Control and Cybernetics

vol. **38** (2009) No. 4B

Convergence of approximate solutions of variational problems*

by

Alexander J. Zaslavski

Department of Mathematics, The Technion-Israel Institute of Technology
32000 Haifa, Israel
e-mail: ajzasl@tx.technion.ac.il

Abstract: We study the structure of approximate solutions of autonomous variational problems on large finite intervals. In our previous research, which was summarized in Zaslavski (2006b), we showed that approximate solutions are determined mainly by the integrand, and are essentially independent of the choice of time interval and data, except in regions close to the endpoints of the time interval. In the present paper we establish convergence of approximate solutions in regions close to the endpoints of the time intervals.

Keywords: good function, infinite horizon, integrand, turnpike property.

1. Introduction

The study of variational and optimal control problems defined on infinite (large) intervals has recently been a rapidly growing area of research. See, for example, Baumeister, Leitao and Silva (2007), Blot and Cartigny (2000), Blot and Cretezz (2004), Blot and Michel (2003), Glizer (2007), Lykina, Pickenhain and Wagner (2008), Mordukhovich (1990), Mordukhovich and Shvartsman (2004), Pickenhain and Lukina (2006), Zaslavski (1996, 1998, 1999, 2006a,b, 2008) and the references mentioned therein. These problems arise in engineering (see Anderson and Moore, 1971; Leizarowitz, 1986), in models of economic growth (see Makarov and Rubinov, 1977; Samuelson, 1965; Zaslavski, 2006b), in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals (see Aubry and Le Daeron, 1983) and in the theory of thermodynamical equilibrium for materials (see Coleman, Marcus and Mizel, 1992; Leizarowitz and Mizel, 1989; Marcus and Zaslavski, 1999, 2002).

*Submitted: January 2009; Accepted: August 2009.

In this paper we analyze the structure of extremals of the variational problems

$$\int_0^T f(z(t), z'(t))dt \to \min,\ z(0) = x,\ z(T) = y, \tag{P}$$

$$z:\ [0,T] \to R^n \text{ is an absolutely continuous (a. c.) function,}$$

where $T > 0$ is sufficiently large, $x, y \in R^n$ and $f : R^n \times R^n \to R^1$ is an integrand. In our research, which was summarized in Zaslavski (2006b) we were interested in turnpike properties of the extremals that are independent of the length of the interval, for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the variational problems are determined mainly by the integrand, and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints of the time interval. In the present paper we establish convergence of approximate solutions in regions close to the endpoints of the time intervals.

It should be mentioned that turnpike properties are well known in mathematical economics (see Makarov and Rubinov, 1977; Samuelson, 1965; Zaslavski, 2006b). The term was first coined by Samuelson in 1948 (see Samuelson, 1965) when he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics (see Makarov and Rubinov, 1977). Many turnpike results can be found in Zaslavski (2006b).

Denote by $|\cdot|$ the Euclidean norm in R^n. Let a be a positive constant and let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\psi(t) \to \infty$ as $t \to \infty$. Denote by $\mathcal{A}$ the set of all continuous functions $f :\ R^n \times R^n \to R^1$, which satisfy the following assumptions:

A(i) for each $x \in R^n$ the function $f(x,\cdot) : R^n \to R^1$ is convex;

A(ii) $f(x,u) \geq \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(x,u) \in R^n \times R^n$;

A(iii) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(x_1,u_1) - f(x_2,u_2)| \leq \epsilon \max\{f(x_1,u_1), f(x_2,u_2)\}$$

for each $u_1, u_2, x_1, x_2 \in R^n$, which satisfy

$$|x_i| \leq M,\ i = 1,2,\ |u_i| \geq \Gamma,\ i = 1,2, \quad |x_1 - x_2|,\ |u_1 - u_2| \leq \delta.$$

It is easy to show that an integrand $f = f(x,u) \in C^1(R^{2n})$ belongs to $\mathcal{A}$ if f satisfies assumptions A(i), A(ii) and if there exists an increasing function $\psi_0 :\ [0,\infty) \to [0,\infty)$ such that

$$\max\{|\partial f/\partial x(x,u)|,\ |\partial f/\partial u(x,u)|\} \leq \psi_0(|x|)(1 + \psi(|u|)|u|)$$

for each $x, u \in R^n$.

For the set $\mathcal{A}$ we consider the uniformity, which is determined by the following base:

$$E(N,\epsilon,\lambda) = \{(f,g) \in \mathcal{A}\times\mathcal{A} :\ |f(x,u) - g(x,u)| \le \epsilon \\ \text{for all } u,x \in R^n \text{ satisfying } |x|,|u| \le N\} \\ \cap\{(f,g) \in \mathcal{A}\times\mathcal{A} :\ (|f(x,u)|+1)(|g(x,u)|+1)^{-1} \in [\lambda^{-1},\lambda] \\ \text{for all } x,u \in R^n \text{ satisfying } |x| \le N\},$$

where $N,\epsilon > 0$ and $\lambda > 1$. It was shown in Zaslavski (1996) that the uniform space $\mathcal{A}$ is metrizable and complete.

We consider functionals of the form

$$I^f(T_1,T_2,x) = \int_{T_1}^{T_2} f(x(t),x'(t))dt \tag{1}$$

where $f \in \mathcal{A}$, $-\infty < T_1 < T_2 < \infty$ and $x :\ [T_1,T_2] \to R^n$ is an absolutely continuous (a.c.) function.

For $f \in \mathcal{A}$, $y,z \in R^n$ and real numbers T_1,T_2 satisfying $T_1 < T_2$ we set

$$U^f(T_1,T_2,y,z) = \inf\{I^f(T_1,T_2,x) :\ x :\ [T_1,T_2] \to R^n \\ \text{is an a.c. function satisfying } x(T_1) = y,\ x(T_2) = z\}. \tag{2}$$

It is easy to see that $-\infty < U^f(T_1,T_2,y,z) < \infty$ for each $f \in \mathcal{A}$, each $y,z \in R^n$ and all numbers T_1,T_2 satisfying $-\infty < T_1 < T_2 < \infty$.

Let $f \in \mathcal{A}$. For any a.c. function $x : [0,\infty) \to R^n$ we set

$$J(x) = \liminf_{T\to\infty} T^{-1} I^f(0,T,x). \tag{3}$$

Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\{J(x) :\ x : [0,\infty) \to R^n \text{ is an a.c. function}\}. \tag{4}$$

Clearly $-\infty < \mu(f) < \infty$. By a simple modification of the proof of Proposition 4.4 in Leizarowitz and Mizel (1989) (see Zaslavski, 1996, Theorems 8.1, 8.2) we obtained the representation formula

$$U^f(0,T,x,y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta^f_T(x,y), \\ x,y \in R^n,\ T \in (0,\infty), \tag{5}$$

where $\pi^f : R^n \to R^1$ is a continuous function and $(T,x,y) \to \theta^f_T(x,y) \in R^1$ is a continuous nonnegative function defined for $T > 0$, $x,y \in R^n$,

$$\pi^f(x) = \inf\{\liminf_{T\to\infty}[I^f(0,T,v) - \mu(f)T] :\ v : [0,\infty) \to R^n \\ \text{is an a. c. function satisfying } v(0) = x\},\ x \in R^n \tag{6}$$

and for every $T > 0$, every $x \in R^n$ there is $y \in R^n$ satisfying $\theta^f_T(x,y) = 0$.

In the sequel we use the following helpful result.

Proposition 1 *(Zaslavski, 1999, Proposition 2.1). Let $f \in \mathcal{A}$. Then $\pi^f(x) \to \infty$ as $|x| \to \infty$.*

An a. c. function $x : [0, \infty) \to R^n$ is called (f)-good (see Zaslavski, 2006b) if the function $T \to I^f(0, T, x) - \mu(f)T$, $T \in (0, \infty)$ is bounded. In Zaslavski (1996) we showed that for each $f \in \mathcal{A}$ and each $z \in R^n$ there exists an (f)-good function $v : [0, \infty) \to R^n$ satisfying $v(0) = z$.

Propositions 1.1 and 3.2 of Zaslavski (1996) imply the following result:

Proposition 2 *For any a.c. function $x : [0, \infty) \to R^n$ either $I^f(0, T, x) - T\mu(f) \to \infty$ as $T \to \infty$ or $\sup\{|I^f(0, T, x) - T\mu(f)| : \ T \in (0, \infty)\} < \infty$. Moreover, any (f)-good function $x : [0, \infty) \to R^n$ is bounded.*

We denote $d(x, B) = \inf\{|x - y| : \ y \in B\}$ for $x \in R^n$ and $B \subset R^n$ and by $\mathrm{dist}(A, B)$ the distance in the Hausdorff metric for two sets $A \subset R^n$ and $B \subset R^n$. For every bounded a. c. function $x : [0, \infty) \to R^n$ define

$$\Omega(x) = \{y \in R^n : \ \text{there exists a sequence } \{t_i\}_{i=1}^{\infty} \subset (0, \infty) \text{ for which } t_i \to \infty,\ x(t_i) \to y \text{ as } i \to \infty\}. \tag{7}$$

We say that an integrand $f \in \mathcal{A}$ has an asymptotic turnpike property, or briefly (ATP), if $\Omega(v_2) = \Omega(v_1)$ for all (f)-good functions $v_i : [0, \infty) \to R^n$, $i = 1, 2$ (see Marcus and Zaslavski, 1999; Zaslavski, 1996).

