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Lyapunov functional for a linear system with both lumped and distributed delay*

by

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Abstract: The paper presents a method of determining the Lyapunov quadratic functional for linear time-invariant system with both lumped and distributed delay. The Lyapunov functional is constructed for its given time derivative, which is calculated on the trajectory of the system with both lumped and distributed delay. The method presented gives analytical formulas for the coefficients of the Lyapunov functional.

Keywords: Lyapunov functional, time delay system, LTI system.

1. Introduction

Lyapunov quadratic functionals are used to test the stability of systems, to compute the critical delay values for time delay systems, to compute the exponential estimates for the solutions of time delay systems, to calculate the robustness bounds for uncertain time delay systems, and to calculate a quadratic performance index of quality for the process of parametric optimization for time delay systems. We construct the Lyapunov functionals for the system with time delay with a given time derivative. For the first time such Lyapunov functional was introduced by Repin (1965) for the case of retarded time delay linear systems with one delay. Repin (1965) delivered also the procedure for determination of coefficients of the functional. Duda (1986) used the Lyapunov functional, which was proposed by Repin, for the calculation of the value of a quadratic performance index of quality in the process of parametric optimization for systems with time delay of retarded type and the extended the results to the case of neutral type time delay system in Duda (1988). Duda (2010a) presented a method of determining the Lyapunov functional for a linear dynamic system with two lumped retarded type time delays in the general case with no-commensurate delays and presented a special case with commensurate delays in which the Lyapunov functional could be determined by solving a set of ordinary differential equations.

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Duda (2010b) presented a method of determining the Lyapunov functional for linear dynamic system with two delays both retarded and neutral type time delay, and then in Duda (2010c) presented a method of determining the Lyapunov quadratic functional for linear time-invariant system with k-non-commensurate neutral type time delays. In Infante and Castelan (1978) construction of the Lyapunov functional is based on a solution of a matrix differential-difference equation over a finite time interval. This solution satisfies symmetry and boundary conditions. Kharitonov and Zhabko (2003) extended the results of Infante and Castelan (1978) and proposed a procedure for constructing quadratic functionals for linear retarded type delay systems, which could be used for the robust stability analysis of time delay systems. This functional was expressed by means of the Lyapunov matrix, which depended on the fundamental matrix of the time delay system. Kharitonov (2005) extended some basic results obtained for the case of retarded type time delay systems to the case of neutral type time delay systems, and in Kharitonov (2008) to the neutral type time delay systems with discrete and distributed delay. Kharitonov and Hinrichsen (2004) used the Lyapunov matrix to derive exponential estimates for the solutions of exponentially stable time delay systems. Kharitonov and Plischke (2006) formulated the necessary and sufficient conditions for the existence and uniqueness of the delay Lyapunov matrix for the case of retarded system with one delay. The numerical scheme for construction of the Lyapunov functionals has been proposed in Gu (1997). This method starts with the discretisation of the Lyapunov functional. The scheme is based on linear matrix inequality (LMI) techniques. Fridman (2001) introduced the Lyapunov-Krasovskii functionals for the stability of linear retarded and neutral type systems with discrete and distributed delays, which were based on equivalent descriptor form of the original system and obtained delay-dependent and delay-independent conditions in terms of LMI. Ivanescu et al. (2003) proceeded with the delay-depended stability analysis for linear neutral systems, constructed the Lyapunov functional and derived sufficient delay-dependent conditions in terms of LMIs. Han (2004a) obtained a delay-dependent stability criterion for neutral systems with time varying discrete delay. This criterion was expressed in the form of LMI and was obtained using the Lyapunov direct method. Han (2004b) investigated robust stability of uncertain neutral systems with discrete and distributed delays, which has been based on the descriptor model transformation and the decomposition technique, and formulated the stability criteria in the form of LMIs. Han (2005a) considered the stability for linear neutral systems with norm-bounded uncertainties in all system matrices and derived a new delay-dependent stability criterion. Neither model transformation nor bounding technique for cross terms is involved in the derivation of the stability criterion. Han (2005b) developed the discretized Lyapunov functional approach to investigation of the stability of linear neutral systems with mixed neutral and discrete delays. Stability criteria, which are applicable to linear neutral systems with both small and non-small discrete delays are formulated in the form of LMIs. Han (2009a) studied the problem of stability of linear time delay systems, both retarded and neutral types, using the discrete delay N-decomposition approach to derive some new, more general discrete delay dependent stability criteria. Han (2009b) employed the delay-decomposition approach to derive some improved stability criteria for linear neutral systems and to deduce some sufficient conditions for the existence of a delayed state feedback controller, which ensure asymptotic stability and a prescribed H1 performance level of the corresponding closed-loop system. Gu and Liu (2009) investigated the stability of coupled differential-functional equations using the discretized Lyapunov functional method and delivered the stability condition in the form of LMI, suitable for numerical computation.

