Control and Cybernetics

vol. **38** (2009) No. 4B

The union of uniform closed balls conjecture*

by

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Abstract: The exterior sphere condition is compared to proximal smoothness, and examples are provided, which show that the two properties are not necessarily equivalent. Then conditions are given under which equivalence holds, and an open question involving the union of uniform closed balls property is stated in the form of a conjecture.

Keywords: proximal smoothness, interior and exterior sphere conditions, union of uniform closed balls property, wedged set, nonsmooth analysis.

1. Introduction

In this paper, we denote by $\|\cdot\|$ the Euclidean norm and by $\langle\cdot,\cdot\rangle$ the usual inner product. For $\rho > 0$ we introduce the notation

$$B(0;\rho) := \{x \in \mathbb{R}^n : \|x\| < \rho\} \text{ and } \bar{B}(0;\rho) := \{x \in \mathbb{R}^n : \|x\| \le \rho\}.$$

The open (respectively closed) unit ball in $\mathbb{R}^n$ is denoted by B (respectively $\bar{B}$). For a set $A \subset \mathbb{R}^n$, $\operatorname{comp} A$, $\operatorname{int} A$, $\operatorname{bdry} A$ and $\operatorname{cl} A$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary and the closure A, respectively.

Now let $S \subset \mathbb{R}^n$ be a nonempty closed set and let $x \in S$. We recall that a vector $\zeta \in \mathbb{R}^n$ is said to be a *proximal normal to S at x* provided that there exists $\sigma = \sigma(x,\zeta) \ge 0$ such that

$$\langle \zeta, s - x\rangle \le \sigma\|s - x\|^2 \quad \forall s \in S. \tag{1}$$

*Submitted: December 2008; Accepted: September 2009.

The relation (1) is commonly referred to as the *proximal normal inequality.* No nonzero ζ satisfying (2) exists if $x \in \operatorname{int} S$, but this may also occur for $x \in \operatorname{bdry} S$, as is the case when S is the epigraph of the function $f(z) = -|z|$ and $x = (0, 0)$. For such points, the only proximal normal is $\zeta = 0$. In view of (1), the set of all proximal normals to S at x is a convex cone, and we denote it by $N_S^P(x)$.

Let $x \in \operatorname{bdry} S$, and suppose that $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ are such that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap S = \emptyset. \tag{2}$$

Then, ζ is a proximal normal to S at x and we say that ζ is *realized by an r-sphere.* Note that ζ is then also realized by an r'-sphere for any $0 < r' < r$. One can show that ζ being realized by an r-sphere is equivalent to the proximal normal inequality holding with $\sigma = \frac{1}{2r}$; that is,

$$\langle \frac{\zeta}{\|\zeta\|}, s - x\rangle \leq \frac{1}{2r}\|s - x\|^2 \quad \forall s \in S. \tag{3}$$

Our general reference regarding proximal normals as well as other constructs of proximal analysis is Clarke et al. (1998).

For a point $x \in \operatorname{bdry} S$, if there exists $r > 0$ such that *some* $0 \neq \zeta \in N_S^P(x)$ is realized by an r-sphere, then we say that S satisfies an *exterior r-sphere condition at x.* In terms of spheres, this is equivalent to the existence of $y_x \notin S$ such that

$$B(y_x; r) \cap S = \emptyset \ \text{ and } \ \|x - y_x\| = r.$$

If this holds (for a single $r > 0$) at every boundary point x, then S is said to satisfy an exterior r-sphere condition, and if there exists such an r, we simply say that S satisfies the *uniform exterior sphere condition.*

When $S = \operatorname{cl}(\operatorname{int} S)$ (the closure of the interior), the uniform exterior sphere condition is equivalent to $(\operatorname{int} S)^c$ (the complement of the interior) satisfying a uniform *interior* sphere condition, which is familiar from control theory, where it is required in connection with regularity properties of the minimal time function; see e.g. Cannarsa and Frankowska (2006), Cannarsa and Sinestrari (1995, 2004) and Sinestrari 2004.

