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High-order variations and small-time local attainability*†

by

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Abstract: We study the problem of small-time local attainability (STLA) of a closed set. For doing this, we introduce a new concept of variations of the reachable set, well adapted to a given closed set and prove a new attainability result.

Keywords: small-time local attainability of a set, high-order control variations.

1. Introduction and preliminaries

Small time local controllability is a central property for obtaining optimality conditions, for stabilizing control systems, for studying the regularity of time minimal problems for control systems (in general, it is well known that the minimal time function is only lower semicontinuous, see for example, Cardaliaguet, Quincampoix and Saint Pierre, 1997), etc. There are many possible approaches to study the small-time local controllability, leading to different results and requiring different assumptions. Here we follow the geometrical approach. The underlying philosophy is that the local properties of the reachable set of smooth control systems are determined by the Lie algebra generated by the "admissible" vector fields. So, it is very natural to look for conditions for small-time local controllability which can be expressed in terms of elements of this Lie algebra. Unfortunately, there is a gap between the necessary and the sufficient controllability conditions. Nevertheless, some very general sufficient conditions for small-time local controllability at a point are known (see Agrachev and Gamkrelidze, 1993; Bianchini and Stefani, 1990; Frankowska, 1989; Hermes, 1982; Liverovskij and Petrov, 1988; Sussmann, 1987), as well as some necessary conditions (see Kawski, 1987; Krastanov, 1998; Stefani, 1986; Sussmann, 1983). To our knowledge, necessary and sufficient conditions for small-time local controllability at a point are proved only in some special cases (see for example,

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Aubin, Frankowska, and Olech, 1986; Brunovsky, 1974; Jurdjevic and Kupka, 1985; Krastanov, 2008; Krastanov and Veliov, 2005; Veliov, 1988; Veliov and Krastanov, 1986).

The problem of local attainability of a closed set with respect to the trajectories of a differential inclusion can not be reduced to the problem of small-time local attainability at every point of the set. The reason is that the small-time local attainability depends not only on the dynamics of the control system, but also on the geometry of the considered closed set. So, it needs a specific study. To state this problem, let us consider the following differential inclusion:

$$\dot{x}(t) \in F(x(t)), \tag{1}$$

where $F : \boldsymbol{R}^n \Rightarrow \boldsymbol{R}^n$ is a multivalued map.

Remark 1.1 *We do not impose any continuity assumptions on the multivalued map F because the important tool of our approach is a concept for high-order control variations. Different examples of control systems whose dynamics is governed by differential inclusions or ODE systems are considered in Krastanov and Quincampoix (2001).*

An absolutely continuous function $x(\cdot)$, satisfying (1) for almost every t from $[0, T]$, is called a trajectory of (1) defined on $[0, T]$. For a fixed point x and for $T > 0$, the attainable set $\mathcal{A}(x, T)$ of (1) at the moment $T > 0$ starting from the point x for $t = 0$ is defined as the set of all points that can be reached in time T from x by means of trajectories of (1).

Definition 1.1 *Let S be a closed subset of $\boldsymbol{R}^n$. It is said that S is small-time locally attainable (STLA) with respect to the differential inclusion (1) if for any $T > 0$ there exists a neighborhood Ω of S such that for every point $x \in \Omega$ there exists an admissible trajectory of (1) starting from the point x and reaching the set S in time not greater than T, i.e. $\mathcal{A}(x, t) \cap S \neq \emptyset$ for some $t \in [0, T]$.*

One of the most common conditions to ensure local attainability of a closed set is the so called Petrov condition (see, for example, Petrov, 1976). Roughly speaking, this condition states that at every point of a neighborhood of the target there exists an admissible velocity that "points" toward the target. To formulate this condition in a rigorous way, we follow the notations from Clarke and Wolenski (1996): Let S be a compact subset of $\boldsymbol{R}^n$. We set

$$S_r := \{y \in \boldsymbol{R}^n : \ \mathrm{d}_S(y, S) \ \leq r\}, \quad \text{where} \ \ \mathrm{d}_S(y) := \ \inf\{\|y - s\| : \ s \in S\}.$$

If x is an arbitrary point from $S_r \setminus S$, we denote by

$$\Pi(x) := \{y \in S : \ \|y - x\| = \mathrm{d}_S(x)\}$$

the set of all metric projections of the point x on the set S.

Let y belong to the boundary ∂S of the set S. A vector $\xi \in \boldsymbol{R}^n$ is called a proximal normal to S at y, provided that there exists $r > 0$ so that the point $y + r\xi$ has y as the closest point. The set of all proximal normals at a point y is a cone. This cone is denoted by $N_S^p(y)$ (for a detailed treatment of proximal analysis and some of its applications, consult, for example, the books of Clarke, 1983, and Clarke, Ledyaev, Stern and Wolenski, 1998). Using these notations, the results of Clarke and Wolenski (1996), Veliov (1994 and 1997) can be formulated as follows:

THEOREM 1.1 *Suppose that S is a nonempty and compact subset of $\boldsymbol{R}^n$, and $F : \boldsymbol{R}^n \Rightarrow \boldsymbol{R}^n$ is a continuous multivalued map*[1] *of modulus ω near S with compact and convex values. Suppose that there exists $\delta > 0$ so that, whenever $y \in S$ and $\xi \in N_S^p(y)$, there exists $v \in F(y)$ for which*

$$\langle \xi, v \rangle \ \leq \ -\delta \|\xi\|. \tag{2}$$

Then S is small-time locally attainable with respect to the differential inclusion (1).