This paper presents a method of determining the Lyapunov functional for the linear dynamic system with both lumped and distributed delay. The novelty of the result lies in the extension of the Repin method to the system with both lumped and distributed delay. To the best of author's knowledge, such extension has not been reported in the literature. An example illustrating the method is also presented.

2. Formulation of the problem

Let us consider the linear system with both lumped and distributed delay, whose dynamics is described by equations

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bx_t(-r) + \int_{-r}^{0} Gx_t(\theta)d\theta \\ x(t_0) = x_0 \in \mathbb{R}^n \\ x_{t_0} = \Phi \in W^{1,2}([-r,0),\mathbb{R}^n) \end{cases} \tag{2.1}$$

$$A,\ B,\ G \in \mathbb{R}^{n\times n}, \quad x(t) \in \mathbb{R}^n, \quad x_t \in W^{1,2}([-r,0),\mathbb{R}^n)$$
$$x_t(\xi) = x(t+\xi),\ t \geq t_0,\ \xi \in [-r,0), \quad r > 0.$$

Here, $W^{1,2}([-r,0),\mathbb{R}^n)$ is the space of all absolutely continuous functions defined on the interval $[-r,0)$ with values in $\mathbb{R}^n$, whose derivatives are in $L^2([-r,0),\mathbb{R}^n)$ — a space of a Lebesgue square integrable functions on interval $[-r,0)$ with values in $\mathbb{R}^n$.

The state of the system (2.1) is a vector

$$S(t) = \begin{bmatrix} x(t) \\ x_t \end{bmatrix} \qquad \text{for} \quad t \geq t_0. \tag{2.2}$$

The state space is defined by the formula

$$X = \mathbb{R}^n \times W^{1,2}([-r,0),\mathbb{R}^n). \tag{2.3}$$

On the state space X we define a Lyapunov functional, positively defined, differentiable, with the derivative computed on the trajectory of the system (2.1) being negatively defined.

$$V(S(t)) = x^T(t)\alpha x(t) + \int_{-r}^{0} x^T(t)\beta(\theta)x_t(\theta)d\theta + \int_{-r}^{0} x_t^T(\theta)\gamma(\theta)x_t(\theta)d\theta + \\ + \int_{-r}^{0}\int_{-r}^{0} x_t^T(\theta)\delta(\theta,\sigma)x_t(\sigma)d\sigma d\theta \tag{2.4}$$

for $t \geq t_0$ where $\alpha = \alpha^T \in \mathbb{R}^{n\times n}$, $\beta,\ \gamma \in C^1([-r,0],\mathbb{R}^{n\times n})$, $\gamma(\theta) = \gamma^T(\theta)$, $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$, $\Omega = \{(\theta,\varsigma):\ \theta \in [-r,0],\ \varsigma \in [\theta,0]\}$. C^1 is a space of continuous functions with continuous derivative.

3. Determination of the coefficients of the functional (2.4)

We compute the derivative of the functional (2.4) on the trajectory of the system (2.1) according to the formula

$$\frac{dV(S(t))}{dt} = grad(V(S(t))\frac{dS(t)}{dt} \quad \text{for } t \geq t_0. \tag{3.1}$$

The derivative of the functional (2.4), calculated on the basis of the formula (3.1), is given by the formula

$$\frac{dV(S(t))}{dt} = x^T(t)\left[A^T\alpha + \alpha A + \frac{\beta(0)+\beta^T(0)}{2} + \gamma(0)\right]x(t) + \\ + x_t^T(-r)\left[2B^T\alpha - \beta^T(-r)\right]x(t) - x_t^T(-r)\gamma(-r)x_t(-r) + \\ + \int_{-r}^{0} x^T(t)\left[2\alpha G + A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta,0) + \delta(0,\theta)\right]x_t(\theta)d\theta + \\ + \int_{-r}^{0} x_t^T(-r)[B^T\beta(\theta) - \delta^T(\theta,-r) - \delta(-r,\theta)]x_t(\theta)d\theta - \int_{-r}^{0} x_t^T(\theta)\frac{d\gamma(\theta)}{d\theta}x_t(\theta)d\theta + \\ - \int_{-r}^{0}\int_{-r}^{0} x_t^T(\theta)\left[\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma} - G^T\beta(\sigma)\right]x_t(\sigma)d\sigma d\theta. \tag{3.2}$$

We identify the coefficients of the functional (2.4), assuming that the derivative (3.2) satisfies the relationship

$$\frac{dV(S(t))}{dt} = -x^T(t)Wx(t) \qquad \text{for } t \geq t_0 \tag{3.3}$$

where $W \in \mathbb{R}^{n\times n}$ is a symmetric positively defined matrix.

