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Robust H_∞ control for uncertain neutral time-delay systems with discrete and distributed delays* †

by

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Abstract: In this paper, the robust H_∞ control problem is considered for uncertain neutral systems with discrete and distributed time delays. Some sufficient conditions on H_∞ robust performance analysis are obtained in terms of linear matrix inequalities (LMIs) by using a descriptor model transformation of the system and by applying Park's inequality for bounding cross terms. Based on these conditions the state-feedback H_∞ controller is designed. Numerical examples are included to illustrate the proposed method.

Keywords: time-delay, linear matrix inequalities, uncertain neutral systems, H_∞ control.

1. Introduction

Recently, neutral systems with time delay have attached much attention (see Karimi and Luo, 2008; Xu, Lam and Yang, 2001; Chen et al., 2006). This is because delay phenomena are frequently encountered in mechanics, physics, biology, economics and engineering systems, and are a source of instability and poor performance. Neutral delay systems constitute a more general class than those of only the delay type, and therefore, stability of these systems is a more complex issue because of the derivative of the delayed state involved. Stability conditions on neutral delay system based on LMIs or Riccati equations have

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been obtained for both the delay-independent (Verriest and Niculescu, 1998) and the delay-dependent (Fridman, 2001; Lien, Yu and Hsieh, 2000; Ivanescu et al., 2000; Yue, Won and Kwon, 2003) cases. Note that unlike retarded type systems, neutral systems may be destabilized by small changes in delays (see Logemann and Townley, 1995). Concerning the H_∞ control problem for neutral systems, delay-independent and delay-dependent state-feedback solutions have been achieved in Mahmoud (2000), Fridman and Shaked (2002), Yue, Won and Kwon (2003), Chen et al. (2001), respectively. Recently, a new descriptor model transformation has been introduced in Fridman (2001) for stability analysis of systems with delays. This transformation transforms the original system to an equivalent descriptor form representation, and therefore, does not introduce any additional dynamics in the sense defined in Gu and Niculescu (2001). In addition, fewer bounds are applied in this method. These bounds can now be made tighter using the bound on cross terms that was introduced in Park (1999).

In this paper, we consider the problem of robust H_∞ control for a class of neutral time-delay systems with parameter uncertainties allowed to be time-varying but norm-bounded. Our goal is to design a state-feedback controller such that the closed-loop system is asymptotically stable and guarantees a prescribed H_∞ performance level for all admissible uncertainties. All the conditions are given in terms of LMIs. Some numerical examples illustrate the effectiveness of our solutions as compared to results obtained by other methods.

For simplification, we use the symbol $Sym\{\cdot\}$ to denote $Sym\{X\} \overset{\text{def}}{=} X+X^T$, the symbol $*$ to denote the symmetric part.

2. Problem formulation and preliminaries

Consider the following system with discrete and distributed delays and parameter uncertainties:

$$\begin{aligned} \dot{x}(t) =& Ax(t) + A_1x(t-\tau_1) + A_2\int_{t-\tau_2}^{t} x(s)ds + A_3\dot{x}(t-\tau_3) + B_1w(t) \\ z(t) =& Cx(t) + D_1w(t) \\ x(t) =& \varphi(t), \forall t \in [-\tau, 0] \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state, $z(t) \in R^r$ is the controlled output, $w(t) \in R^l$ is the disturbance of finite energy in the space $L_2[0,\infty)$, and A, A_i, $i = 1,2,3$, B_1, C, D_1 are known constant matrices of appropriate dimensions. The scalars $\tau_i > 0, i = 1,2,3$ are time delays. $\tau = max(\tau_1,\tau_2,\tau_3)$, $\varphi(t)$ is a real-valued continuous initial function on $[-\tau, 0]$. T_{zw} is the transfer function from w to z. Given a positive scalar $\gamma > 0$, define, for system (1), the performance index $J(w) := \int_0^\infty z^T z dt - \int_0^\infty \gamma^2 w^T w dt$, then $\|T_{zw}\|_\infty < \gamma$ iff $J(w) < 0$, $\forall w \in L_2[0,\infty)$.