REMARK 1.2 *We would like to point out that the condition (2) implies that*

$$\langle \chi, v \rangle \ \leq \ -\delta \tag{3}$$

for each $\chi \in N_S^p(y)$ with $\|\chi\| = 1$, i.e. this scalar product is uniformly bounded away from zero. Recently, a sufficient condition for attainability of a closed set with respect to the trajectories of a smooth nonlinear system has been proved in Marigonda (2006) (see also Colombo, Marigonda and Wolenski, 2006). Under a suitable assumption for regularity of the closed set S, the positive number δ in the Petrov condition (2) is replaced by a continuous nondecreasing function $\mu(\cdot)$ such that $\mu(\rho) > 0$ for $\rho > 0$ and $\lim_{\rho \to 0} \mu(\rho) = 0$. The idea of the corresponding proof is new and uses control variations of zero and first order.

The condition (2) is very strong. As it is proved in Veliov (1997), this condition is equivalent to Lipschitz continuity of the minimal time function up to the boundary of the set S. Unfortunately, if the inequality (2) is violated at some boundary point y of S (for example, when all admissible velocities are "tangent" to the closed set S at y), we can not apply Theorem 1.1 (see the illustrative examples in Krastanov, 2002).

The traditional approach towards proving sufficient local controllability conditions has been to construct "high-order control variation". Heuristically, if one can construct variations in all possible directions, then the reachable set contains a neighborhood of the starting point. The so called "high-order conditions" for attainability of a closed set are also known. Usually the "high-order" term is used for conditions involving Lie brackets of the original vector fields (see, for

[1]See Clarke and Wolenski (1996) for the definition for continuity of modulus ω.

example, Bianchini and Stefani, 1990, and Coron, 1994, for the case of attainability of a point, and Bacciotti and Stefani, 1980; Bardi and Capuzzo-Dolcetta, 1997; Krastanov and Quincampoix, 2001; Krastanov, 2002; and Soravia, 1978, in the general case). To present the main idea, let us consider the following nonlinear control system:

$$\dot{x}(t) = \sum_{j=1}^{m} u_j(t) f_j(x(t)),$$

where f_j, $j = 1, \ldots, m$, are smooth vector fields defined on $\boldsymbol{R}^n$ and each control function $u_j(\cdot)$, $j = 1, \ldots, m$, takes its values from the interval $[-1, 1]$. If at a point $\bar{y} \in S$ the Petrov condition does not hold for some element $\xi \in N^p_S(\bar{y})$, then the existence of a Lie bracket Λ of the vector fields f_j, $j = 1, \ldots, m$, is required such that

$$\langle \xi, \Lambda(\bar{y}) \rangle \;\le\; -\delta \|\xi\|.$$

This condition can be viewed as a Petrov type condition of higher order. Usually, high-order conditions ensure local attainability of a closed set and imply Hölder continuity of the minimal-time function, where the modulus of continuity is determined by the "length" of the Lie brackets, which are involved.

In the present paper, we remind some concepts of high-order control variations defined in Krastanov and Quincampoix (2001), and present some of their properties. Next we propose a larger class of high-order variations *well adapted* to the considered problem of local attainability of a closed set. This allows us to obtain a unified treatment of the STLA problem. Moreover, we extend the main result of Marigonda (2006) in two directions. First, we do not impose any regularity assumptions for the set S, and second, we prove a Petrov type condition of higher order. The paper is organized as follows. The main result is formulated in Section 3. The corresponding proof is given in Section 4. An illustrative example is also presented.

2. High-order control variations

Following the approach proposed in Krastanov and Quincampoix (2001), we present some concepts of high-order control variations: Let us fix a positive real number $T > 0$, an open subset Ω of $\boldsymbol{R}^n$ and assume that the right-hand side F of the differential inclusion (1) is defined on Ω. We also assume that the attainable set $\mathcal{A}(x, t)$ of (1) is non empty for each $0 < t \le T$ and for each point $x \in \Omega$.

By $\mathcal{V}$ we denote the linear space of all smooth vector fields on $\boldsymbol{R}^n$, considered as a Lie algebra with the Lie product

$$[X, Y] := \frac{\partial Y}{\partial x} X - \frac{\partial X}{\partial x} Y. \tag{4}$$

Given a family of smooth vector fields Z_μ parameterized by $\mu \geq 0$ and a positive real s, we denote by exp (Z_s) (x) the value of the solution of the following ODE system

$$\dot{x}(t) \; = \; Z_s(x(t)), \; x(0) = x, \; t \in [0,1], \tag{5}$$

at time $t = 1$.

Let r be a positive number, S be a closed subset of the set Ω and $\bar{x}$ be an arbitrary point of the set $S_r \setminus S$. Next we define a smooth high-order variation A to the attainable set $\mathcal{A}(\bar{x}, \cdot)$ of the differential inclusion (1).

Definition 2.1 *Let $\Omega_{\bar{x}}$ be a neighborhood of the point $\bar{x}$, $A : \Omega_{\bar{x}} \setminus S \to R^n$ be a smooth function, and $\alpha > 0$, $\theta > 0$ and $\beta > \alpha$ be positive numbers. It is said that A belongs to the set $\mathcal{E}^{\bar{x}}_{\alpha,\beta,\theta}$ of high-order variations to the attainable set $\mathcal{A}(\bar{x}, \cdot)$ of the differential inclusion (1) if for each $t \in [0,T]$ and for each point $x \in \Omega_{\bar{x}}$ the following inclusion holds true*

$$exp\,(t^\alpha A + a(t) + b(t))\,(x) \;\in \mathcal{A}(x, p(t)), \tag{6}$$

where $p : [0,T] \to \boldsymbol{R}$ is an increasing continuous function with $p(0) = 0$ and the smooth maps $a(\cdot,\cdot) : [0,T] \times \Omega_{\bar{x}} \to R^n$ and $b(\cdot,\cdot) : [0,T] \times \Omega_{\bar{x}} \to R^n$ satisfy the inequalities

$$\|a(t,x)\| \;\leq\; M\; t^\theta\; d_S(x) \quad and \quad \|b(t,x)\| \;\leq\; N\; t^\beta,\; t \in [0,T],\; x \in \Omega_{\bar{x}},$$

for some positive constants M and N.