When we know the Lyapunov functional and the relationship (3.3) holds, we can easily determine the value of a square indicator of quality of the parametric optimization, because

$$J = \int_{t_0}^{\infty} x^T(t) W x(t) dt = V(S(t_0)). \tag{3.4}$$

From equations (3.2) and (3.3) we obtain the system of equations (3.5) to (3.11):

$$A^T\alpha + \alpha A + \frac{\beta(0) + \beta^T(0)}{2} + \gamma(0) = -W \tag{3.5}$$

$$2B^T\alpha - \beta^T(-r) = 0 \tag{3.6}$$

$$\gamma(-r) = 0 \tag{3.7}$$

$$A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta(0,\theta) + \delta^T(\theta,0) + 2\alpha G = 0 \tag{3.8}$$

$$B^T\beta(\theta) - \delta(-r,\theta) - \delta^T(\theta,-r) = 0 \tag{3.9}$$

$$\frac{d\gamma(\theta)}{d\theta} = 0 \tag{3.10}$$

$$\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma} - G^T\beta(\sigma) = 0 \tag{3.11}$$

for $\theta \in [-r,0]$, $\sigma \in [-r,0]$.

From equations (3.7) and (3.10) it results that

$$\gamma(\theta) = 0 \qquad \text{for } \theta \in [-r,0]. \tag{3.12}$$

Let us consider that the solution of the equation (3.11) is as below

$$\delta(\theta,\sigma) = \varphi(\theta-\sigma) + \varphi^T(\sigma-\theta) + \int_0^{\sigma} G^T\beta(\xi)d\xi \tag{3.13}$$

where $\varphi \in C^1([-r,r],\mathbb{R}^{n\times n})$.

From equation (3.13) we obtain

$$\delta^T(\theta,0) = \varphi(-\theta) + \varphi^T(\theta) \tag{3.14}$$

$$\delta(0,\theta) = \varphi(-\theta) + \varphi^T(\theta) + \int_0^{\theta} G^T\beta(\xi)d\xi \tag{3.15}$$

$$\delta(-r,\theta) = \varphi(-\theta-r) + \varphi^T(\theta+r) + \int_0^{\theta} G^T\beta(\xi)d\xi \tag{3.16}$$

$$\delta^T(\theta,-r)=\varphi^T(\theta+r)+\varphi(-\theta-r)+\int_0^{-r}\beta^T(\xi)Gd\xi. \tag{3.17}$$

When we put (3.14) and (3.15) into (3.8), we get the formula

$$-\frac{d\beta(\theta)}{d\theta}+A^T\beta(\theta)+2\varphi^T(\theta)+2\varphi(-\theta)+\int_0^{\theta}G^T\beta(\xi)d\xi+2\alpha G=0 \tag{3.18}$$

for $\theta\in[-r,0]$.

We put (3.16) and (3.17) into (3.9). After some calculations we obtain

$$2\varphi^T(\theta)+2\varphi(-\theta)=\beta^T(-\theta-r)B-\int_0^{-\theta-r}\beta^T(\xi)Gd\xi-\int_0^{-r}G^T\beta(\xi)d\xi. \tag{3.19}$$

We put (3.19) into (3.18). After some calculations we get

$$\frac{d\beta(\theta)}{d\theta}=A^T\beta(\theta)+\beta^T(-\theta-r)B+\int_{-r}^{\theta}\beta^T(-\xi-r)Gd\xi+\int_{-r}^{\theta}G^T\beta(\xi)d\xi+2\alpha G. \tag{3.20}$$

We introduce a new function

$$\kappa(\theta)=\beta(-\theta-r)\qquad\text{for}\quad\theta\in[-r,0]. \tag{3.21}$$

Now we can write down the formula (3.20) in the form

$$\frac{d\beta(\theta)}{d\theta}=A^T\beta(\theta)+\kappa^T(\theta)B+\int_{-r}^{\theta}\kappa^T(\xi)Gd\xi+\int_{-r}^{\theta}G^T\beta(\xi)d\xi+2\alpha G. \tag{3.22}$$

We calculate the derivative of the function κ given by formula (3.21), taking into account the relation (3.22)

$$\frac{d\kappa(\theta)}{d\theta}=-A^T\kappa(\theta)-\beta^T(\theta)B+\int_0^{\theta}G^T\kappa(\xi)d\xi+\int_0^{\theta}\beta^T(\xi)Gd\xi-2\alpha G. \tag{3.23}$$

We introduce two new functions

$$\eta(\theta)=A^T\beta(\theta)+\kappa^T(\theta)B+\int_{-r}^{\theta}\kappa^T(\xi)Gd\xi+\int_{-r}^{\theta}G^T\beta(\xi)d\xi+2\alpha G \tag{3.24}$$

$$\vartheta(\theta)=-A^T\kappa(\theta)-\beta^T(\theta)B+\int_0^{\theta}G^T\kappa(\xi)d\xi+\int_0^{\theta}\beta^T(\xi)Gd\xi-2\alpha G. \tag{3.25}$$

Functions η and ϑ are not independent. It is easy to check that they are linked by formula

$$\eta(-\theta - r) = -\vartheta(\theta) \quad for\ \theta \in [-r, 0]\,. \tag{3.26}$$