In Zaslavski (1996, Theorem 2.1) we established the following result:

Theorem 1 *There exists a set $\mathcal{F} \subset \mathcal{A}$, which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that each integrand $f \in \mathcal{F}$ possesses (ATP).*

By Proposition 2 for each integrand $f \in \mathcal{A}$, which possesses (ATP) there exists a compact set $H(f) \subset R^n$ such that $\Omega(v) = H(f)$ for each (f)-good function $v : [0, \infty) \to R^n$. In the sequel we always use this notation.

Let $f \in \mathcal{A}$. We say that the integrand f has the strong turnpike property, or briefly (STP), with a turnpike $D \subset R^n$, where D is a nonempty compact subset of R^n, if for each $\epsilon, K > 0$ there exist real numbers $\delta > 0$ and $l_0 > l > 0$ such that the following assertion holds:

For each $T \geq 2l_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \leq K,\ I^f(0, T, v) \leq U^f(0, T, v(0), v(T)) + \delta$$

the inequality

$$\mathrm{dist}(D, \{v(t) : \ t \in [\tau, \tau + l]\}) \leq \epsilon \tag{8}$$

holds for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), D) \leq \delta$, then (8) holds for all $\tau \in [0, T - l_0]$ and if $d(v(T), D) \leq \delta$, then (8) holds for each $\tau \in [l_0, T - l]$.

Note that (STP) describes the structure of approximate solutions of problem (P) except in regions close to the endpoints of the time interval $[0,T]$.

Denote by $\mathcal{M}$ the set of all functions $f \in C^1(R^{2n})$ which satisfy the following assumptions:

$$\partial f/\partial u_i \in C^1(R^{2n}) \text{ for } i = 1,\dots,n;$$

the matrix $(\partial^2 f/\partial u_i \partial u_j)(x,u)$, $i,j = 1,\dots,n$ is positive definite for all $(x,u) \in R^{2n}$;

$$f(x,u) \geq \max\{\psi(|x|),\ \psi(|u|)|u|\} - a \text{ for all } (x,u) \in R^n \times R^n;$$

there exist a number $c_0 > 1$ and monotone increasing functions $\phi_i : [0,\infty) \to [0,\infty)$, $i = 0,1,2$ such that

$$\begin{aligned}
&\phi_0(t)/t \to \infty \text{ as } t \to \infty,\\
&f(x,u) \geq \phi_0(c_0|u|) - \phi_1(|x|),\ x,u \in R^n,\\
&\max\{|\partial f/\partial x_i(x,u)|,\ |\partial f/\partial u_i(x,u)|\} \leq \phi_2(|x|)(1+\phi_0(|u|)),\\
&\qquad\qquad x,u \in R^n,\ i = 1,\dots,n.
\end{aligned}$$

It is easy to see that $\mathcal{M} \subset \mathcal{A}$. In Zaslavski (1999, Theorem 1.2) we established the following result:

Theorem 2 *Assume that an integrand $f \in \mathcal{M}$ has (ATP). Then f possesses (STP) with the set $H(f)$ being the turnpike.*

The following fact was established in Zaslavski (2008):

Theorem 3 *Assume that $f \in \mathcal{M}$ possesses (STP). Then f possesses (ATP).*

Let $f \in \mathcal{A}$. For each pair of real numbers $T_2 > T_1$ and each a. c. function $v : [T_1,T_2] \to R^n$ set

$$\Gamma^f(T_1,T_2,v) = I^f(T_1,T_2,v) - (T_2 - T_1)\mu(f) - \pi^f(v(T_1)) + \pi^f(v(T_2)). \tag{9}$$

By (9), (2) and (5),

$$\Gamma^f(T_1,T_2,v) \geq 0$$

$$\text{for each } T_1 \in R^1,\ \text{each } T_2 > T_1 \text{ and each a. c. function } v : [T_1,T_2] \to R^n. \tag{10}$$

The following useful result was established in Zaslavski (1996, Theorem 8.3).

Proposition 3 *For every $x \in R^n$ there exists an (f)-good function $v : [0,\infty) \to R^n$ such that $v(0) = x$ and $\Gamma^f(T_1,T_2,v) = 0$ for each $T_1 \geq 0$ and each $T_2 > T_1$.*

An (f)-good function $v : [0,\infty) \to R^n$ is called (f)-perfect if $\Gamma^f(T_1,T_2,v) = 0$ for all $T_1 \geq 0$ and $T_2 > T_1$ (see Marcus and Zaslavski, 2002; Zaslavski, 2006a).

In view of Theorem 1, most integrands of the space $\mathcal{A}$ possess (ATP). By Theorem 2, if an integrand $f \in \mathcal{M}$ possesses (ATP), then f also possesses (STP) which describes the structure of approximate solutions of problem (P) in the region $[l_0, T - l_0]$ (see the definition of (STP)). In the present paper we study the structure of approximate solutions in the regions $[0,l_0]$ and $[T - l_0, T]$.

2. Main results

Assume that $f \in \mathcal{M}$ possesses (ATP) with the turnpike $H(f)$. It means that $\Omega(v) = H(f)$ for any (f)-good $v : [0, \infty) \to R^n$. Set

$$\bar{f}(x, y) = f(x, -y),\ (x, y) \in R^{2n}. \tag{11}$$

Clearly, $\bar{f} \in \mathcal{M}$. Let $v : [0, T] \to R^n$ be an a.c. function. Put

$$\bar{v}(t) = v(T - t),\ t \in [0, T]. \tag{12}$$

It is easy to see that

$$\int_0^T \bar{f}(\bar{v}(t), \bar{v}'(t))dt = \int_0^T f(v(T - t), v'(T - t))dt = \int_0^T f(v(t), v'(t))dt. \tag{13}$$

THEOREM 4 *$\bar{f}$ possesses (ATP) and for each $(\bar{f})$-good function $v : [0, \infty) \to R^n$, $\Omega(v) = H(f)$.*

Proof. By Theorem 2, f possesses (STP) with the turnpike $H(f)$. In view of (11)-(13) it is not difficult to see that $\bar{f}$ possesses (STP) with the turnpike $H(f)$. Together with Theorem 3 this implies that $\bar{f}$ possesses (ATP) and $\Omega(v) = H(f)$ for each $(\bar{f})$-good function $v : [0, \infty) \to R^n$. ∎

In this paper we study problem (P) and the following two problems

$$I^f(0, T, z) \to \min,\ z : [0, T] \to R^n \text{ is an a. c. function such that } z(0) = x, \tag{P_1}$$

$$I^f(0, T, z) \to \min,\ z : [0, T] \to R^n \text{ is an a. c. function,} \tag{P_2}$$

where $T > 0$ is sufficiently large and $x \in R^n$.

For $g \in \mathcal{A}$, $x \in R^n$ and $T > 0$ put

$$U^g(T, x) = \inf\{I^g(0, T, w) :\ w :\ [0, T] \to R^n \text{ is an a.c. function satisfying } w(0) = x\}, \tag{14}$$

$$U^g(T) = \inf\{I^g(0, T, w) :\ w :\ [0, T] \to R^n \text{ is an a.c. function}\}. \tag{15}$$

The following theorem establishes the convergence of approximate solutions of problem (P) in the regions which contain zero. It will be proved in Section 4.

THEOREM 5 *Let $x, y \in R^n$, a sequence of positive numbers $T_i \to \infty$ as $i \to \infty$ and let $v_i : [0, T_i] \to R^n$, $i = 1, 2, \ldots$ be an a. c. function such that*

$$v_i(0) = x,\ v_i(T_i) = y,\ I^f(0, T_i, v_i) - U^f(0, T_i, x, y) \to 0 \text{ as } i \to \infty. \tag{16}$$

Then there exist an (f)-perfect function $w : [0, \infty) \to R^n$ and a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ such that $w(0) = x$ and that for each $T > 0$,

$$v_{i_k} \to w \text{ as } k \to \infty \text{ uniformly on } [0, T].$$

Theorem 5 and (11)-(13) imply the following result, which establishes the convergence of approximate solutions of problem (P) in the regions containing the right end points of the time intervals $[0, T]$.