We also introduce a set $\mathcal{T}^{\bar{x}}$ of the "tangent" vector fields to the attainable set $\mathcal{A}(\bar{x}, \cdot)$ of the differential inclusion (1) which can be used for constructing new elements of the set $\mathcal{E}^{\bar{x}}_{\alpha,\beta,\theta}$, provided that some elements of this set are already known:

Definition 2.2 *It is said that the smooth vector field Z belongs to the set $\mathcal{T}^{\bar{x}}$ if there exist a neighborhood $\Omega_{\bar{x}}$ of $\bar{x}$ and positive real numbers C and T, such that for every point $x \in \Omega_{\bar{x}}$ and for each $t \in [0,T]$ the following inclusion holds true*

$$exp(tZ)(x) \;\in\; \mathcal{A}(x, Ct). \tag{7}$$

This definition implies the inclusion $\mathcal{T}^{\bar{x}} \subset \mathcal{E}^{\bar{x}}_{\alpha,\beta,\theta}$ for each real $\alpha > 0$, $\theta > 0$ and $\beta > \alpha$.

The origin of the above definitions can be found in the papers by Hermes (1978), Sussmann (1978), Hirshorn (1989) and Kunita (1979), where some of the following propositions were proved. Next, these concepts were used in Krastanov (2008), Krastanov and Veliov (2005), Veliov (1988), and Veliov and Krastanov (1986) for studying controllability of systems with linear dynamics, and in Krastanov (2002) and Krastanov and Quincampoix (2001) – for studying local attainability of a closed set.

PROPOSITION 2.1 *Let $A_1, A_2, ..., A_k$, belong to $\mathcal{T}^{x_0}$ and $A_1(x_0) + A_2(x_0) + ... + A_k(x_0) = 0$. Then $[A_i, A_j], i, j = 1, ..., k$, belong to $\mathcal{E}^{x_0}_{2,3,1}$.*

PROPOSITION 2.2 *Let A_1 and A_2 belong to $\mathcal{T}^{x_0}$ and $A_1(x_0)+A_2(x_0) = 0$. Then $[A_1, [A_1, A_2]] + [A_2, [A_2, A_1]]$ belongs to $\mathcal{E}^{x_0}_{3,4,1}$.*

PROPOSITION 2.3 *The set $\mathcal{E}^{x_0}_{\alpha,\beta,\theta}$ is a convex cone for each $\beta > \alpha > 0$ and $\theta > 0$.*

PROPOSITION 2.4 *Let A_1 and A_2 belong to $\mathcal{E}^{x_0}_{\alpha,\beta,\theta}$, $A_1(x_0) + A_2(x_0) = 0$ and B belong to $\mathcal{T}^{x_0}$ with $B(x_0) = 0$. Then there exist positive real numbers $\bar{\alpha}$, $\bar{\theta}$ and $\bar{\beta}$ with $\bar{\beta} > \bar{\alpha}$ such that $[B, A_1]$ and $[B, A_2]$ belong to $\mathcal{E}^{x_0}_{\bar{\alpha},\bar{\beta},\bar{\theta}}$.*

The proofs of these propositions, as well as some other useful assertions, can be found in Krastanov and Quincampoix (2001).

The main result of this paper is a sufficient condition for local attainability of a closed set with respect to the trajectories of the differential inclusion (1). It is based on a suitable class of high-order variations:

DEFINITION 2.3 *Let $r_0 > 0$, $\alpha > 0$, $\theta > 0$ and $\beta > \alpha$, and let $\mathcal{E}_{\alpha,\beta,\theta}$ be the family of continuous functions $A_\gamma : \Omega_\gamma \to \boldsymbol{R}^n$, $\gamma \in \Gamma$, where each Ω_γ is an open subset of $S_{r_0} \setminus S$. It is said that $\mathcal{E}_{\alpha,\beta,\theta}$ is a regular set of high-order variations to the attainable set of the differential inclusion (1) provided that there exist positive constants T_0, M, N, C and P such that $\|A_\gamma(x)\| \le C$ and the following inclusion holds true*

$$x + t^\alpha A_\gamma(x) + a_\gamma(t, x) + O_\gamma(t^\beta, x) \ \in \mathcal{A}(x, p_\gamma(t)) \tag{8}$$

for each $t \in [0, T_0]$, for each point $x \in \Omega_\gamma$ and for each $\gamma \in \Gamma$, where $p_\gamma : [0, T] \to \boldsymbol{R}$ is an increasing continuous function with $p_\gamma(0) = 0$ and the continuous maps $a_\gamma(\cdot, \cdot) : [0, T] \times \Omega_\gamma \to R^n$ and $O_\gamma(\cdot, \cdot) : [0, T] \times \Omega_\gamma \to R^n$ satisfy the inequalities

$$\|a_\gamma(t, x)\| \le M t^\theta d_S(x) \quad and \quad O_\gamma(t^\beta, x) \le N t^\beta, \ t \in [0, T], \ x \in \Omega_\gamma, \ \gamma \in \Gamma.$$

REMARK 2.1 *We would like to point out that the sets of variations $\mathcal{E}_{\alpha,\beta,\theta}$ are closely related to the high-order variations introduced in Krastanov and Quincampoix (2001). Let us fix an index $\gamma \in \Gamma$. Then the term $O_\gamma(t^\beta, x)$ is of higher order with respect to t^α uniformly for $x \in \Omega_\gamma$. From here and since $\|a_\gamma(t, x)\|$ tends to zero as $r > 0$ tends to zero, we can conclude that the "leading term" in (8) is $t^\alpha A_\gamma(x)$ for small values of $t > 0$ and for all points x sufficiently close to the set S. So, it is natural to check the Petrov condition for the variation A_γ.*

