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Lyapunov functional for a linear system with two delays*

by

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Abstract: The paper presents a method of determining the Lyapunov functional for linear time-invariant LTI system with two lumped delays.

Keywords: Lyapunov functional, time delay system, LTI system.

1. Introduction

Lyapunov quadratic functionals are used to test the stability of systems, to compute the critical delay values for time delay systems, to compute the exponential estimates for the solutions of time delay systems, to calculate the robustness bounds for uncertain time delay systems, and to calculate a quadratic performance index for the process of parametric optimization for time delay systems. We construct the Lyapunov functionals for the systems with time delay with a given time derivative. For the first time such Lyapunov functional was introduced by Repin (1965) for the case of retarded time delay linear systems with one delay. Repin (1965) provided also the procedure for determining the coefficients of functional. Duda (1986) used the Lyapunov functional for the calculation of the value of a quadratic performance index in the process of parametric optimization for systems with time delay of retarded type and extended the results to the case of neutral type time delay systems in Duda (1988). In Infante and Castelan (1978), construction of the Lyapunov functional is based on a solution of a matrix differential-difference equation on a finite time interval. This solution satisfies symmetry and boundary conditions. Kharitonov and Zhabko (2003) extended the results of Infante and Castelan (1978) and proposed a procedure of constructing quadratic functionals for linear retarded type delay systems, which could be used for the robust stability analysis of time delay systems. This functional was expressed by means of the

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Lyapunov matrix, which depended on the fundamental matrix of the time delay system. Kharitonov (2005) extended some basic results obtained for the case of retarded type time delay systems to the case of neutral type time delay systems, and then, in Kharitonov (2008), to the neutral type time delay systems with discrete and distributed delay. Kharitonov and Hinrichsen (2004) used the Lyapunov matrix to derive exponential estimates for the solutions of exponentially stable time delay systems. Kharitonov and Plischke (2006) obtained the necessary and sufficient conditions for the existence and uniqueness of the delay Lyapunov matrix for the case of retarded system with one delay.

A numerical scheme for construction of the Lyapunov functionals has been proposed in Gu (1997). The method starts with the discretisation of the Lyapunov functional. The scheme is based on linear matrix inequality (LMI) techniques. Fridman (2001) introduced the Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems with discrete and distributed delays, which were based on equivalent descriptor form of the original system and obtained delay-dependent and delay-independent conditions in terms of LMI. Ivanescu et al. (2003) proceeded with the delay-depended stability analysis for linear neutral systems, constructed the Lyapunov functional and derived sufficient delay-dependent conditions in terms of linear matrix inequalities (LMIs). Han (2004a) obtained a delay-dependent stability criterion for neutral systems with time varying discrete delay. This criterion was expressed in the form of LMI and was obtained using the Lyapunov direct method. Han (2004b) investigated robust stability of uncertain neutral systems with discrete and distributed delays, basing on the descriptor model transformation and the decomposition technique, and formulated the stability criteria in the form of LMIs. Han (2005) developed the discretized Lyapunov functional approach to investigation of stability of linear neutral systems with mixed neutral and discrete delays. Stability criteria, which are applicable to linear neutral systems with both *small* and *non-small* discrete delays are formulated in the form of LMIs. Han (2009) studied the problem of stability of linear time delay systems of both retarded and neutral types, using the discrete delay N-decomposition approach to derive some new more general discrete delay dependent stability criteria. Gu and Liu (2009) investigated the stability of coupled differential-functional equations using the discretized Lyapunov functional method and delivered the stability condition in the form of LMI, suitable for numerical computation.

This paper presents a method of determining the Lyapunov functional for a linear dynamic system with two delays in the general case with non-commensurate delays and presents a special case with commensurate delays in which the Lyapunov functional can be determined by solving a set of ordinary differential equations. The novelty of the result lies in the extension of the Repin's method to the system with two delays. To the best of the author's knowledge, such extension has not been reported in the literature. An example, illustrating the new method, is also presented.