THEOREM 6 *Let $x, y \in R^n$, a sequence of positive numbers $T_i \to \infty$ as $i \to \infty$ and let $v_i : [0, T_i] \to R^n$, $i = 1, 2, \dots$ be an a. c. function such that*

$$v_i(0) = x,\ v_i(T_i) = y,\ I^f(0, T_i, v_i) - U^f(0, T_i, x, y) \to 0 \text{ as } i \to \infty.$$

Then there exist an $(\bar{f})$-perfect function $\bar{w} : [0, \infty) \to R^n$ and a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ such that $\bar{w}(0) = y$ and that for each $T > 0$,

$$v_{i_k}(T_{i_k} - t) \to \bar{w}(t) \text{ as } k \to \infty \text{ uniformly on } [0, T].$$

The following result establishes the convergence of approximate solutions of problem (P_1) in the regions which contain the right end points of the time intervals $[0, T]$. It will be proved in Section 5.

THEOREM 7 *Let $M > 0$, $x_i \in R^n$ satisfy $|x_i| \le M$ for all $i = 1, 2, \dots$, a sequence of positive numbers $T_i \to \infty$ as $i \to \infty$ and let $v_i : [0, T_i] \to R^n$, $i = 1, 2, \dots$ be an a. c. function such that*

$$v_i(0) = x_i \text{ and } I^f(0, T_i, v_i) - U^f(T, x_i) \to 0 \text{ as } i \to \infty. \tag{17}$$

Then there exist an $(\bar{f})$-perfect function $\bar{w} : [0, \infty) \to R^n$ and a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ such that for each $T > 0$,

$$v_{i_k}(T_{i_k} - t) \to \bar{w}(t) \text{ as } k \to \infty \text{ uniformly on } [0, T],$$
$$\pi^{\bar{f}}(\bar{w}(0)) = \inf\{\pi^{\bar{f}}(z) :\ z \in R^n\}.$$

The following theorem establishes the convergence of approximate solutions of problem (P_2) in the regions which contain zero. It will be proved in Section 6.

THEOREM 8 *Let a sequence of positive numbers $T_i \to \infty$ as $i \to \infty$ and let $v_i : [0, T_i] \to R^n$, $i = 1, 2, \dots$ be an a.c. function such that*

$$I^f(0, T_i, v_i) - U^f(T_i) \to 0 \text{ as } i \to \infty. \tag{18}$$

Then there exist an (f)-perfect function $w : [0, \infty) \to R^n$ and s strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ such that for each $T > 0$,

$$v_{i_k} \to w \text{ as } k \to \infty \text{ uniformly on } [0, T],$$
$$\pi^f(w(0)) = \inf\{\pi^f(z) :\ z \in R^n\}.$$

Theorem 8 and (11)-(13) imply the following result, which establishes the convergence of approximate solutions of problem (P_2) in the regions containing the right end points of the time intervals $[0, T]$.

THEOREM 9 *Let a sequence of positive numbers $T_i \to \infty$ as $i \to \infty$ and let $v_i : [0, T_i] \to R^n$, $i = 1, 2, \dots$ be an a.c. function such that $I^f(0, T_i, v_i) - U^f(T_i) \to 0$ as $i \to \infty$. Then there exist an $(\bar{f})$-perfect function $\bar{w} : [0, \infty) \to R^n$ and a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ such that for each $T > 0$,*

$$v_{i_k}(T_{i_k} - t) \to \bar{w}(t) \text{ as } k \to \infty \text{ uniformly on } [0, T],$$
$$\pi^{\bar{f}}(\bar{w}(0)) = \inf\{\pi^{\bar{f}}(z) : \; z \in R^n\}.$$

The following two theorems describe the limiting behavior of the value-functions $U^f(0, T, x, y)$, $U^f(T, x)$ and $U^f(T)$ as $T \to \infty$. They will be proved in Section 7.

THEOREM 10 *Let $K > 0$, $\epsilon \in (0, 1)$. Then there exists $T_0 > 0$ such that for each $T \geq T_0$ and each $x, y \in R^n$ satisfying $|x|, |y| \leq K$ the following inequality holds:*

$$|U^f(0, T, x, y) - (\pi^f(x) + \pi^{\bar{f}}(y) + T\mu(f) - \inf\{\pi^f(z) : \; z \in H(f)\})| \leq \epsilon. \quad (19)$$

THEOREM 11 *Let $K > 0$, $\epsilon > 0$. Then there exists $T_0 > 0$ such that the following assertions hold:*

(i) for each $T \geq T_0$ and each $x \in R^n$ satisfying $|x| \leq K$,

$$|U^f(T, x) - (\pi^f(x) + \inf\{\pi^{\bar{f}}(z) : z \in R^n\} + T\mu(f) - \inf\{\pi^f(z) : z \in H(f)\})| \leq \epsilon;$$

(ii) for each $T \geq T_0$,

$$|U^f(T) - (\inf\{\pi^f(z) : \; z \in R^n\} + \inf\{\pi^{\bar{f}}(z) : z \in R^n\}$$
$$+ T\mu(f) - \inf\{\pi^f(z) : z \in H(f)\})| \leq \epsilon.$$

3. Auxiliary results

In order to prove our main theorems we need the following auxiliary results.

PROPOSITION 4 *Let $g \in \mathcal{M}$ and M_1, M_2, c be positive numbers. Then there exist a number $S > 0$ such that for each each $T_1 \in R^1$ and each $T_2 \in [T_1 + c, \infty)$ the following properties hold:*

(i) if $x, y \in R^n$ satisfy $|x|, |y| \leq M_1$ and if an a.c. function $v : [T_1, T_2] \to R^n$ satisfies

$$v(T_1) = x, \; v(T_2) = y, \; I^g(T_1, T_2, v) \leq U^g(T_1, T_2, x, y) + M_2, \quad (20)$$

then

$$|v(t)| \leq S, \; t \in [T_1, T_2]; \quad (21)$$

(ii) if $x \in R^n$ satisfies $|x| \le M_1$ and if an a.c. function $v : [T_1, T_2] \to R^n$ satisfies $v(T_1) = x$, $I^g(T_1, T_2, v) \le U^g(T_2 - T_1, x) + M_2$, then the inequality (21) is valid;

(iii) if an a.c. function $v : [T_1, T_2] \to R^n$ satisfies $I^g(T_1, T_2, v) \le U^g(T_2 - T_1) + M_2$, then the inequality (21) is valid.

The properties (i) and (ii) were established in Zaslavski (1998, Theorem 1.3). The property (iii) is proved analogously to the properties (i) and (ii).

PROPOSITION 5 *(Zaslavski, 1998, Proposition 2.5). Assume that $f \in \mathcal{A}$, $M_1 > 0$, $-\infty < T_1 < T_2 < \infty$, $x_i : [T_1, T_2] \to R^n$, $i = 1, 2, \dots$ is a sequence of a.c. functions such that $I^f(T_1, T_2, x_i) \le M_1$, $i = 1, 2, \dots$. Then there exist a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ and an a.c. function $x : [T_1, T_2] \to R^n$ such that $I^f(T_1, T_2, x) \le \liminf_{i\to\infty} I^f(T_1, T_2, x_i)$,*

$$x_{i_k} \to x(t) \text{ as } k \to \infty \text{ uniformly in } [T_1, T_2] \text{ and}$$
$$x'_{i_k} \to x' \text{ as } k \to \infty \text{ weakly in } L^1(R^n; (T_1, T_2)).$$

LEMMA 1 *(Zaslavski, 2006a, Lemma 4.2). Let $g \in \mathcal{M}$ possess (ATP), $\epsilon \in (0, 1)$. Then there exist numbers $q, \delta > 0$ such that for each $h_1, h_2 \in R^n$ satisfying $d(h_i, H(g)) \le \delta$, $i = 1, 2$ and each $T \ge q$ there exists an a.c. function $v : [0, T] \to R^n$, which satisfies $v(0) = h_1$, $v(T) = h_2$, $\Gamma^g(0, T, v) \le \epsilon$.*

PROPOSITION 6 *(Zaslavski, 1998, Theorem 1.2). Let $g \in \mathcal{A}$. Then there exists a a number $M > 0$ such that for each (g)-good function $x : [0, \infty) \to R^n$, $\limsup_{t\to\infty} |x(t)| < M$.*

PROPOSITION 7 *(Zaslavski, 1996, Proposition 2.5). Let $f \in \mathcal{A}$, $S_0, S_1 > 0$ and let $x : [0, \infty) \to R^n$ be an a.c. function such that $|x(t)| \le S_0$ for all $t \in [0, \infty)$ and $I^f(0, i, x) \le U^f(0, i, x(0), x(i)) + S_1$, $i = 1, 2 \dots$. Then x is an (f)-good funtion.*

PROPOSITION 8 *Let $g \in \mathcal{M}$ possess (ATP) and $v : [0, \infty) \to R^n$ be an a.c. function such that*

$$\sup\{|v(t)| : t \in [0, \infty)\} < \infty, \tag{22}$$
$$I^g(0, T, v) = U^g(0, T, v(0), v(T)) \text{ for all } T > 0. \tag{23}$$

Then the function v is (g)-perfect.