3. The main result

First, we introduce the following notations

$$\sigma_1(\alpha, \theta, \beta) := \begin{cases} \beta, & \text{if } \alpha = \theta; \\ \dfrac{\alpha}{\alpha - \theta}, & \text{if } \alpha > \theta; \end{cases}, \ \sigma_3(\alpha, \beta) := \begin{cases} \beta, & \text{if } \beta \ge 2\alpha; \\ \dfrac{\alpha}{2\alpha - \beta}, & \text{if } \beta < 2\alpha; \end{cases},$$

$$\sigma_2(\alpha,\beta,\theta) := \begin{cases} \beta, & \text{if } \beta+\theta \geq 2\alpha; \\ \dfrac{\alpha}{2\alpha-\beta-\theta}, & \text{if } \beta+\theta < 2\alpha; \end{cases}.$$

then the main result of this paper can be formulated as follows:

Theorem 3.1 *Suppose that S is a nonempty closed subset of $\boldsymbol{R}^n$, $r_0 > 0$, $\alpha \geq 1$, $\alpha \geq \theta > 0$ and $\beta > \alpha$, and let $\mathcal{E}_{\alpha,\beta,\theta}$ be a regular set of high-order variations to the attainable set of the differential inclusion (1). Assume that for each point $x \in S_{r_0} \setminus S$ there exist a point $\pi(x) \in \Pi(x)$ and an element $A \in \mathcal{E}_{\alpha,\beta,\theta}$ defined on an open neighborhood of x such that*

$$\langle x - \pi(x), A(x)\rangle \;\leq\; -\delta\, d_S(x)^{\lambda}, \tag{9}$$

where δ and λ are positive real numbers satisfying

$$1 \leq \lambda \;<\; \min\left(\sigma_1(\alpha,\theta),\ \sigma_2(\alpha,\beta,\theta),\ \sigma_3(\alpha,\beta),\ \frac{2\alpha}{2\alpha-\theta},\ \frac{2\alpha}{2\alpha-1}\right). \tag{10}$$

Then S is locally attainable with respect to the differential inclusion (1).

Remark 3.1 *The condition (9) can be written as follows:*

$$\langle \chi, A(x)\rangle \;\leq\; -\delta\, d_S(x)^{\hat{\lambda}}, \tag{11}$$

where $\chi := \dfrac{x-\pi(x)}{\|x-\pi(x)\|}$ and $\hat{\lambda} := \lambda - 1 \geq 0$. Clearly, $\|\chi\| = 1$ and for $\hat{\lambda} > 0$ the scalar product in (11) tends to zero as the point x tends to the set S in contrast to the usual condition (3). The illustrative example, presented below, shows that the case $\hat{\lambda} > 0$ is possible. Also, we would like to point out that the condition (10) is technical and gives a relation between the geometry of the set S and the properties of the high-order variations we need in the proof of Theorem 3.1.

Let $\Theta(x)$ be the minimal time of steering to the set S from the point x by means of a trajectory of the differential inclusion (1), i.e. $\Theta(x) := \inf\{t \geq 0,\ \text{such that } z(0) = x,\ z(t) \in S$ for some trajectory $z(\cdot)$ of the differential inclusion (1)$\}$. The map $\Theta(\cdot)$ is called time optimal map of reaching the set S. The proof of Theorem 3.1 implies directly the following corollary:

Corollary 3.1 *Suppose that the assumptions of Theorem 3.1 hold true. Then there exist $\gamma > 0$ and $\eta > 0$ such that*

$$\Theta(x) \leq \gamma\, d_S^{\eta}(x) \tag{12}$$

for every x in some neighborhood U of S.

Proposition 3.1 *Suppose that there exist some positive constants* r, γ, η, K *and* σ *for which the following conditions hold true:*

i) $\Theta(x) \;\leq\; \gamma\, d_S^{\eta}(x)$ *for every* x *in* S_r.

ii) if $z(.)$ *is a trajectory of the control system (1) defined on* $[0,T]$ *such that* $z(T) \in S$, *and if* y *is a point in* $S_r \setminus S$ *such that* $\|y - z(0)\| \leq \sigma$, *then there exists a trajectory* $z_y(.)$ *of (1) with*

$$z_y(0) \;=\; y \quad \textit{and} \quad \|z_y(t) - z(t)\| \;\leq\; e^{Kt}\|z_y(0) - z(0)\|$$

for every $t \in [0,T]$.

Then Θ *is* η*-Hölder continuous in* S_r.

Remark 3.2 *Under the assumptions of Theorem 3.1, the condition i) holds always. The condition ii) is satisfied for control systems with Lipschitz continuous right-hand side.*

Example 3.1 *Let us consider the following example: Let* $S := S^1 \cup S^2$ *(see Fig. 1), where*

$$S^1 := \{(x,y,z):\; y \geq x^{5/4},\; |x| \leq \frac{1}{2},\; z = 0\}$$

and

$$S^2 := \{(x,y,z):\; x = 0,\; y \leq 0,\; z = 0\},$$

and let us consider the following control system Σ

$$\left|\begin{array}{l} \dot{x}(t) = z(t), \\[2ex] \dot{y}(t) = -x^2(t), \quad u(t) \in [-1,1], \\[2ex] \dot{z}(t) = u(t). \end{array}\right.$$

This example is interesting because the Petrov condition does not hold true for all admissible velocities at all points of the form $(x,0,0)$. But, there exists a high-order variation satisfying the Petrov condition. We would like to point out that the scalar product of this variation and the corresponding normal to the set S tends to zero as the point tends to the set S.

