : [0,1] \to \mathbb{R}^n$ satisfying $\nu(s) \in N_K(z(s)) \cap B \quad \mu$-almost everywhere.

When f and ℓ are merely locally Lipschitz with respect to x and φ is locally Lipschitz, then in the above the derivatives are replaced by some elements in the corresponding generalized gradients and generalized jacobians and the maximum principle can be written in a similar way.

More precisely, let $M(n \times n)$ denote the set of all $n \times n$ matrices. If $(z, \bar{u})$ is a strong local minimizer of the above Bolza problem, then under some regularity assumptions, there exist $\pi_0, \pi_1 \in \mathbb{R}^n$, $A(\cdot) \in L^1(0,1; M(n \times n)), \pi(\cdot) \in L^1(0,1;\mathbb{R}^n)$ and λ, p, ψ as above such that $(\lambda, p, \psi) \neq 0$ and $p(\cdot)$ is a solution to the linear system

$$-p'(s) = A(s)^*(p(s) + \psi(s)) - \lambda\pi(s),$$

satisfying the transversality condition

$$(p(0), -p(1) - \psi(1)) \in \lambda(\pi_0, \pi_1) + N_C(z(0)) \times \{0\},$$

the maximum principle (1.3) and (1.4) with μ and ν as above.

Elements π, π_0, π_1 are in the generalized gradients of the corresponding functions of cost and $A(s)$ are matrices in the generalized jacobian of $f(s, \cdot, \bar{u}(s))$.

We refer to the monograph of Vinter (2000) for various forms of the nonsmooth constrained maximum principle. In this paper we do not need to be precise about π, π_0, π_1 and $A(\cdot)$ since we do not derive any maximum principle. Our aim is to provide sufficient conditions under which a maximum is normal. The elements π, π_0, π_1 and $A(\cdot)$ do not play any role in these conditions. For instance, matrices $A(s)$ can be related to the Hamiltonian form of the maximum principle (see Remark 1 below) and π, π_0, π_1 may belong to some sets different from the Clarke generalized gradients.

Rampazzo and Vinter (1999) considered time independent U and assumed the existence of a continuous $v : [0,1] \times \mathbb{R}^n \to U$, such that for all $x \in \partial K$, $f(s, x, v(s,x))$ points strictly inside of the state constraints. Then they proved that every optimal trajectory-control pair satisfies a normal maximum principle, see also Galbraith and Vinter (2003). In Frankowska and Bettiol (2007) a more general situation was considered and it was shown that under some weaker assumptions every nondegenerate maximum principle is normal. However, a weak point of this last result is that it was proved only for Lipschitz optimal trajectories. For this reason, applications to the problem of Lipschitz regularity of optimal trajectories, as in Frankowska and Marchini (2006), are not possible.

We propose here a new inward pointing condition to handle the unbounded case when the boundary of constraints is C^1. Further developments for more general set of constraints is postponed to a future work. A discussion on this condition is provided at the end of the paper.

Under this inward pointing condition we get two results about normality of the maximum principle when sets of admissible velocities are unbounded. The first one (Theorem 1) deals with nondegenerate maximum principles for arbitrary closed set of initial states. The second one (Theorem 2) can be applied also to degenerate maximum principles, but requires an additional assumption on the initial conditions.

Normality is useful, in particular, for investigation of regularity of optimal trajectories and optimal controls, see Galbraith and Vinter (2003), Frankowska and Marchini (2006), Bettiol and Frankowska (2008), Cannarsa, Frankowska and Marchini (2009).

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2. Preliminaries

Let B denote the closed unit ball in $\mathbb{R}^n$. For a subset $Y \subset \mathbb{R}^n$, $\overline{co}\, Y$ denotes the closed convex hull of Y and ∂Y the boundary of Y.

Let $K \subset \mathbb{R}^n$. The distance function from a point $x \in \mathbb{R}^n$ to K is defined by $\mathrm{dist}(x;K) := \inf_{y\in K} |x-y|$, while the oriented distance $d(\cdot)$ is defined by

$$d(x) = \mathrm{dist}(x;K) - \mathrm{dist}(x;\mathbb{R}^n \backslash K)$$

for $x \in \mathbb{R}^n$.

In this paper we always assume that d is continuously differentiable on a neighborhood of the boundary of K. By $T_K(x)$ (respectively $N_K(x)$) we denote the tangent (respectively normal) cone to K at $x \in K$. In this case the normal cone coincides with Clarke's normal cone and so our notations are consistent. If $x \in \mathrm{Int}(K)$, then $T_K(x) = \mathbb{R}^n$ and for all $x \in \partial K$ we have $T_K(x) = \{v \mid \langle \nabla d(x), v\rangle \le 0\}$.

For any $A \in M(n \times n)$, A^* denotes its adjoint and $||A||$ its norm.

The space of absolutely continuous functions from $[0,1]$ to $\mathbb{R}^n$ is denoted by $W^{1,1} := W^{1,1}([0,1];\mathbb{R}^n)$, while $NBV := NBV([0,1];\mathbb{R}^n)$ denotes the space of normalized functions of bounded variation on $[0,1]$ with values in $\mathbb{R}^n$, i.e. the space of functions with bounded total variation, vanishing at zero and right-continuous on $(0,1)$. For any $\psi \in NBV$, the right (left) limit of ψ at $t \in [0,1)$ (respectively $t \in (0,1]$) is denoted by $\psi(t+)$ (respectively $\psi(t-)$). For properties of the space $NBV([0,1],\mathbb{R}^n)$ see, for instance, Luenberger (1969).