2. Formulation of the problem

Let us consider the linear system with two delays τ and r, whose dynamics is described by equations

$$\begin{cases} \dfrac{dx(t)}{dt} = A \cdot x(t) + B \cdot x_t(-r) + C \cdot x_t(-\tau) \\ x(t_0) = x_0 \in R^n \\ x_{t_0} = \Phi \in L^2([-r,0), R^n) \end{cases} \tag{1}$$

$$r > \tau > 0; \;\; x(t) \in R^n; \;\; A, B, C \in R^{n \times n}; \;\; x_t \in L^2([-r,0), R^n)$$

$$x_t(\theta) = x(t+\theta), \;\; t \geq t_0, \;\; \theta \in [-r, 0)\,.$$

The state of the system (1) is a vector

$$S(t) = \begin{bmatrix} x(t) \\ x_t \end{bmatrix} \quad \text{for} \quad t \geq t_0\,. \tag{2}$$

The state space is defined by the formula

$$X = R^n \times L^2([-r,0), R^n)\,. \tag{3}$$

On the state space X we define a Lyapunov functional, positively defined, differentiable, with the derivative computed on the trajectory of the system (1) being negatively defined.

$$\begin{aligned} V(S(t)) &= x^T(t) \cdot \alpha \cdot x(t) + \int_{-r}^{0} x^T(t) \cdot \beta(\theta) \cdot x_t(\theta) d\theta + \\ &+ \int_{-r}^{0} x_t^T(\theta) \cdot \gamma_1(\theta) \cdot x_t(\theta) d\theta + \int_{-\tau}^{0} x^T(t) \cdot \kappa(\sigma) \cdot x_t(\sigma) d\sigma + \\ &+ \int_{-\tau}^{0} x_t^T(\sigma) \cdot \gamma_2(\sigma) \cdot x_t(\sigma) d\sigma + \int_{-r}^{0} \int_{\theta}^{0} x_t^T(\theta) \cdot \delta_1(\theta, \xi) \cdot x_t(\xi) d\xi d\theta + \\ &+ \int_{-\tau}^{0} \int_{\sigma}^{0} x_t^T(\sigma) \cdot \delta_2(\sigma, \zeta) \cdot x_t(\zeta) d\zeta d\sigma + \int_{-r}^{0} \int_{-\tau}^{0} x_t^T(\theta) \cdot \delta_3(\theta, \sigma) \cdot x_t(\sigma) d\sigma d\theta \end{aligned} \tag{4}$$

$$\alpha = \alpha^T \in R^{n \times n}; \quad \beta, \gamma_1 \in C^1([-r,0), R^{n \times n}); \quad \kappa, \gamma_2 \in C^1([-\tau,0), R^{n \times n});$$
$$\delta_1 \in C^1(\Omega_1, R^{n \times n}); \quad \delta_2 \in C^1(\Omega_2, R^{n \times n}); \quad \delta_3 \in C^1(\Omega_3, R^{n \times n});$$

$$\Omega_1 = \{(\theta, \xi) : \theta \in [-r, 0), \theta \leq \xi \leq 0\}; \; \Omega_2 = \{(\sigma, \zeta) : \sigma \in [-\tau, 0), \sigma \leq \zeta \leq 0\};$$
$$\Omega_3 = \{(\theta, \sigma) : \theta \in [-r, 0), \sigma \in [-\tau, 0)\}\,.$$

3. Determination of the coefficients of the functional (4)

We compute the derivative of the functional (4) on the trajectory of the system (1) according to the formula

$$\frac{dV(S(t))}{dt} = grad(V(S(t)) \cdot \frac{dS(t)}{dt} \quad \text{for} \quad t \geq t_0\,. \tag{5}$$

The derivative of the functional (4), calculated on the basis of the formula (5), is given by the formula