Proof. By Proposition 3 there exists a (g)-perfect function $u : [0, \infty) \to R^n$ such that

$$u(0) = v(0). \tag{24}$$

In view of (22), (23) and Proposition 7, v is (g)-good function.

Assume that v is not (g)-perfect. Then there is $T_0 > 0$ such that

$$\Delta_0 := \Gamma^g(0, T_0, v) > 0. \tag{25}$$

By Lemma 1 there exist $q, \delta > 0$ such that the following property holds:

(C1) for each $h_1, h_2 \in R^n$ satisfying $d(h_i, H(g)) \le \delta$, $i = 1, 2$ and each $T \ge q$ there exists an a.c. function $\xi : [0, T] \to R^n$, which satisfies $\xi(0) = h_1$, $\xi(T) = h_2$, $\Gamma^g(0, T, \xi) \le \Delta_0/4$.

Since the integrand g possesses (ATP), there exists

$$T_1 > q + T_0 \tag{26}$$

such that for all $t \ge T_1$,

$$d(v(t), H(g)) \le \delta,\ d(u(t), H(g)) \le \delta. \tag{27}$$

By (25), (9), (10) and (26),

$$\Gamma^g(0, T_1, v) \ge \Gamma^g(0, T_0, v) = \Delta_0. \tag{28}$$

By (C1) and (27), there exists an a. c. function $\xi_1 : [T_1, T_1 + q] \to R^n$ such that

$$\xi_1(T_1) = u(T_1),\ \xi_1(T_1 + q) = v(T_1 + q),\ \Gamma^g(T_1, T_1 + q, \xi_1) \le \Delta_0/4. \tag{29}$$

Put

$$\begin{aligned} &w(t) = u(t),\ t \in [0, T_1],\ w(t) = \xi_1(t),\ t \in (T_1, T_1 + q], \\ &w(t) = v(t),\ t \in (T_1 + q, \infty). \end{aligned} \tag{30}$$

In view of (29), (30) and (24), w is a well-defined a. c. function. By (30) and (29),

$$w(0) = u(0) = v(0),\ w(T_1 + q) = \xi_1(T_1 + q) = v(T_1 + q). \tag{31}$$

Since u is (g)-perfect, it follows from (9), (31), (29), (30) and (30) that

$$\begin{aligned} &I^g(0, T_1 + q, v) - I^g(0, T_1 + q, w) = \Gamma^g(0, T_1 + q, v) - \Gamma^g(0, T_1 + q, w) \\ &= \Gamma^g(0, T_1, v) + \Gamma^g(T_1, T_1 + q, v) - \Gamma^g(0, T_1, w) - \Gamma^g(T_1, T_1 + q, w) \\ &\ge \Delta_0 - \Gamma^g(0, T_1, u) - \Gamma^g(T_1, T_1 + q, \xi_1) \ge \Delta_0 - \Delta_0/4. \end{aligned}$$

Combined with (31), this contradicts (23). The contradiction we have reached proves that the function v is (g)-perfect. Proposition 8 is proved. ∎

LEMMA 2 *(Zaslavski, 2002a, Lemma 3.3). Let $g \in \mathcal{M}$ possess (ATP). Then for each $z \in H(g)$ there exists an a.c. function $v : R^1 \to H(g)$ such that $v(0) = z$ and $\Gamma^g(-T, T, v) = 0$ for all $T > 0$.*

LEMMA 3 *Let $g \in \mathcal{M}$ possess (ATP). Then,* $\sup\{\pi^g(z) : z \in H(g)\} = 0$.

Proof. Let $\widehat{z} \in H(g)$ satisfy

$$\pi^g(\widehat{z}) = \sup\{\pi^g(z) : z \in H(g)\}. \tag{32}$$

By Lemma 2 there exists $\widehat{v} : R^1 \to H(g)$ such that

$$\widehat{v}(0) = \widehat{z},\ \Gamma^g(-T, T, \widehat{v}) = 0 \text{ for all } T > 0. \tag{33}$$

It follows from (33), (6), (9), (ATP) and (32) that

$$\begin{aligned}&\pi^g(\widehat{z}) \le \liminf_{T\to\infty}[I^g(0,T,\widehat{v}) - T\mu(g)]\\ &= \liminf_{T\to\infty}[\pi^g(\widehat{z}) - \pi^g(\widehat{v}(T))] = \pi^g(\widehat{z}) - \sup\{\pi^g(z) : z \in H(g)\} = 0.\end{aligned} \tag{34}$$

On the other hand, by (6), Proposition 2, (9), (10), (ATP) and (32),

$$\begin{aligned}&\pi^g(\widehat{z}) = \inf\{\liminf_{T\to\infty}[I^g(0,T,v) - T\mu(g)] :\\ &v : [0,\infty) \to R^n \text{ is a } (g) - \text{good function such that } v(0) = \widehat{z}\}\\ &\qquad \ge \inf\{\liminf_{T\to\infty}[\pi^g(\widehat{z}) - \pi^g(v(t))] :\\ &v : [0,\infty) \to R^n \text{ is a } (g) - \text{good function such that } v(0) = \widehat{z}\}\\ &\qquad \ge \pi^g(\widehat{z}) - \sup\{\pi^g(z) : z \in H(g)\} = 0.\end{aligned}$$

Together with (34), this implies that $\pi^g(\widehat{z}) = 0$. Lemma 3 is proved. ∎

PROPOSITION 9 *Let $g \in \mathcal{M}$ possess (ATP), $\bar{g}(x,y) = g(x,-y),\ x, y \in R^n$,*

$$\begin{aligned}&z_1, z_2 \in H(g),\ \pi^g(z_1) = \inf\{\pi^g(z) : z \in H(g)\},\\ &\qquad\qquad \pi^g(z_2) = \sup\{\pi^g(z) : z \in H(g)\}.\end{aligned} \tag{35}$$

Then $\mu(\bar{g}) = \mu(g)$ and

$$\pi^{\bar{g}}(z_1) = \sup\{\pi^{\bar{g}}(z) : z \in H(g)\},\ \pi^{\bar{g}}(z_2) = \inf\{\pi^{\bar{g}}(z) : z \in H(g)\}. \tag{36}$$

Proof. By Lemma 2 there exists an a.c. function $v : R^1 \to H(g)$ such that

$$v(0) = \widehat{z}_1,\ \Gamma^g(-T, T, v) = 0 \text{ for all } T > 0. \tag{37}$$

Put

$$\tilde{v}(t) = v(-t),\ t \in R^1. \tag{38}$$

It is easy to see that for all pairs $S < T$,

$$\int_S^T \bar{g}(\tilde{v}(t), \tilde{v}'(t))dt = \int_{-T}^{-S} g(v(t), v'(t))dt \tag{39}$$

and that in view of (37), $I^{\bar{g}}(-T,T,\tilde{v}) = U^{\bar{g}}(-T,T,\tilde{v}(-T),\tilde{v}(T)),\ T>0$. Since $\bar{g}$ possesses (ATP), it follows from the inequality above and Proposition 3.8 that

$$\Gamma^{\bar{g}}(-T,T,\tilde{v}) = 0 \text{ for all } T>0. \tag{40}$$

In view of (39)) and (37),

$$\mu(\bar{g}) = \lim_{k\to\infty}(2k)^{-1}I^{\bar{g}}(-k,k,\tilde{v}) = \lim_{k\to\infty}(2k)^{-1}I^{g}(-k,k,v) = \mu(g). \tag{41}$$

By (40), (9), (10), (41), (39) and (37),

$$\begin{aligned}&\sup\{\pi^{\bar{g}}(\tilde{v}(S)) - \pi^{\bar{g}}(\tilde{v}(T)) :\ -\infty<S<T<\infty\}\\ &= \sup\{I^{\bar{g}}(S,T,\tilde{v}) - (T-S)\mu(g) :\ -\infty<S<T<\infty\}\\ &= \sup\{I^{g}(S,T,v) - (T-S)\mu(g) :\ -\infty<S<T<\infty\}\\ &\sup\{\pi^{g}(v(S)) - \pi^{g}(v(T)) :\ -\infty<S<T<\infty\}.\end{aligned} \tag{42}$$