Define the Hamiltonian $H : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \times \{0,1\} \to \mathbb{R} \cup \{+\infty\}$ associated to the above Bolza problem as follows:

$$H(s,x,p,\lambda) = \sup_{u \in U(s)} (\langle p, f(s,x,u)\rangle - \lambda \ell(s,x,u)). \tag{2.1}$$

Definition 1 *An admissible trajectory/control pair $(z,\bar{u})$ of system (1.2) is called extremal for a triple (λ,p,ψ) if $\lambda \in \{0,1\}$, $\psi \in NBV$ and $p \in W^{1,1}$ are such that $(\lambda,p,\psi) \neq 0$ and for some integrable mappings $A : [0,1] \to M(n\times n)$, $\pi : [0,1] \to \mathbb{R}^n$, and some $\pi_0, \pi_1 \in \mathbb{R}^n$ the following relations hold true :*

$$-p'(s) = A^*(s)(p(s)+\psi(s)) - \lambda\pi(s) \ \text{ a.e. in } [0,1], \tag{2.2}$$

$$(p(0), -p(1)-\psi(1)) \in \lambda(\pi_0,\pi_1) + N_C(z(0)) \times \{0\}, \tag{2.3}$$

$$\langle p(s)+\psi(s), z'(s)\rangle - \lambda\ell(s,z(s),\bar{u}(s)) = H(s,z(s),p(s)+\psi(s),\lambda) \ \text{ a.e.}, \tag{2.4}$$

$$\psi(t) = \int_{[0,t]} \nu(s)d\mu(s) \ \ \forall\, t \in (0,1] \tag{2.5}$$

for a positive (scalar) Radon measure μ on $[0,1]$ and a Borel measurable $\nu(\cdot):[0,1]\to\mathbb{R}^n$ satisfying

$$\nu(s)\in N_K(z(s))\cap B \quad \mu-a.e. \tag{2.6}$$

A triple (λ,p,ψ) as above is called nondegenerate if $\lambda+\sup_{t\in(0,1)}|p(t)+\psi(t)|\neq 0$. If $\lambda=1$, then (λ,p,ψ) is called normal.

In this paper we shall always assume that $d\in C^1$ on a neighborhood of ∂K. Then it follows from Cernea and Frankowska (2006) that (2.6) implies the following jump conditions

$$\psi(0+)\in N_K(z(0)),\ \psi(t)-\psi(t-)\in N_K(z(t)) \quad \forall\, t\in(0,1]. \tag{2.7}$$

Remark 1 *Assume that for some $\varepsilon>0,\ k\in L^1(0,1;\mathbb{R}_+)$, for almost all $s\in[0,1]$ and all $u\in U(s)$, the mappings $f(s,\cdot,u)$ and $\ell(s,\cdot,u)$ are $k(s)-$Lipschitz on $B(z(s),\varepsilon)$ (closed ball of center $z(s)$ and radius ε).*

Then, under some additional technical assumptions, every strong local minimizer $(z,\bar u)$ of the above Bolza problem satisfies a constrained maximum principle in jacobian form, e.g. Vinter (2000). Relations (2.2), (2.3), (2.4), (2.5), (2.6) correspond to such maximum principle and conditions for nondegeneracy can be found in Vinter (2000).

However, some other maximum principles can be reduced to this form with the matrix $A(s)$ not necessarily related to the Jacobian of $f(s,\cdot,\bar u(s))$. This happens, for instance, if the adjoint variable $p(\cdot)$ satisfies the Hamiltonian inclusion

$$-p'(s)\in\partial_x H(s,z(s),p(s)+\psi(s),\lambda) \ \ a.e.\ in\ [0,1],$$

where $\partial_x H(s,z(s),p(s)+\psi(s),\lambda)$ is the Clarke's generalized gradient of

$$H(s,\cdot,p(s)+\psi(s),\lambda)$$

at $z(s)$. Then such matrix can be defined, if in addition, it is assumed that for all $s\in[0,1]$ and $x\in B(z(s),\varepsilon)$ there exists $u\in U(s)$ satisfying

$$H(s,x,p(s)+\psi(s),\lambda)=\langle p(s)+\psi(s),f(s,x,u)\rangle-\lambda L(s,x,u).$$

Indeed, then it is not difficult to check that for any $q\in\partial_x H(s,z(s),p(s)+\psi(s),\lambda)$

$$|q|\le k(s)|p(s)+\psi(s)|+\lambda k(s)\le k(s)n\max_{1\le j\le n}|p_j(s)+\psi_j(s)|+\lambda k(s).$$

Hence, for some measurable $\xi:[0,1]\to B$ we have

$$-p'(s)+\lambda k(s)\xi(s)\in k(s)n\max_{1\le j\le n}|p_j(s)+\psi_j(s)|B \ \ a.e.\ in\ [0,1].$$

Set $\pi(s):=k(s)\xi(s)$. For every $s\in[0,1]$ define the matrix $A(s)=(a_{j,i}(s))$ as follows:

If $p(s)+\psi(s)=0$, then $A(s)=0$. Otherwise consider the smallest j_0 with

$$|p_{j_0}(s)+\psi_{j_0}(s)| = \max_{1\le j\le n} |p_j(s)+\psi_j(s)|$$

and for all i set $a_{j,i}(s)=0$ for every $j\neq j_0$,

$$a_{j_0,i}(s) = \frac{-p_i'(s)+\lambda\pi_i(s)}{p_{j_0}(s)+\psi_{j_0}(s)}.$$

Notice that $A(\cdot)$ is integrable and (2.2) holds true.

