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Stabilization of fractional positive continuous-time linear systems with delays in sectors of left half complex plane by state-feedbacks*

by

Tadeusz Kaczorek

Białystok Technical University, Faculty of Electrical Engineering
Wiejska 45D, 15-351 Białystok, Poland
e-mail: kaczorekisep.pw.edu.pl

Abstract: The problem of stabilization of fractional positive linear continuous-time linear systems with delays by state-feedbacks is addressed. The gain matrix of the state feedback is chosen so that the zeros of the closed-loop polynomial are located in a sector of the left half of complex plane. Necessary and sufficient conditions for the solvability of the problem are established and a procedure for computation of a gain matrix of the feedback is proposed. The considerations are illustrated by a numerical example.

Keywords: fractional, positive, continuous-time, linear, system, state-feedback, stabilization.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems is given in the monographs of Farina and Rinaldi (2000)and Kaczorek (2002).

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19^{th} century (Nishimoto, 1984; Oustaloup, 1993; Podlubny, 1999). This idea has been used by engineers for modelling different processes in the late 1960s (Nishimoto, 1984; Oldham and Spanier, 1974;

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Ortigueira, 1997; Ostalczyk, 2000, 2004a,b, 2008; Oustaloup, 1993, 1995; Podlubny, 1999, 2002; Podlubny, Dorcak and Kostial, 1999; Sierociuk, 2007; Sierociuk and Dzieliński, 2006; Vinagre, Monje and Calderon, 2002). Mathematical fundamentals of fractional calculus are given in Nishimoto (1984), Oldham and Spanier (1974), Ostalczyk (2008) and Oustaloup (1993). The fractional order controllers have been developed in Oustaloup (1993) and Podlubny, Dorcak and Kostial (1999). A generalization of the Kalman filter for fractional order systems has been proposed in Sierociuk and Dzieliński (2006). Some other applications of fractional order systems can be found in Ortigueira (1997), Ostalczyk (2000, 2004a,b), Podlubny (2002), Podlubny, Dorcak and Kostial (1999), Sierociuk (2007), Sierociuk and Dzieliński (2006) and Vinagre, Monje and Calderon (2002). Fractional polynomials and nD systems have been investigated in Gałkowski and Kummert (2005), and the stability of the fractional continuous-time system with delay in Busłowicz (2008). The concept of positive fractional discrete-time linear systems was introduced in Kaczorek (2008b) and the reachability and controllability to zero of positive fractional system was investigated in Kaczorek (2007). The concept of fractional positive continuous-time linear systems was introduced in Kaczorek (2008c). The stabilization problem of fractional discrete-time linear systems by state-feedback was considered in Kaczorek (2009a). The problem of positivity and stabilization of 2D linear systems by state-feedbacks have been analysed in Kaczorek (2009c).

In this paper the problem of stabilization of fractional positive linear continuous-time systems with delays by state-feedbacks will be addressed. The gain matrix of the state feedback will be chosen so that the zeros of the closed-loop polynomial are located in a sector of the left half of complex plane.

The paper is organized as follows. The basic definitions and theorems concerning the positive fractional systems are recalled in Section 2 and for positive fractional systems with delays in Section 4. The main results of the paper are presented in Sections 3 and 5. In Section 3 the stability of the positive fractional continuous-time linear systems is discussed and the equilibrium point of the systems is introduced. The stabilization problem by state-feedbacks is formulated and solved in Section 5. Concluding remarks are given in Section 6.

The following notation will be used in the paper. The set of $n \times m$ real matrices will be denoted $\Re^{n\times m}$ and $\Re^n := \Re^{n\times 1}$. The set of $m \times n$ real matrices with nonnegative entries will be denoted by $\Re_+^{m\times n}$ and $\Re_+^n := \Re_+^{n\times 1}$. A matrix A (a vector x) with positive entries (positive components) will be denoted by $A > 0$ ($x > 0$). The set of nonnegative integers will be denoted by Z_+ and the $n \times n$ identity matrix by I_n.

2. Positive fractional continuous-time linear systems

In this paper the following Caputo definition of the fractional derivative will be used (Ostalczyk, 2008; Podlubny, 1999; Vinagre, Monje and Calderon, 2002):

$$
{}_0^cD_t^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau,
$$
$$
n-1 < \alpha \le n \in N = \{1, 2, ...\} \tag{1}
$$

where $\alpha \in \Re$ is the order of fractional derivative and $f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}$.