$$\frac{dV(S(t))}{dt} = x^T(t)\cdot[A^T\cdot\alpha+\alpha\cdot A+\frac{\beta(0)+\beta^T(0)}{2}+\gamma_1(0)+\frac{\kappa(0)+\kappa^T(0)}{2}$$
$$+\gamma_2(0)]\cdot x(t)+x^T(t)[2\alpha\cdot B-\beta(-r)]\cdot x_t(-r)+x^T(t)[2\alpha\cdot C-\kappa(-\tau)]x_t(-\tau)$$
$$-x_t^T(-r)\cdot\gamma_1(-r)\cdot x_t(-r)-x_t^T(-\tau)\cdot\gamma_2(-\tau)\cdot x_t(-\tau)$$
$$+\int_{-r}^{0}x^T(t)\cdot[A^T\beta(\theta)-\frac{d\beta(\theta)}{d\theta}+\delta_1^T(\theta,0)+\delta_3^T(\theta,0)]x_t(\theta)d\theta$$
$$+\int_{-r}^{0}x_t^T(-r)[C^T\beta(\theta)-\delta_3^T(\theta,-\tau)]x_t(\theta)d\theta-\int_{-r}^{0}x_t^T(\theta)\cdot\frac{d\gamma_1(\theta)}{d\theta}\cdot x_t(\theta)d\theta$$
$$+\int_{-\tau}^{0}x^T(t)\cdot[A^T\cdot\kappa(\sigma)-\frac{d\kappa(\sigma)}{d\sigma}+\delta_2^T(\sigma,0)+\delta_3(0,\sigma)]\cdot x_t(\sigma)d\sigma$$
$$+\int_{-\tau}^{0}x_t^T(-\tau)\cdot[B^T\cdot\kappa(\sigma)-\delta_3(-r,\sigma)]\cdot x_t(\sigma)d\sigma$$
$$-\int_{-\tau}^{0}x_t^T(\sigma)\cdot\frac{d\gamma_2(\sigma)}{d\sigma}\cdot x_t(\sigma)d\sigma-\int_{-r}^{0}\int_{\theta}^{0}x_t^T(\theta)\cdot[\frac{\partial\delta_1(\theta,\xi)}{\partial\theta}+\frac{\partial\delta_1(\theta,\xi)}{\partial\xi}]x_t(\xi)d\xi d\theta$$
$$-\int_{-\tau}^{0}\int_{\sigma}^{0}x_t^T(\sigma)\cdot[\frac{\partial\delta_2(\sigma,\zeta)}{\partial\sigma}+\frac{\partial\delta_2(\sigma,\zeta)}{\partial\zeta}]\cdot x_t(\zeta)d\zeta d\sigma$$
$$-\int_{-r}^{0}\int_{-\tau}^{0}x_t^T(\theta)\cdot[\frac{\partial\delta_3(\theta,\sigma)}{\partial\theta}+\frac{\partial\delta_3(\theta,\sigma)}{\partial\sigma}]\cdot x_t(\sigma)d\sigma d\theta \qquad t\ge t_0\,. \tag{6}$$

We identify the coefficients of the functional (4), assuming that the derivative (6) satisfies the relationship

$$\frac{dV(S(t))}{dt} = -x^T(t)\cdot x(t) \quad \text{for} \quad t\ge t_0\,. \tag{7}$$

When the system (1) is asymptotically stable and the relationship (7) holds, one can easily determine the value of a square indicator of quality of parametric optimization, knowing the Lyapunov functional (4), because

$$J=\int_{t_0}^{\infty}x^T(t)x(t)dt = V(S(t_0))\,. \tag{8}$$

From equations (6) and (7) we obtain the system of equations (9) to (24):

$$\mathrm{A}^{\mathrm{T}}\cdot\alpha+\alpha\cdot A+\frac{\beta(0)+\beta^T(0)}{2}+\gamma_1(0)+\frac{\kappa^T(0)+\kappa(0)}{2}+\gamma_2(0)=-I \tag{9}$$
$$2\alpha\cdot B-\beta(-r)=0 \tag{10}$$
$$2\alpha\cdot C-\kappa(-\tau)=0 \tag{11}$$

$$\gamma_1(-r) = 0 \tag{12}$$

$$\gamma_2(-\tau) = 0 \tag{13}$$

$$A^T \cdot \beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta_1^T(\theta, 0) + \delta_3^T(\theta, 0) = 0 \tag{14}$$

$$B^T \cdot \beta(\theta) - \delta_1(-r, \theta) = 0 \tag{15}$$

$$C^T \cdot \beta(\theta) - \delta_3^T(\theta, -\tau) = 0 \tag{16}$$

$$\frac{d\gamma_1(\theta)}{d\theta} = 0 \tag{17}$$

$$A^T \cdot \kappa(\sigma) - \frac{d\kappa(\sigma)}{d\sigma} + \delta_2^T(\sigma, 0) + \delta_3(0, \sigma) = 0 \tag{18}$$

$$C^T \cdot \kappa(\sigma) - \delta_2(-\tau, \sigma) = 0 \tag{19}$$

$$B^T \cdot \kappa(\sigma) - \delta_3(-r, \sigma) = 0 \tag{20}$$

$$\frac{d\gamma_2(\sigma)}{d\sigma} = 0 \tag{21}$$

$$\frac{\partial \delta_1(\theta, \xi)}{\partial \theta} + \frac{\partial \delta_1(\theta, \xi)}{\partial \xi} = 0 \tag{22}$$

$$\frac{\partial \delta_2(\sigma, \eta)}{\partial \sigma} + \frac{\partial \delta_2(\sigma, \eta)}{\partial \eta} = 0 \tag{23}$$

$$\frac{\partial \delta_3(\theta, \sigma)}{\partial \theta} + \frac{\partial \delta_3(\theta, \sigma)}{\partial \sigma} = 0 \tag{24}$$

$$\text{for} \quad \theta \in [-r, 0], \quad \sigma \in [-\tau, 0], \quad \xi \in [\theta, 0], \quad \eta \in [\sigma, 0]\,.$$