LEMMA 1 *(Park, 1999) Assume that $a(\alpha) \in R^{n_x}$ and $b(\alpha) \in R^{n_y}$ are given for $\alpha \in \Omega$. Then, for any positive definite matrix $X \in R^{n_x \times n_x}$ and any matrix $M \in R^{n_y \times n_y}$, the following holds:*

$$-2\int_{\Omega} b^T(\alpha)a(\alpha)d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & XM \\ * & (2,2) \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} d\alpha \tag{2}$$

where $(2,2)$ denotes $(M^TX+I)X^{-1}(XM+I)$.

LEMMA 2 *Let D, S, F be real matrices of appropriate dimensions and F satisfying $F^TF \leq I$. Then the following statements hold:*

For any scalar $\varepsilon > 0$ and vectors $x, y \in R^n$,

$$2x^TDFSy \leq \varepsilon^{-1}x^TDD^Tx + \varepsilon y^TS^TSy.$$

3. Main results

3.1. H_∞ performance analysis

THEOREM 1 *Assume $\tau_1 > 0$, $\tau_2 > 0$, $\gamma > 0$ are given positive scalars. The system (1) is asymptotically stable and satisfies $\|T_{zw}\|_\infty < \gamma$, if there exist symmetric and positive-definite matrices $P_1 > 0$, $S > 0$, $H > 0$, $U > 0$, $R_1 > 0$, $R_3 > 0$ and matrices P_2, P_3, R_2, W_i, $i = 1, \cdots, 4$ such that the following LMI holds*

$$\begin{bmatrix} (1,1) & (1,2) & -W_3^TA_1 & P_2^TA_2 & P_2^TA_3 & \tau_1(W_1^T+P_1) & (1,7) & P_2^TB_1 & C^T \\ * & (2,2) & -W_4^TA_1 & P_3^TA_2 & P_3^TA_3 & \tau_1W_2^T & (2,7) & P_3^TB_1 & 0 \\ * & * & -S & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -H & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -U & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\tau_1R_1 & -\tau_1R_2 & 0 & 0 \\ * & * & * & * & * & * & -\tau_1R_3 & 0 & 0 \\ * & * & * & * & * & * & * & -\gamma^2I & D_1^T \\ * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{3}$$

where

$$(1,1) = Sym\{P_2^T(A+A_1)+W_3^TA_1\} + S + \tau_2^2H$$
$$(1,2) = P_1 - P_2^T + (A+A_1)^TP_3 + A_1^TW_4$$
$$(2,2) = -P_3^T - P_3 + U + \tau_1A_1^TR_3A_1$$
$$(1,7) = \tau_1(W_3^T + P_2^T)$$
$$(2,7) = \tau_1(W_4^T + P_3^T)$$

Proof. Represent (1) in the equivalent descriptor form:

$$\dot{x}(t) = y(t)$$
$$y(t) = Ax(t) + A_1x(t-\tau_1) + A_2\int_{t-\tau_2}^{t} x(s)ds + A_3\dot{x}(t-\tau_3) + B_1w(t). \qquad (4)$$

From the Newton-Leibniz Formula, we can get

$$y(t) = (A+A_1)x(t) - A_1\int_{t-\tau_1}^{t} y(s)ds + B_1w(t) + A_2\int_{t-\tau_2}^{t} x(s)ds + A_3y(t-\tau_3).$$

Then, we can obtain

$$\begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ A+A_1 & -I \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} w(t) - \begin{bmatrix} 0 \\ A_1 \end{bmatrix}\int_{t-\tau_1}^{t} y(s)ds$$
$$+ \begin{bmatrix} 0 \\ A_2 \end{bmatrix}\int_{t-\tau_2}^{t} x(s)ds + \begin{bmatrix} 0 \\ A_3 \end{bmatrix} y(t-\tau_3). \qquad (5)$$

