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Boundary control of retarded parabolic systems*

by

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Abstract: Optimal boundary control problems for distributed parabolic systems in which retarded arguments appear in the integral form in the state equations are presented. Necessary and sufficient conditions of optimality are derived for the non-homogeneous Dirichlet problem. A simple example of application is also provided.

Keywords: boundary control, retarded argument, parabolic system.

1. Introduction

In this paper we consider linear retarded parabolic systems with non-homogeneous Dirichlet boundary conditions. The retarded argument appears in the integral form with $h \in (0,c)$ in the state equation. Using the transposition method and some interpolation theorems sufficient conditions for the existence of unique solutions for such retarded parabolic systems are proved. The performance functional has the quadratic form. The time horizon T is fixed. Finally, we impose some constraints on the boundary control. Necessary and sufficient conditions of optimality for the non-homogeneous Dirichlet problem with the quadratic performance functional and constrained control are derived. A simple example of application is also presented.

2. Existence of solutions in the space $H^{2,1}(Q)$

Consider the distributed-parameter system described by the following parabolic delay equation:

$$\frac{\partial y}{\partial t} + A(t)y + \int_0^c b(x,t)y(x,t-h)dh = u \qquad x \in \Omega,\ t \in (0,T),\ h \in (0,c) \tag{2.1}$$

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$$y(x,t') = \Phi_o(x,t') \qquad x\in \Omega,\ t' \in [-c,0) \tag{2.2}$$

$$y(x,0) = y_0(x) \qquad x \in \Omega \tag{2.3}$$

$$y(x,t) = v(x,t) \qquad x\in \Gamma,\ t\in (0,T), \tag{2.4}$$

where $\Omega \subset R^n$ is a bounded, open set with boundary Γ, which is a C^∞ manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ;

$$y \equiv y(x,t;u), \qquad u \equiv u(x,t), \qquad v \equiv v(x,t), \qquad \Sigma \equiv \Gamma \times (0,T),$$
$$Q = \Omega \times (0,T), \qquad Q_0 = \Omega \times [-c,0),$$

where:
T is a specified positive number representing time horizon,
b is a given real C^∞ function defined on $\bar{Q}$,
h is a retarded argument such that $h \in (0,c)$,
Φ_0 is a initial function defined on Q_o.

The parabolic operator $\frac{\partial}{\partial t} + A(t)$ in the state equation (2.1) satisfies the hypothesis of Lions and Magenes (1972, Vol. 2, p. 2) and A(t) is given by

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right) \tag{2.5}$$

where the functions $a_{ij}(x,t)$ are real C^∞ functions defined on $\bar{Q}$ (closure of Q) satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\varphi_i\varphi_j \geq \alpha \sum_{i=1}^{n}\varphi_i^2, \alpha > 0, \qquad \forall (x,t) \in \bar{Q}, \forall \varphi_i \in R. \tag{2.6}$$

Equations (2.1) - (2.4) constitute a Dirichlet problem.

First we shall prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (2.1) - (2.4) in the space $H^{2,1}(Q)$. For this purpose, for any pair of real numbers $r, s \geq 0$, we introduce the Sobolev space $H^{r,s}(Q)$ (Lions and Magenes, 1972, Vol. 2, p. 6) defined by

$$H^{r,s}(Q) = H^0(0,T;H^r(\Omega)) \cap H^s(0,T;H^0(\Omega)) \tag{2.7}$$

which is a Hilbert space normed by

$$\left(\int_0^T \parallel y(t) \parallel^2_{H^r(\Omega)} dt + \parallel y \parallel^2_{H^s(0,T;H^0(\Omega))}\right)^{1/2} \tag{2.8}$$

where the spaces $H^r(\Omega)$ and $H^s(0,T;H^0(\Omega))$ are defined in Lions and Magenes (1972, Vol. 1, Chapter 1).

THEOREM 2.1 *Let y_0, Φ_0, v, u be given with $y_0 \in H^1(\Omega)$, $\Phi_0 \in H^{2,1}(Q_0)$, $v \in H^{\frac{3}{2},\frac{3}{4}}(\sum)$ and $u \in L^2(Q)$ and the following compatibility relation is satisfied:*

$$y_0(x) = v(x,0) \quad \text{on } \Gamma \qquad (\mathcal{R}.\mathcal{C}.).$$

Then, there exists a unique solution $y \in H^{2,1}(Q)$ for the mixed initial-boundary value problem (2.1) - (2.4).