From equations (12) and (17) it results that

$$\gamma_1(\theta) = 0 \quad \text{for} \quad \theta \in [-r, 0]\,. \tag{25}$$

From equations (13) and (21) we get that

$$\gamma_2(\sigma) = 0 \quad \text{for} \quad \sigma \in [-\tau, 0]\,. \tag{26}$$

The solutions of equations (22)–(24) are functions of

$$\delta_i(\theta, \sigma) = \varphi_i(\theta - \sigma) \quad \text{where} \quad \varphi_i \in C^1(\,[-r, \tau]\,) \quad \text{for} \quad i = 1, 2, 3\,. \tag{27}$$

From equations (15) and (27) we obtain

$$\delta_1(-r, \theta) = \varphi_1(-\theta - r) = B^T \cdot \beta(\theta)\,. \tag{28}$$

Hence

$$\delta_1^T(\theta, 0) = \varphi_1^T(\theta) = \beta^T(-\theta - r) \cdot B\,. \tag{29}$$

From equations (16) and (27) we obtain

$$\delta_3^T(\theta, -\tau) = \varphi_3^T(\theta + \tau) = C^T \cdot \beta(\theta)\,. \tag{30}$$

Hence

$$\delta_3^T(\theta,0)=\varphi_3^T(\theta)=C^T\cdot\beta(\theta-\tau)\,. \tag{31}$$

When we put (29) and (31) into (14), we get the formula

$$\frac{d\beta(\theta)}{d\theta}=A^T\beta(\theta)+\beta^T(-\theta-r)\cdot B+C^T\beta(\theta-\tau)\quad\text{for}\quad\theta\in[-r,0]\,. \tag{32}$$

From equation (19) we obtain

$$\delta_2(-\tau,\sigma)=\varphi_2(-\sigma-\tau)=C^T\cdot\kappa(\sigma)\,. \tag{33}$$

Hence

$$\delta_2^T(\sigma,0)=\varphi_2^T(\sigma)=\kappa^T(-\sigma-\tau)\cdot C\,. \tag{34}$$

From equation (20) we obtain

$$\delta_3(-r,\sigma)=\varphi_3(-\sigma-r)=B^T\cdot\kappa(\sigma)\,. \tag{35}$$

Hence

$$\delta_3(0,\sigma)=\varphi_3(-\sigma)=B^T\cdot\kappa(\sigma-r)\,. \tag{36}$$

When we put (34) and (36) into (18), we get the formula

$$\frac{d\kappa(\sigma)}{d\sigma}=A^T\kappa(\sigma)+\kappa^T(-\sigma-\tau)C+B^T\kappa(\sigma-r)\quad\text{for}\quad\sigma\in[-\tau,0]\,. \tag{37}$$

We introduce two new functions

$$\eta(\theta)=\beta(-\theta\text{−r})\quad\text{for}\quad\theta\in[-r,0] \tag{38}$$

$$\vartheta(\sigma)=\kappa(-\sigma-\tau)\quad\text{for}\quad\sigma\in[-\tau,0]\,. \tag{39}$$

We calculate the derivatives of (38) and (39)

$$\frac{d\eta(\theta)}{d\theta}=-\beta^T(\theta)B-A^T\eta(\theta)-C^T\eta(\theta+\tau)\quad\text{for}\quad\theta\in[-r,0] \tag{40}$$

$$\frac{d\vartheta(\sigma)}{d\sigma}=-\kappa^T(\sigma)C-A^T\vartheta(\sigma)-B^T\vartheta(\sigma+r)\quad\text{for}\quad\sigma\in[-\tau,0]\,. \tag{41}$$

We obtained the system of differential equations

$$\begin{cases}\dfrac{d\beta(\theta)}{d\theta}=A^T\cdot\beta(\theta)+\eta^T(\theta)\cdot B+C^T\cdot\beta(\theta-\tau)\\ \dfrac{d\eta(\theta)}{d\theta}=-\beta^T(\theta)\cdot B-A^T\cdot\eta(\theta)-C^T\cdot\eta(\theta+\tau)\\ \dfrac{d\kappa(\sigma)}{d\sigma}=A^T\cdot\kappa(\sigma)+\vartheta^T(\sigma)\cdot C+B^T\cdot\kappa(\sigma-r)\\ \dfrac{d\vartheta(\sigma)}{d\sigma}=-\kappa^T(\sigma)\cdot C-A^T\cdot\vartheta(\sigma)-B^T\cdot\vartheta(\sigma+r)\\ \theta\in[-r,0]\,,\quad\sigma\in[-\tau,0]\,.\end{cases} \tag{42}$$