Consider the following Lyapunov-Krasovskii functional candidate of the form

$$V(t) = \sum_{i=1}^{6} V_i(t)$$

where

$$V_1(t) = \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$
$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$$
$$V_2(t) = \int_{t-\tau_1}^{t} x^T(s)Sx(s)ds$$
$$V_3(t) = \int_{-\tau_1}^{0}\int_{t+\theta}^{t} y^T(s)A_1^TR_3A_1y(s)dsd\theta$$
$$V_4(t) = \int_{t-\tau_2}^{t} [\int_{s}^{t} x^T(\theta)d\theta]H[\int_{s}^{t} x(\theta)d\theta]ds$$
$$V_5(t) = \int_{0}^{\tau_2} ds\int_{t-s}^{t} (\theta - t + s)x^T(\theta)Hx(\theta)d\theta$$
$$V_6(t) = \int_{t-\tau_3}^{t} y^T(s)Uy(s)ds.$$

The time derivative of $V(t)$ along the trajectory of the system (1) is given by

$$\dot{V}(t) = \sum_{i=1}^{6} \dot{V}_i(t) \tag{6}$$

where

$$\dot{V}_1(t) = 2 \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} \tag{7}$$
$$\dot{V}_2(t) = x^T(t)Sx(t) - x^T(t-\tau_1)Sx(t-\tau_1)$$
$$\dot{V}_3(t) = \tau_1 y^T(t)A_1^T R_3 A_1 y(t) - \int_{t-\tau_1}^{t} y^T(s)A_1^T R_3 A_1 y(s)ds$$
$$\dot{V}_4(t) = 2\int_{t-\tau_2}^{t} (\theta - t + \tau_2)x^T(t)Hx(\theta)d\theta - \left[\int_{t-\tau_2}^{t} x^T(\theta)d\theta\right] H \left[\int_{t-\tau_2}^{t} x(\theta)d\theta\right]$$
$$\dot{V}_5(t) = \frac{1}{2}\tau_2^2 x^T(t)Hx(t) - \int_{t-\tau_2}^{t} (\theta - t + \tau_2)x^T(\theta)Hx(\theta)d\theta$$
$$\dot{V}_6(t) = y^T(t)Uy(t) - y^T(t-\tau_3)Uy(t-\tau_3).$$

Substitute (5) into (7). Define $\xi = \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix}$. From Lemma 1, we can get the following inequalities:

$$\begin{aligned} -2\xi P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \int_{t-\tau_1}^{t} y(s)ds \le & \int_{t-\tau_1}^{t} y^T(s) \begin{bmatrix} 0 & A_1^T \end{bmatrix} R \begin{bmatrix} 0 \\ A_1 \end{bmatrix} y(s)ds \\ & +2\int_{t-\tau_1}^{t} y^T(s) \begin{bmatrix} 0 & A_1^T \end{bmatrix} RMP\xi^T ds \\ & + \int_{t-\tau_1}^{t} \xi P^T (M^T R + I)R^{-1}(RM + I)P\xi^T ds \end{aligned} \tag{8}$$

where

$$R = \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} > 0.$$

Rearrange the terms of (8) and define the following equation

$$W = RMP = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}.$$

Then we can get

$$\begin{aligned}
-2\xi P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \int_{t-\tau_1}^{t} y(s)ds \leq & \int_{t-\tau_1}^{t} y^T(s) A_1^T R_3 A_1 y(s) ds \\
& + 2(x(t) - x(t-\tau_1)) \begin{bmatrix} 0 & A_1^T \end{bmatrix} W \xi^T \\
& + \tau_1 \xi (W^T + P^T) R^{-1} (W + P) \xi^T. \qquad (9)
\end{aligned}$$

Now, by Lemma 2, it can be shown that

$$2x^T(t) Q_2 x(\theta) \leq x^T(t) Q_2 x(t) + x^T(\theta) Q_2 x(\theta).$$

Therefore

$$\begin{aligned}
\dot{V}_4(t) \leq & \int_{t-\tau_2}^{t} (\theta - t + \tau_2) x^T(\theta) H x(\theta) d\theta + \frac{1}{2}\tau_2^2 x^T(t) H x(t) \\
& - \left[\int_{t-\tau_2}^{t} x^T(\theta) d\theta \right] H \left[\int_{t-\tau_2}^{t} x(\theta) d\theta \right].
\end{aligned}$$