If, for a point $x \in \operatorname{bdry} S$, $r > 0$ is such that *every* $0 \neq \zeta \in N_S^P(x)$ is realized by an r-sphere, then S is said to be *r-proximally smooth* at x. Analogously to the preceding terminology, if this holds at every boundary point x for some positive r, then we say that S is r-proximally smooth, and if there exists such an r, S is simply said to be *uniformly proximally smooth.*

Uniform proximal smoothness of S implies that $N_S^P(x) \neq \{0\}$ for all $x \in \operatorname{bdry} S$. Furthermore, if S is closed and convex, then the proximal normal inequality holds at every $x \in S$ with $\sigma = 0$; hence this class of sets is uniformly proximally smooth, and every $x \in \operatorname{bdry} S$ is realized by an r-sphere of arbitrarily large radius. It appears that uniform proximal smoothness was first studied by Federer (1959), who referred to the property as *positive reach*. Another reference in this vein is Gorniewicz (1995), who called proximally smooth sets "proximal retracts". In Clarke, Stern and Wolenski (1995) (see also Canino, 1998, and Shapiro, 1994), proximal smoothness was studied in detail in a Hilbert space setting, but here we will not require those results. Related properties such as *prox-regularity* and φ*-convexity* are investigated in Poliquin and Rockafellar (1996), Poliquin, Rockafellar and Thibault (2000), Rockafellar and Wets (1998), Colombo and Marigonda (2005) and Colombo, Marigonda and Wolenski (2006).

One goal of the present expository note is to compare the exterior sphere condition with proximal smoothness. Obviously, if S is r-proximally smooth, then it satisfies the exterior r-sphere condition. We will answer the following two questions concerning possible reverse implications:

($*$) *If S satisfies an exterior r-sphere condition and S is known to be uniformly proximally smooth, is it necessarily r-proximally smooth?*

($**$) *If S satisfies a uniform exterior sphere condition, is S necessarily uniformly proximally smooth?* (*Here there is no mention of radius.*)

We will also discuss the equivalence between S satisfying the uniform interior sphere condition and S being the union of uniform spheres, and thereby clarify a semantic ambiguity in the literature concerning these properties. A conjecture regarding these properties will be stated.

In the next section, we shall see, by means of counter examples, that the answer to both questions ($*$) and ($**$) is "no". Then, in Section 3 geometric conditions will be provided, under which equivalence between the uniform exterior sphere condition and uniform proximal smoothness *does* hold. Section 4 is devoted to the comparison of the uniform interior sphere condition and the union of uniform balls property, and the framing of an open question in the form of a conjecture.

Full details and proofs of our results can be found in Nour, Stern and Takche (2009).

2. Examples

As mentioned above, it is clear from the definitions that if S is r-proximally smooth, then it possesses the exterior r-sphere condition. That the reverse implication is not necessarily true is illustrated by the following simple example.

Example 1 Let $S := \{(x, |x|) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. This set possesses the exterior r-sphere condition for any $r > 0$, but fails to be r-proximally smooth for any $r > 0$, and thereby provides a negative answer to Question ($**$). Indeed, for each $x \in]0, +\infty[$ the vector $\zeta = (-1, 1)$ is a proximal normal to S at (x, x), but the radius of the sphere which realizes ζ must approach 0 as $x \downarrow 0$.

In the preceding example, the set S has an empty interior. We now will focus our attention on sets S satisfying $S = \mathrm{cl}\,(\mathrm{int}\, S)$, which are of the type commonly used in control theory as targets in minimal time optimal control problems. We shall refer to such sets as *standard* sets. Consider the following.

Example 2 Let S be the standard region inside the rectangle and outside the two large circles in Fig. 1. This set satisfies an exterior 1-sphere condition. (Observe that the non-vertically oriented circle has 1 as radius.) But, while S is clearly uniformly proximally smooth, it fails to be 1-proximally smooth, since the unit vector $(0, -1)$ normal to S at $(0, \frac{1}{2})$ cannot be realized by 1-sphere. This shows that Question $(*)$ has a negative answer.