Since the integrands g and $\bar{g}$ possess (STP) with the turnpike $H(g)$, it follows from (40), (37), (42) and Lemma 3 that

$$0 = \sup\{\pi^{g}(z) :\ z\in H(g)\} = \sup\{\pi^{\bar{g}}(z) :\ z\in H(g)\}, \tag{43}$$

$$\inf\{\pi^{\bar{g}}(z) :\ z\in H(g)\} = \inf\{\pi^{g}(z) :\ z\in H(g)\} \tag{44}$$

and that there exist negative numbers $S_i\to-\infty$ as $i\to\infty$ and positive numbers $T_i\to\infty$ as $i\to\infty$ such that

$$\lim_{i\to\infty} v(S_i) = z_1,\ \lim_{i\to\infty} v(T_i) = z_2. \tag{45}$$

By (35), Lemma 3, (45), (37), (39), (41), (40) and (38),

$$\begin{aligned}&\pi^{g}(z_1) = \pi^{g}(z_1) - \pi^{g}(z_2) = \lim_{i\to\infty}[\pi^{g}(v(S_i)) - \pi^{g}(v(T_i))] \qquad (46)\\ &= \lim_{i\to\infty}[I^{g}(S_i,T_i,v) - (T_i-S_i)\mu(g)] = \lim_{i\to\infty}[I^{\bar{g}}(-T_i,-S_i,\tilde{v}) - (T_i-S_i)\mu(g)]\\ &= \lim_{i\to\infty}[\pi^{\bar{g}}(\tilde{v}(-T_i)) - \pi^{\bar{g}}(\tilde{v}(-S_i))] - \lim_{i\to\infty}[\pi^{\bar{g}}(v(T_i)) - \pi^{\bar{g}}(v(S_i))]\\ &= \pi^{\bar{g}}(z_2) - \pi^{\bar{g}}(z_1).\end{aligned}$$

By (46), (15), Lemma 3, (43) and (44),

$$\begin{aligned}&\pi^{\bar{g}}(z_2) - \pi^{\bar{g}}(z_1) = \inf\{\pi^{g}(z) :\ z\in H(g)\} - \sup\{\pi^{g}(z) :\ z\in H(g)\}\\ &= \inf\{\pi^{\bar{g}}(z) :\ z\in H(g)\} - \sup\{\pi^{\bar{g}}(z) :\ z\in H(g)\}.\end{aligned}$$

Hence

$$\pi^{\bar{g}}(z_2) = \inf\{\pi^{\bar{g}}(z) :\ z\in H(g)\},\ \pi^{\bar{g}}(z_1) = \sup\{\pi^{\bar{g}}(z) :\ z\in H(g)\}.$$

Proposition 9 is proved. ■

PROPOSITION 10 *(Zaslavski, 1998, Corollary 2.1). For each $f\in\mathcal{M}$, each pair of numbers T_1, T_2 satisfying $T_1<T_2$ and each $z_1, z_2\in R^n$ there exists an a.c. function $x : [T_1,T_2]\to R^n$ such that $x(T_i) = z_i,\ i=1,2,\ I^f(T_1,T_2,x) = U^f(T_1,T_2,z_1,z_2)$.*

4. Proof of Theorem 5

By Proposition 4 and (16) there is $M_0 > 0$ such that

$$|v_i(t)| \leq M_0 \text{ for all } t \in [0, T_i] \text{ and all } i = 1, 2, \ldots \tag{47}$$

Since the function $U^f(0, T, \cdot, \cdot)$ is continuous for any $T > 0$ (see Zaslavski, 2006b, pages 83-87) it follows from (47) and (16) that for any $T > 0$ the sequence $\{I^f(0, T, v_i)\}_{i=1}^{\infty}$ is bounded. Now, it follows from Proposition 5 that there exist a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ and an a.c. function $v : [0, \infty) \to R^n$ such that for each integer $m \geq 1$,

$$v_{i_k} \to v \text{ as } k \to \infty \text{ uniformly on } [0, m], \tag{48}$$

$$I^f(0, m, v) \leq \liminf_{k \to \infty} I^f(0, m, v_{i_k}). \tag{49}$$

Relations (48) and (16) imply that $v(0) = x$. In view of (49), (16) and (48) for each integer $m \geq 1$,

$$I^f(0, m, v) \leq \liminf_{k \to \infty} U^f(0, m, v_{i_k}(0), v_{i_k}(m)) = U^f(0, m, v(0), v(m)).$$

Thus, for each integer $m \geq 1$,

$$I^f(0, m, v) = U^f(0, m, v(0), v(m)). \tag{50}$$

By (47) and (48) $|v(t)| \leq M_0$ for all $t \geq 0$. Together with (50) and Proposition 8 this implies that v is (f)-perfect. Theorem 5 is proved.

5. Proof of Theorem 7

By (17) and Proposition 4 there is $M_0 > 0$ such that

$$|v_i(t)| \leq M_0 \text{ for all } t \in [0, T_i] \text{ and all } i = 1, 2, \ldots. \tag{51}$$

For $i = 1, 2, \ldots$ set

$$\bar{v}_i(t) = v(T_i - t), \ t \in [0, T_i]. \tag{52}$$

In view of (52), (17), (12) and (13),

$$I^{\bar{f}}(0, T_i, \bar{v}_i) - U^{\bar{f}}(0, T_i, \bar{v}_i(0), \bar{v}_i(T_i)) \to 0 \text{ as } i \to \infty. \tag{53}$$

It follows from (53), (51) and Proposition 5 that there exists a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ and an a. c. function $u : [0, \infty) \to R^n$ such that for each integer $m \geq 1$,

$$\bar{v}_{i_k} \to u \text{ as } k \to \infty \text{ uniformly on } [0, m], \tag{54}$$

$$I^{\bar{f}}(0, m, u) \leq \liminf_{k \to \infty} I^{\bar{f}}(0, m, \bar{v}_{i_k}). \tag{55}$$

By (55), (53) and (54) for each natural number m,

$$I^{\bar{f}}(0,m,u) \le \liminf_{k\to\infty} U^{\bar{f}}(0,m,\bar{v}_{i_k}(0),\bar{v}_{i_k}(m)) = U^{\bar{f}}(0,m,u(0),u(m)). \tag{56}$$

Relations (54), (51) and (52) imply that

$$|u(t)| \le M_0,\ t \in [0,\infty). \tag{57}$$

By (56), (57) and Proposition 8, u is $(\bar{f})$-perfect.

In order to complete the proof it is sufficient to show that

$$\pi^{\bar{f}}(u(0)) = \inf\{\pi^{\bar{f}}(z):\ z \in R^n\}.$$

Assume the contrary. Then, by Proposition 1, there is $\widehat{z} \in R^n$ such that

$$\pi^{\bar{f}}(\widehat{z}) = \inf\{\pi^{\bar{f}}(z):\ z \in R^n\} < \pi^{\bar{f}}(u(0)). \tag{58}$$

In view of Proposition 3 there is an $(\bar{f})$-perfect function $\widehat{u} : [0,\infty) \to R^n$ such that

$$\widehat{u}(0) = \widehat{z}. \tag{59}$$

Choose a positive number

$$\Delta < (\pi^{\bar{f}}(u(0)) - \pi^{\bar{f}}(\widehat{z}))/4. \tag{60}$$

By Lemma 1 there exist $q, \delta > 0$ such that the following property holds:

(C2) for each $h_1, h_2 \in R^n$ satisfying $d(h_i, H(f)) \le \delta$, $i = 1,2$ and each $T \ge q$ there exists an a. c. function $\xi : [0,T] \to R^n$ such that $\xi(0) = h_1$, $\xi(T) = h_2$, $\Gamma^{\bar{f}}(0,T,\xi) \le \Delta/4$.

Since the functions $u, \widehat{u}$ are $(\bar{f})$-perfect and $\bar{f}$ possesses (ATP), there exists $L_0 > q$ such that

$$d(\widehat{u}(t), H(f)),\ d(u(t), H(f)) \le \delta/8 \text{ for all } t \ge L_0. \tag{61}$$

By (17), (54), the continuity of $\pi^{\bar{f}}$ and (55), there exists a natural number k such that

$$T_{i_k} > 4(L_0+q+1),\ I^f(0,T_{i_k},v_{i_k}) \le U^f(T_{i_k},x_{i_k}) + \Delta/8, \tag{62}$$

$$|\bar{v}_{i_k}(t) - u(t)| \le \delta/8,\ t \in [0, 4(L_0+q+1)], \tag{63}$$

$$|\pi^{\bar{f}}(u(L_0+q)) - \pi^{\bar{f}}(\bar{v}_{i_k}(L_0+q))| \le \Delta/8, \tag{64}$$

$$I^{\bar{f}}(0,L_0+q,\bar{v}_{i_k}) + \Delta/8 > I^{\bar{f}}(0,L_0+q,u). \tag{65}$$

Relations (61)-(63) imply that

$$d(\bar{v}_{i_k}(L_0+q), H(f)) \le \delta/4. \tag{66}$$

By (C2), (61) and (66) there is an a. c. function $\xi : [L_0, L_0+q] \to R^n$ such that

$$\xi(L_0) = \widehat{u}(L_0),\ \xi(L_0+q) = \bar{v}_{i_k}(L_0+q),\ \Gamma^{\bar{f}}(L_0, L_0+q, \xi) \le \Delta/4. \tag{67}$$