Proof. The parabolic delay equations (2.1) with boundary conditions (2.2)–(2.4) may be rewritten as

$$\frac{\partial y}{\partial t} + A(t)y = Ny + f \tag{2.9}$$

where

$$f(x,t) := u(x,t) + b(x,t) \int\limits_{\min(0,t-c)}^{0} \Phi_0(x,\tau)\, d\tau \tag{2.10}$$

$$Ny(x,t) := b(x,t) \int\limits_{\max(0,t-c)}^{t} y(x,\tau)\, d\tau. \tag{2.11}$$

Let

$$G_0 : H^1(\Omega) \to H^{2,1}(Q) \tag{2.12a}$$

$$G_1 : H^{\frac{3}{2},\frac{3}{4}}(\Sigma) \to H^{2,1}(Q) \tag{2.12b}$$

$$S : L^2(Q) \to H^{2,1}(Q) \tag{2.12c}$$

denote the continuous solution operators provided by Theorem 6.1 and Remark 6.3 of Lions and Magenes (1972, Vol. 2, pp. 33 and 37). Then the problem (2.1) - (2.4) is equivalent to the fixed point equation

$$y = G_0 y_0 + G_1 v + Sf + SNy. \tag{2.13}$$

We need to find an estimate for $\|SN\|_{\mathcal{L}(H^{2,1}(Q),H^{2,1}(Q))}$. We have

$$\|SNy\|_{H^{2,1}(Q)} \leq \|S\|_{\mathcal{L}(L^2(Q),H^{2,1}(Q))} \|Ny\|_{L^2(Q)}$$

$$\leq c \int\limits_{\max(o,t-c)}^{t} \|y(x,\tau)\|_{H^{2,1}(Q)}\, d\tau \leq cT \|y\|_{H^{2,1}(Q)}. \tag{2.14}$$

From (2.14) we deduce

$$\|SN\|_{\mathcal{L}(H^{2,1}(Q),H^{2,1}(Q))} < 1 \quad \text{if} \quad T < \frac{1}{c}.$$

Evidently, we can extend our result to any $T < +\infty$. ∎

3. The adjoint problem

Consider now the adjoint problem in the context of the Theorem 2.1, that is

$$A^*(t)p - p' + \int_0^c b(x,t+h)p(x,t+h)dh = \varphi \qquad in\ Q, \tag{3.1}$$

$$p(x,t') = 0, \qquad x \in \Omega, \qquad t' \in (T,c], \tag{3.2}$$

$$p(x,t) = 0 \qquad on\ \Sigma, \tag{3.3}$$

$$p(x,T) = 0, \qquad x \in \Omega, \tag{3.4}$$

where

$$A^*(t)p = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial p}{\partial x_i}\right).$$

The problem (3.1) - (3.4) can be solved backwards in time. For this purpose, we may apply Theorem 2.1.

The following result can be proved

LEMMA 3.1 *Let φ be given in $L^2(Q)$. Then, there exists a unique solution $p \in H^{2,1}(Q)$ for the problem (3.1) - (3.4).*

Let us denote by $X(Q)$ the space described by the solutions of (3.1) - (3.4) as φ describes $L^2(Q)$.

We have

$$X(Q) \subset H^{2,1}(Q).$$

We can equivalently define

$$X(Q) = \left\{ p | p \in H^{2,1}(Q) : p(x,t) = 0\ on\ \Sigma, p(x,T) = 0, \right.$$
$$\left. A^*p - p' + \int_0^c b(x,t+h)p(x,t+h)dh \in L^2(Q) \right\}.$$

Providing $X(Q)$ with the norm of the graph we get

$$P^*(= A^*p - p' + \int_0^c b(x,t+h)p(x,t+h)dh)$$
$$\text{is an isomorphism of } X(Q) \text{ onto } L^2(Q). \tag{3.5}$$

4. Transposition of the adjoint isomorphism

By transposition we deduce from (3.5)

LEMMA 4.1 *Let $p \to L(p)$ be a continuous linear form on $X(Q)$. Then, there exists a unique $y \in L^2(Q)$ such that*

$$\langle y, A^*p - p' + \int_0^c b(x,t+h)p(x,t+h)dh\rangle = L(p) \qquad \forall p \in X(Q) \tag{4.1}$$