Then we can get

$$\begin{aligned}
&\dot{V}(t) \leq 2\xi P^T \begin{bmatrix} 0 & I \\ A + A_1 & -I \end{bmatrix} \xi^T + x^T(t) S x(t) \\
&+ 2\xi P^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \int_{t-\tau_2}^{t} x(s) ds + 2\xi P^T \begin{bmatrix} 0 \\ A_3 \end{bmatrix} y(t-\tau_3) \\
&- x^T(t-\tau_1) S x(t-\tau_1) + \tau_1 y^T(t) A_1^T R_3 A_1 y(t) \\
&+ 2(x(t) - x(t-\tau_1)) \begin{bmatrix} 0 & A_1^T \end{bmatrix} W \xi^T + y^T(t) U y(t) \\
&+ \tau_1 \xi (W + P)^T R^{-1} (W + P) \xi^T - y^T(t-\tau_3) U y(t-\tau_3) + \tau_2^2 x^T(t) H x(t) \\
&- \left(\int_{t-\tau_2}^{t} x^T(s) ds \right) H \left(\int_{t-\tau_2}^{t} x(s) ds \right) + 2\xi P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} w(t).
\end{aligned}$$

When $w(t) = 0$, we can get $\dot{V}(t) \leq \beta^T \Xi \beta$, where

$$\beta = \begin{bmatrix} \xi & x^T(t-\tau_1) & \int_{t-\tau_2}^{t} x^T(s) ds & y^T(t-\tau_3) \end{bmatrix}$$

$$\Xi = \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & \tau_1 (W + P)^T \\ * & -S & 0 & 0 & 0 \\ * & * & -H & 0 & 0 \\ * & * & * & -U & 0 \\ * & * & * & * & -\tau_1 R \end{bmatrix}$$

$$(1,1) = Sym\left\{P^T\begin{bmatrix} 0 & I \\ A + A_1 & -I \end{bmatrix}\right\} + Sym\left\{W^T\begin{bmatrix} 0 & 0 \\ A_1 & 0 \end{bmatrix}\right\}$$
$$+\begin{bmatrix} S + \tau_2^2 H & 0 \\ * & U + \tau_1 A_1^T R_3 A_1 \end{bmatrix}$$
$$(1,2) = -W^T\begin{bmatrix} 0 \\ A_1 \end{bmatrix}, (1,3) = P^T\begin{bmatrix} 0 \\ A_2 \end{bmatrix}, (1,4) = P^T\begin{bmatrix} 0 \\ A_3 \end{bmatrix}.$$

Now, from the LMI in (3), it is easy to see, by the Schur complement formula, that (3) implies $\Xi < 0$. Then we can have $\dot{V}(t) < 0$ for all $\alpha(t) \neq 0$ when $w(t) = 0$. Therefore, the system (1) is asymptotically stable.

Next, we shall establish the H_∞ performance of the system (1) under the zero initial condition. Noting the zero initial condition, it can be shown that

$$\begin{aligned} J(w) &= \int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{V}(t)]dt - V(\infty) \\ &\leq \int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{V}(t)]dt. \end{aligned}$$

Note

$$z^T(t)z(t) = \left[x^T(t)C^T + w^T(t)D_1^T\right]\left[Cx(t) + D_1 w(t)\right]. \tag{10}$$

Then we can get

$$z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{V}(t) \leq \alpha^T \Pi \alpha \tag{11}$$

where,

$$\alpha = \begin{bmatrix} \beta & w(t) \end{bmatrix}, \quad \Pi = \begin{bmatrix} \Xi & (1,2) \\ * & -\gamma^2 I + D_1^T D_1 \end{bmatrix},$$
$$(1,2) = \begin{bmatrix} P^T\begin{bmatrix} 0 \\ B_1 \end{bmatrix} + C^T D_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Pi < 0$ implies that $J(w) < 0$. By Schur complement Lemma, the inequality $\Pi < 0$ can be equivalently changed to (3). This completes the proof. ■