Set

$$\begin{aligned} &w(t) = v_{i_k}(t),\ t \in [0, T_{i_k} - L_0 - q],\\ &w(t) = \xi(T_{i_k} - t),\ t \in (T_{i_k} - L_0 - q, T_{i_k} - L_0),\\ &w(t) = \widehat{u}(T_{i_k} - t),\ t \in [T_{i_k} - L_0, T_{i_k}]. \end{aligned} \tag{68}$$

By (68), (67), (52) and (17), the a. c. function $w : [0, T_{i_k}] \to R^n$ is well-defined,

$$w(0) = v_{i_k}(0) = x_{i_k}. \tag{69}$$

In view of (69) and (62),

$$I^f(0, T_{i_k}, w) - I^f(0, T_{i_k}, v_{i_k}) \ge U^f(T_{i_k}, x_{i_k}) - I^f(0, T_{i_k}, v_{i_k}) \ge -\Delta/8. \tag{70}$$

Since the functions u, $\widehat{u}$ are $(\bar{f})$-perfect, it follows from (67), (68), (65), (64), (59), (60) and (9) that

$$\begin{aligned} &I^f(0, T_{i_k}, w) - I^f(0, T_{i_k}, v_{i_k}) = I^f(T_{i_k} - L_0 - q, T_{i_k}, w) - I^f(T_{i_k} - L_0 - q, T_{i_k}, v_{i_k})\\ &= I^f(T_{i_k} - L_0 - q, T_{i_k} - L_0, w) + I^f(T_{i_k} - L_0, T_{i_k}, w) - I^f(T_{i_k} - L_0 - q, T_{i_k}, v_{i_k})\\ &= I^{\bar{f}}(L_0, L_0+q, \xi) + I^{\bar{f}}(0, L_0, \widehat{u}) - I^{\bar{f}}(0, L_0+q, \bar{v}_{i_k})\\ &\qquad \le I^{\bar{f}}(L_0, L_0+q, \xi) + I^{\bar{f}}(0, L_0, \widehat{u}) - I^{\bar{f}}(0, L_0+q, u) + \Delta/8\\ &= \Gamma^{\bar{f}}(L_0, L_0+q, \xi) + \pi^{\bar{f}}(\xi(L_0)) - \pi^{\bar{f}}(\xi(L_0+q)) + q\mu(f)\\ &\qquad + \Gamma^{\bar{f}}(0, L_0, \widehat{u}) + L_0\mu(f) + \pi^{\bar{f}}(\widehat{u}(0)) - \pi^{\bar{f}}(\widehat{u}(L_0))\\ &\qquad - (\Gamma^{\bar{f}}(0, L_0+q, u) + (L_0+q)\mu(f) + \pi^{\bar{f}}(u(0)) - \pi^{\bar{f}}(u(L_0+q))) + \Delta/8\\ &\qquad \le \Delta/4 + \pi^{\bar{f}}(\xi(L_0)) - \pi^{\bar{f}}(\xi(L_0+q)) + \pi^{\bar{f}}(\widehat{u}(0)) - \pi^{\bar{f}}(\widehat{u}(L_0))\\ &\qquad - \pi^{\bar{f}}(u(0)) + \pi^{\bar{f}}(u(L_0+q)) + \Delta/8\\ &= 8^{-1}(3\Delta) + \pi^{\bar{f}}(\widehat{u}(0) - \pi^{\bar{f}}(u(0)) + \pi^{\bar{f}}(u(L_0+q)) - \pi^{\bar{f}}(v_{i_k}(L_0+q))\\ &\qquad \le \pi^{\bar{f}}(\widehat{u}(0)) - \pi^{\bar{f}}(u(0)) + \Delta/2 \le -3\Delta. \end{aligned}$$

This contradicts (70). The contradiction we have reached proves Theorem 7.

6. Proof of Theorem 8

By (18) and Proposition 4 there is $M_0 > 0$ such that

$$|v_i(t)| \le M_0 \text{ for all } t \in [0, T_i] \text{ and all } i = 1, 2, \dots. \tag{71}$$

It follows from (71), (18), Proposition 5 and the continuity of U^f that there exist a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ and an a.c. function $u : [0, \infty) \to R^n$ such that for each integer $m \geq 1$,

$$v_{i_k} \to u \text{ as } k \to \infty \text{ uniformly on } [0, m], \tag{72}$$

$$I^f(0, m, u) \leq \liminf_{k \to \infty} I^f(0, m, v_{i_k}). \tag{73}$$

By (73), (18), (72) and the continuity of U^f, for each natural number m,

$$I^f(0, m, u) \leq \liminf_{k \to \infty} U^f(0, m, v_{i_k}(0), v_{i_k}(m)) = U^f(0, m, u(0), u(m)). \tag{74}$$

Relations (72) and (71) imply that

$$|u(t)| \leq M_0, \ t \in [0, \infty). \tag{75}$$

By (75), (74) and Proposition 8, u is (f)-perfect.

In order to complete the proof it is sufficient to show that

$$\pi^f(u(0)) = \inf\{\pi^f(z) : \ z \in R^n\}.$$

Assume the contrary. Then by Proposition 1 there is $\widehat{z} \in R^n$ such that

$$\pi^f(\widehat{z}) = \inf\{\pi^f(z) : \ z \in R^n\} < \pi^f(u(0)). \tag{76}$$

In view of Proposition 3 there is an (f)-perfect function $\widehat{u} : [0, \infty) \to R^n$ such that

$$\widehat{u}(0) = \widehat{z}. \tag{77}$$

Choose a positive number

$$\Delta < (\pi^{\bar{f}}(u(0)) - \pi^f(\widehat{z}))/4. \tag{78}$$

By Lemma 1 there exist $q, \delta > 0$ such that the following property holds:

(C3) for each $h_1, h_2 \in R^n$ satisfying $d(h_i, H(f)) \leq \delta$, $i = 1, 2$ and each $T \geq q$ there exists an a.c. function $\xi : [0, T] \to R^n$ such that $\xi(0) = h_1$, $\xi(T) = h_2$, $\Gamma^f(0, T, \xi) \leq \Delta/4$.

Since the functions $u, \widehat{u}$ are (f)-perfect and f possesses (ATP), there exists $L_0 > q$ such that

$$d(\widehat{u}(t), H(f)), \ d(u(t), H(f)) \leq \delta/8 \text{ for all } t \geq L_0. \tag{79}$$

By (18), (72), (73) and the continuity of $\pi^{\bar{f}}$ there exists a natural number k such that

$$T_{i_k} > 4(L_0 + q + 1), \ I^f(0, T_{i_k}, v_{i_k}) \leq U^f(T_{i_k}) + \Delta/8, \tag{80}$$

$$|v_{i_k}(t) - u(t)| \leq \delta/8, \ t \in [0, 4(L_0 + q + 1)], \tag{81}$$

$$|\pi^f(u(L_0 + q)) - \pi^f(v_{i_k}(L_0 + q))| \leq \Delta/8, \tag{82}$$

$$I^f(0, L_0 + q, v_{i_k}) + \Delta/8 > I^f(0, L_0 + q, u). \tag{83}$$

By (81) and (79),

$$d(v_{i_k}(L_0+q),H(f))\le \delta/4. \tag{84}$$

By (C3), (84) and (79) there exists an a. c. function $\xi:[L_0,L_0+q]\to R^n$ such that

$$\xi(L_0)=\widehat{u}(L_0),\ \xi(L_0+q)=v_{i_k}(L_0+q),\ \Gamma^f(L_0,L_0+q,\xi)\le \Delta/4. \tag{85}$$

Set

$$\begin{aligned}&w(t)=\widehat{u}(t),\ t\in[0,L_0],\ w(t)=\xi(t),\ t\in(L_0,L_0+q],\\&w(t)=v_{i_k}(t),\ t\in(L_0+q,T_{i_k}].\end{aligned} \tag{86}$$

By (85) and (86) the a. c. function $w:[0,T_{i_k}]\to R^n$ is well-defined. By (80),

$$I^f(0,T_{i_k},w)-I^f(0,T_{i_k},v_{i_k})\ge U^f(T_{i_k})-I^f(0,T_{i_k},v_{i_k})\ge -\Delta/8. \tag{87}$$

Since the functions u and $\widehat{u}$ are (f)-perfect, it follows from (86), (83), (85), (77) and (9) that

$$\begin{aligned}&I^f(0,T_{i_k},w)-I^f(0,T_{i_k},v_{i_k})=I^f(0,L_0+q,w)-I^f(0,L_0+q,v_{i_k})\\&=I^f(0,L_0,\widehat{u})+I^f(L_0,L_0+q,\xi)-I^f(0,L_0+q,v_{i_k})\\&\quad\le I^f(0,L_0,\widehat{u})+I^f(L_0,L_0+q,\xi)-I^f(0,L_0+q,u)+\Delta/8\\&=\Gamma^f(0,L_0,\widehat{u})+L_0\mu(f)+\pi^f(\widehat{u}(0))-\pi^f(\widehat{u}(L_0))\\&\quad+\Gamma^f(L_0,L_0+q,\xi)+q\mu(f)+\pi^f(\xi(L_0))-\pi^f(\xi(L_0+q))\\&\quad-[\Gamma^f(0,L_0+q,u)+(L_0+q)\mu(f)+\pi^f(u(0))-\pi^f(u(L_0+q))]+\Delta/8\\&\quad\le \Delta/4+\pi^f(\widehat{u}(0))-\pi^f(\xi(L_0+q))+\pi^f(u(0))-\pi^f(u(L_0+q))+\Delta/8\\&\quad\le 3\Delta/8-4\Delta+\Delta/8\le -3\Delta.\end{aligned}$$

This contradicts (87). The contradiction we have reached proves Theorem 8.