We choose L in the form

$$L(p) = \int_\Omega \int_0^c \int_{-h}^0 \Phi_0(t,x)b(x,t+h)p(x,t+h)dtdhdx + \langle u,p\rangle + \langle v, p_{|\Sigma}\rangle + \langle y_0, p(x,0)\rangle. \tag{4.2}$$

We take

$$\Phi_0 \in L^2(Q_0) \tag{4.3}$$

$$u \in (H^{2,1}(Q))'. \tag{4.4}$$

Since $p \to p_{|\Sigma}$ is a continuous linear mapping of $X(Q) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ we may take

$$v \in (H^{\frac{1}{2},\frac{1}{4}}(\Sigma))'. \tag{4.5}$$

Similarly, since $p \to p(x,0)$ is a continuous linear mapping of $X(Q) \to H_0^1(\Omega)$, we may take

$$y_0 \in H^{-1}(\Omega). \tag{4.6}$$

According to Lemma 4.1 we have

THEOREM 4.1 *Let Φ_0, u, v, y_0 be given with (4.3),(4.4),(4.5), and (4.6). There exists a unique $y \in L^2(Q)$ such that (4.1) holds with (4.2).*

5. Existence of solutions in the space $H^{\frac{1}{2},\frac{1}{4}}(Q)$

We shall now apply interpolation theory (Lions and Magenes, 1972, Vol. 1, Chapter 1, Section 5).

We consider the mapping $\mathcal{G}$

$$\mathcal{G} : \{\Phi_0, u, v, y_0\} \to y = \mathcal{G}(\Phi_0, u, v, y_0), \tag{5.1}$$

then from Theorem 2.1 and Theorem 4.1 it follows that it is a continuous mapping of

$$H^{2,1}(Q) \times H^{0}(Q) \times (H^{\frac{3}{2},\frac{3}{4}}(\Sigma) \times H^{1}(\Omega), \mathcal{R}.\mathcal{C}.) \to H^{2,1}(Q) \tag{5.2}$$

and

$$(H^{0}(Q) \times H^{2,1}(Q))' \times (H^{\frac{1}{2},\frac{1}{4}}(\Sigma))' \times (H^{1}(\Omega))' \to H^{0}(Q). \tag{5.3}$$

We shall now interpolate between (5.2) and (5.3). We set

$$A_0 = (H^{\frac{3}{2},\frac{3}{4}}(\Sigma) \times H^{1}(\Omega), \mathcal{R}.\mathcal{C}.) \tag{5.4}$$

$$A_1 = (H^{\frac{1}{2},\frac{1}{4}}(\Sigma))' \times (H^{1}(\Omega))' \tag{5.5}$$

Using the Theorem 14.1 of Lions and Magenes (1972, Vol. 2, p. 68) we have

$$[A_0, A_1]_{\Theta=\frac{3}{4}} = H^{0}(\Sigma) \times (H^{\frac{1}{2}}(\Omega))'. \tag{5.6}$$

According to the results of Lions and Magenes (1972, vol.2, Chapter 4, Section 15.1 and 2.1) we have

$$[H^{0}(Q), (H^{2,1}(Q))']_{\Theta=\frac{3}{4}} = (H^{\frac{3}{2},\frac{3}{4}}(Q))', \tag{5.7}$$

$$[H^{2,1}(Q), H^{0}(Q)]_{\Theta=\frac{3}{4}} = H^{\frac{1}{2},\frac{1}{4}}(Q). \tag{5.8}$$

From (5.1) - (5.8) we deduce

$$\left.\begin{array}{l} \mathcal{G}\textit{ defined by } (5.1) \textit{ is a continuous linear mapping of} \\ H^{\frac{1}{2},\frac{1}{4}}(Q) \times (H^{\frac{3}{2},\frac{3}{4}}(Q))' \times H^{0}(\Sigma) \times (H^{\frac{1}{2}}(\Omega))' \to H^{\frac{1}{2},\frac{1}{4}}(Q) \end{array}\right\}. \tag{5.9}$$

From (5.9) we obtain

THEOREM 5.1 *Let Φ_0, u, v and y_0 be given with $\Phi_0 \in H^{\frac{1}{2},\frac{1}{4}}(Q)$, $u \in (H^{\frac{3}{2},\frac{3}{4}}(Q))'$, $v \in L^2(\Sigma)$, $y_0 \in (H^{\frac{1}{2}}(\Omega))'$. Then, there exists a unique solution $y \in H^{\frac{1}{2},\frac{1}{4}}(Q)$ for the mixed initial - boundary value problem (2.1) - (2.4) (in the sense of Theorem 4.1).*

6. Optimal boundary control

We shall now formulate the optimal boundary control problem for the Dirichlet problem (2.1) - (2.4) in the context of Theorem 5.1. Let us denote by $U = L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem.