Functions β, η, κ, ϑ are not independent, β and η are linked by formula (38), κ and ϑ by formula (39). The functions β and κ are also combined. This is implied by formulas (31) and (35). From (31) we obtain

$$\varphi_3(\theta) = \beta^T(\theta - \tau)C$$

and from (35) we have

$$\varphi_3(\sigma) = B^T\kappa(-\sigma - r)\,.$$

According to (27), function φ_3 is defined on the interval $[-r, \tau]$.

Now we can write down the following functional interdependences between the functions β, η, κ, ϑ

$$C^T \cdot \beta(\theta - \tau) = \vartheta^T(\theta + r - \tau) \cdot B \quad \text{for} \quad \theta \in [-r, -r + \tau] \tag{43}$$

$$C^T \cdot \eta(\theta + \tau) = \kappa^T(\theta) \cdot B \quad \text{for} \quad \theta \in [-\tau, 0] \tag{44}$$

$$B^T \cdot \kappa(\sigma - r) = \eta^T(\sigma - r + \tau) \cdot C \quad \text{for} \quad \sigma \in [-\tau, -\tau + r - \tau] \tag{45}$$

$$B^T \cdot \vartheta(\sigma + r) = \beta^T(\sigma) \cdot C \quad \text{for} \quad \sigma \in [-\tau, 0]\,. \tag{46}$$

Upon taking the relations (25), (26) and (38), (39) into account, equations (9)–(11) take the form of

$$A^T \cdot \alpha + \alpha \cdot A + \frac{\beta(0) + \beta^T(0)}{2} + \frac{\kappa(0) + \kappa^T(0)}{2} = -I \tag{47}$$

$$2\alpha \cdot B = \eta(0) \tag{48}$$

$$2\alpha \cdot C = \vartheta(0)\,. \tag{49}$$

The solution of the differential equations (42) with regard to relations (43)–(46) satisfies the conditions

$$\beta\left(\frac{-r}{2}\right) = \eta\left(\frac{-r}{2}\right) \tag{50}$$

$$\kappa\left(\frac{-\tau}{2}\right) = \vartheta\left(\frac{-\tau}{2}\right)\,. \tag{51}$$

Formula (50) was obtained from (38) and formula (51) from (39).

The set of algebraic equations (47)–(51) allows for determination of the matrix α and the initial conditions of the functions $\beta(\theta)$, $\eta(\theta)$, $\kappa(\sigma)$, $\vartheta(\sigma)$ *for* $\theta \in [-r, 0]$, $\sigma \in [-\tau, 0]$.

From equations (27), (29) and (38) we get the formula

$$\delta_1(\theta, \sigma) = B^T \cdot \eta(\theta - \sigma) \quad \text{for} \quad \theta \in [-r, 0], \quad \sigma \in [-\tau, 0]\,. \tag{52}$$

From equations (27), (34), (39) we obtain the formula

$$\delta_2(\theta, \sigma) = C^T \cdot \vartheta(\theta - \sigma) \quad \text{for} \quad \theta \in [-r, 0], \quad \sigma \in [-\tau, 0]\,. \tag{53}$$

From equations (27), (31), (36) we obtain the formula

$$\delta_3(\theta,\sigma) = \beta^T(\theta-\sigma-\tau)\cdot C = B^T\cdot\kappa(\sigma-\theta-r) \quad \text{for} \quad \theta\in[-r,0], \quad \sigma\in[-\tau,0]\,. \tag{54}$$

In this way we obtained all parameters of the Lyapunov functional.