More generally, if the adjoint variable $p(\cdot)$ is so that for some $k\in L^1(0,1;\mathbb{R}_+)$, $|p'(s)|\le k(s)|p(s)+\psi(s)|+\lambda k(s)$ a.e., then the corresponding integrable matrix valued mapping $A(\cdot)$ can be defined as above.

3. Normality of the Maximum Principle

From now on we always assume that $U:[0,1]\rightsquigarrow \mathcal{Z}$ is measurable with nonempty closed values, that $C\subset\mathbb{R}^n$ is closed, $f,\ \ell$ are measurable with respect to s and continuous with respect to (x,u). Denote by ϖ the Lebesgue measure.

We also assume that the oriented distance $d(\cdot)$ is C^1 on a neighborhood of ∂K. We impose next some qualification assumptions on the constrained control system:

$$\left\{\begin{array}{l} \forall\ R>0, \exists\ \eta_R>0,\ M_R>0, \rho_R>0,\ d(\cdot)\in C^1 \text{ on } \partial K\cap RB+\eta_R B,\\ \ \forall\ y\in K \text{ satisfying } y\in\partial K\cap RB+\eta_R B \quad\text{and}\quad \forall\ s\in[0,1],\\ \forall\ u\in U(s) \text{ with } \langle\nabla d(y), f(s,y,u)\rangle\ge 0, \exists\ v_y\in\overline{co}\ f(s,y,U(s))\\ \ \text{such that } |v_y-f(s,y,u)|\le M_R,\ \langle\nabla d(y), v_y-f(s,y,u)\rangle\le -\rho_R. \end{array}\right. \tag{3.1}$$

Remark 2 *Assume that f and U are time independent and for every $r>0$ there exist $T_r>0,\eta_r>0,k_r>0$ such that for all $x\in\partial K\cap rB$, $f(x,U)\subset T_rB$ and for all $u\in U$, $f(\cdot,u)$ is k_r-Lipschitz on $B(x,\eta_r)$. Then (3.1) is equivalent to the usual inward pointing condition: for every $x\in\partial K$ there exists $u_x\in U$ such that*

$$\langle\nabla d(x), f(x,u_x)\rangle<0. \tag{3.2}$$

The main interest in introducing (3.1) is to handle the case of unbounded sets $f(x,U)$ and also measurable with respect to time f and U.

Our main results concern arbitrary extremals. If one is interested by a particular extremal $(z,\bar u)$, then, by the proofs given below, (3.1) may be localized around $z([0,1])$.

Theorem 1 *Assume (3.1). If $(z,\bar u)$ is extremal for a nondegenerate triple (λ,p,ψ), then $\lambda=1$. In other words, every nondegenerate triple (λ,p,ψ) is normal.*

THEOREM 2 *Assume (3.1). If $(z, \bar{u})$ is extremal for a triple (λ, p, ψ) and*

$$C_C(z(0)) \cap \mathrm{Int}\,(T_K(z(0))) \neq \emptyset, \tag{3.3}$$

where $C_C(z(0))$ denotes Clarke's tangent cone to C at $z(0)$, then $\lambda = 1$. In other words, every extremal starting at $z(0)$ and satisfying (3.3) has only normal triples.

COROLLARY 1 *Consider a trajectory/control pair $(z, \bar{u})$ that is extremal for a triple (λ, p, ψ) and such that $z(0) \in \mathrm{Int}(K)$. Then $\lambda = 1$.*

LEMMA 1 *Assume (3.1). Let $(z, \bar{u})$ be extremal for a triple (λ, p, ψ).*

i) If $(z, \bar{u})$ is nondegenarate and there exists a solution $\bar{w} \in W^{1,1}$ to the viability problem

$$\begin{cases} w'(s) \in A(s)w(s) + T_{\overline{co}\, f(s,z(s),U(s))}(z'(s)) & \text{a.e. in } [0,1] \\ w(0) = 0 \\ w(s) \in \mathrm{Int}(T_K(z(s))) \text{ for all } s \in (0,1], \end{cases}$$

then $\lambda = 1$.

ii) If $\Gamma := C_C(z(0)) \cap \mathrm{Int}\,(T_K(z(0))) \neq \emptyset$ and there exists a solution $\bar{w} \in W^{1,1}$ to the viability problem

$$\begin{cases} w'(s) \in A(s)w(s) + T_{\overline{co}\, f(s,z(s),U(s))}(z'(s)) & \text{a.e. in } [0,1] \\ w(0) \in \Gamma \\ w(s) \in \mathrm{Int}(T_K(z(s))) \text{ for all } s \in [0,1], \end{cases}$$

then $\lambda = 1$.