From equations (3.22) and (3.24) it results that

$$\frac{d\beta(\theta)}{d\theta} = \eta(\theta). \tag{3.27}$$

From equations (3.23) and (3.25) it follows that

$$\frac{d\kappa(\theta)}{d\theta} = \vartheta(\theta). \tag{3.28}$$

We calculate the derivatives of (3.24) and (3.25). Upon taking the relations (3.27) and (3.28) into account, we get the formulas

$$\frac{d\eta(\theta)}{d\theta} = A^T\eta(\theta) + \vartheta^T(\theta)B + G^T\beta(\theta) + \kappa^T(\theta)G \tag{3.29}$$

$$\frac{d\vartheta(\theta)}{d\theta} = -A^T\vartheta(\theta) - \eta^T(\theta)B + G^T\kappa(\theta) + \beta^T(\theta)G. \tag{3.30}$$

We obtained the system of differential equations

$$\begin{cases} \frac{d\beta(\theta)}{d\theta} = \eta(\theta) \\ \frac{d\kappa(\theta)}{d\theta} = \vartheta(\theta) \\ \frac{d\eta(\theta)}{d\theta} = A^T\eta(\theta) + \vartheta^T(\theta)B + G^T\beta(\theta) + \kappa^T(\theta)G \\ \frac{d\vartheta(\theta)}{d\theta} = -A^T\vartheta(\theta) - \eta^T(\theta)B + G^T\kappa(\theta) + \beta^T(\theta)G \end{cases} \tag{3.31}$$

for $\theta \in [-r, 0]$.

The solution of the differential equations (3.31) satisfies the conditions

$$\beta(\theta)\,|_{\theta=-\frac{r}{2}} = \kappa(\theta)\,|_{\theta=-\frac{r}{2}} \tag{3.32}$$

$$\eta(\theta)\,|_{\theta=-\frac{r}{2}} = -\vartheta(\theta)\,|_{\theta=-\frac{r}{2}}\,. \tag{3.33}$$

Formula (3.32) was obtained from (3.21) and formula (3.33) from (3.26).

Now we obtain the initial conditions of the differential equations (3.31).

From equation (3.21) it results that

$$\kappa(-r) = \beta(0). \tag{3.34}$$

From equation (3.24) we get

$$\eta(-r) = A^T\beta(-r) + \kappa^T(-r)B + 2\alpha G. \tag{3.35}$$

Upon taking the relations (3.34) and (3.12) into account, equations (3.5) and (3.6) assume the form

$$A^T\alpha + \alpha A + \frac{\kappa(-r) + \kappa^T(-r)}{2} = -W \tag{3.36}$$

$$2B^T\alpha - \beta^T(-r) = 0. \tag{3.37}$$

We obtained the system of algebraic equations

$$\begin{cases} A^T\alpha + \alpha A + \frac{\kappa(-r)+\kappa^T(-r)}{2} = -W \\ 2B^T\alpha - \beta^T(-r) = 0 \\ -\eta(-r) + A^T\beta(-r) + \kappa^T(-r)B + 2\alpha G = 0 \\ \beta\,(\theta)\ |_{\theta=-\frac{r}{2}} = \kappa(\theta)\ |_{\theta=-\frac{r}{2}} \\ \eta\,(\theta)\ |_{\theta=-\frac{r}{2}} = -\vartheta(\theta)\ |_{\theta=-\frac{r}{2}}\ . \end{cases} \tag{3.38}$$

The set of algebraic equations (3.38) allows for determination of the matrix α and the initial conditions of the system of differential equations (3.31).

From equations (3.18) and (3.27) we obtain

$$\varphi^T(\theta) + \varphi(-\theta) = -\alpha G - \frac{1}{2}A^T\beta(\theta) + \frac{1}{2}\eta(\theta) - \frac{1}{2}\int_0^\theta G^T\beta(\xi)d\xi. \tag{3.39}$$

When we put (3.39) into (3.13), we get the matrix $\delta(\theta,\sigma)$

$$\delta(\theta,\sigma) = -\frac{1}{2}\beta^T(\theta-\sigma)A + \frac{1}{2}\eta^T(\theta-\sigma) - \frac{1}{2}\int_0^{\theta-\sigma}\beta^T(\xi)Gd\xi + \int_0^\sigma G^T\beta(\xi)d\xi - G^T\alpha \tag{3.40}$$

for $\theta \in [-r,0]$, $\sigma \in [-r,0]$.

In this way we obtained all parameters of the Lyapunov functional.