7. Proofs of Theorems 10 and 11

Proof of Theorem 10 We may assume that

$$K>\sup\{|z|:\ z\in H(f)\}+1. \tag{88}$$

By Proposition 4 there exists $S_0>K$ such that the following property holds:

(C4) for each $T\ge 1$ and each a. c. function $v:[0,T]\to R^n$ satisfying

$$|v(0)|,\ |v(T)|\le K, \tag{89}$$

$$I^f(0,T,v)\le U^f(0,T,v(0),v(T))+1 \tag{90}$$

the following inequality holds:

$$|v(t)| \le S_0,\ t \in [0,T]. \tag{91}$$

By Proposition 3 and Theorem 4 for each $x \in R^n$ there exist an (f)-perfect function $v_x : [0,\infty) \to R^n$ and an $(\bar{f})$-perfect function $u_x : [0,\infty) \to R^n$ such that

$$v_x(0) = u_x(0) = x. \tag{92}$$

By (C4), (88) and (ATP) for the integrands f and $\bar{f}$,

$$|u_x(t)|,\ |v_x(t)| \le S_0 \text{ for all } t \in [0,\infty) \text{ and all } x \in R^n \text{ satisfying } |x| \le K. \tag{93}$$

Since the functions π^f, $\pi^{\bar{f}}$ are continuous, there is $\delta_0 \in (0, 4^{-1}\epsilon)$ such that for each $x, y \in R^n$ satisfying

$$|x|, |y| \le 4 \sup\{|z| :\ z \in H(f)\},\ |x-y| \le 8\delta_0 \tag{94}$$

the following inequalities hold:

$$|\pi^f(x) - \pi^f(y)|,\ |\pi^{\bar{f}}(x) - \pi^{\bar{f}}(y)| \le 16^{-1}\epsilon. \tag{95}$$

By Lemma 1 there are $\delta \in (0, \delta_0)$, $T_1 > 0$ such that the following property holds:

(C5) for each $x, y \in R^n$ satisfying $d(x, H(f)),\ d(y, H(f)) \le \delta$ and each $T \ge T_1$ there is an a. c. function $\xi : [0,T] \to R^n$, which satisfies

$$\xi(0) = x,\ \xi(T) = y,\ \Gamma^f(0,T,\xi) \le \epsilon/8.$$

By Theorem 2 f has (STP) with the turnpike $H(f)$ and therefore there exist $T_2 > L > 0$ such that the following property holds:

(C6) for each $T > 2T_2$, each a. c. function $v : [0,T] \to R^n$, which satisfies

$$|v(0)|,\ |v(T)| \le S_0,\ I^f(0,T,v) = U^f(0,T,v(0),v(T))$$

and each $\tau \in [T_2, T - T_2]$,

$$\mathrm{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + L]\}) \le \delta/2.$$

Choose

$$T_0 > 8(T_1 + T_2 + 1). \tag{96}$$

Let

$$z_0, z_1 \in H(f),\ \pi^f(z_0) = \sup\{\pi^f(z) :\ z \in H(f)\}, \tag{97}$$
$$\pi^f(z_1) = \inf\{\pi^f(z) :\ z \in H(f)\}.$$

By Lemma 3, Propositions 9 and (97),

$$\pi^f(z_0) = 0,\ \pi^{\bar f}(z_0) = \inf\{\pi^{\bar f}(z) : \in H(f)\},$$
$$\pi^{\bar f}(z_1) = \sup\{\pi^{\bar f}(z) :\ z \in H(f)\} = 0. \tag{98}$$

Assume that

$$T \geq T_0,\ x, y \in R^n,\ |x|, |y| \leq K. \tag{99}$$

We show that (19) holds. By Proposition 8 there exists an a. c. function $v : [0, T] \to R^n$ such that

$$v(0) = x,\ v(T) = y,\ I^f(0, T, v) = U^f(0, T, x, y). \tag{100}$$

By (99), (100), (96) and (C6) there exist

$$t_0 \in [T_2, T_2 + L],\ t_1 \in [T - T_2 - L, T - T_2] \tag{101}$$

such that

$$|v(t_0) - z_0|,\ |v(t_1) - z_1| \leq \delta/2. \tag{102}$$

We estimate $I^f(0, t_0, v)$, $I^f(t_0, t_1, v)$ and $I^f(t_1, T, v)$. Relations (9), (10) and (100) imply that

$$I^f(0, t_0, v) \geq t_0\mu(f) + \pi^f(x) - \pi^f(v(t_0)). \tag{103}$$

By the definition of δ_0 (see (94), (95)), (102), (97) and (98),

$$|\pi^f(v(t_0)) - \pi^f(z_0)| \leq \epsilon/16,\ |\pi^f(v(t_0))| \leq \epsilon/16, \tag{104}$$
$$|\pi^f(v(t_1)) - \pi^f(z_1)| \leq \epsilon/16,\ |\pi^{\bar f}(v(t_1)) - \pi^{\bar f}(z_1)| \leq \epsilon/16.$$

It follows from (103) and (104) that

$$I^f(0, t_0, v) \geq t_0\mu(f) + \pi^f(x) - \epsilon/16. \tag{105}$$

By (9), (10) and (104),

$$I^f(t_0, t_1, v) \geq (t_1 - t_0)\mu(f) + \pi^f(v(t_0)) - \pi^f(v(t_1)). \tag{106}$$
$$\geq (t_1 - t_0)\mu(f) - \pi^f(z_1) - 8^{-1}\epsilon.$$

Finally we estimate $I^f(t_1, T, v)$. Consider the function $\widehat{v} : [t_1, T] \to R^n$ defined by

$$\widehat{v}(t) = v(T - t + t_1),\ t \in [t_1, T]. \tag{107}$$

In view of (107), (9), (10), (100), (96) and (104),

$$I^f(t_1,T,v) = I^{\bar f}(t_1,T,\widehat{v}) \geq (T-t_1)\mu(f) + \pi^{\bar f}(\widehat{v}(t_1)) - \pi^{\bar f}(\widehat{v}(T)) \tag{108}$$
$$= (T-t_1)\mu(f) + \pi^{\bar f}(v(T)) - \pi^{\bar f}(v(t_1)) \geq (T-t_1)\mu(f) + \pi^{\bar f}(y) - \pi^{\bar f}(z_1) - \epsilon/16$$
$$= (T-t_1)\mu(f) + \pi^{\bar f}(y) - \epsilon/16.$$

By (100), (105), (106), (108) and (97),

$$U^f(0,T,x,y) = I^f(0,T,v) \geq t_0\mu(f) + \pi^f(x) - \epsilon/16 + (t_1-t_0)\mu(f)$$
$$-\pi^f(z_1) - 8^{-1}\epsilon + (T-t_1)\mu(f) + \pi^{\bar f}(y) - \epsilon/16$$
$$T\mu(f) + \pi^f(x) + \pi^{\bar f}(y) - \inf\{\pi^f(z) : \ z \in H(f)\} - \epsilon/4. \tag{109}$$

Consider an (f)-perfect function $v_x : [0,\infty) \to R^n$ and an $(\bar f)$-perfect function $u_y : [0,\infty) \to R^n$ such that

$$v_x(0) = x, \ u_y(0) = y. \tag{110}$$

Define $\widehat{u} : (-\infty, T] \to R^n$ by

$$\widehat{u}(t) = u_y(T-t), \ t \in (-\infty, T]. \tag{111}$$

It is not difficult to see that for each pair of numbers S_1, S_2 such that $S_1 < S_2 \leq T$,

$$U^{\bar f}(S_1, S_2, \widehat{u}(S_1), \widehat{u}(S_2)) = I^{\bar f}(S_1, S_2, \widehat{u}). \tag{112}$$

It follows from (111) and (110) that

$$\widehat{u}(T) = y. \tag{113}$$

By (93), (110), (99) and (111),

$$|\widehat{u}(t)| \leq S_0, \ t \in (-\infty, T], \ |v_x(t)| \leq S_0, \ t \in [0,\infty). \tag{114}$$