3.2. H_∞ performance analysis for uncertain systems

Consider the following system with discrete and distributed delays and parameter uncertainties:

$$\begin{aligned}\dot{x}(t) =& [A + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-\tau_1)\\ &+ [A_2 + \Delta A_2(t)]\int_{t-\tau_2}^{t} x(s)ds + B_1 w(t) + [A_3 + \Delta A_3(t)]\dot{x}(t-\tau_3)\\ z(t) =& Cx(t) + D_1 w(t)\\ x(t) =& \varphi(t), \forall t \in [-\tau, 0]\end{aligned} \tag{12}$$

where $\Delta A(t)$, $\Delta A_1(t)$, $\Delta A_2(t)$, $\Delta A_3(t)$ are unknown matrices representing time-varying parameter uncertainties, the other signals are the same as in system (1). In this paper, the parameter uncertainties are assumed to be of the form

$$\begin{bmatrix} \Delta A(t) & \Delta A_1(t) & \Delta A_2(t) & \Delta A_3(t) \end{bmatrix} = DF(t)\begin{bmatrix} E & E_1 & E_2 & E_3 \end{bmatrix} \tag{13}$$

where D, E, E_1, E_2, E_3 are known real constant matrices of appropriate dimensions. $F(\cdot) : R \to R^{k\times l}$ is an unknown time-varying matrix function satisfying

$$F^T(t)F(t) \le I, \forall t. \tag{14}$$

Assume that all the elements of $F(t)$ are Lebesgue measurable. The uncertain matrices $\Delta A(t)$, $\Delta A_1(t)$, $\Delta A_2(t)$, $\Delta A_3(t)$ are said to be admissible if both (13) and (14) hold.

THEOREM 2 *Assume $\tau_1 > 0$, $\tau_2 > 0$, $\gamma > 0$ are given positive scalars. The system (12) is robustly asymptotically stable and satisfies $\|T_{zw}\|_\infty < \gamma$ for all admissible uncertainties, if there exist positive scalar $\varepsilon > 0$, symmetric and positive-definite matrices $P_1 > 0$, $S > 0$, $H > 0$, $U > 0$, $R_1 > 0$, $R_3 > 0$ and matrices P_2, P_3, R_2, W_i, $i = 1, \cdots, 4$ such that the following LMI holds,*

$$\begin{bmatrix} (1,1) & (1,2) & -W_3^TA_1 & P_2^TA_2 & P_2^TA_3 & (1,6) & (1,7) & P_2^TB_1 & C^T & P_2^TD & \varepsilon E^T\\ * & (2,2) & -W_4^TA_1 & P_3^TA_2 & P_3^TA_3 & \tau_1 W_2^T & (2,7) & P_3^TB_1 & 0 & P_3^TD & 0\\ * & * & -S & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon E_1^T\\ * & * & * & -H & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon E_2^T\\ * & * & * & * & -U & 0 & 0 & 0 & 0 & 0 & \varepsilon E_3^T\\ * & * & * & * & * & -\tau_1 R_1 & -\tau_1 R_2 & 0 & 0 & 0 & 0\\ * & * & * & * & * & * & -\tau_1 R_3 & 0 & 0 & 0 & 0\\ * & * & * & * & * & * & * & -\gamma^2 I & D_1^T & 0 & 0\\ * & * & * & * & * & * & * & * & -I & 0 & 0\\ * & * & * & * & * & * & * & * & * & -\varepsilon I & 0\\ * & * & * & * & * & * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0 \tag{15}$$

where

$$
\begin{aligned}
&(1,1) = Sym\{P_2^T(A+A_1) + W_3^T A_1\} + S + \tau_2^2 H \\
&(1,2) = P_1 - P_2^T + (A+A_1)^T P_3 + A_1^T W_4 \\
&(1,6) = \tau_1(W_1^T + P_1), (1,7) = \tau_1(W_3^T + P_2^T) \\
&(2,2) = -P_3^T - P_3 + U + \tau_1 A_1^T R_3 A_1, (2,7) = \tau_1(W_4^T + P_3^T)
\end{aligned}
$$