4. Examples

1. Let us consider the system described by

$$\begin{cases} \frac{dx(t)}{dt} = ax(t) + bx_t(-r) + \int_{-r}^0 gx_t(\theta)d\theta \\ x(t_0) = x_0 \in \mathbb{R} \\ x_{t_0} = \Phi \in W^{1,2}([-r,0),\mathbb{R}) \end{cases} \tag{4.1}$$

$$t \ge t_0,\ x(t) \in \mathbb{R}, \quad \theta \in [-r,0), \quad a,\ b,\ g \in \mathbb{R}, \quad r > 0$$

$$x_t \in W^{1,2}([-r,0),\mathbb{R}), \quad x_t(\xi) = x(t+\xi),\ \xi \in [-r,0).$$

The Lyapunov functional is defined by the formula

$$V(S(t)) = \alpha x^2(t) + \int_{-r}^{0} x(t)\beta(\theta)x_t(\theta)d\theta + \int_{-r}^{0} \gamma(\theta)x_t^2(\theta)d\theta + \\ + \int_{-r}^{0}\int_{-r}^{0} x_t(\theta)\delta(\theta,\sigma)x_t(\sigma)d\sigma d\theta. \quad (4.2)$$

The set of equations (3.31) becomes

$$\begin{bmatrix} \frac{d\beta(\theta)}{d\theta} \\ \frac{d\eta(\theta)}{d\theta} \\ \frac{d\kappa(\theta)}{d\theta} \\ \frac{d\vartheta(\theta)}{d\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ g & a & g & b \\ 0 & 0 & 0 & 1 \\ g & -b & g & -a \end{bmatrix} \begin{bmatrix} \beta(\theta) \\ \eta(\theta) \\ \kappa(\theta) \\ \vartheta(\theta) \end{bmatrix}. \quad (4.3)$$

Eigenvalues of the matrix of equation (4.3) are as follows

$$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = \sqrt{a^2 + 2g - b^2},\ \lambda_4 = -\sqrt{a^2 + 2g - b^2}. \quad (4.4)$$

Now we give the formulas for determination of the set of initial conditions of equation (4.3) and the coefficient α

$$\begin{cases} 2a\alpha + \kappa(-r) = -w \\ 2b\alpha - \beta(-r) = 0 \\ -\eta(-r) + a\beta(-r) + b\kappa(-r) + 2g\alpha = 0 \\ \beta(\theta)\mid_{\theta=-\frac{r}{2}} = \kappa(\theta)\mid_{\theta=-\frac{r}{2}} \\ \eta(\theta)\mid_{\theta=-\frac{r}{2}} = -\vartheta(\theta)\mid_{\theta=-\frac{r}{2}}. \end{cases} \quad (4.5)$$

Having the solution of equations (4.3) and the coefficient α we obtain $\delta(\theta,\sigma)$

$$\delta(\theta,\sigma) = -ga - \frac{1}{2}a\beta(\theta-\sigma) + \frac{1}{2}\eta(\theta-\sigma) - \frac{1}{2}\int_{0}^{\theta-\sigma} g\beta(\xi)d\xi + \int_{0}^{\sigma} g\beta(\xi)d\xi. \quad (4.6)$$

Fig. 1 shows the graphs of functions $\beta(\theta)$, $\eta(\theta)$, $\kappa(\theta)$, $\vartheta(\theta)$ and α, obtained with the Matlab code, for given values of parameters a, b, g, w, r of the system (4.1).

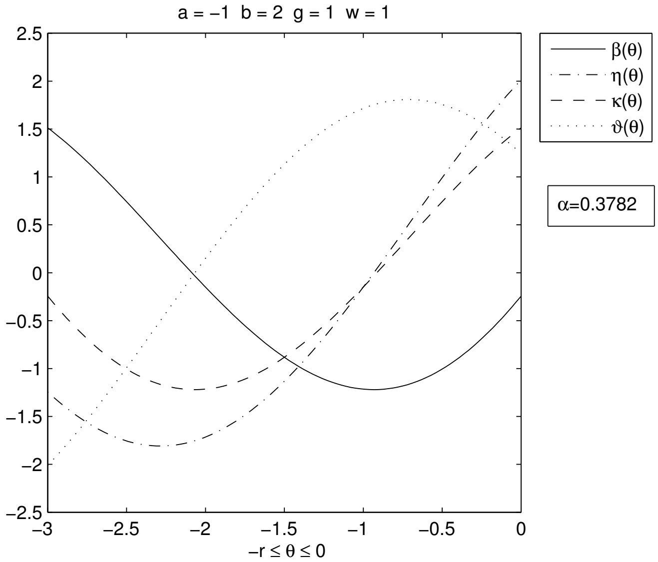

Figure 1.