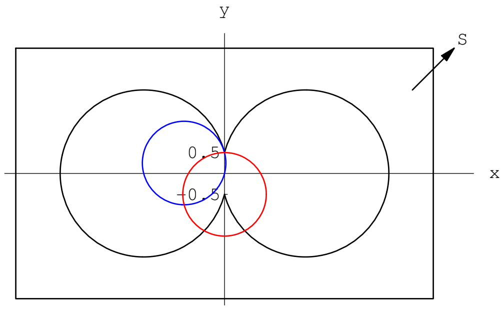

Figure 1. Example 2

While it is true that the set S in the previous example is not 1-proximally smooth, it *is* r-proximally smooth for any $0 < r \leq \frac{1}{2}$, and therefore it does not address Question $(**)$. The following example does so.

Example 3 Let S be the standard region inside the infinite rectangle and outside the circles (of radius 2) of Fig. 2. The intersection of two consecutive circles C_n and C_{n+1} consists of two points of the form $p_n = (a_n, \frac{1}{2n})$ and $q_n = (a_n, -\frac{1}{2n})$,

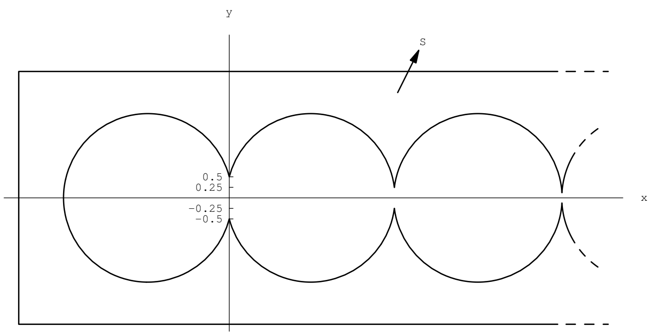

Figure 2. Example 3

where $a_n \in \mathbb{R}$ and $n \geq 1$. Then S satisfies the exterior 1-sphere condition, but fails to be r-proximally smooth for any $r>0$, since vertical proximal normals at p_n and q_n are realizable only by spheres of radius equal at most $\frac{1}{2n}$.

The set S of the previous example is connected, but it fails to be compact. The following is a two dimensionsal compact counterexample for Question $(**)$ in which the set S is not connected. Example 4 is due to Zvi Artstein (private communication). A similar example, but in another context, can be also found in Marigonda (2006).

Example 4 Let S be the infinite union of the "curved" triangles of Fig. 3. The curved sides of these triangles are arcs of unit circles tangent to the horizontal bases of the triangles. The points a_n and b_n are chosen in such a way that the sequences $|b_n - a_n|$ and $|a_{n+1} - b_n|$ converge to 0 and such that the curved triangles converge to a point (included in the set S); note that S is therefore compact. Clearly, S satisfies the exterior 1-sphere condition but fails to be r-proximally smooth for any $r > 0$. Indeed, the radius of the spheres, which realize horizontal proximal normals at the points b_n must approach 0.

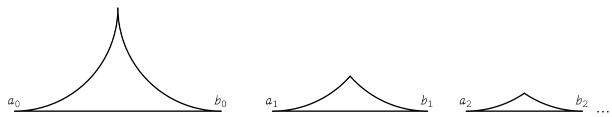

Figure 3. Example 4

Remark 1 It is an open question whether a two dimensional connected and compact counterexample to Question $(**)$ exists.

We shall conclude this section with a third negative example for Question $(**)$, but where S is a three dimensional compact and connected set.

Example 5 Consider the following three surfaces in $\mathbb{R}^3$, shown in the left picture of Fig. 4:

- S_1 is the part of the sphere $x^2 + y^2 + (z-2)^2 = 4$ with $x \leq 0$, $y \leq 0$ and $z \leq 2$.
- S_2 is the part of the cylinder $y^2 + (z-2)^2 = 4$ with $0 \leq x \leq 2$, $-2 \leq y \leq 0$ and $z \leq 2$.
- S_3 is the part of the cylinder $x^2 + (z-2)^2 = 4$ with $-2 \leq x \leq 0$, $0 \leq y \leq 2$ and $z \leq 2$.