Proof. i) Set $\mathcal{C} = \{w \in C([0,1]) \mid w(s) \in \mathrm{Int}(T_K(z(s)))\ \forall\, s \in (0,1],\ w(0) = 0\}$,

$$S = \{w \in W^{1,1} \mid w'(s) \in A(s)w(s) + T_{\overline{co}\, f(s,z(s),U(s))}(z'(s)) \text{ a.e. in } [0,1]\}.$$

Assume for a moment that $\lambda = 0$. Then, by (2.3), $p(1) + \psi(1) = 0$. By (2.5) for all $w \in C([0,1])$, $\int_0^1 w(s)d\psi(s) = \int_{[0,1]} w(s)\nu(s)d\mu(s)$. From (2.6) it follows that $\int_0^1 w(s)d\psi(s) \leq 0$ for every $w \in \mathcal{C}$. On the other hand, by (2.2), for every $w \in S$ and integrable selection $v(s) \in T_{\overline{co}\, f(s,z(s),U(s))}(z'(s))$, satisfying $w'(s) = A(s)w(s) + v(s)$ a.e. in $[0,1]$, we have

$$\begin{aligned} \int_0^1 (p'w + pw' + \psi w')(s)ds &= \int_0^1 (-A^*(p+\psi)w + p(Aw+v) + \psi(Aw+v))(s)ds \\ &= \int_0^1 \langle p(s) + \psi(s), v(s)\rangle ds. \end{aligned}$$

By (2.4), $\langle p(s) + \psi(s), v(s)\rangle \leq 0$ a.e. Thus $\int_0^1 (p'w + pw' + \psi w')(s)ds \leq 0$ and therefore

$$\langle p(1), w(1)\rangle - \langle p(0), w(0)\rangle + \int_0^1 \psi(s)w'(s)ds \leq 0. \tag{3.4}$$

Integrating by parts we get $\int_0^1 w(s)d\psi(s) \geq -\langle p(0), w(0)\rangle$. Consequently, for every $w \in S \cap \mathcal{C}$, $\int_0^1 w(s)d\psi(s) = 0$.

Let $\bar{w}$ be as in assumption i). Then, for all $s \in (0,1]$ such that $\nu(s) \neq 0$ we have $\langle \nu(s), \bar{w}(s)\rangle < 0$. Therefore, $\mu(\{s \in (0,1] \mid \nu(s) \neq 0\}) = 0$ and $\psi \equiv -p(1)$ on $(0,1]$. Thus, $\phi := p + \psi$ is absolutely continuous on $(0,1]$, vanishes for $t = 1$ and satisfies the adjoint system $-\phi'(s) = A(s)^*\phi$ a.e. in $(0,1]$. Hence $p + \psi = 0$ on $(0,1]$. This contradicts the nondegeneracy and ends the proof of i).

ii) Define $\mathcal{C} = \{w \in C([0,1]) \mid w(s) \in \mathrm{Int}(T_K(z(s)))\ \forall\, s \in [0,1]\}$,

$$S = \{w \in W^{1,1} \mid w'(s) \in A(s)w(s) + T_{\overline{co}\, f(s,z(s),U(s))}(z'(s)) \text{ a.e., } w(0) \in \Gamma\}.$$

Assume for a moment that $\lambda = 0$. Then, by (2.3), $p(1)+\psi(1) = 0$. Exactly as in the proof of i) we deduce that for every $w \in S$, (3.4) holds true. Integrating by parts we obtain $\langle p(0), w(0)\rangle + \int_0^1 w(s)d\psi(s) \geq 0$. By (2.3), $p(0) \in N_C(z(0))$ and therefore $\langle p(0), w(0)\rangle \leq 0$. Then $\int_0^1 w(s)d\psi(s) \geq 0$. On the other hand, for every $w \in \mathcal{C}$, $\int_0^1 w(s)d\psi(s) \leq 0$. Thus, for every $w \in S \cap \mathcal{C}$, $\int_0^1 w(s)d\psi(s) = 0$. In the same way as before, using the assumption ii), we deduce that $p+\psi = 0$ on $(0,1]$. So $p(0) + \psi(0+) = 0$. Let $\bar{w}$ be as in the assumption ii). Since $p(0) \in N_C(z(0)$ we obtain $\langle p(0), \bar{w}(0)\rangle \leq 0$. From (2.7) we know that $\psi(0+) \in N_K(z(0))$ and therefore $\langle \psi(0+), \bar{w}(0)\rangle < 0$. Then, $\langle p(0) + \psi(0+), \bar{w}(0)\rangle < 0$, contradicting $p(0) + \psi(0+) = 0$. Therefore $\lambda = 1$. ∎

Hence, to prove our main theorems we have to find solutions to the two systems appearing in Lemma 1. Set $R = \|z\|_\infty$ and let $\eta = \eta_R,\ M = M_R,\ \rho = \rho_R$ be as in (3.1). For any $t \in [0,1]$, we denote by $Y(\cdot;t)$ the matrix solution to

$$\left\{\begin{array}{l} X'(s) = A(s)X(s) \\ X(t) = \mathbb{I}. \end{array}\right. \tag{3.5}$$