2. Let us consider the system described by

$$\begin{cases} \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_1(t-r) \\ x_2(t-r) \end{bmatrix} + \\ \qquad + \int_{-r}^{0} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} x_1(t+\theta) \\ x_2(t+\theta) \end{bmatrix} d\theta \\ \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\ \begin{bmatrix} x_1(t_0+\theta) \\ x_2(t_0+\theta) \end{bmatrix} = \begin{bmatrix} \Phi_1(\theta) \\ \Phi_2(\theta) \end{bmatrix}. \end{cases} \tag{4.7}$$

The Lyapunov functional is defined by the formula

$$\begin{aligned} V(S(t)) &= \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \\ &+ \int_{-r}^{0} \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} \beta_{11}(\theta) & \beta_{12}(\theta) \\ \beta_{21}(\theta) & \beta_{22}(\theta) \end{bmatrix} \begin{bmatrix} x_1(t+\theta) \\ x_2(t+\theta) \end{bmatrix} d\theta + \\ &+ \int_{-r}^{0}\int_{-r}^{0} \begin{bmatrix} x_1(t+\theta) & x_2(t+\theta) \end{bmatrix} \begin{bmatrix} \delta_{11}(\theta,\sigma) & \delta_{12}(\theta,\sigma) \\ \delta_{21}(\theta,\sigma) & \delta_{22}(\theta,\sigma) \end{bmatrix} \begin{bmatrix} x_1(t+\sigma) \\ x_2(t+\sigma) \end{bmatrix} d\sigma d\theta. \end{aligned} \tag{4.8}$$

The set of equations (3.31) becomes

$$\frac{d}{d\theta}\begin{bmatrix} col\,\beta(\theta) \\ col\,\eta(\theta) \\ col\,\kappa(\theta) \\ col\,\vartheta(\theta) \end{bmatrix} = Q \begin{bmatrix} col\,\beta(\theta) \\ col\,\eta(\theta) \\ col\,\kappa(\theta) \\ col\,\vartheta(\theta) \end{bmatrix} \tag{4.9}$$

for $\theta \in [-r, 0]$, where

$$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \tag{4.10}$$

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ g_{11} & g_{21} & 0 & 0 & a_{11} & a_{21} & 0 & 0 \\ g_{12} & g_{22} & 0 & 0 & a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & g_{11} & g_{21} & 0 & 0 & a_{11} & a_{21} \\ 0 & 0 & g_{12} & g_{22} & 0 & 0 & a_{12} & a_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_{11} & g_{21} & 0 & 0 & -b_{11} & -b_{21} & 0 & 0 \\ 0 & g_{21} & g_{11} & 0 & 0 & 0 & -b_{11} & -b_{21} \\ g_{12} & g_{22} & 0 & 0 & -b_{12} & -b_{22} & 0 & 0 \\ 0 & 0 & g_{12} & g_{22} & 0 & 0 & -b_{12} & -b_{22} \end{bmatrix} \tag{4.11}$$

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_{11} & g_{21} & 0 & 0 & b_{11} & b_{21} & 0 & 0 \\ 0 & 0 & g_{11} & g_{21} & 0 & 0 & b_{11} & b_{21} \\ g_{12} & g_{22} & 0 & 0 & b_{12} & b_{22} & 0 & 0 \\ 0 & 0 & g_{12} & g_{22} & 0 & 0 & b_{12} & b_{22} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ g_{11} & g_{21} & 0 & 0 & -a_{11} & -a_{21} & 0 & 0 \\ g_{12} & g_{22} & 0 & 0 & -a_{12} & -a_{22} & 0 & 0 \\ 0 & 0 & g_{11} & g_{21} & 0 & 0 & -a_{11} & -a_{21} \\ 0 & 0 & g_{12} & g_{22} & 0 & 0 & -a_{12} & -a_{22} \end{bmatrix} \tag{4.12}$$

$$\begin{bmatrix} col\,\beta(\theta) \\ col\,\eta(\theta) \\ col\,\kappa(\theta) \\ col\,\vartheta(\theta) \end{bmatrix} = e^{Q(\theta+r)} \begin{bmatrix} col\,\beta(-r) \\ col\,\eta(-r) \\ col\,\kappa(-r) \\ col\,\vartheta(-r) \end{bmatrix} \tag{4.13}$$

for $\theta \in [-r, 0]$.

We introduce

$$e^{Q\frac{r}{2}} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}. \tag{4.14}$$

Now we give the formulas for determination of the set of initial conditions of equation (4.9) and the matrix α:

$$Z \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{22} \\ col\,\beta(-r) \\ col\,\eta(-r) \\ col\,\kappa(-r) \\ col\,\vartheta(-r) \end{bmatrix} = \begin{bmatrix} -w_{11} \\ -w_{12} \\ -w_{22} \\ 0_{(16,1)} \end{bmatrix} \tag{4.15}$$

where

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ 0_{(8,3)} & Z_{22} & Z_{23} \end{bmatrix} \tag{4.16}$$

$$Z_{11} = \begin{bmatrix} 2a_{11} & 2a_{21} & 0 \\ a_{12} & a_{11}+a_{22} & a_{21} \\ 0 & 2a_{12} & 2a_{22} \\ b_{11} & b_{21} & 0 \\ b_{12} & b_{22} & 0 \\ 0 & b_{11} & b_{21} \\ 0 & b_{12} & b_{22} \\ 2g_{11} & 2g_{21} & 0 \\ 2g_{12} & 2g_{22} & 0 \\ 0 & 2g_{11} & 2g_{21} \\ 0 & 2g_{12} & 2g_{22} \end{bmatrix} \tag{4.17}$$

$$Z_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ a_{11} & a_{21} & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & a_{21} & 0 & 0 & -1 & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & a_{12} & a_{22} & 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.18}$$

$$Z_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{11} & b_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{12} & b_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{11} & b_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{12} & b_{22} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.19}$$

$$Z_{22} = \begin{bmatrix} p_{11} - p_{31} & p_{12} - p_{32} \\ p_{21} + p_{41} & p_{22} + p_{42} \end{bmatrix} \tag{4.20}$$

$$Z_{23} = \begin{bmatrix} p_{13} - p_{33} & p_{14} - p_{34} \\ p_{23} + p_{43} & p_{24} + p_{44} \end{bmatrix}. \tag{4.21}$$