Consider the fractional continuous-time linear system described by the state equations

$$
\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha \le 1 \tag{2a}
$$
$$
y(t) = Cx(t) + Du(t) \tag{2b}
$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are, respectively, the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Theorem 1 *The solution of equation (2a) is given by*

$$
x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \quad x(0) = x_0 \tag{3}
$$

where

$$
\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{4}
$$
$$
\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{5}
$$

and $E_\alpha(At^\alpha)$ is the Mittage-Leffler matrix function, $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function.

Proof is provided in Kaczorek (2008c).

Remark 1 *From* (4) *and* (5) *for $\alpha = 1$ we have $\Phi_0(t) = \Phi(t) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(k+1)} = e^{At}$.*

Definition 1 *The fractional system* (2) *is called (internally) positive if and only if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$ for $t \ge 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \ge 0$.*

A square real matrix $A = [a_{ij}]$ is called the Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \ge 0$ for $i \ne j$ (Farina and Rinaldi, 2000; Kaczorek, 2002). The set of $n \times n$ Metzler matrices will be denoted M_n.

Lemma 1 *(Kaczorek, 2008c) Let $A \in \Re^{n\times n}$ and $0 < \alpha \le 1$. Then*

$$
\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \in \Re_+^{n\times n} \quad \text{for } t \ge 0 \tag{6}
$$

and

$$\Phi(t)=\sum_{k=0}^{\infty}\frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\in\Re_+^{n\times n}\quad\text{for}\ \ t\geq 0 \tag{7}$$

if and only if A is a Metzler matrix, i.e. $A\in M_n$.

THEOREM 2 *The fractional continuous-time system* (2) *is internally positive if and only if*

$$A\in M_n,\quad B\in\Re_+^{n\times m},\quad C\in\Re_+^{p\times n},\quad D\in\Re_+^{p\times m}. \tag{8}$$

Proof is provided in Kaczorek (2008c).

3. Stability of the positive fractional systems

DEFINITION 2 *The positive fractional system*

$$\frac{d^\alpha x(t)}{dt^\alpha}=Ax(t),\quad A\in M_n,\quad 0<\alpha\leq 1 \tag{9}$$

is called asymptotically stable (shortly: stable) if and only if

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}\Phi_0(t)x_0=0 \tag{10}$$

for all $x_0\in\Re_+^n$.

The characteristic polynomial of (9) has the form

$$\det[I_n s^\alpha-A]=(s^\alpha)^n+a_{n-1}(s^\alpha)^{n-1}+...+a_1 s^\alpha+a_0. \tag{11}$$

Substitution of

$$\lambda=s^\alpha \tag{12}$$

into (11) yields

$$\det[I_n\lambda-A]=\lambda^n+a_{n-1}\lambda^{n-1}+...+a_1\lambda+a_0. \tag{13}$$

Let us denote $\arg s=\phi$ and $\arg\lambda=\varphi$. Then from (12) we have

$$\varphi=\alpha\phi. \tag{14}$$

From (11), (13) and (14) for $\varphi=\frac{\pi}{2}$ we have the following corollary:

COROLLARY 1 *If the zeros of the characteristic polynomial* (13) *are located in the left half of complex plane then the zeros of the characteristic polynomial* (11) *are located in the sector defined by $\phi=\frac{\pi}{2\alpha}$ in the left half complex plane (see the Fig. 1).*

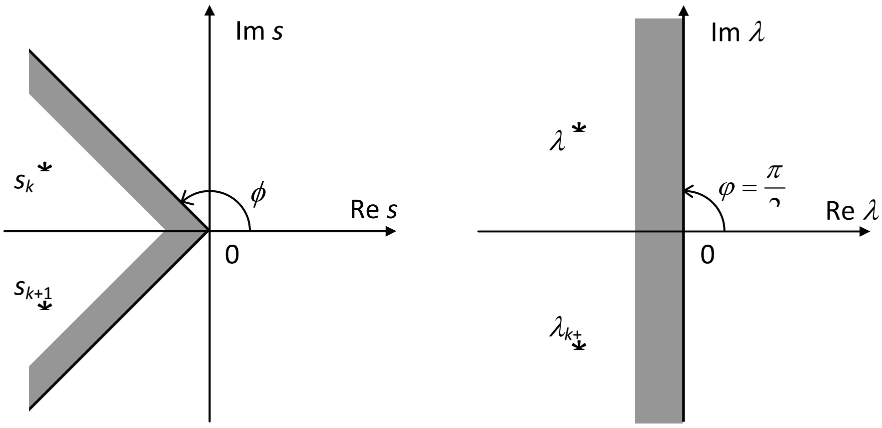

Figure 1. Illustration for the Corollary 1

THEOREM 3 *The zeros of the characteristic polynomial* (11) *are located in the sector* $\phi = \frac{\pi}{2\alpha}$ *if and only if one of the following equivalent conditions is satisfied:*

1) All coefficients of the characteristic polynomial (13) *are positive, i.e.* $a_i \geq 0$ *for* $i = 0, 1, ..., n-1$.