4. A specific case, the system with two commensurate delays

Let us consider a special case, in which the system of equations (42) will be transformed into the set of ordinary differential equations. We assume that the quotient of delays τ and r is a positive rational number which is less than or equal to 1. Hence, there exist two natural numbers m and n such that there does not exist a natural number not equal to 1 that divides the numbers m and n. The following relationships hold

$$\tau = m\cdot h; \quad r = n\cdot h; \quad m,n\in N; \quad 0<h\in R\,. \tag{55}$$

We introduce the functions

$$\begin{cases} \beta_i(\xi),\ \eta_i(\xi),\ \kappa_j(\xi),\ \vartheta_j(\xi) \\ \quad \text{for } \xi\in[-h,0];\ i=1,...,n; \quad j=1,...,m \\ \beta(\theta)=\beta_i(\theta),\ \eta(\theta)=\eta_i(\theta) \\ \quad \text{for } \theta\in[-r+(i-1)\cdot h,\ -r+i\cdot h];\ i=1,...,n \\ \kappa(\sigma)=\kappa_j(\sigma),\ \vartheta(\sigma)=\vartheta_j(\sigma) \\ \quad \text{for } \sigma\in[-\tau+(j-1)\cdot h,\ -\tau+j\cdot h];\ j=1,...,m\,. \end{cases} \tag{56}$$

These functions satisfy the following set of conditions

$$\begin{cases} \beta_1(-h)=\beta(-r)=\eta(0) \\ \beta_i(-h)=\beta_{i-1}(0) \quad \text{for} \quad i=2,...,n \\ \eta_1(-h)=\eta(-r)=\beta(0) \\ \eta_i(-h)=\eta_{i-1}(0) \quad \text{for} \quad i=2,...,n \\ \kappa_1(-h)=\kappa(-\tau)=\vartheta(0) \\ \kappa_j(-h)=\kappa_{j-1}(0) \quad \text{for} \quad j=2,...,m \\ \vartheta_1(-h)=\vartheta(-\tau)=\kappa(0) \\ \vartheta_j(-h)=\vartheta_{j-1}(0) \quad \text{for} \quad j=2,...,m\,. \end{cases} \tag{57}$$

We can write equations (42) with regard to dependencies (43)–(46) for the functions (56) in the form

$$\begin{cases} \dfrac{d\beta_i(\xi)}{d\xi} = A^T \cdot \beta_i(\xi) + \eta_i^T(\xi) \cdot B + \vartheta_i^T(\xi) \cdot B \\ \quad \text{for} \quad i = 1, ..., m \\ \dfrac{d\beta_i(\xi)}{d\xi} = A^T \cdot \beta_i(\xi) + \eta_i^T(\xi) \cdot B + C^T \cdot \beta_{i-m}(\xi) \\ \quad \text{for} \quad i = m+1, ..., n \\ \dfrac{d\eta_i(\xi)}{d\xi} = -\beta_i^T(\xi) \cdot B - A^T \cdot \eta_i(\xi) - C^T \cdot \eta_{i+m}(\xi) \\ \quad \text{for} \quad i = 1, ..., n-m \\ \dfrac{d\eta_i(\xi)}{d\xi} = -\beta_i^T(\xi) \cdot B - A^T \cdot \eta_i(\xi) - \kappa_{i-(n-m)}^T(\xi) \cdot B \\ \quad \text{for} \quad i = n-m+1, ..., n \\ \dfrac{d\kappa_j(\xi)}{d\xi} = A^T \cdot \kappa_j(\xi) + \vartheta_j^T(\xi) \cdot C + \eta_j^T(\xi) \cdot C \\ \quad \text{for} \quad j = 1, ..., n-m \\ \dfrac{d\kappa_j(\xi)}{d\xi} = A^T \cdot \kappa_j(\xi) + \vartheta_j^T(\xi) \cdot C + B^T \cdot \kappa_{j-(n-m)}(\xi) \\ \quad \text{for} \quad j = n-m+1, ..., m \\ \dfrac{d\vartheta_j(\xi)}{d\xi} = -\kappa_j^T(\xi) \cdot C - A^T \cdot \vartheta_j(\xi) - \beta_{j+n-m}^T(\xi) \cdot C \\ \quad \text{for} \quad j = 1, ..., m \qquad \xi \in [-h, 0]\,. \end{cases} \tag{58}$$

There are relationships between the initial conditions of the system (58) as below

$$\begin{cases} \beta_i(0) = \eta_{n-i}(0) \quad \text{for} \quad i = 1, ..., n-1 \\ \beta_n(0) = \beta(0) \\ \vartheta_j(0) = \kappa_{m-j}(0) \quad \text{for} \quad j = 1, ..., m-1 \\ \vartheta_m(0) = \vartheta(0)\,. \end{cases} \tag{59}$$