The performance functional is given by

$$I(v) = \lambda_1 \left| y(x,T;v) - z_d \right|^2_{H^{-1}(\Omega)} + \lambda_2 \int_{\Sigma} (Nv) v \, d\Gamma dt \tag{6.1}$$

where: $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 > 0$; z_d is a given element in $H^{-1}(\Omega)$; N is given,

$$N \in \mathcal{L}(L^2(\Sigma), L^2(\Sigma)), \quad (Nv, v)_{L^2(\Sigma)} \geq \mu \|v\|^2_{L^2(\Sigma)}, \mu > 0.$$

Finally, we assume the following constraint on controls $v \in U_{ad}$, where

$$U_{ad} \text{ is a closed, convex subset of } U. \tag{6.2}$$

Let $y(x, t; v)$ denote the solution of the mixed initial-boundary value problem (2.1) - (2.4) corresponding to a given control $v \in U_{ad}$. The starting point for our considerations will be following theorem, which can be found in Lions (1971, p.10).

Theorem 6.1 *Assume that the function $v \to I(v)$ is strictly convex and differentiable such that $I(v) \to +\infty$ as $\|v\| \to +\infty$, $v \in U_{ad}$ (the last hypothesis may be omitted if U_{ad} is bounded). Then the unique element $v_0 \in U_{ad}$ satisfying $I(v_0) = inf_{v \in U_{ad}} I(v)$ is characterized by*

$$I'(v_0) \cdot (v - v_0) \geq 0 \qquad \forall v \in U_{ad}. \tag{6.3}$$

The solving of the formulated optimal control problem is equivalent to seeking a $v_0 \in U_{ad}$ such that $I(v_0) \leq I(v)$ $\forall v \in U_{ad}$.
Then from Theorem 6.1 it follows that for $\lambda_2 > 0$ a unique optimal control v_0 exists; moreover, v_0 is characterized by the condition (6.3).

Using the form of the performance functional (6.1) we can express (6.3) in the following form

$$\lambda_1 \langle y(x, T; v_0) - z_d, y(x, T; v) - y(x, T; v_0)\rangle_{H^{-1}(\Omega)} + \lambda_2 \int_\Sigma N v_0 (v - v_0) d\Gamma dt \geq 0 \qquad \forall v \in U_{ad}. \tag{6.4}$$

To simplify (6.4), we introduce the adjoint equation and for every $v \in U_{ad}$, we define the adjoint variable $p = p(v) = p(x, t; v)$ as the solution of the equation

$$-\frac{\partial p(v)}{\partial t} + A^*(t) p(v) + \int_0^c b(x, t+h) p(x, t+h; v) dh = 0,$$

$$x \in \Omega, \ t \in (0, T), \ h \in (0, c), \tag{6.5}$$

$$p(x, t; v) = 0 \qquad x \in \Omega, \ t \in (T, T+c), \tag{6.6}$$

$$p(x, T; v) = -\lambda_1 (-\Delta + I)^{-1} (y(x, T; v) - z_d) \qquad x \in \Omega, \tag{6.7}$$

$$p(x, t; v) = 0, \qquad x \in \Gamma, \ t \in (0, T). \tag{6.8}$$

The norm in the space $H^{-1}(\Omega)$ is given by the following formula

$$\|f\|^2_{H^{-1}(\Omega)} = \int_\Omega ((-\Delta + I)^{-1} f) f dx \tag{6.9}$$

where:

$(-\Delta+I)^{-1}f=\Phi$ - the solution of the problem

$(-\Delta+I)\Phi=f$ in Ω and $\Phi=0$ on Γ.

Moreover, Δ is a Laplace operator on Ω.

The following lemma can be proved.