Since the function v_x is (f)-perfect, it follows from (102), (114), (99), (96) and (C6) that there are

$$t_0 \in [T_2, T_2+L], \ t_1 \in [T-T_2-L, T-T_2] \tag{115}$$

such that

$$|v_x(t_0) - z_0| \leq \delta/2, \ |\widehat{u}(t_1) - z_1| \leq \delta/2. \tag{116}$$

By (P5), (116), (97), (115), (96) and (99) there is an a. c. function $h : [t_0, t_1] \to R^n$ such that

$$h(t_0) = v_x(t_0), \ h(t_1) = \widehat{u}(t_1), \ \Gamma^f(t_0, t_1, h) \leq \epsilon/8. \tag{117}$$

Set

$$v_*(t) = v_x(t),\ t \in [0, t_0],\ v_*(t) = h(t),\ \ t \in (t_0, t_1],$$
$$v_*(t) = \widehat{u}(t),\ t \in (t_1, T]. \tag{118}$$

In view of (117) and (118) the a. c. function $v_* : [0, T] \to R^n$ is well-defined. By (118), (110) and (113),

$$v_*(0) = x_1,\ v_*(T) = y. \tag{119}$$

Since v_x is (f)-perfect and u_y is $(\bar{f})$-perfect it follows from (119), (118), (110), (111), (117), (9) and (10) that

$$\begin{aligned} &U^f(0, T, x, y) \le I^f(0, T, v_*) = I^f(0, t_0, v_x) + I^f(t_0, t_1, h) + I^f(t_1, T, \widehat{u}) \\ &= t_0\mu(f) + \pi^f(x) - \pi^f(v_x(t_0)) + \Gamma^f(t_0, t_1, h) + (t_1 - t_0)\mu(f)) \\ &\quad + \pi^f(v_x(t_0)) - \pi^f(\widehat{u}(t_1)) + I^{\bar{f}}(0, T - t_1, u_y \\ &\le t_1\mu(f) + \pi^f(x) + \epsilon/8 - \pi^f(\widehat{u}(t_1)) + (T - t_1)\mu(f) \\ &\quad + \pi^{\bar{f}}(u_y(0)) - \pi^{\bar{f}}(u_y(T - t_1)) \\ &= T\mu(f) + \pi^f(x) + \pi^{\bar{f}}(y) + \epsilon/8 - \pi^f(\widehat{u}(t_1)) - \pi^{\bar{f}}(u_y(T - t_1)). \end{aligned} \tag{120}$$

By (116), the choice of δ_0 (see (94), (95)) and (97),

$$|\pi^f(\widehat{u}(t_1)) - \pi^f(z_1)| \le \epsilon/16. \tag{121}$$

In view of (111) and (116), $u_y(T - t_1) = \widehat{u}(t_1)$ and $|u_y(T - t_1) - z_1| \le \delta/2$. Combined with the choice of δ_0 (see (94), (95)), (97) and (98) this implies that

$$|\pi^{\bar{f}}(u_y(T - t_1))| = |\pi^{\bar{f}}(u_y(T - t_1)) - \pi^{\bar{f}}(z_1)| \le \epsilon/16.$$

Together with (120) and (121) this implies that

$$\begin{aligned} &U^f(0, T, x, y) \le T\mu(f) + \pi^f(x) + \pi^{\bar{f}}(y) - \pi^f(z_1) + \epsilon/4 \\ &= T\mu(f) + \pi^f(x) + \pi^{\bar{f}}(y) - \inf\{\pi^f(z) :\ z \in R^n\} + \epsilon/4. \end{aligned}$$

Combined with (109) this implies (19). Theorem 10 is proved. ■

Proof of Theorem 11. By Proposition 1 there are $z_0, z_1 \in R^n$ such that

$$\pi^f(z_0) = \inf\{\pi^f(z) :\ z \in R^n\},\ \pi^{\bar{f}}(z_1) = \inf\{\pi^{\bar{f}}(z) :\ z \in R^n\}. \tag{122}$$

We may assume that $|z_0|, |z_1| \le K$. By Proposition 4 there exists $S_0 > K$ such that the following properties hold:

(C7) for each $T \ge 1$, each $x \in R^n$ satisfying $|x| \le K$ and each a. c. function $v : [0, T] \to R^n$ satisfying $v(0) = x$ and $I^f(0, T, v) \le U^f(T, x) + 1$ the inequality $|v(t)| \le S_0$ holds for all $t \in [0, T]$;

(C8) for each $T \ge 1$ and each a. c. function $v : [0, T] \to R^n$ satisfying $I^f(0, T, v) \le U^f(T) + 1$ the inequality $|v(t)| \le S_0$ holds for all $t \in [0, T]$.

Theorem 11 now follows form Theorem 10 and properties (C7) and (C8). ■

Acknowledgement

The author thanks the referee for useful comments.

References

Anderson, B.D.O. and Moore, J.B. (1971) *Linear Optimal Control*. Prentice-Hall, Englewood Cliffs, NJ.

Aubry, S. and Le Daeron, P.Y. (1983) The discrete Frenkel-Kontorova model and its extensions I. *Physica D* **8**, 381-422.

Baumeister, J., Leitao, A. and Silva, G.N. (2007) On the value function for nonautonomous optimal control problem with infinite horizon. *Systems Control Lett.* **56**, 188-196.

Blot, J. and Cartigny, P. (2000) Optimality in infinite-horizon variational problems under sign conditions. *J. Optim. Theory Appl.* **106**, 411-419.

Blot, J. and Cretezz, B. (2004) On the smoothness of optimal paths. *Decis. Econ. Finance* **27**, 1-34.

Blot, J. and Michel, P. (2003) The value-function of an infinite-horizon linear quadratic problem. *Appl. Math. Lett.* **16**, 71-78.

Coleman, B.D., Marcus, M. and Mizel, V.J. (1992) On the thermodynamics of periodic phases. *Arch. Rational Mech. Anal.* **117**, 321-347.

Glizer, V. (2007) Infinite horizon quadratic control of linear singularly perturbed systems with small state delays: an asymptotic solution of Riccatti-type equation. *IMA J. Math. Control Inform.* **24**, 435-459.

Leizarowitz, A. (1986) Tracking nonperiodic trajectories with the overtaking criterion. *Appl. Math. and Opt.* **14**, 155-171.

Leizarowitz, A. and Mizel, V.J. (1989) One dimensional infinite horizon variational problems arising in continuum mechanics. *Arch. Rational Mech. Anal.* **106**, 161-194.

Lykina, V., Pickenhain, S. and Wagner, M. (2008) Different interpretations of the improper integral objective in an infinite horizon control problem. *J. Math. Anal. Appl.* **340**, 498-510.

Makarov, V.L. and Rubinov, A.M. (1977) *Mathematical Theory of Economic Dynamics and Equilibria*. Springer-Verlag, New York.

Marcus, M. and Zaslavski, A.J. (1999) The structure of extremals of a class of second order variational problems. *Ann. Inst. H. Poincare, Anal. Non Lineaire* **16**, 593-629.

Marcus, M. and Zaslavski, A.J. (2002) The structure and limiting behavior of locally optimal minimizers. *Ann. Inst. H. Poincare, Anal. Non Lineaire* **19**, 343-370.

Mordukhovich, B. (1990) Minimax design for a class of distributed parameter systems. *Automat. Remote Control* **50**, 1333-1340.

Mordukhovich, B. and Shvartsman, I. (2004) Optimization and feedback control of constrained parabolic systems under uncertain perturbations.

Optimal Control, Stabilization and Nonsmooth Analysis. Lecture Notes Control Inform. Sci. Springer, 121-132.

PICKENHAIN, S. and LUKINA, V. (2006) Sufficiency conditions for infinite horizon optimal control problems. *Recent Advances in Optimization. Proceedings of the 12th French-German-Spanish Conference on Optimization, Avignon.* Springer, 217-232.

SAMUELSON, P.A. (1965) A catenary turnpike theorem involving consumption and the golden rule. *American Economic Review* **55**, 486-496.

ZASLAVSKI, A.J. (1996) Dynamic properties of optimal solutions of variational problems. *Nonlinear Analysis: Theory, Methods and Applications* **27**, 895-931.

ZASLAVSKI, A.J. (1998) Existence and uniform boundedness of optimal solutions of variational problems. *Abstract and Applied Analysis* **3**, 265-292.

ZASLAVSKI, A.J. (1999) Turnpike property for extremals of variational problems with vector-valued functions. *Transactions of the AMS* **351**, 211-231.

ZASLAVSKI, A.J. (2006a) A nonintersection property for extremals of variational problems with vector-valued functions. *Ann. Inst. H. Poincaré, Anal. non linéaize* **23**, 929-948.

ZASLAVSKI, A.J. (2006b) *Turnpike Properties in the Calculus of Variations and Optimal Control.* Springer, New York.

ZASLAVSKI, A.J. (2008) Turnpike properties of approximate solutions of autonomous variational problems. *Control and Cybernetics* **37**, 491-512.