It can be directly checked that the main result of Marigonda (2006) can not be applied. On the other hand, Theorem 3.1 is applicable. By applying the approach proposed in Krastanov and Quincampoix (2001), we can construct a high-order variation of the attainable set (satisfying conditions (9) and (10)) at every point belonging to some neighborhood of S. By applying these variations, we can move towards the set S according to Theorem 3.1.

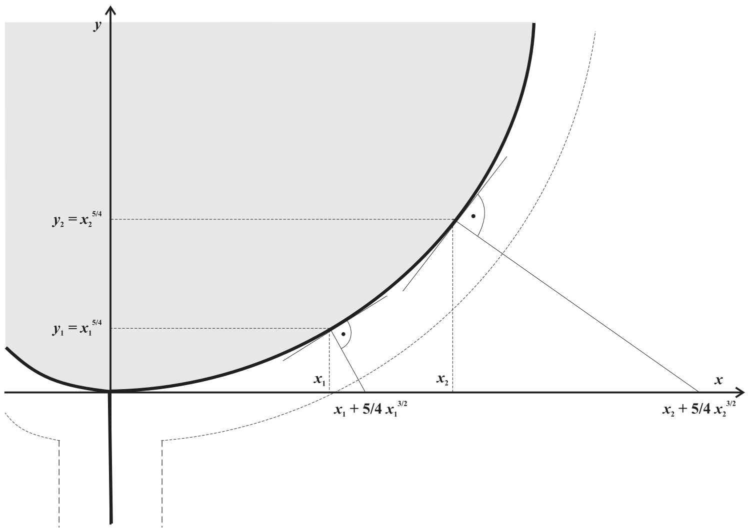

Figure 1. The intersection of the set S and the hyperplane $\{(x, y, z) : z = 0\}$.

To make this in a rigorous way, we choose the positive number r so small that the inclusion $\eta := (x, 0, 0) \in S_r$ implies that $x \le 1$. We have to check conditions (9) and (10) for each starting point $\eta_0 \in S_r$. To avoid technicalities, we consider only the most complicate case. Namely, we assume that the starting point is of the form $\eta_0 := (x_0 + 5/4 x_0^{3/2}, 0, 0)$ with $x_0 > 0$. For every real number $T \in [0, 1]$ we choose a real $t \in (0, T]$ and set

$$u_t(s) = \begin{cases} -1, & \text{if } s \in [0, t/2]; \\ 1, & \text{if } s \in [t/2, t]. \end{cases} \tag{13}$$

It can be directly checked that the trajectory $\eta_t(\cdot) = (x_t(\cdot), y_t(\cdot), z_t(\cdot))$ starting from the point η_0 and corresponding to the control $u_t(\cdot)$ is well defined on $[0, t]$ and

$$x_t(t) = x_0 + \frac{5}{4} x_0^{3/2} - t^2/4, \;\; y_t(t) = -t \left(x_0 + \frac{5}{4} x_0^{3/2} \right)^2 + O(t^3, \eta_0), \;\; z_t(t) = 0. \tag{14}$$

We set $A(\eta_0) := (-1, 0, 0)$ and denote by $\pi(\eta_0)$ the metric projection of the point η_0 on the set S. It can be directly verified (see, also Fig. 1) that $\pi(\eta_0) = (x_0, x_0^{5/4}, 0)$. Because the point $\eta_0 \in S_r$, we have $x_0 \le 1$, and hence $x_0^{5/4} < d_S(\eta_0) < 2 x_0^{5/4}$. Also,

$$\langle A(\eta_0), \eta_0 - \pi(\eta_0) \rangle = -\frac{5}{4}\, x_0^{3/2} < -\frac{5}{4} \left(\frac{d_S(\eta_0)^{4/5}}{2^{4/5}} \right)^{3/2} = -\frac{5}{4} \frac{1}{2^{6/5}}\, d_S(\eta_0)^{6/5}.$$

This inequality shows that the condition (9) holds true. Taking into account (14), we can represent the end point $\eta_t(t)$ of the trajectory $\eta_t(\cdot)$ as follows

$$\eta_t(t) = \eta_0 + \frac{t^2}{4} A(\eta_0) + a(t,\eta_0) + O(t^3,\eta_0),$$

where $\|a(t,\eta_0)\| \le c_1\, t\, d_S(\eta_0)^{8/5}$ and $\|O(t^3,\eta_0)\| \le c_2\, t^3$ for some positive constants c_1 and c_2. Thus, $A \in \mathcal{V}^{\eta_0}_{2,3,1}$. Clearly, the condition (10) also holds true for $\lambda = \frac{6}{5}$.

Similarly, we can construct a high order variation of the attainable set at every other point belonging to S_r. Moreover, it can be shown that the set of these variations is regular. Applying Theorem 3.1, Corollary 3.1 and Proposition 3.1, we obtain that the set S is small-time locally attainable and the time-optimal map is Hölder continuous.

4. Proofs

Proof of Theorem 3.1. The regularity of $\mathcal{E}_{\alpha,\beta,\theta}$ implies the existence of positive constants T_0, M, N, C and P, for which the conditions of Definition 2.3 are satisfied.