2) All leading principle minors of the matrix

$$-A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{15}$$

are positive, i.e.

$$|a_{11}| > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ \dots\ , \ \det[-A] > 0 \tag{16}$$

3) There exists a strictly positive vector $\lambda > 0$ $(\lambda \in \Re^n_+)$ *such that*

$$A\lambda < 0 \ \textit{(strictly negative)}. \tag{17}$$

Proof. For a positive fractional system $A \in M_n$ it is well known (Kaczorek, 2002, p. 64) that the system is stable if and only if the polynomial (13) has positive coefficients a_i, $i = 0, 1, ..., n-1$.

In Kaczorek (2008a) it was also shown that conditions 1) and 2) are equivalent. It is also well known (Kaczorek, 2002, 2008a) that if $A \in M_n$ then the conditions 2) and 3) are also equivalent. ■

DEFINITION 3 *The vector* $x_e \in \Re^n_+$ *is called the equilibrium point of the stable positive system (2a) for constant input* $u \in \Re^m_+$ $(u(t) = u)$ *if and only if*

$$Ax_e + Bu = 0. \tag{18}$$

If $Bu = \mathbf{1}_n = [1 \;...\; 1]^{\mathrm{T}} \in \Re_+^n$ (T denotes the transpose) then from (18) we have

$$x_e = -A^{-1}\mathbf{1}_n > 0 \tag{19}$$

since for a stable system $-A^{-1} \in \Re_+^{n\times n}$ (Kaczorek, 2002).

REMARK 2 *As strictly positive vector* λ *in* (17) *the equilibrium point* (19) *can be chosen since*

$$A\lambda = A(-A^{-1}\mathbf{1}_n) = -\mathbf{1}_n. \tag{20}$$

4. Stability of positive continuous-time systems with delays

Consider a continuous-time linear system with q delays in state and inputs

$$\dot{x}(t) = \sum_{k=0}^{q}[A_k x(t-d_k) + B_k u(t-d_k)] \tag{21a}$$

$$y(t) = Cx(t) + D(u(t) \tag{21b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ and $y(t) \in \Re^p$ are, respectively, the state, input and output vectors, A_k, B_k, $k = 0, 1, ..., q$, C, D are real matrices of appropriate dimensions and d_k, $k = 1, 2, ...q$, are the delays ($d_k \geq 0,\ d_0 = 0$).

Initial conditions for (21a) have the form

$$x(t) = x_0(t) \;\text{ for }\; t \in [-d, 0], \quad d = \max_k d_k \tag{22}$$

where $x_0(t)$ is a given vector function.

The system (21) is called (internally) positive if and only if $x(t) \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \geq 0$ for any $x_0(t) \in \Re_+^n$ and for all inputs $u(t) \in \Re_+^m$, $t \geq -d$.

THEOREM 4 *The system* (21) *is (internally) positive if and only if*

$$A_0 \in M_n, \quad A_k \in \Re_+^{n\times n}, \quad k = 1, ...q, \quad B_l \in \Re_+^{n\times m}, \quad l = 0, 1, ...q,$$
$$C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \tag{23}$$

The proof is provided in Kaczorek (2009b).

THEOREM 5 *The positive system* (21) *is asymptotically stable if and only if there exists a strictly positive vector* $\lambda \in \Re_+^n$ *satisfying the equality*

$$A\lambda < 0, \quad A = \sum_{k=0}^{q} A_k. \tag{24}$$

The proof is provided in Kaczorek (2009b).

THEOREM 6 *The positive system with delays* (21) *is asymptotically stable if and only if the positive system without delays*

$$\dot{x} = Ax, \quad A = \sum_{k=0}^{q} A_k \in M_n \tag{25}$$

is asymptotically stable.

The proof is provided in Kaczorek (2009b).

It follows from Theorem 6 that the checking of the asymptotic stability of positive systems with delays (21) can be reduced to checking the asymptotic stability of corresponding positive systems without delays (25). To check the asymptotic stability of positive systems (21) the following theorem can be used (Kaczorek, 2002, 2008a).