Since $A(\cdot)$ is integrable and $d \in C^1$ on $\partial K \cap RB + \eta B$, there exists $\varepsilon_1 > 0$ such that whenever $0 \leq s < t \leq 1$ and $t - s \leq \varepsilon_1$, $||Y(t;t_0)Y^{-1}(s;t_0) - \mathbb{I}|| \leq \frac{\rho}{4M}$ and

$$(z(t),\ z(s) \in \partial K \cap RB + \eta B) \implies |\nabla d(z(t)) - \nabla d(z(s))| \leq \frac{\rho}{4M}.$$

Define $\mathcal{D} := \{s \in [0,1] \mid z(s) \in \partial K + \eta B,\ \langle \nabla d(z(s)), z'(s)\rangle \geq 0\}$. By assumption (3.1) and the measurable selection theorem there exists a measurable selection $v(s) \in \overline{co}\, f(s, z(s), U(s))$ on $\mathcal{D}$ such that for a.e. $s \in \mathcal{D}$

$$\langle \nabla d(z(s)), v(s) - z'(s)\rangle \leq -\rho \ \ \&\ \ |v(s) - z'(s)| \leq M. \tag{3.6}$$

Set $v(s) := z'(s)$ for all $s \in [0,1] \setminus \mathcal{D}$ and $L := \max_{0 \leq s \leq t \leq 1} ||Y(t;s)||$.

Lemma 2 *If $z((0,1]) \cap \partial K \neq \emptyset$, then there exists $\theta > 0$ with $z(\theta) \in \partial K$ and a solution to the system*

$$\left\{\begin{array}{ll} w'(s) = A(s)w(s) + v(s) - z'(s) & \text{for a.e.} \quad s \in [0,\theta], \\ w(0) = 0, & \\ w(s) \in \mathrm{Int}(T_K(z(s)))\ \ \forall\, s \in (0,\theta]. & \end{array}\right. \tag{3.7}$$

Proof. Define $t_1 := \inf\{s \in [0,1] \mid z(s) \in \partial K)\}$.

If $t_1 > 0$, then $z([0,t_1)) \subset \mathrm{Int}(K)$. Let $0 < t_0 < t_1$ be such that $t_1 - t_0 \leq \varepsilon_1$, $z([t_0,t_1]) \subset \partial K + \eta B$. Since $d(z(t_1)) = d(z(t_0)) + \int_{t_0}^{t_1} \langle \nabla d(z(s)), z'(s)\rangle ds$ and $d(z(t_1)) = 0$, $d(z(t_0)) < 0$, the set $\mathcal{A} := [t_0,t_1] \cap \mathcal{D}$ has a positive Lebesgue measure. Consider the system

$$\begin{cases} w'(s) = A(s)w(s) + v(s) - z'(s) \\ w(t_0) = 0. \end{cases} \tag{3.8}$$

Then $w \equiv 0$ solves $w' = A(s)w$ and $w(t) \in \mathrm{Int}\,(T_K(z(t))) = \mathbb{R}^n$ on $(0,t_0]$. Furthermore, for $Z(\cdot) = Y(\cdot;t_0)$ the solution to (3.8) on $[t_0,t_1]$ is given by

$$w(t) = \int_{t_0}^{t} Z(t)Z^{-1}(s)(v(s) - z'(s))\, ds.$$

Then

$$\begin{aligned} &\langle \nabla d(z(t_1)), w(t_1)\rangle = \textstyle\int_{t_0}^{t_1} \left\langle \nabla d(z(t_1)), Z(t_1)Z^{-1}(s)(v(s) - z'(s))\right\rangle ds \\ &\leq \textstyle\int_{t_0}^{t_1} \langle \nabla d(z(t_1)), v(s) - z'(s)\rangle\, ds + \\ &+ \textstyle\int_{t_0}^{t_1} ||Z(t_1)Z^{-1}(s) - \mathbb{I}|||v(s) - z'(s)|ds \\ &\leq \textstyle\int_{\mathcal{A}} \langle \nabla d(z(s)), v(s) - z'(s)\rangle\, ds + M\int_{\mathcal{A}} |\nabla d(z(t_1)) - \nabla d(z(s))|ds + \\ &+M\textstyle\int_{\mathcal{A}} ||Z(t_1)Z^{-1}(s) - \mathbb{I}||ds \leq -\rho\,\varpi(\mathcal{A}) + \frac{\rho}{2}\,\varpi(\mathcal{A}) < 0. \end{aligned} \tag{3.9}$$

Setting $\theta = t_1$ we end the proof for $t_1 > 0$.

If $t_1 = 0$ and for some $\varepsilon > 0$, $z((0,\varepsilon]) \subset \mathrm{Int}(K)$, then it is enough to apply the same arguments as before with t_1 replaced by $t_2 := \inf\{s \in [\varepsilon,1] \mid z(s) \in \partial K)\}$ and put $\theta = t_2$.