Let us fix an arbitrary T from the interval $(0, T_0]$. First, we determine the positive number r so that the neighborhood S_r of the set S be sufficiently small. To do this, we note that the condition (10) implies that

$$2 - 2\lambda + \frac{\lambda}{\alpha} > 0,\ 2 + \lambda\left(\frac{\theta}{\alpha} - 2\right) > 0$$
$$1 + \lambda\left(\frac{\beta+\theta}{\alpha} - 2\right) \text{ and } 1 + \lambda\left(\frac{\beta}{\alpha} - 2\right) > 0.$$

This and the inequality $\beta > \alpha \ge 1$ imply that we can find a positive real r for which the following inequalities hold true:

$$\begin{aligned}
&\Bigg[M^2\left(\frac{\delta}{C^2}\right)^{2\theta/\alpha} r^{2+2\lambda(\theta/\alpha-1)} + N^2\left(\frac{\delta}{C^2}\right)^{2\beta/\alpha} r^{2\lambda(\beta/\alpha-1)}\\
&+2MN\left(\frac{\delta}{C^2}\right)^{(\beta+\theta)/\alpha} r^{1+\lambda((\beta+\theta)/\alpha-2)} + 2MC\left(\frac{\delta}{C^2}\right)^{1+\theta/\alpha} r^{1+\lambda(\theta/\alpha-1)}\\
&+2NC\left(\frac{\delta}{C^2}\right)^{1+\beta/\alpha} r^{\lambda(\beta/\alpha-1)} + 2M\left(\frac{\delta}{C^2}\right)^{\theta/\alpha} r^{2+\lambda(\theta/\alpha-2)}\\
&+2N\left(\frac{\delta}{C^2}\right)^{\beta/\alpha} r^{1+\lambda(\beta/\alpha-2)}\Bigg] < \frac{\delta^2}{2C^2}\\
&\text{and } \frac{4P}{C^2}\left(\frac{\delta}{C^2}\right)^{1/\alpha-2} \frac{r^{2-2\lambda+\lambda/\alpha}}{2-2\lambda+\lambda/\alpha} < T.
\end{aligned} \tag{15}$$

This choice of r provides for all estimates we need in the proof. Next, we fix an arbitrary point x from $S_r \setminus S$. Let $\pi(x)$ be an arbitrary point from the set $\Pi(x)$. Then $\|x - \pi(x)\| = \mathrm{d}_S(x)$, $0 \neq x - \pi(x) \in N_S^p(\pi(x))$, and according to (9), there exists an element A from $\mathcal{E}_{\alpha,\beta,\theta}$ for which

$$C \;\geq\; \|A(x)\| \;\geq\; \left\langle -\frac{x-\pi(x)}{\|x-\pi(x)\|}, A(x) \right\rangle \;\geq\; \delta\, \mathrm{d}_S^{\lambda-1}(x).$$

It follows from here that

$$\mathrm{d}_S^2(x) - \frac{\delta^2}{2C^2}\mathrm{d}_S^{2\lambda}(x) = \mathrm{d}_S^2(x) - \frac{1}{2}\frac{\delta^2\, \mathrm{d}_S^{2\lambda-2}(x)}{C^2}\mathrm{d}_S^2(x) \geq \frac{1}{2}\mathrm{d}_S^2(x). \tag{16}$$

According to Definition 2.3, the relation $A \in \mathcal{E}_{\alpha,\beta,\theta}$ implies the existence of $p_{x,\pi(x)}(\cdot)$, $a(\cdot,\cdot)$ and $O(\cdot,\cdot)$ such that for all $t \in [0,T]$

$$z_{x,\pi(x)}(x,t) \;:=\; x + a(t,x) + t^\alpha A(x) + O(t^\beta, x) \;\in\; \mathcal{A}(x, p_{x,\pi(x)}(t)). \tag{17}$$

Moreover, the regularity of $\mathcal{E}_{\alpha,\beta,\theta}$ implies that for all $t \in [0,T]$

$$\|a(t,x)\| \;\leq\; M\, t^\theta\, \mathrm{d}_S(x),\;\; \|O(t^\beta,x)\| \;\leq\; N\, t^\beta \text{ and } |p_{x,y}(t)| \;\leq\; P\, t. \tag{18}$$

Then for every $t \in [0,T]$ we have that

$$\begin{aligned}
& \mathrm{d}_S^2(z_{x,\pi(x)}(x,t)) \\
\leq\;& \|z_{x,\pi(x)}(x,t) - \pi(x)\|^2 \;=\; \|(z_{x,\pi(x)}(x,t) - x) + (x - \pi(x))\|^2 \\
\leq\;& \big(\|a(t,x)\| + t^\alpha\|A(x)\| + \|O(t^\beta,x)\|\big)^2 \\
& +2\left\langle a(t,x) + t^\alpha A(x) + O(t^\beta,x), x - \pi(x)\right\rangle + \|x-\pi(x)\|^2 \\
\leq\;& \big(Mt^\theta \mathrm{d}_S(x) + t^\alpha C + t^\beta N\big)^2 + 2t^\alpha < A(x), x - \pi(x) > \\
& +2\left\langle a(t,x) + O(t^\beta,x), x-\pi(x)\right\rangle + \mathrm{d}_S(x)^2 \\
\leq\;& M^2 t^{2\theta}\mathrm{d}_S^2(x) + t^{2\alpha}C^2 + N^2t^{2\beta} + 2MCt^{\theta+\alpha}\mathrm{d}_S(x) + 2MNt^{\theta+\beta}\mathrm{d}_S(x) \\
& +2NCt^{\alpha+\beta} - 2t^\alpha\delta\,\mathrm{d}_S^\lambda(x) + 2Mt^\theta\mathrm{d}_S^2(x) + 2t^\beta N\mathrm{d}_S(x) + \mathrm{d}_S^2(x).
\end{aligned}$$