THEOREM 7 *The positive system with delays* (21) *is asymptotically stable if and only if one of the following equivalent conditions holds:*

1) *Eigenvalues* $s_1, s_2, \dots, s_n$ *of the matrix* A *have negative real parts,* $Re\, s_k < 0$, $k = 1, ..., n$.
2) *All coefficients of the characteristic polynomial of the matrix* A *are positive.*
3) *All leading principal minors of the matrix*

$$-A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \tag{26}$$

are positive, i.e.

$$|a_{11}| > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \, \dots, \, \det[-A] > 0. \tag{27}$$

THEOREM 8 *The positive system with delays* (21) *is unstable for any matrices* A_k, $k = 1, ..., q$, *if the positive system* $\dot{x} = A_0 x$ *is unstable.*

The proof is provided in Kaczorek (2009b).

THEOREM 9 *If at least one diagonal entry of matrix* A_0 *is positive then the positive system* (21) *is unstable for any* A_k, $k = 1, ..., q$.

The proof is provided in Kaczorek (2009b).

5. Stabilization of the fractional linear systems with delays by state-feedbacks

Consider the fractional linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = \sum_{k=0}^{q} [A_k x(t - d_k) + B_k u(t - d_k)] \tag{28}$$

with the state-feedback

$$u(t) = Kx(t) \tag{29}$$

where $K \in \Re^{m\times n}$ is a gain matrix.

By substituting (29) in (28) we obtain the closed-loop system

$$\frac{d^\alpha x(t)}{dt^\alpha} = \sum_{k=0}^{q}(A_k + B_k K)x(t-d_k), \quad 0 < \alpha \le 1. \tag{30}$$

The positive system with delays (30) is asymptotically stable if and only if the positive system without delays is asymptotically stable. This follows from the fact that stability of positive continuous-time linear systems with delays is independent of the number and values of delays (Kaczorek, 2009b) and the fact that asymptotic stability is considered in the sectors of the left half complex plane

$$\frac{d^\alpha x(t)}{dt^\alpha} = (A + BK)x(t), \quad A = \sum_{k=0}^{q} A_k, \quad B = \sum_{k=0}^{q} B_k, \tag{31}$$

is asymptotically stable.

We are looking for a gain matrix K such that the closed-loop system (30) is positive and the zeros of the characteristic polynomial

$$\det[I_n s^\alpha - (A+BK)] = (s^\alpha)^n + \bar{a}_{n-1}(s^\alpha)^{n-1} + \ldots + \bar{a}_1 s^\alpha + \bar{a}_0 \tag{32}$$

are located in the sector $\phi = \frac{\pi}{2\alpha}$.

THEOREM 10 *The closed-loop fractional system* (30) *is positive and the zeros of the polynomial* (32) *are located in the sector* $\phi = \frac{\pi}{2\alpha}$ *if and only if there exists a diagonal matrix*

$$\Lambda = diag[\lambda_1,\ ...,\ \lambda_n] \ \ with\ \ \lambda_k > 0, k = 1, ..., n \tag{33}$$

and a real matrix $D \in \Re^{m\times n}$ *such that the following conditions are satisfied*

$$A\Lambda + BD \in M_n \tag{34}$$

$$(A\Lambda + BD)\mathbf{1}_n < 0. \tag{35}$$

The gain matrix K *is given by the formula*

$$K = D\Lambda^{-1}. \tag{36}$$

Proof. First, we shall show that the closed-loop system (30) is positive if and only if (34) holds. Using (30), (31) and (36) we obtain

$$\sum_{k=0}^{q}(A_k + B_k K) = A + BK = A + BD\Lambda^{-1} = (A\Lambda + BD)\Lambda^{-1} \in M_n \tag{37}$$

if and only if the condition (34) is satisfied.

Taking into account that

$$K\Lambda\mathbf{1}_n = D\Lambda^{-1}\Lambda\mathbf{1}_n = D\mathbf{1}_n \text{ and } \Lambda\mathbf{1}_n = \lambda = [\lambda_1, \dots, \lambda_n]^{\mathrm{T}} \tag{38}$$

and using (17) we obtain

$$(A + BK)\lambda = (A + BK)\Lambda\mathbf{1}_n = (A\Lambda + BD)\mathbf{1}_n < 0. \tag{39}$$

Therefore, by Theorem 3, the zeros of the characteristic polynomial (32) are located in the sector $\phi = \frac{\pi}{2\alpha}$ if and only if the condition (35) is met. ∎

If the conditions of Theorem 10 are satisfied then the problem of stabilization can be solved by using the following procedure:

Procedure

Step 1. Choose a diagonal matrix (33) with $\lambda_k > 0$, $k = 1, ..., n$ and a real matrix $D \in \Re^{m\times n}$ satisfying the condition (34) and (35).