Proof. Similarly to the proof of Theorem 1, system (12) is represented in the equivalent descriptor form. Note that

$$
\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} = &\begin{bmatrix} 0 & I \\ A+A_1 & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} w(t) \\
&- \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \int_{t-\tau_1}^{t} y(s)ds + \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \int_{t-\tau_2}^{t} x(s)ds \\
&+ \begin{bmatrix} 0 \\ A_3 \end{bmatrix} y(t-\tau_3) + \begin{bmatrix} 0 & 0 \\ \Delta A + \Delta A_1(t) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ \Delta A_2(t) \end{bmatrix} \int_{t-\tau_2}^{t} x(s)ds + \begin{bmatrix} 0 \\ \Delta A_3(t) \end{bmatrix} y(t-\tau_3) \\
&- \begin{bmatrix} 0 \\ \Delta A_1(t) \end{bmatrix} (x(t) - x(t-\tau_1)).
\end{aligned} \tag{16}
$$

Substitute (16) to (7). Define

$$
\begin{aligned}
&\xi = \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} \\
&\beta = \begin{bmatrix} \xi & x^T(t-\tau_1) & \int_{t-\tau_2}^{t} x^T(s)ds & y^T(t-\tau_3) \end{bmatrix}^T \\
&\Gamma = \varepsilon \beta^T \begin{bmatrix} E^T \\ 0 \\ E_1^T \\ E_2^T \\ E_3^T \end{bmatrix} \begin{bmatrix} E & 0 & E_1 & E_2 & E_3 \end{bmatrix} \beta.
\end{aligned}
$$

Note that

$$
\begin{aligned}
2\xi P^T \Bigg\{ &\begin{bmatrix} 0 & 0 \\ \Delta A + \Delta A_1(t) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} - \begin{bmatrix} 0 \\ \Delta A_1(t) \end{bmatrix} (x(t) - x(t-\tau_1)) \\
&+ \begin{bmatrix} 0 \\ \Delta A_2(t) \end{bmatrix} \int_{t-\tau_2}^{t} x(s)ds + \begin{bmatrix} 0 \\ \Delta A_3(t) \end{bmatrix} y(t-\tau_3) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\xi P^T \left\{ \begin{bmatrix} 0 \\ DF(t)E \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ DF(t)E_1 \end{bmatrix} x(t-\tau_1) \right. \\
&\qquad \left. + \begin{bmatrix} 0 \\ DF(t)E_2 \end{bmatrix} \int_{t-\tau_2}^{t} x(s)ds + \begin{bmatrix} 0 \\ DF(t)E_3 \end{bmatrix} y(t-\tau_3) \right\} \\
&= 2\xi P^T \begin{bmatrix} 0 \\ D \end{bmatrix} F(t) \begin{bmatrix} E & 0 & E_1 & E_2 & E_3 \end{bmatrix} \beta \\
&\leq \varepsilon^{-1} \xi P^T \begin{bmatrix} 0 \\ D \end{bmatrix} \begin{bmatrix} 0 & D^T \end{bmatrix} P\xi + \Gamma
\end{aligned}
$$