We obtain matrix α and the initial conditions of the system (58) by solving the set of algebraic equations

$$\begin{cases} A^T \cdot \alpha + \alpha \cdot A + \dfrac{\beta(0) + \beta^T(0)}{2} + \dfrac{\kappa(0) + \kappa^T(0)}{2} = -I \\ 2\alpha \cdot B = \eta(0) \\ 2\alpha \cdot C = \vartheta(0) \\ \beta_i\left(-\dfrac{h}{2}\right) = \eta_{n+1-i}\left(-\dfrac{h}{2}\right) \quad \text{for} \quad i = 1, ..., n \\ \kappa_j\left(-\dfrac{h}{2}\right) = \vartheta_{m+1-j}\left(-\dfrac{h}{2}\right) \quad \text{for} \quad j = 1, ..., m\,. \end{cases} \tag{60}$$

Having a solution of the set of equations (58) we can obtain the matrices $\beta(\theta)$, $\eta(\theta)$, $\kappa(\sigma)$, $\vartheta(\sigma)$ from equations (56) and the matrices $\delta_1(\theta,\sigma)$, $\delta_2(\theta,\sigma)$, $\delta_3(\theta,\sigma)$ from equations (52)–(54).

5. An example

Let us consider the system described by equation

$$\begin{cases} \dfrac{dx(t)}{dt} = a \cdot x(t) + b \cdot x_t(-2h) + c \cdot x_t(-h) \\ x(0) = x_0 \in R \\ x_t(0) = \Phi \in L^2([-2h,\, 0), R) \end{cases} \tag{61}$$

$$t \geq 0;\ x(t) \in R;\ x_t \in L^2([-2h,\, 0), R);\ x_t(\theta) = x(t+\theta); \tag{62}$$
$$a, b, c \in R;\ h > 0\,.$$

The Lyapunov functional is defined by the formula

$$\begin{aligned} V(S(t)) &= \alpha \cdot x^2(t) + \int_{-2h}^{0} x(t) \cdot \beta(\theta) \cdot x_t(\theta) d\theta + \int_{-2h}^{0} \gamma_1(\theta) \cdot x_t^2(\theta) d\theta + \\ &+ \int_{-h}^{0} x(t) \cdot \kappa(\sigma) \cdot x_t(\sigma) d\sigma + \int_{-h}^{0} \gamma_2(\sigma) \cdot x_t^2(\sigma) d\sigma + \\ &+ \int_{-2h}^{0} \int_{\theta}^{0} x_t(\theta) \cdot \delta_1(\theta,\xi) \cdot x_t(\xi) d\xi d\theta + \int_{-h}^{0} \int_{\sigma}^{0} x_t(\sigma) \cdot \delta_2(\sigma,\zeta) \cdot x_t(\zeta) d\zeta d\sigma + \\ &+ \int_{-2h}^{0} \int_{-h}^{0} x_t(\theta) \cdot \delta_3(\theta,\sigma) \cdot x_t(\sigma) d\sigma d\theta\,. \end{aligned} \tag{63}$$

The set of equations (58) becomes

$$\begin{bmatrix} \dfrac{d\beta_1(\xi)}{d\xi} \\ \dfrac{d\beta_2(\xi)}{d\xi} \\ \dfrac{d\eta_1(\xi)}{d\xi} \\ \dfrac{d\eta_2(\xi)}{d\xi} \\ \dfrac{d\kappa(\xi)}{d\xi} \\ \dfrac{d\vartheta(\xi)}{d\xi} \end{bmatrix} = \begin{bmatrix} a & 0 & b & 0 & 0 & b \\ c & a & 0 & b & 0 & 0 \\ -b & 0 & -a & -c & 0 & 0 \\ 0 & -b & 0 & -a & -b & 0 \\ 0 & 0 & c & 0 & a & c \\ 0 & -c & 0 & 0 & -c & -a \end{bmatrix} \cdot \begin{bmatrix} \beta_1(\xi) \\ \beta_2(\xi) \\ \eta_1(\xi) \\ \eta_2(\xi) \\ \kappa(\xi) \\ \vartheta(\xi) \end{bmatrix}, \tag{64}$$
$$\xi \in [-h, 0]\,.$$

Eigenvalues of the matrix of equation (64) are as follows

$$\lambda_1 = a \quad \lambda_2 = -a \quad \lambda_3 = \sqrt{g+d} \quad \lambda_4 = -\sqrt{g+d} \quad \lambda_5 = \sqrt{g-d}$$
$$\lambda_6 = -\sqrt{g-d} \quad where \quad g = a^2 - b^2 - \frac{c^2}{2}, \quad d = c\cdot\sqrt{\frac{c^2}{4} + 2b^2 - 2a\cdot b}\,.$$