LEMMA 6.1 *For given $z_d \in L^2(Q)$ and any $v \in L^2(\Sigma)$, there exists a unique solution $p(v) \in H^{\frac{1}{2},\frac{1}{4}}(Q)$ for the problem (6.5) - (6.8).*

We simplify (6.4) using the adjoint equation (6.5) - (6.8). For this purpose, setting $v=v_0$ in (6.5) - (6.8), multiplying both sides of (6.5) by $(y(v)-y(v_0))$, then integrating over Q, we obtain

$$\int_Q \Big(-\frac{\partial p(v_0)}{\partial t}+A^*(t)p(v_0)+$$

$$+\int_0^c b(x,t+h)p(x,t+h;v_0)dh\Big)(y(x,t;v)-y(x,t;v_0))dxdt=$$

$$=-\int_\Omega p(x,T;v_0)(y(x,T;v)-y(x,T;v_0))dx+$$

$$+\int_Q p(v_0)\frac{\partial}{\partial t}(y(v)-y(v_0))dxdt+\int_Q A^*(t)p(v_0)(y(v)-y(v_0))dxdt+$$

$$+\int_\Omega\int_0^T\int_0^c b(x,t+h)p(x,t+h;v_0)(y(v)-y(v_0))dhdtdx=$$

$$=\lambda_1\int_\Omega(-\Delta+I)^{-1}((y(x,T;v_0)-z_d)(y(x,T;v)-y(x,T,v_0))dx+$$

$$+\int_Q p(v_0)\frac{\partial}{\partial t}(y(v)-y(v_0))dxdt+\int_Q A^*(t)p(v_0)(y(v)-y(v_0))dxdt+$$

$$+\int_\Omega\int_0^T\int_0^c b(x,t+h)p(x,t+h;v_0)(y(v)-y(v_0))dhdtdx=$$

$$=\lambda_1\langle y(x,T;v_0)-z_d,y(x,T;v)-y(x,T;v_0)\rangle_{H^{-1}(\Omega)}+$$

$$+\int_Q p(v_0)\frac{\partial}{\partial t}(y(v)-y(v_0))dxdt+\int_Q A^*(t)p(v_0)(y(v)-y(v_0))dxdt+$$

$$+\int_0^c\int_\Omega\int_0^T b(x,t+h)p(x,t+h;v_0)(y(x,t;v)-y(x,t;v_0))dtdxdh=0. \quad (6.10)$$

Using equation (2.1), the second term on the right-hand side of (6.10) can be rewritten as

$$\int_Q p(v_0)\frac{\partial}{\partial t}(y(v)-y(v_0))dxdt = -\int_Q p(v_0)A(t)(y(v)-y(v_0))dxdt-$$

$$-\int_0^c\int_\Omega\int_0^T p(x,t;v_0)b(x,t)(y(x,t-h;v)-y(x,t-h;v_0))dtdxdh =$$

$$= -\int_Q p(v_0)A(t)(y(v)-y(v_0))dxdt-$$

$$-\int_0^c\int_\Omega\int_{-h}^{T-h} p(x,t'+h;v_0)b(x,t'+h)(y(x,t';v)-y(x,t';v_0))dt'dxdh =$$

$$= -\int_Q p(v_0)A(t)(y(v)-y(v_0))dxdt-$$

$$-\int_0^c\int_\Omega\int_{-h}^{0} p(x,t'+h;v_0)b(x,t'+h)(y(x,t';v)-y(x,t';v_0))dt'dxdh-$$

$$-\int_0^c\int_\Omega\int_0^{T-h} p(x,t'+h;v_0)b(x,t'+h)(y(x,t';v)-y(x,t';v_0))dt'dxdh =$$

$$= -\int_Q p(v_0)A(t)(y(v)-y(v_0))dxdt-$$

$$-\int_0^c\int_\Omega\int_0^{T-h} p(x,t'+h;v_0)b(x,t'+h)(y(x,t';v)-y(x,t';v_0))dt'dxdh. \quad (6.11)$$

The third term on the left-hand side of (6.10), in view of Green's formula, can be expessed as

$$\int_Q A^*(t)p(v_0)(y(v)-y(v_0))dxdt = \int_Q p(v_0)A(t)(y(v)-y(v_0))dxdt+$$

$$+\int_0^T\int_\Gamma p(v_0)(\frac{\partial y(v)}{\partial \eta_A}-\frac{\partial y(v_0)}{\partial \eta_A})d\Gamma dt - \int_0^T\int_\Gamma \frac{\partial p(v_0)}{\partial \eta_{A^*}}(y(v)-y(v_0)d\Gamma dt. \quad (6.12)$$