Now we fix the positive number t to be sufficiently small. To do this, let us consider the terms $-2t^\alpha\delta\,\mathrm{d}_S^\lambda(x)$ and $t^{2\alpha}C^2$. By setting

$$t(x) \;:=\; \left(\frac{\delta\,\mathrm{d}_S^\lambda(x)}{C^2}\right)^{1/\alpha},$$

these terms will be of one and the same degree 2λ with respect to $\mathrm{d}_S(x)$. Moreover, by applying the inequalities (15), we obtain that

$$
\begin{aligned}
&\mathrm{d}_S(z_{x,\pi(x)}(x,t(x)))^2 \\
&\leq \mathrm{d}_S^2(x) - \frac{\delta^2}{C^2}\,\mathrm{d}_S^{2\lambda}(x) + \mathrm{d}_S(x)^{2\lambda}\left[M^2\left(\frac{\delta}{C^2}\right)^{2\theta/\alpha}\mathrm{d}_S^{2+2\lambda(\theta/\alpha-1)}(x)\right. \\
&+N^2\left(\frac{\delta}{C^2}\right)^{2\beta/\alpha}\mathrm{d}_S^{2\lambda(\beta/\alpha-1)}(x) + 2MN\left(\frac{\delta}{C^2}\right)^{(\beta+\theta)/\alpha}\mathrm{d}_S^{1+\lambda((\beta+\theta)/\alpha-2)}(x) \\
&+2MC\left(\frac{\delta}{C^2}\right)^{1+\theta/\alpha}\mathrm{d}_S^{1+\lambda(\theta/\alpha-1)}(x) + 2NC\left(\frac{\delta}{C^2}\right)^{1+\beta/\alpha}\mathrm{d}_S^{\lambda(\beta/\alpha-1)}(x) \\
&\left. 2M\left(\frac{\delta}{C^2}\right)^{\theta/\alpha}\mathrm{d}_S^{2+\lambda(\theta/\alpha-2)}(x) + 2N\left(\frac{\delta}{C^2}\right)^{\beta/\alpha}\mathrm{d}_S^{1+\lambda(\beta/\alpha-2)}(x)\right] \\
&< \mathrm{d}_S^2(x) - \frac{\delta^2}{2C^2}\mathrm{d}_S(x)^{2\lambda}.
\end{aligned} \tag{19}
$$

Hence, taking into account (16), we obtain that for every point x from $S_r \setminus S$ the following estimation holds true:

$$
\mathrm{d}_S(z_{x,\pi(x)}(x,t(x))^2 \ < \mathrm{d}_S^2(x) - \frac{\delta^2}{2C^2}\mathrm{d}_S^{2\lambda}(x)
$$

with $z_{x,\pi(x)}(x,t(x)) \in S_r$. Using this estimate, we can define the sequence $\{x_k\}_{k=0}^{\infty}$ of points of S_r tending to the set S. We set

$$
x_k := \begin{cases} x, & \text{if } k=0; \\ z_{x_{k-1},\pi(x_{k-1})}(x_{k-1},t(x_{k-1})), & \text{if } k\geq 1 \text{ and } x_{k-1}\in S_r\setminus S; \\ x_{k-1}, & \text{if } k\geq 1 \text{ and } x_{k-1}\in S. \end{cases}
$$

Then for each positive integer $k \geq 0$, for which the point x_k belongs to the set $S_r \setminus S$ the following estimate holds true:

$$
\mathrm{d}_S^2(x_{k+1}) = \mathrm{d}_S^2(z_{x_k,\pi(x_k)}(x_k,t(x_k)) \ < \mathrm{d}_S^2(x_k) - \frac{\delta^2}{2C^2}\mathrm{d}_S^{2\lambda}(x_k). \tag{20}
$$

The sequence $\{\mathrm{d}_S(x_k)\}_{k=0}^{\infty}$ is decreasing and bounded from below. Hence, it is convergent. Let $d := \lim_{k\to\infty} \mathrm{d}_S(x_k)$. If we assume that $d > 0$, then for each positive $\epsilon > 0$ there exists a positive integer k_ϵ such that $d < \mathrm{d}_S(x_k) < d+\epsilon$ for each positive integer $k \geq k_\epsilon$. By applying the estimate (20), we obtain that

$$
\begin{aligned}
\mathrm{d}_S^2(x_{k_\epsilon+1}) &= \mathrm{d}_S^2(z_{x_{k_\epsilon},\pi(x_{k_\epsilon})}(x_{k_\epsilon},t(x_{k_\epsilon})) < \mathrm{d}_S^2(x_{k_\epsilon}) - \frac{\delta^2}{2C^2}\mathrm{d}_S^{2\lambda}(x_{k_\epsilon}) \\
&< (d+\epsilon)^2 - \frac{\delta^2}{2C^2}d^{2\lambda} < d^2
\end{aligned}
$$

whenever $\epsilon > 0$ is sufficiently small. The contradiction obtained shows that $d = 0$.

We claim that $\sum_{x_k \in S_r \setminus S} p_{x_k,\pi(x_k)}(t(x_k)) < T$, i.e. the set S is attainable from the point x via a trajectory of the differential inclusion (1) in finite time which is less than T. Let us assume that all points x_k, $k = 0, 1, 2, \ldots$ belong to the set $S_r \setminus S$ (the case when only a finite number of points x_k belong to the set $S_r \setminus S$ is simpler and can be considered in the same way). Application of the estimate (20) and the equalities

$$\begin{aligned}\frac{\delta^2}{2C^2}\mathrm{d}_S(x_k)^{2\lambda} &= \frac{C^2}{2}\left(\frac{\delta\,\mathrm{d}_S^\lambda(x_k)}{C^2}\right)^{1/\alpha}\left(\frac{\delta}{C^2}\right)^{2-1/\alpha}\mathrm{d}_S^{2\lambda-\lambda/\alpha}(x_k)\\ &= \frac{C^2}{2}t(x_k)\left(\frac{\delta}{C^2}\right)^{2-1/\alpha}\mathrm{d}_S^{2\lambda-\lambda/\alpha}(x_k),\end{aligned}$$

yields

$$\mathrm{d}_S^2(x_{k+1}) < \mathrm{d}_S^2(x_k) - \frac{C^2}{2}t(x_k)\left(\frac{\delta}{C^2}\right)^{2-1/\alpha}\mathrm{d}_S^{2\lambda-\lambda/\alpha}(x_k),$$