Now, define S to be the region between the surface $S_1 \cup S_2 \cup S_3$ and the plane $z = 0$, as is shown in the right picture of Fig. 4. Clearly, S is a standard set. Moreover, S satisfies the exterior 1-sphere condition, but it fails to be uniformly

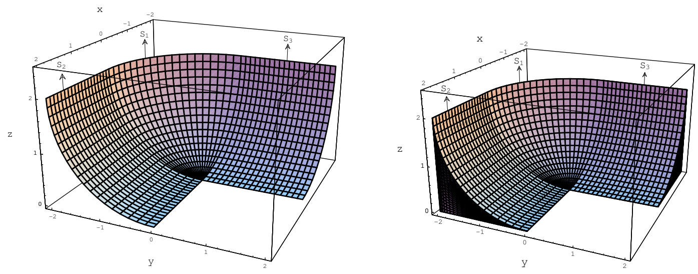

Figure 4. Example 5

proximally smooth. Indeed, similarly to Example 1, for each $x \in]0, 2[$ the vector $\zeta = (0, 1, 0)$ is a proximal normal to S at $(x, 0, 0)$, but the radius of the sphere which realizes ζ necessarily approaches 0 as $x \downarrow 0$.

3. An equivalence result

Recall that a closed set S is said to be *wedged* (or *epi-Lipschitz*) at a boundary point x, if near x the set S can be viewed, after application of an orthogonal matrix, as the epigraph of a Lipschitz continuous function. Specifically, there exists an open neighborhood V of x, a unit vector e, and for the hyperplane

$$H := \{x' : \langle e, x' - x \rangle = 0\}$$

through x, a Lipschitz continuous function $f : H \cap V \longrightarrow \mathbb{R}$ such that for some open neighborhood W of x one has

$$W \cap S = W \cap \{x' + te : x' \in H \cap V \text{and } f(x') \leq t < \infty\}.$$

This geometric definition was introduced by Rockafellar (1979). The property is also characterizable in terms of the nonemptiness of the topological interior of the Clarke tangent cone, which is also equivalent to the pointedness of the Clarke normal cone; see Clarke et al. (1998) and Rockafellar and Wets (1998). If S is wedged at x for all $x \in \text{bdry}\, S$, then we simply say that S is wedged.

In Nour, Stern and Takche (2009) various results (some of them local in nature and quite technical) were provided concerning the equivalence of the exterior sphere condition and proximal smoothness. For our purposes in this expository note, the following (non-local) result is the easiest to state and also of the greatest interest:

THEOREM 1 *Let $S \subset \mathbb{R}^n$ be a wedged set with compact boundary. Then S satisfies a uniform exterior sphere condition if and only if S is uniformly proximally smooth.*

Remark 2 The set of Example 3 is wedged but does not have a compact boundary. This shows that the preceding corollary fails if we drop that compactness assumption. On the other hand, the sets of Example 4 and Example 5 are not wedged but possess compact boundary. This shows that the corollary also fails if we drop the wedgedness assumption.

4. Interior sphere condition and a conjecture

We continue to assume that our set S is standard; that is, $S = \mathrm{cl}\,(\mathrm{int}\, S)$. In the control theoretic literature on can find two definitions of the interior r-sphere condition. The first one (see Alvarez, Cardaliaguet and Monneau, 2005; Cannarsa and Cardaliaguet, 2006, and Cannarsa and Frankowska, 2006) is complementary to the notion of exterior r-sphere condition which we have been using; that is, for each $x \in \mathrm{bdry}\, S$ there exists $y_x \in S$ such that

$$x \in \bar{B}(y_x; r) \subset S.$$

The second one (see Cannarsa and Sinestrari, 1995, 2004, and Sinestrari, 2004) says that for all $x \in S$ there exists $y_x \in S$ such that

$$x \in \bar{B}(y_x; r) \subset S.$$

This means that S is the union of closed r-balls. Equivalently, there exists $S_0 \subset S$ such that $S_0 + r\bar{B} = S$. Clearly, if S is the union of closed r-balls then it satisfies the interior r-sphere condition. The following example shows that the reverse implication is not necessarily true and then the two definitions are *not equivalent.*