Using the boundary condition (2.4), the last component on the right-hand side of (6.12) can be written as

$$\int_0^T \int_\Gamma \frac{\partial p(v_0)}{\partial \eta_{A^*}} (y(v) - y(v_0) d\Gamma dt = \int_0^T \int_\Gamma \frac{\partial p(v_0)}{\partial \eta_{A^*}} (v - v_0) d\Gamma dt. \tag{6.13}$$

Substituting (6.8) and (6.13) into (6.12) and then (6.11) and (6.12) into (6.10) we obtain

$$\begin{aligned}
&\lambda_1 \langle y(x,T;v_0) - z_d, y(x,T;v) - y(x,T;v_0)\rangle_{H^{-1}(\Omega)} = \\
&= \int_Q p(v_0) A(t)(y(v) - y(v_0)) dx dt + \\
&+ \int_0^c \int_\Omega \int_0^{T-h} p(x,t+h;v_0) b(x,t+h)(y(x,t;v) - y(x,t;v_0)) dt dx dh - \\
&- \int_Q p(v_0) A(t)(y(v) - y(v_0)) dx dt + \int_0^T \int_\Gamma \frac{\partial p(v_0)}{\partial \eta_{A^*}} (v - v_0) d\Gamma dt - \\
&- \int_0^c \int_\Omega \int_0^T b(x,t+h) p(x,t+h;v_0)(y(x,t;v) - y(x,t;v_0)) dt dx dh = \\
&= - \int_0^c \int_\Omega \int_{T-h}^T p(x,t+h;v_0) b(x,t+h)(y(x,t;v) - y(x,t;v_0)) dt dx dh + \\
&+ \int_0^T \int_\Gamma \frac{\partial p(v_0)}{\partial \eta_{A^*}} (v - v_0) d\Gamma dt = \\
&- \int_0^c \int_\Omega \int_T^{T+h} p(x,t;v_0) b(x,t)(y(x,t-h;v) - y(x,t-h;v_0)) dt dx dh = \\
&= \int_0^T \int_\Gamma \frac{\partial p(v_0)}{\partial \eta_{A^*}} (v - v_0) d\Gamma dt.
\end{aligned} \tag{6.14}$$

Upon substituting (6.14) into (6.4) we obtain the formula

$$\int_\Sigma \left(\frac{\partial p(v_0)}{\partial \eta_{A^*}} + \lambda_2 N v_0 \right) (v - v_0) d\Gamma dt \geq 0 \qquad \forall v \in U_{ad}. \tag{6.15}$$

THEOREM 6.2 *For the problem (2.1) - (2.4) with the performance functional (6.1) with $z_d \in H^{-1}(\Omega)$ and $\lambda_2 > 0$ and with constraints on controls (6.2),* *there exists a unique optimal control v_0 which satisfies the maximum condition (6.15).*

We can also consider an analogous optimal control problem where the performance functional is given by

$$\hat{I}(v) = \lambda_1 \left\| \frac{\partial y(v)}{\partial \eta_A}\bigg|_\Sigma - z_{\Sigma d} \right\|^2_{H^{-1}(\Sigma)} + \lambda_2 \int\limits_\Sigma (Nv)v d\Gamma dt \tag{6.16}$$

where $z_{\Sigma d}$ is a given element in $H^{-1}(\Sigma)$.
The norm in the space $H^{-1}(\Sigma)$ is given by the following formula

$$\|g\|^2_{H^{-1}(\Sigma)} = \int\limits_\Sigma [(-\Delta_\Sigma + I)^{-1} g] g d\Gamma dt \tag{6.17}$$

where Δ_Σ is a Laplace - Beltrami operator on Σ. The Laplace-Beltrami operator is taken with homogeneous boundary conditions at $t = 0$ and $t = T$.
The optimal control v_0 is characterized by

$$\lambda_1 \left\langle \frac{\partial y(v_0)}{\partial \eta_A}\bigg|_\Sigma - z_{\Sigma d}, \frac{\partial y(v)}{\partial \eta_A} - \frac{\partial y(v_0)}{\partial \eta_A} \right\rangle_{H^{-1}(\Sigma)} +$$
$$+\lambda_2 \int\limits_\Sigma (Nv_0)(v - v_0) d\Gamma dt \geq 0 \qquad \forall v \in U_{ad}. \tag{6.18}$$