Now we obtain the matrix $\delta(\theta, \sigma)$:

$$\begin{aligned} \delta_{11}(\theta,\sigma) = & -\frac{1}{2}a_{11}\beta_{11}(\theta-\sigma) - \frac{1}{2}a_{21}\beta_{21}(\theta-\sigma) + \frac{1}{2}\eta_{11}(\theta-\sigma) + \\ & -\frac{1}{2}\int\limits_0^{\theta-\sigma} \left[g_{11}\beta_{11}(\xi) + g_{21}\beta_{21}(\xi)\right] d\xi + \int\limits_0^{\sigma} \left[g_{11}\beta_{11}(\xi) + g_{21}\beta_{21}(\xi)\right] d\xi + \\ & -g_{11}\alpha_{11} - g_{21}\alpha_{12} \end{aligned} \tag{4.22}$$

$$\begin{aligned} \delta_{12}(\theta,\sigma) = & -\frac{1}{2}a_{12}\beta_{11}(\theta-\sigma) - \frac{1}{2}a_{22}\beta_{21}(\theta-\sigma) + \frac{1}{2}\eta_{21}(\theta-\sigma) + \\ & -\frac{1}{2}\int\limits_0^{\theta-\sigma} \left[g_{12}\beta_{11}(\xi) + g_{22}\beta_{21}(\xi)\right] d\xi + \int\limits_0^{\sigma} \left[g_{11}\beta_{12}(\xi) + g_{21}\beta_{22}(\xi)\right] d\xi + \\ & -g_{11}\alpha_{12} - g_{21}\alpha_{22} \end{aligned} \tag{4.23}$$

$$\delta_{21}(\theta,\sigma) = -\frac{1}{2}a_{11}\beta_{12}(\theta-\sigma) - \frac{1}{2}a_{21}\beta_{22}(\theta-\sigma) + \frac{1}{2}\eta_{12}(\theta-\sigma) +$$
$$-\frac{1}{2}\int\limits_0^{\theta-\sigma}[g_{11}\beta_{12}(\xi)+g_{21}\beta_{22}(\xi)]\,d\xi + \int\limits_0^{\sigma}[g_{12}\beta_{11}(\xi)+g_{22}\beta_{21}(\xi)]\,d\xi +$$
$$-g_{12}\alpha_{11} - g_{22}\alpha_{12} \tag{4.24}$$
$$\delta_{22}(\theta,\sigma) = -\frac{1}{2}a_{12}\beta_{12}(\theta-\sigma) - \frac{1}{2}a_{22}\beta_{22}(\theta-\sigma) + \frac{1}{2}\eta_{22}(\theta-\sigma) +$$
$$-\frac{1}{2}\int\limits_0^{\theta-\sigma}[g_{12}\beta_{12}(\xi)+g_{22}\beta_{22}(\xi)]\,d\xi + \int\limits_0^{\sigma}[g_{12}\beta_{12}(\xi)+g_{22}\beta_{22}(\xi)]\,d\xi +$$
$$-g_{12}\alpha_{12} - g_{22}\alpha_{22}. \tag{4.25}$$

Figs. 2, 3, 4, 5 show graphs of functions $\beta(\theta)$, $\eta(\theta)$, $\kappa(\theta)$, $\vartheta(\theta)$ obtained with the Matlab code, for given values of matrices A, B, G, W of the system (4.7)

$$A = \begin{bmatrix} -1 & 0.3 \\ 0.5 & -2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0.4 \\ 0.1 & 2 \end{bmatrix},\ G = \begin{bmatrix} 1 & 0.7 \\ 0.3 & 2 \end{bmatrix},\ W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.26}$$

Matrix α obtained for the values (4.26) is given below

$$\alpha = \begin{bmatrix} 0.1833 & 0.0281 \\ 0.0281 & 0.1600 \end{bmatrix}. \tag{4.27}$$