3.3. H_∞ control for uncertain systems

Consider the controlled form of a system with discrete and distributed delays and parameter uncertainties:

$$
\begin{aligned}
\dot{x}(t) &= [A + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-\tau_1) + B_1 w(t) + B_2 u(t) \\
&\quad + [A_2 + \Delta A_2(t)] \int_{t-\tau_2}^{t} x(s)ds + [A_3 + \Delta A_3(t)]\dot{x}(t-\tau_3) \\
z(t) &= Cx(t) + D_1 w(t) + D_2 u(t) \qquad (17)
\end{aligned}
$$

where $u(t) \in R^{n_u}$ is the control input. Take the state-feedback control law

$$u(t) = Kx(t). \qquad (18)$$

Then the corresponding closed-loop system can be written down as

$$
\begin{aligned}
\dot{x}(t) &= [A_c + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-\tau_1) + B_1 w(t) \\
&\quad + [A_2 + \Delta A_2(t)] \int_{t-\tau_2}^{t} x(s)ds + [A_3 + \Delta A_3(t)]\dot{x}(t-\tau_3) \\
z(t) &= C_c x(t) + D_1 w(t) \qquad (19)
\end{aligned}
$$

where

$$A_c = A + B_2 K, C_c = C + D_2 K. \qquad (20)$$

THEOREM 3 *Assume $\tau_1 > 0$, $\tau_2 > 0$, $\gamma > 0$ are given positive scalars and λ is a given scalar. There exists a state-feedback controller of the form (18) such that the closed-loop system (19) is robustly asymptotically stable and satisfies $\|T_{zw}\|_\infty < \gamma$ for all admissible uncertainties, if there exist positive scalar $\varepsilon > 0$, symmetric and positive-definite matrices $Q_1 > 0$, $\bar{S} > 0$, $\bar{H} > 0$, $\bar{U} > 0$, $\bar{R}_1 > 0$,*

$\bar{R}_3 > 0$ and matrices Q_2, Q_3, $\bar{R}_2$, Y such that the following LMI holds,

$$\begin{bmatrix} (1,1) & (1,2) & \tau_1(\lambda+1)\bar{R}_3 & (1,4) & (1,5) & (1,6) & \begin{bmatrix} 0 & Q_1E^T \\ \bar{\varepsilon}D & 0 \end{bmatrix} \\ * & (2,2) & 0 & 0 & 0 & 0 & \begin{bmatrix} 0 & E_1^T \\ 0 & E_2^T \\ 0 & E_3^T \end{bmatrix} \\ * & * & -\tau_1\begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ * & \bar{R}_3 \end{bmatrix} & 0 & 0 & 0 & 0 \\ * & * & * & (4,4) & 0 & 0 & 0 \\ * & * & * & * & -\tau_1\bar{R}_3 & 0 & 0 \\ * & * & * & * & * & (6,6) & 0 \\ * & * & * & * & * & * & -\bar{\varepsilon}I \end{bmatrix} < 0 \qquad (21)$$

where

$$(1,1) = \begin{bmatrix} Q_2+Q_2^T & Q_3-Q_2^T+Q_1(A+A_1)^T+\lambda Q_1A_1^T+Y^TB_2^T \\ * & -Q_3-Q_3^T \end{bmatrix}$$

$$(1,2) = \begin{bmatrix} 0 & 0 & 0 \\ -\lambda A_1\bar{S} & A_2\bar{H} & A_3\bar{U} \end{bmatrix}, (1,4) = \begin{bmatrix} Q_1 & \tau_2Q_1 & Q_2^T \\ 0 & 0 & Q_3^T \end{bmatrix}$$

$$(1,5) = \tau_1\begin{bmatrix} Q_2^T \\ Q_3^T \end{bmatrix}A_1^T, (1,6) = \begin{bmatrix} 0 & Q_1C^T+Y^TD_2^T \\ B_1 & 0 \end{bmatrix}$$

$$(2,2) = (4,4) = -diag\{\bar{S},\bar{H},\bar{U}\}, (6,6) = \begin{bmatrix} -\gamma^2I & D_1^T \\ * & -I \end{bmatrix}.$$

Furthermore, if the above condition holds, the state-feedback gain is then given by $K = YQ_1^{-1}$.