It remains to consider the case $z(0) \in \partial K$ and for every $\varepsilon > 0$, $z((0,\varepsilon]) \cap \partial K \neq \emptyset$. Let $0 < \theta < \varepsilon_1$ be such that $z(\theta) \in \partial K$. Fix any $t \in [0,\theta]$ with $z(t) \in \partial K$ and notice that the set $\mathcal{A} := [0,t] \cap \mathcal{D}$ has a positive Lebesgue measure. Consider again the system (3.8) with $t_0 = 0$. Similarly as for estimate (3.9) we get $\langle \nabla d(z(t)), w(t)\rangle < 0$. ∎

Lemma 3 *Let $t_0 \in [0,1)$ be such that $z(t_0) \in \partial K$ and $w_0 \in \mathrm{Int}(T_K(z(t_0)))$. Define $t_1 = \max\{s \in [t_0, t_0 + \varepsilon_1] \mid z(s) \in \partial K + \eta B\}$. Then, there exists a solution to the differential inclusion*

$$\begin{cases} w'(s) \in A(s)w(s) + \mathbb{R}_+(v(s) - z'(s)) \;\text{ a.e. in }\; [t_0,t_1] \\ w(t_0) = w_0 \\ w(s) \in \mathrm{Int}(T_K(z(s))) \;\text{ for all }\; s \in [t_0,t_1]. \end{cases} \tag{3.10}$$

Proof. Define $Z(\cdot) := Y(\cdot;t_0)$. Let $t_0 < \bar{t} \leq t_1$ be such that $\bar{t} - t_0 < \varepsilon_1$ and $\langle \nabla d(z(s)), Z(s)w_0\rangle \leq 0$ for all $t_0 \leq s \leq \bar{t}$.

Case 1. If $z(\bar{t}) \in \partial K$, then the set $\mathcal{A}_1 := [t_0, \bar{t}] \cap \mathcal{D}$ has a positive Lebesgue measure. Consider the solution w to

$$\begin{cases} w'(s) = A(s)w(s) + \frac{4L|w_0|}{\rho\varpi(\mathcal{A}_1)}(v(s) - z'(s)) \text{ a.e. in } [t_0, t_1] \\ w(t_0) = w_0. \end{cases} \tag{3.11}$$

We claim that $w(s) \in \mathrm{Int}(T_K(z(s)))$ for all $s \in [t_0, t_1]$. Indeed, for any $t \in [t_0, \bar{t}]$ such that $z(t) \in \partial K$,

$$\begin{aligned} &\langle \nabla d(z(t)), w(t) \rangle = \\ &\langle \nabla d(z(t)), Z(t)w_0 \rangle + \tfrac{4L|w_0|}{\rho\varpi(\mathcal{A}_1)} \textstyle\int_{t_0}^t \left\langle \nabla d(z(t)), Z(t)Z^{-1}(s)(v(s) - z'(s)) \right\rangle ds \\ &\leq \tfrac{4L|w_0|}{\rho\varpi(\mathcal{A}_1)} \textstyle\int_{t_0}^t \left\langle \nabla d(z(t)), Z(t)Z^{-1}(s)(v(s) - z'(s)) \right\rangle \, ds. \end{aligned}$$

As in the proof of Lemma 2 we show that

$$\int_{t_0}^t \left\langle \nabla d(z(t)), Z(t)Z^{-1}(s)(v(s) - z'(s)) \right\rangle \, ds < 0.$$

Thus, $w(t) \in \mathrm{Int}(T_K(z(t)))$.

Consider next $t \in (\bar{t}, t_1]$ such that $z(t) \in \partial K$. Define $\mathcal{A} := [t_0, t] \cap \mathcal{D}$. Then $\varpi(\mathcal{A}) \geq \varpi(\mathcal{A}_1) > 0$. Similarly to the proof of Lemma 2 we get

$$\begin{aligned} &\langle \nabla d(z(t)), w(t) \rangle = \\ &\langle \nabla d(z(t)), Z(t)w_0 \rangle + \tfrac{4L|w_0|}{\rho\varpi(\mathcal{A}_1)} \textstyle\int_{t_0}^t \left\langle \nabla d(z(t)), Z(t)Z^{-1}(s)(v(s) - z'(s)) \right\rangle ds \\ &\leq L|w_0| + \tfrac{4L|w_0|}{\rho\varpi(\mathcal{A}_1)} \textstyle\int_{t_0}^t \langle \nabla d(z(s)), v(s) - z'(s) \rangle \, ds + \tfrac{4L|w_0|}{\rho\varpi(\mathcal{A}_1)} \tfrac{\rho}{2} \varpi(\mathcal{A}) \\ &\leq L|w_0| - \tfrac{4L|w_0|}{\varpi(\mathcal{A}_1)} \varpi(\mathcal{A}) + \tfrac{2L|w_0|}{\varpi(\mathcal{A}_1)} \varpi(\mathcal{A}) = L|w_0| - \tfrac{2L|w_0|\varpi(\mathcal{A})}{\varpi(\mathcal{A}_1)} \leq -L|w_0|. \end{aligned}$$

Thus, $w(t) \in \mathrm{Int}(T_K(z(t)))$.