and hence

$$\begin{aligned}t(x_k) &< \frac{2}{C^2}\left(\frac{\delta}{C^2}\right)^{1/\alpha-2}\frac{\mathrm{d}_S^2(x_k)-\mathrm{d}_S^2(x_{k+1})}{\mathrm{d}_S^{2\lambda-\lambda/\alpha}(x_k)}\\ &= \frac{2}{C^2}\left(\frac{\delta}{C^2}\right)^{1/\alpha-2}\frac{(\mathrm{d}_S(x_k)-\mathrm{d}_S(x_{k+1}))(\mathrm{d}_S(x_k)+\mathrm{d}_S(x_{k+1}))}{\mathrm{d}_S^{2\lambda-\lambda/\alpha}(x_k)}\\ &\le \frac{4\mathrm{d}_S(x_k)}{C^2}\left(\frac{\delta}{C^2}\right)^{1/\alpha-2}\frac{\mathrm{d}_S(x_k)-\mathrm{d}_S(x_{k+1})}{\mathrm{d}_S^{2\lambda-\lambda/\alpha}(x_k)}\\ &= \frac{4}{C^2}\left(\frac{\delta}{C^2}\right)^{1/\alpha-2}\frac{\mathrm{d}_S(x_k)-\mathrm{d}_S(x_{k+1})}{\mathrm{d}_S^{2\lambda-1-\lambda/\alpha}(x_k)}.\end{aligned}$$

From here it follows that

$$\begin{aligned}&\sum_{k=0}^{\infty} p_{x_k,\pi(x_k)}(t(x_k))\\ \le\ &\sum_{k=0}^{\infty} P\,t(x_k) \le \frac{4P}{C^2}\left(\frac{\delta}{C^2}\right)^{1/\alpha-2}\sum_{k=0}^{\infty}\frac{\mathrm{d}_S(x_k)-\mathrm{d}_S(x_{k+1})}{\mathrm{d}_S^{2\lambda-1-\lambda/\alpha}(x_k)}\\ \le\ &\frac{4P}{C^2}\left(\frac{\delta}{C^2}\right)^{1/\alpha-2}\int_0^{d_S(x)}\frac{1}{\mu^{2\lambda-1-\lambda/\alpha}}\,d\mu = \frac{4P}{C^2}\left(\frac{\delta}{C^2}\right)^{1/\alpha-2} d_S^{2-2\lambda+\lambda/\alpha}(x)\\ \le\ &\frac{4P}{C^2}\left(\frac{\delta}{C^2}\right)^{1/\alpha-2} r^{2-2\lambda+\lambda/\alpha} < T.\end{aligned}$$

The last inequality is implied by (15). This completes the proof. ∎

Proof of Proposition 3.1. Our proof follows the original proof from Bianchini and Stefani (1990), where the case is studied when S is a point and the control system is determined by a differential equation. We present it for completeness. First, we set

$$D \ := \ e^{K\gamma r^{\eta}}. \tag{21}$$

Let x belong to the interior of the set S_r and let U be a neighborhood of x, such that

$$U \ \subset \ S_r \ \cap \ \left\{ y \in \boldsymbol{R}^n : \ \|y - x\| \ \leq \ \frac{r}{2D} \right\} \cap \ \left\{ y \in \boldsymbol{R}^n : \ \|y - x\| \ \leq \ \frac{\sigma}{2} \right\}.$$

Let y_1 and y_2 be arbitrary points from U. Suppose $\Theta(y_1) < \Theta(y_2)$. Fix an arbitrary ε from the interval $(0, \ \Theta(y_2) - \Theta(y_1))$. Since $0 \leq \Theta(y_1) < \Theta(y_1) + \varepsilon$, there exists a trajectory $z_1(\cdot)$ of (1) starting from y_1 that reach S in some time τ with

$$\Theta(y_1) \leq \tau < \Theta(y_1) + \varepsilon \ < \ \Theta(y_2).$$

According to the assumption ii) there exists a trajectory $z_2(\cdot)$ of (1) starting from y_2, defined on $[0, \tau]$ and such that for every $t \in [0, \tau]$

$$\|z_2(t) - z_1(t)\| \ \leq \ e^{Kt}\|y_2 - y_1\|.$$

Because $y_2 \in S_r$, we obtain that

$$\tau \ < \ \Theta(y_2) \ \leq \ \gamma\, r^{\eta}.$$

Our choice of U, the equality (21) and the inclusion $z_1(\tau) \in S$ imply that

$$\|\mathrm{d}_S(z_2(\tau))\| \ \leq \ \|z_2(\tau) - z_1(\tau)\| \ \leq \ e^{K\tau}\|y_2 - y_1\| \ \leq \ D\|y_2 - y_1\| \ \leq \ r. \tag{22}$$

So, $z_2(\tau) \in S_r$ and by the assumption i)

$$\Theta(z_2(\tau)) \leq \gamma\, d_S^{\eta}(z_2(\tau)).$$

Then, according to (22),

$$\Theta(y_2) \leq \tau + \Theta(z_2(\tau)) \leq \Theta(y_1) + \epsilon + \gamma\, d_S^{\eta}(z_2(\tau)) \leq$$

$$\leq \Theta(y_1) + \gamma\, D^{\eta}\|y_2 - y_1\|^{\eta} + \epsilon.$$

Since $\Theta(y_1)$ and $\Theta(y_2)$ do not depend on ϵ, we obtain that

$$\|\Theta(y_2) - \Theta(y_1)\| \ \leq \ \hat{\gamma}\|y_2 - y_1\|^{\eta} \ \text{ with } \ \hat{\gamma} \ := \ \gamma\, D^{\eta}.$$

This completes the proof. ∎

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