Now we give the formulas for determination of the set of initial conditions of equation (64). Relations (57) take the form as below:

$$\begin{cases} \beta_1(-h) = \eta(0) \\ \beta_2(-h) = \beta_1(0) \\ \eta_1(-h) = \beta(0) \\ \eta_2(-h) = \eta_1(0) \\ \kappa(-h) = \vartheta(0) \\ \vartheta(-h) = \kappa(0) \end{cases} . \tag{65}$$

Among the initial conditions there are relations as below:

$$\begin{cases} \beta_1(0) = \eta_1(0) \\ \beta_2(0) = \beta(0) \end{cases} . \tag{66}$$

Relations (60) become

$$\begin{cases} 2\alpha\cdot a + \beta(0) + \kappa(0) = -1 \\ 2\alpha\cdot b = \eta(0) \\ 2\alpha\cdot c = \vartheta(0) \\ \eta_1(-\frac{h}{2}) = \beta_2(-\frac{h}{2}) \\ \beta_1(-\frac{h}{2}) = \eta_2(-\frac{h}{2}) \\ \kappa(-\frac{h}{2}) = \vartheta(-\frac{h}{2}) \end{cases} . \tag{67}$$

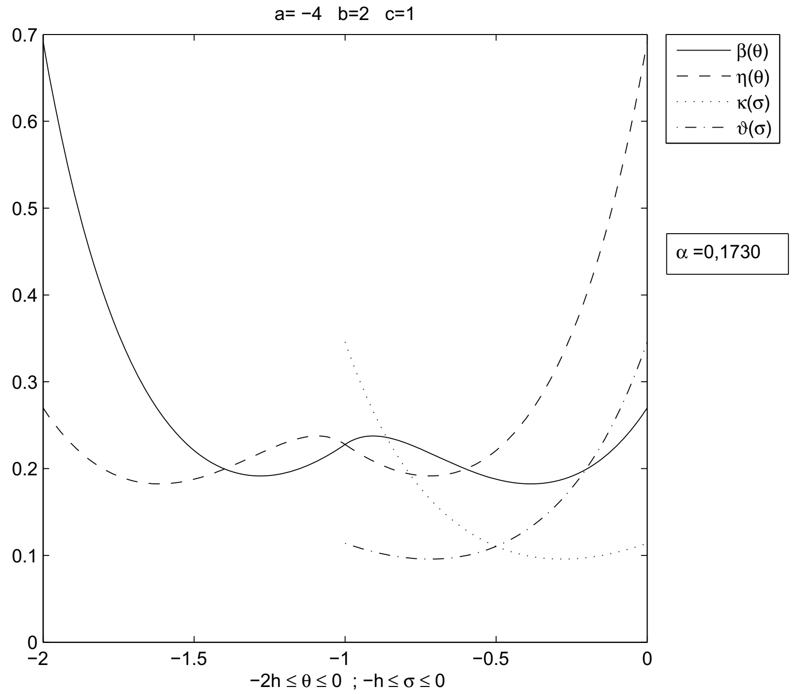
Having the solution of equations (64)

$$\beta_1(\xi),\ \beta_2(\xi),\ \eta_1(\xi),\ \eta_2(\xi),\ \kappa(\xi),\ \vartheta(\xi), \quad \xi\in[-h,\ 0] \quad \text{and the matrix } \alpha$$

we obtain

$$\beta(\theta),\ \eta(\theta),\ \kappa(\sigma),\ \vartheta(\sigma)\ and\ \delta_1(\theta,\sigma) = b\cdot\eta(\theta-\sigma),$$
$$\delta_2(\theta,\sigma) = c\cdot\vartheta(\theta-\sigma),\ \delta_3(\theta,\sigma) = c\cdot\beta(\theta-\sigma-\tau)\,.$$

The figure below shows the graphs of functions $\beta(\theta)$, $\eta(\theta)$, $\kappa(\sigma)$, $\vartheta(\sigma)$ and α, obtained with the Matlab code, for given values of parameters a, b and c of the system (61).



6. Conclusions

The paper presents the procedure for determining of the coefficients of the Lyapunov functional, given by the formula (4), for the linear system with two delays, described by equation (1). This paper extended the method due to Repin to the systems with two delays. The method presented allows for obtaining the analytical formula for the factors occurring in the Lyapunov functional, which can be used to examine the stability and in the process of parametric optimization to determine the square quality index, given by formula (8).

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