Case 2. Assume next that $z(\bar{t}) \in \mathrm{Int}(K)$. If $z([\bar{t}, t_1]) \subset \mathrm{Int(K)}$, then the proof ends. Otherwise define $t_2 := \min\{s \in [\bar{t}, t_1] \mid z(s) \in \partial K\}$. Then the set $\mathcal{A}_1 := [\bar{t}, t_2] \cap \mathcal{D}$ has a positive Lebesgue measure. Consider the solution w to (3.11). In the same way as in Case 1 it follows that for all $t \in [t_0, t_1]$ with $z(t) \in \partial K$, we have $w(t) \in \mathrm{Int}(T_K(z(t)))$. ∎

Lemma 4 *Let $t_0 \in [0, 1)$ be such that $z(t_0) \in \mathrm{Int}(K)$, $z([t_0, 1]) \cap \partial K \neq \emptyset$ and $w_0 \in \mathbb{R}^n$. Define $t_1 = \min\{s \in [t_0, 1] \mid z(s) \in \partial K\}$. Then, there exists a solution to the differential inclusion (3.10).*

Proof. Let $t_0 < \bar{t} < t_1$ be such that $z([\bar{t}, t_1]) \in \partial K + \eta B$. It is enough to apply the same arguments as in the proof of Lemma 3, Case 2. ∎

Proof of Theorem 1. By Lemma 1 we have to show that there exists a solution $\bar{w} : [0, 1] \to \mathbb{R}^n$ to

$$\begin{cases} w'(s) \in A(s)w(s) + \mathbb{R}_+(v(s) - z'(s)) \text{ a.e. in } [0,1] \\ w(0) = 0 \\ w(s) \in \mathrm{Int}\,(T_K(z(s))) \text{ for all } s \in (0,1]. \end{cases} \tag{3.12}$$

If $z((0,1]) \subset \mathrm{Int}(K)$, then it is enough to consider the solution w to (3.8) with $t_0 = 0$. Assume next that $z((0,1]) \cap K \neq \emptyset$. Since z is absolutely continuous, there exists $0 < \varepsilon < \varepsilon_1$ such that for any $t \in [0,1)$ with $z(t) \in \partial K$, $z([t, (t+\varepsilon) \wedge 1]) \subset \partial K + \eta B$, where $(t + \varepsilon) \wedge 1 := \max\{t + \varepsilon, 1\}$.

Claim. For all $t_0 \in [0, 1)$ and $w_0 \in \mathrm{Int}\,(T_K(z(t_0)))$ there exists $t_0 < \delta(t_0) \leq 1$ such that either $\delta(t_0) = 1$ or $\delta(t_0) - t_0 \geq \varepsilon$ and differential inclusion (3.10) has a solution for $t_1 = \delta(t_0)$.

Indeed, fix $t_0 \in [0, 1)$ and $w_0 \in \mathrm{Int}\,(T_K(z(t_0)))$. If $z((t_0, 1]) \subset \mathrm{Int}(K)$, then it is enough to consider the solution w to

$$\begin{cases} w'(s) = A(s)w(s) + v(s) - z'(s) \text{ a.e. in } [t_0, 1] \\ w(t_0) = w_0. \end{cases}$$

If $z(t_0) \in \partial K$, then, by Lemma 3 and the absolute continuity of z, such $\delta(t_0)$ does exist. If $z(t_0) \in \mathrm{Int}(K)$, then consider t_1 and $w(\cdot)$ as in Lemma 4 and set $w_1 = w(t_1)$. Applying Lemma 3 with t_0 replaced by t_1 and w_0 by w_1, we prove that there exists a solution w to (3.10) on some time interval $[t_0, \delta(t_0)]$ with $\delta(t_0)$ either equal to 1 or satisfying $\delta(t_0) - t_0 \geq \varepsilon$. This proves our claim.

From Lemma 2 we deduce that there exist $\theta > 0$ and a solution $\bar{w}$ to system (3.12) on $[0, \theta]$ such that $z(\theta) \in \partial K$. Consider any finite sequence $\tau_0 = \theta < \tau_1 < \ldots < \tau_k = 1$ such that $\tau_{i+1} - \tau_i \leq \varepsilon$. We define $\bar{w}$ using an induction argument. Set $w_0 := \bar{w}(\theta)$, $t_0 = \theta$. By our claim there exists a solution to (3.12) on $[\tau_0, \tau_1]$.

Assume that we already constructed $\bar{w}$ on $[0, \tau_j]$ and $j < k$. Setting $t_0 = \tau_j$ and $w_0 = \bar{w}(\tau_j)$ and using again the above claim we extend $\bar{w}$ on $[\tau_j, \tau_{j+1}]$. ∎

To prove Theorem 2 it is enough to apply the same reasoning as for the proof of Theorem 1 with $\theta = 0$ and $\bar{w}(\theta)$ replaced by w_0.

Remark 3 *We discuss next the role of the convex hull in condition (3.1). In general (3.1) is not equivalent to the same condition without the convex hull:*

$$\begin{cases} \forall\, R > 0, \exists\, \eta_R > 0, M_R > 0, \rho_R > 0, d(\cdot) \in C^1 \text{ on } \partial K \cap RB + \eta_R B, \\ \forall\, y \in K \text{ satisfying } y \in \partial K \cap RB + \eta_R B \text{ and } \forall\, s \in [0,1], \\ \forall\, u \in U(s) \text{ with } \langle \nabla d(y), f(s,y,u) \rangle \geq 0, \exists\, u_{s,y} \in U(s) \text{ such that} \\ |f(s,y,u_{s,y}) - f(s,y,u)| \leq M_R \text{ and} \\ \langle \nabla d(y), f(s,y,u_{s,y}) - f(s,y,u) \rangle \leq -\rho_R. \end{cases} \tag{3.13}$$

Clearly, (3.13) yields (3.1), but the converse implication is in general false.