Example 6 Let S be the closed region inside the three circles of Fig. 5. Clearly, this set satisfies the interior 1-sphere condition (in the first sense) since the three circles are of radius 1. But the origin cannot be covered by a 1-ball contained in S; in fact, the maximal radius for a family of covering balls is $\frac{1}{\sqrt{3}}$. Therefore the interior sphere condition does not hold for S in the second sense.

If a closed set $C \subset \mathbb{R}^n$ is r-proximally smooth, then the complement of its interior, $(\mathrm{int}\, C)^c$, is the union of closed r-balls. To see why, consider any $x \in (\mathrm{int}\, C)^c$. If $d_C(x) > r$, then clearly there is an r-ball centered at x which is contained in $(\mathrm{int}\, C)^c$. If $d_C(x) \leq r$, consider any closest point $s \in C$ to x. Then $\zeta := x - s$ is a proximal normal to C at s. Since ζ is realizable by an r-sphere, there is a closed r-ball centered at $s + r\frac{\zeta}{\|\zeta\|}$, which is contained in $(\mathrm{int}\, C)^c$, and x is in this ball.

Therefore we have

$(\mathrm{int}\, S)^c$ is uniformly proximally smooth $\Rightarrow$
S is the union of uniform closed balls $\Rightarrow$
S has the uniform interior sphere condition.

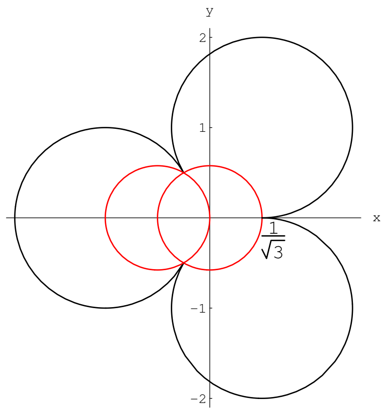

Figure 5. Example 6

The reverse implications are not necessarily true, as shown by Examples 3, 4 and 5. Indeed, in those examples the set $(\operatorname{int} S)^c$ has the union of uniform balls property, but S is not uniformly proximally smooth.

From Theorem 1 we obtain the following corollary, which asserts that the wedgedness of S together with boundary compactness guarantee the equivalence between the three properties under consideration.

Corollary 1 *Assume that S is wedged and that* $\operatorname{bdry} S$ *is compact. Then the following assertions are equivalent:*

(i) $(\operatorname{int} S)^c$ *is proximally smooth.*
(ii) S *is the union of uniform closed balls.*
(iii) S *possesses the uniform interior sphere condition.*

Remark 3 Let us reconsider Example 6. We noted that while S has the interior 1-sphere property, it is not the union of closed 1-balls. But it certainly *is* the union of closed r-balls for $r \leq \frac{1}{\sqrt{3}}$. It remains an open question as to whether the uniform interior sphere condition for S implies that S is a union of uniform closed balls. We conclude by expressing this question as a formal conjecture, and in a way that is free of terminology.

Conjecture 1 Suppose that S is a closed set and that there exists $r > 0$ as follows: For each $x \in \operatorname{bdry} S$ there exists $y_x \in S$ for which $x \in \bar{B}(y_x; r) \subset S$. Then there exists $r' > 0$ such that S is the union of balls of radius r'.

Acknowledgements

The authors wish to thank Professors Zvi Artstein and Piermarco Cannarsa for useful discussions. A part of this research was conducted while the first author visited Concordia University, Montreal, Canada. The research of the first author was supported by the Lebanese American University under the URC grant number URC-T-2008-25. The research of the second author was supported by the Natural Sciences Engineering Research Council of Canada.

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