Step 2. Using the formula (36) compute the gain matrix K.

Example

Given is the fractional system (21a) with $\alpha = 0.8$, $q = 2$ and the matrices

$$A_0 = \begin{bmatrix} 0.5 & 0.3 & -0.2 \\ 0.2 & -1 & 0 \\ 0 & -0.2 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.3 & 0.4 & -0.3 \\ 0.1 & -0.5 & 0 \\ 0 & -0.1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0.3 & -0.5 \\ 0.7 & -1.5 & 0 \\ 0 & -0.7 & 0.5 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \\ 0.2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \\ 0.3 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \\ 0.5 & 0 \end{bmatrix}. \tag{40}$$

Find a gain matrix $K \in \Re^{2\times 3}$ such that the closed-loop system is positive and the zeros of its characteristic polynomial are located in the sector $\phi = \frac{5}{8}\pi$.

Note that the fractional system with (40) is not positive since the matrices A_0, A_1 and A_2 have negative off-diagonal entries. In this case

$$A = \sum_{k=0}^{2} A_k, = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 0 \\ 0 & -1 & 2.5 \end{bmatrix}, \quad B = \sum_{k=0}^{2} B_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{41}$$

Using Procedure and (41) we obtain the following

Step 1. We choose

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.5 & 2 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix} \tag{42}$$

and check condition (34)

$$A\Lambda + BD = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 0 \\ 0 & -1 & 2.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 2 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix} =$$

$$= \begin{bmatrix} -3 & 2 & 0.4 \\ 1 & -6 & 0 \\ 0.5 & 0 & -1 \end{bmatrix} \in M_3$$

and condition (35)

$$(A\Lambda + BD)\mathbf{1}_n = \begin{bmatrix} -3 & 2 & 0.4 \\ 1 & -6 & 0 \\ 0.5 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -5 \\ -0.5 \end{bmatrix}.$$

Therefore, the conditions are satisfied.

Step 2. Using (36) we obtain the gain matrix

$$K = D\Lambda^{-1} = \begin{bmatrix} 0.5 & 2 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 1 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix}.$$

The closed-loop system is positive, since the matrix

$$A_c = A + BK = \begin{bmatrix} -3 & 1 & 0.4 \\ 1 & -3 & 0 \\ 0.5 & 0 & -1 \end{bmatrix}$$

is a Metzler matrix.

The characteristic polynomial

$$\det\left[I_n\lambda - A_c\right] = \begin{vmatrix} \lambda+3 & -1 & -0.4 \\ -1 & \lambda+3 & 0 \\ -0.5 & 0 & \lambda+1 \end{vmatrix} = \lambda^3 + 7\lambda^2 + 13.8\lambda + 7.4$$

has positive coefficients. Therefore, zeros of the characteristic polynomial of the closed system are located in the desired sector $\phi = \frac{5}{8}\pi$.

These considerations can be extended to the fractional system

$$\frac{d^\alpha x(t)}{dt^\alpha} = \sum_{k=0}^{q} A_k x(t - d_k) + Bu(t) \tag{43}$$

with the state-feedbacks of the form

$$u(t) = \sum_{k=0}^{q} K_k x(t - d_k). \tag{44}$$

Substituting (44) into (43) we obtain

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = \sum_{k=0}^{q} \bar{A}_k x(t-d_k) \tag{45}$$

where

$$\bar{A}_k = A_k + BK_k.$$

6. Concluding remarks

The problem of stabilization of fractional positive linear continuous-time systems with delays by state-feedbacks so that the zeros of the closed-loop polynomial are located in the sector of the left half of complex plane has been addressed. Necessary and sufficient conditions for the solvability of the problem have been established. A procedure for computation of a gain matrix of the feedback has been proposed and illustrated by a numerical example. These considerations can be easily extended for fractional positive 2D hybrid linear systems. An extension of this approach for fractional positive 2D continuous-time linear systems is an open problem.

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References

Busłowicz, M. (2008a) Stability of linear continuous-time fractional order systems with delays of the retarded type. *Bull. Pol. Acad. Sci. Techn.*, **56** (4), 319-324.