We introduce the following adjoint equation

$$-\frac{\partial p(v_0)}{\partial t} + A^*(t)p(v_0) + \int\limits_0^c b(x, t+h)p(x, t+h; v_0) = 0,$$
$$x \in \Omega,\ t \in (0, T),\ h \in (0, c), \tag{6.19}$$
$$p(x, t; v_0) = 0 \qquad x \in \Omega,\ t \in (T, T+c), \tag{6.20}$$
$$p(x, T; v_0) = 0 \qquad x \in \Omega, \tag{6.21}$$
$$p(x, t; v_0) = \lambda_1(-\Delta_\Sigma + I)^{-1} \left(\frac{\partial y(v_0)}{\partial \eta_A}\bigg|_\Sigma - z_{\Sigma d} \right),$$
$$x \in \Gamma,\ t \in (0, T) \tag{6.22}$$
$$p(x, t; v_0) = 0 \qquad x \in \Gamma,\ t = 0 \text{ and } t = T. \tag{6.23}$$

The following lemma can be proved:

Lemma 6.2 *For given $z_{\Sigma d} \in H^{-1}(\Sigma)$ and any $v_0 \in L^2(\Sigma)$, there exists a unique solution $p(v_0) \in H^{\frac{1}{2}, \frac{1}{4}}(Q)$ to the problem (6.19)-(6.23).*

In this case the condition (6.18) can be also rewritten in the form (6.15). The following theorem now holds:

THEOREM 6.3 *For the problem (2.1) - (2.4) with the performance functional (6.16) with* $z_{\Sigma d} \in H^{-1}(\Sigma)$ *and* $\lambda_2 > 0$ *and with constraints on control (6.2), there exists a unique optimal control* v_0 *which satisfies the maximum condition (6.15).*

We must notice that the conditions of optimality derived above (Theorems 6.2 and 6.3) allow us for obtaining an analytical formula for the optimal control in particular cases only (e.g. there are no constraints on controls). This results from the following: determining the function $p(v_0)$ in the maximum condition from the adjoint equation is possible if and only if we know y_0 which corresponds to the control v_0. These mutual connections make the practical use of the derived optimization formulas difficult. Therefore we resign from the exact determining of the optimal control and we use approximation methods.

In the case of performance functionals (6.1), (6.16), with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to minimizing the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one which can be solved by the use of the well-known algorithms, e.g. Gilbert's (1966).

7. Application

EXAMPLE 1 To illustrate the practical applications of the algorithm mentioned above we shall formulate the following control problem as an example: equation of the system control (2.1) - (2.4), performance functional (6.1) with $\lambda_1 = 1$ and $\lambda_2 = 0$, i.e.

$$I(v) = \|y(T) - z_d\|^2_{H^{-1}(\Omega)}, \tag{7.1}$$

and constraint on controls

$$U_{ad} = \{v \in L^2(\Sigma) : \|v(x,t)\|_{L^2(\Sigma)} \le 1\}. \tag{7.2}$$

We shall define the attainable set Y_{ad}

$$Y_{ad} = \{y(T,v) : \frac{\partial y(v)}{\partial t} + A(t)y(v) + \int_0^c b(x,t)y(x,t-h)dh = u \text{ in } Q,$$

$$y(x,t') = \Phi_0(x,t'),\ y(x,0;v) = y_0(x),\ y(x,t) = v \text{ in } \Sigma,\ v \in U_{ad}\}. \tag{7.3}$$

The following result can be proved.

THEOREM 7.1 *The set* Y_{ad} *is a closed, convex and a bounded one in the space* $Y = H^{-1}(\Omega)$.

The proof of this fact is obtained in a similar way as in the case of a parabolic equation which is given in Malanowski (1974).

The control problem (2.1) - (2.4), (7.1), (7.2) can be considered as the one of seeking an element y_0, more precisely — the corresponding v_0, belonging to a closed, convex and a bounded set Y_{ad} in a certain Hilbert space, whose distance from a given element z_d is minimal. Thus, it is a quadratic programming problem in a Hilbert space.

Now we shall describe a certain iteration procedure for solving the quadratic programming problem.