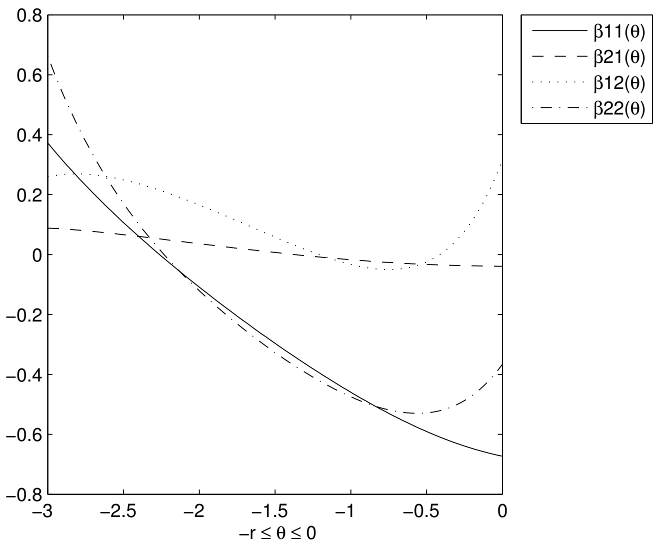

Figure 2.

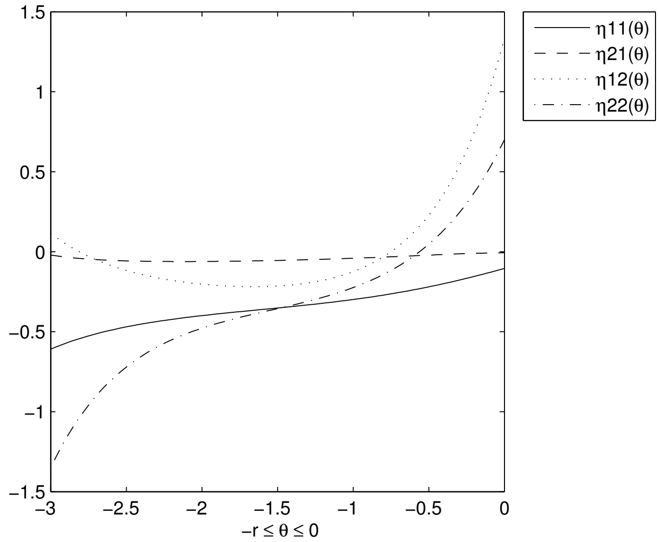

Figure 3.

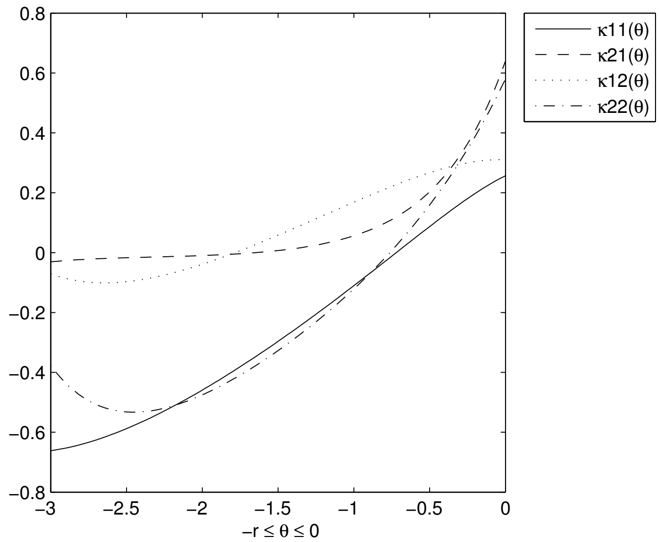

Figure 4.

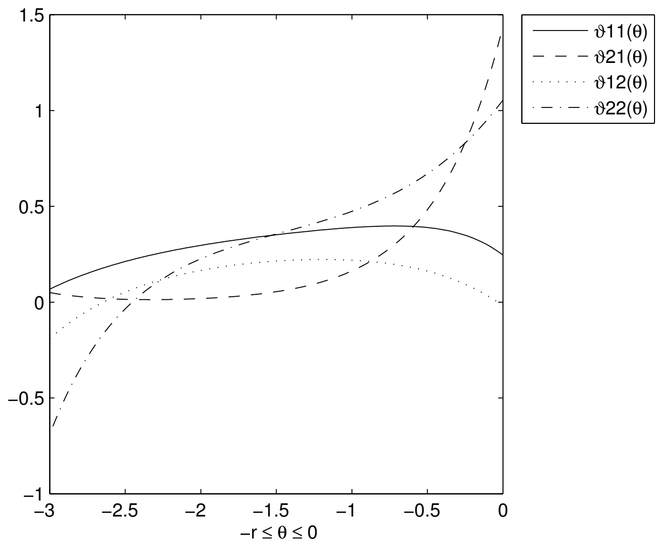

Figure 5.

5. Conclusions

The paper presents the procedure for determining the coefficients of the Lyapunov functional, given by the formula (2.4), for the linear system with both lumped and distributed delay, described by equation (2.1). This paper extended the method due to Repin to the systems with both lumped and distributed delay. The method presented allows for obtaining the analytical formula for the factors occurring in the Lyapunov functional, which can be used to examine the stability and in the process of parametric optimization to determine the square quality index, given by formula (3.4).

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