Busłowicz, M. (2008b) Frequency domain method for stability analysis of linear continuous-time fractional systems. In: K. Malinowski, L. Rutkowski, eds., *Recent Advances in Control and Automation.* Acad. Publ. House EXIT, Warsaw, 83-92.

Farina, L. and Rinaldi, S. (2000) *Positive Linear Systems; Theory and Applications.* J. Wiley, New York.

Gałkowski, K. and Kummert, A. (2005) Fractional polynomials and nD systems. *Proc IEEE Int. Symp. Circuits and Systems, ISCAS'2005*, Kobe, Japan, CD-ROM.

Kaczorek, T. (2002) *Positive 1D and 2D Systems.* Springer-Verlag, London.

Kaczorek, T. (2007) Reachability and controllability to zero of positive fractional discrete-time systems. *Machine Intelligence and Robotic Control*, **6** (4) (2004), 139-143.

Kaczorek, T. (2008a) Asymptotic stability of positive 1D and 2D linear systems. In: K. Malinowski, L. Rutkowski, eds., *Recent Advances in Control and Automation*. Acad. Publ. House EXIT, Warsaw, 41-52.

Kaczorek, T. (2008b) Practical stability of positive fractional discrete-time systems. *Bull. Pol. Acad. Sci. Techn.*, **56** (4), 313-318.

Kaczorek, T. (2008c) Fractional positive continuous-time linear systems and their reachability. *Int. J. Appl. Math. Comput. Sci.*, **18** (2), 223-228.

Kaczorek, T. (2009a) Stabilization of fractional discrete-time linear systems using state-feedback. *Proc. Conf. LOGITRANS*, April 15-17, Szczyrk 2009.

Kaczorek, T. (2009b) Stability of positive continuous-time linear systems with delays. *Bull. Pol. Acad. Sci. Techn.*, **57** (4), 395-398.

Kaczorek, T. (2009c) Positivity and stabilization of 2D linear systems with delays. *Proc. MMAR Conference 2009*, Szczecin (CD-ROM).

Nishimoto, K. (1984) *Fractional Calculus*. Descartes Press, Koriama.

Oldham, K.B. and Spanier, J. (1974) *The Fractional Calculus*. Academic Press, New York.

Ortigueira, M.D. (1997) Fractional discrete-time linear systems. *Proc. of the IEE-ICASSP 97*, Munich, Germany. IEEE, New York, **3**, 2241-2244.

Ostalczyk, P. (2000) The non-integer difference of the discrete-time function and its application to the control system synthesis. *Int. J. Syst.* Sci., **31** (12), 1551-1561.

Ostalczyk, P. (2004a) Fractional-Order Backward Difference Equivalent Forms Part I – Horner's Form. *Proc. 1st IFAC Workshop on Fractional Differentiation and its Applications, FDA'04*, Bordeaux, France, 342-347.

Ostalczyk, P. (2004b) Fractional-Order Backward Difference Equivalent Forms Part II – Polynomial Form. *Proc. 1st IFAC Workshop on Fractional Differentiation and its Applications*, *FDA'04*, Bordeaux, France, 348-353.

Ostalczyk, P. (2008) *Epitome of the Fractional Calculus*. Technical University of Lodz Publishing House, Lodz (in Polish).

Oustaloup, A. (1993) *Commande CRONE*. Hermés, Paris.

Oustaloup, A. (1995) *La dérivation non entiére*. Hermés, Paris.

Podlubny, I. (1999) *Fractional Differential Equations*. Academic Press, San Diego.

Podlubny, I. (2002) Geometric and physical interpretation of fractional integration and fractional differentation. *Fract. Calc. Appl. Anal.* **5** (4), 367-386.

Podlubny, I., Dorcak, L. and Kostial, I. (1999) On fractional derivatives, fractional order systems and $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$-controllers. *Proc. 36th IEEE Conf. Decision and Control*, San Diego, CA, 4985-4990.

Sierociuk, D. (2007) Estimation and control of discrete-time fractional systems described by state equations. PhD thesis, Warsaw University of Technology, Warsaw (in Polish).

SIEROCIUK, D. and DZIELIŃSKI, D. (2006) Fractional Kalman filter algorithm for the states, parameters and order of fractional system estimation. *Int. J. Appl. Math. Comp. Sci.*, **16** (1), 129-140.

VINAGRE, B.M., MONJE, C.A. and CALDERON, A.J. (2002) Fractional order systems and fractional order control actions. *Lecture 3 IEEE CDC'02 TW#2: Fractional Calculus Applications in Automatic Control and Robotics.*