Let $\{Y_{ad}^i\}$ be a system of closed and convex subsets of the set Y_{ad}. We denote by $y^i \in Y_{ad}^i$ an element whose distance from element z_d is minimal, i.e. the following condition is fulfilled:

$$\left\| y^i(T) - z_d \right\| = min_{y \in Y_{ad}^i} \left\| y(T) - z_d \right\| . \tag{7.4}$$

By $\overline{y}^{i+1}(T)$ we denote the element such that

$$\left\langle y^i(T) - z_d, y(T) - \overline{y}^{i+1}(T) \right\rangle_{H^{-1}(\Omega)} \geq 0, \qquad \forall y \in Y_{ad}. \tag{7.5}$$

The point $\overline{y}^{i+1}(T)$ is a support of the set Y_{ad}^i determined by the hyperplane M^i orthogonal to the vector $(z_d - y^i(T))$.

In Malanowski (1974) it is shown that if the system of sets $\{Y_{ad}^i\}$ has the structure

$$Y_{ad}^{i+1} \supset y^i \cup \overline{y}^{i+1} \tag{7.6}$$

then the sequence $\{y^i\}$ is strongly convergent to y_0 in the space Y.

The step-by-step algorithms for finding the sequence y_i convergent to y_0 differ from each other by the construction of the sets Y_{ad}^i, only. The simplest one of them has been proposed by Gilbert (1966) and applied in Kowalewski and Duda (1992) for distributed parabolic systems with the Neumann boundary conditions involving time delays.

Now we describe the method of determining the element $\overline{y}^{i+1}(T)$ for the optimal control problem (2.1) -(2.4), (4.1) and (4.2).

We introduce the following notation:

$$y^i(T) = y(T; v^i), \qquad \overline{y}^{i+1}(T) = y(T; \overline{v}^{i+1}), \qquad p^i = p(v^i). \tag{7.7}$$

Here, we introduce the adjoint equation

$$-\frac{\partial p^i}{\partial t} + A^*(t)p^i + \int_0^c b(x, t+h)p(x, t+h; v^i)dh = 0,$$

$$(x,t) \in \Omega \times (0,T), \; h \in (0,c), \tag{7.8}$$

$$p^i(x,t) = 0 \qquad (x,t) \in \Omega \times (T, T+c), \tag{7.9}$$

$$p^i(x,T) = -\lambda_1(-\Delta + I)^{-1}(y^i(x,T) - z_d), \qquad x \in \Omega \tag{7.10}$$

$$p^i(x,t) = 0 \qquad x \in \Gamma, \; t \in (0,T). \tag{7.11}$$

Proceeding in a similar way as in deriving the formula (6.15), the condition (7.5) is written as

$$\int_0^T\int_\Gamma \frac{\partial p}{\partial \eta_{A^*}}(v-\overline{v}^{i+1})d\Gamma dt = \left\langle \frac{\partial p}{\partial \eta_{A^*}}, v-\overline{v}^{i+1}\right\rangle_{L^2(\Sigma)} \geq 0 \qquad \forall v \in U_{ad}. \tag{7.12}$$

Taking into account the form of the set U_{ad} from the formula (7.12) we get

$$\overline{v}^{i+1} = -\frac{\frac{\partial p}{\partial \eta_{A^*}}}{\left\|\frac{\partial p}{\partial \eta_{A^*}}\right\|_{L^2(\Sigma)}}. \tag{7.13}$$

Now, it is easy to notice that there are no mutual connections between the equation of the system control, the adjoint equation and the maximum condition, which made impossible the determination of the optimal control, earlier. Hence, from the formula (7.13) we find out $\overline{v}^{i+1}$ for p^i which we determine from (7.8) - (7.11) knowing $y^i(T)$ from previous iteration. Then, having $\overline{v}^{i+1}$ we compute $\overline{y}^{i+1}(T)$ from (2.1) - (2.4).

8. Conclusions

In this paper we have considered the optimal retarded parabolic systems with the non-homogeneous Dirichlet boundary conditions.

The results presented in the paper can be treated as a generalization of the results obtained by Kowalewski and Krakowiak (2001) onto the case of retarded argument appearing in the integral form with $h \in (0, c)$ in the state equations.

The existence and uniqueness of solutions for such retarded parabolic systems are proved — Theorems 2.1 and 5.1.

Necessary and sufficient conditions of optimality with the quadratic performance functionals and constrained control are derived — Theorems 6.2 and 6.3.

A simple example of application is also presented — Example 7.1.

We can also consider boundary control problems for retarded parabolic systems with the Dirichlet boundary conditions involving retarded arguments given in the integral form.

The ideas mentioned above will be developed in forthcoming papers.

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