Proof. Applying Theorem 2 to the closed-loop system (19). In order to obtain an LMI, we have to restrict ourselves to the case of $W = \lambda P$, where $\lambda \in R$ is a scalar parameter. It is obvious from the requirement of $P_1 > 0$ and the fact that in (15) $-P_3 - P_3^T$ must be negative definite, that P is nonsingular. Defining

$$Q = P^{-1} = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}$$

$$\Delta = diag\{Q, S^{-1}, H^{-1}, U^{-1}, R_3^{-1}, R_3^{-1}, I, I, \varepsilon^{-1}I, I\}$$

we multiply (15) by Δ^T and Δ on the left and on the right, respectively. Applying the Schur complement Lemma to the quadratic term in Q, substitute (20) into the obtained equation, defining $\bar{S} = S^{-1}$, $\bar{H} = H^{-1}$, $\bar{U} = U^{-1}$, $\bar{R}_1 = R_3^{-1}R_1R_3^{-1}$, $\bar{R}_2 = R_3^{-1}R_2R_3^{-1}$, $\bar{R}_3 = R_3^{-1}$ and denoting KQ_1 by Y, we can obtain (21). ∎

4. Illustrative examples

Example 1 Consider the system (1), with

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0 & -0.9 \end{bmatrix}, A_1 = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, A_3 = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.2 \end{bmatrix}$$
$$A_2 = 0, B_1 = C = D_1 = 0$$

which was presented in Chen et al. (2001). By the criteria in Chen et al. (2001) and Yue, Won and Kwon (2003), the nominal system is asymptotically stable for any h satisfying $h \leq 0.5658$ and $h \leq 1.5687$, respectively. However, using Theorem 1, we found the admissible bound of τ_1 to be 1.71.

Example 2 Consider system (1) shown in Park and Won (2000) with

$$A = \begin{bmatrix} -d_1 & 0 \\ 0 & -d_2 \end{bmatrix}, A_1 = \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix}, A_2 = \begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix}$$
$$A_3 = B_1 = C = D_1 = 0$$

where

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0 & -0.9 \end{bmatrix}, A_1 = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, A_2 = 0,$$
$$A_3 = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.2 \end{bmatrix}$$

and d_i, b_i, c_i, $i = 1, 2$ are constant coefficients. For the case when $d_1 = d_2 = 0.9$, $b_1 = -1$, $b_2 = c_1 = c_2 = -0.12$ and $\tau_2 = 1$, using the discrete delay dependent result given in Theorem 2 of Chen et al. (2001), it was found that the upper bound of τ_1 must be less than 0.9086 because of the constraint condition (8a) of Chen et al. (2001). And using Theorem 1 of Yue, Won and Kwon (2003), it was found that the upper bound of τ_1 must be less than 1.8302. However, using Theorem 1, we found the admissible bound of τ_1 to be 2.2.

EXAMPLE 3 Consider the uncertain delay system (17), where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 0.1 & -0.3 \\ 0 & 0.1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D_1 = 0, D_2 = 0.1$$

$$E = \begin{bmatrix} -0.1 & 0.1 \end{bmatrix}, E_1 = \begin{bmatrix} 0.1 & 0.5 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0.3 & -0.1 \end{bmatrix}, E_3 = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}$$

$$\tau_2 = 0.5, \lambda = -0.3$$

In this example, the H_∞ performance level γ is specified to be 0.5. We can solve the LMIs in Theorem 3, and obtain the admissible bound of τ_1 to be 0.9 and the solution as follows:

$$Q_1 = \begin{bmatrix} 7.8910 & 0.0308 \\ 0.0308 & 0.0792 \end{bmatrix} \times 10^3$$

$$Y = \begin{bmatrix} -311.4860 & -767.3463 \end{bmatrix}.$$

Then we can get the gain matrix of the stabilizing state-feedback controller for the system (17):

$$K = \begin{bmatrix} -0.0017 & -9.6865 \end{bmatrix}.$$

5. Conclusions

In this paper, we have considered the design problem of robust H_∞ state-feedback controller for a class of neutral systems with discrete and distributed time delays and time-varying norm-bounded parameter uncertainties. Robust H_∞ performance analysis conditions and state-feedback solutions are given in terms of LMIs. Examples have been provided to illustrate the effectiveness of the proposed approach.

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