Consider the following simple example: $K = \mathbb{R}\times\mathbb{R}_-$ *and* $f : [0,1]\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $f(s,y,u) = u$ *and for all* $s \in [0,1]$,

$$U(s) = U := \{(\alpha, 1/\alpha) \mid \alpha > 0\} \cup \{0\} \times [0, -\infty).$$

Let $\varepsilon > 0$. *Fix any* $M > 0$. *Then there exists* $\alpha_0 > M + 1/\varepsilon$ *such that for all* $\alpha \geq \alpha_0$ *and every* $(u_1, u_2) \in U$ *the inequality* $|(\alpha, 1/\alpha) - (u_1, u_2)| \leq M$ *yields* $u_1 \geq \alpha_0 - M \geq 1/\varepsilon$. *Consequently* $\langle (0,1), (u_1,u_2) - (\alpha, 1/\alpha)\rangle \geq -\varepsilon$. *This implies that condition (3.13) does not hold. On the other hand, since* $\overline{co}\, U = \mathbb{R}_+ \times \mathbb{R}$, *(3.1) is satisfied.*

REMARK 4 *In the present paper we have considered only the case when the boundary of* K *is smooth. However, the construction made in Theorem 1 may be adapted to state constraints* K *that are intersection of a finite number of sets with smooth boundaries, see Frankowska (2006), where normality of extremals was proved under some stronger assumptions on the initial states. Then* K *has a nonsmooth boundary and* $\nabla d(y)$ *have to be replaced by unit vectors from Clarke's normal cone. For instance, for time independent* f, U, *(3.2) becomes: for every* $x \in \partial K$, *there exists* $u_x \in U$ *such that*

$$\sup_{n\in N_K(x),\ |n|=1} \langle n, f(x, u_x)\rangle < 0. \tag{3.14}$$

It may happen then that (3.14) does not hold and still a nonsmooth analogue of (3.1) is satisfied. Indeed, consider f, U *as in Remark 3 and* $K = \mathbb{R}_+ \times \mathbb{R}_-$. *Then, (3.14) is not true at* 0 *and at the same time*

$$\begin{cases} \forall\, x \in \partial K,\ \forall\, u \in U,\ u_x := u + (1,-1) \in \overline{co}\, U \text{ satisfies} \\ |u_x - u| \leq 2 \text{ and } \sup_{n\in N_K(x),\ |n|=1} \langle n, u_x - u\rangle \leq -1. \end{cases} \tag{3.15}$$

Notice that for this example construction of $\bar{w}$ *as in Lemma 1 is elementary, similarly as verification of normality. Investigation of nonsmooth constraints in the general nonlinear case is postponed to a future work.*

REMARK 5 *It is not difficult to realize that for smooth* ∂K, *(3.1) implies that for every* $x \in \partial K$ *and every* $s \in [0,1]$, *there exists* $u_{s,x} \in U(s)$ *such that*

$$\langle \nabla d(x), f(s, x, u_{s,x})\rangle < 0. \tag{3.16}$$

When the boundary of K *is nonsmooth, this is not always the case (see Remark 4 above).*

Notice that if K *and* f *are as in Remark 3 and* $U(\cdot)$ *is given by: for all* $s \in [0,1)$, $U(s) = [0,1] \times \{1-s, s-1\}$ *and* $U(1) = \{(0,-1)\}$, *then (3.16) holds true,* $\sup_{s\in[0,1]} \inf_{u\in U(s)} \langle (0,1), u\rangle = 0$ *and (3.1) is not satisfied. For this reason, in the measurable case it is usual to replace (3.16) by a stronger assumption* $\langle \nabla d(x), f(s, x, u_{s,x})\rangle < -\rho$ *for some* $\rho > 0$ *independent of* s, x.

Sufficiency of (3.16) for normality when z is Lipschitz, f, U are regular enough and U is bounded is known even for nonsmooth state constraints, see for instance Frankowska (2006).

The proofs of this paper do not apply when only (3.16) is imposed and z' is unbounded, for the following reason. Even though for all $u \in U(s)$ such that $\langle \nabla d(x), f(s,x,u)\rangle \geq 0$ we have $\langle \nabla d(x), f(s,x,u_{s,x}) - f(s,x,u)\rangle < 0$, the existence of $M_R > 0$ such that in addition $|f(s,x,u_{s,x}) - f(s,x,u)| \leq M_R$ for all $s \in [0,1]$ and $x \in \partial K \cap RB$ in the unbounded case is in general false. This last inequality is important in our proofs.

To get such bounds it is more appropriate to allow the choice of $u_{s,y}$ to be dependent on $u \in U(s)$ and even more, to look for $v(s,y,u) \in \overline{co}\, f(s,y,U(s))$ such that $\langle \nabla d(y), v(s,y,u) - f(s,y,u)\rangle \leq -\rho_R$ and $|v(s,y,u) - f(s,y,u)| \leq M_R$. By Remark 3, invoking the convex hull leads to a more general result.

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