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# Optimality conditions for a class of optimal boundary control problems with quasilinear elliptic equations*†

by

**Eduardo Casas**[1] **and Vili Dhamo**[2]

[1]Dpto. de Matemática Aplicada y Ciencias de la Computación,
E.T.S.I. Industriales y de Telecommunicación, Universidad de Cantabria
39005 Santander, Spain

[2]Institut für Mathematik, Technische Universität Berlin
D-10623 Berlin, Str. des 17. Juni 136, Germany

e-mail: eduardo.casas@unican.es, vdhamo@math.tu-berlin.de

**Abstract:** First- and second-order optimality conditions are established for the boundary optimal control of quasilinear elliptic equations with pointwise constraints on the control. The theory is developed for Neumann controls in polygonal domains of dimension two. For the derivation of second-order sufficient optimality conditions, which is the main goal of this paper, the regularity of the solutions to the state equation and its linearization is studied in detail. Moreover, a Pontryagin principle is proved. The main difficulty in the analysis of these problems is the nonmonotone character of the state equation.

**Keywords:** optimal control, Neumann boundary control, quasilinear elliptic equation, Pontryagin principle, second order optimality conditions.

## 1. Introduction

This paper is concerned with a class of optimal control problems for the following quasilinear elliptic equation

$$\begin{cases} -\operatorname{div}\left[a(x,y(x))\nabla y(x)\right] + f(x,y(x)) = 0 & \text{in } \Omega\,, \\ a(x,y(x))\nabla y(x)\cdot\vec{n}(x) = u(x) & \text{on } \Gamma\,, \end{cases} \tag{1.1}$$

where $\vec{n}(x) = (n_1(x), n_2(x)) \in \mathbb{R}^2$ is the outward unit normal vector to $\Gamma$ at $x$.

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Problems of this type occur in many practical applications of optimal control theory to problems in engineering and medical science; for instance, in models of heat conduction, where the heat conductivity $a$ depends on the spatial coordinate $x$ and on the temperature $y$. As outlined in Casas and Tröltzsch (2009), the heat conductivity of carbon steel depends on the temperature and also on the alloying additions contained; see Bejan (1995). If the different alloys of steel are distributed smoothly in the domain, then $a = a(x, y)$ should depend in a sufficiently smooth way on $(x, y)$. Similarly, the heat conductivity depends on $(x, y)$ in the growth of silicon carbide bulk single crystals, see Klein et al. (2001).

Although $f$ is considered monotone nondecreasing with respect to $y$, the above equation is not of monotone type because the coefficients of the operator depend on the state $y$. This causes the main difficulty in deducing regularity properties for the solution to (1.1).

There has been some recent progress in the case of optimal control problems governed by quasilinear equations. The first step towards a corresponding analysis was recently made by Casas and Tröltzsch (2009), where first- and second-order optimality conditions for the distributed optimal control of quasilinear elliptic equations are discussed. For other classes of quasilinear equations, in which $a$ depends on the gradient of $y$, we refer to, for instance, Lions (1969). To our knowledge the state equation (1.1) with Neumann boundary control has not yet been investigated in the context of optimal control. The present paper extends the theory developed in Casas and Tröltzsch (2009) to the case of Neumann boundary controls. However, the analysis is more difficult, since the regularity of the states is lower than that for distributed controls. Although our equation has a particular type, the control of (1.1) is - with respect to the analysis - of model character for optimal boundary control problems with more general quasilinear equations or systems.

For semilinear elliptic and parabolic equations there exists a very extensive literature about optimization. For instance, the Pontryagin principle was discussed for different elliptic problems in Bonnans and Casas (1991), Casas (1996), while the parabolic case was investigated in Casas (1997), Casas, Raymond and Zidani (2000), and Raymond and Zidani (1999). Problems with quasilinear equations with nonlinearity of gradient type were considered by Casas and Fernández (1993, 1995), Casas, Fernández and Yong (1995), and Casas and Yong (1995).

While first-order optimality conditions are useful to deduce regularity properties for optimal controls, the second-order optimality conditions are very important to analyze the convergence properties of numerical optimization algorithms applied to control problems. They are also a key tool to derive error estimates for local solutions of the finite element approximations of the optimal control problem.

The outline of the paper is as follows. In the next two sections, we discuss the well-posedness of equation (1.1) in different spaces and derive some important differentiability properties of the control-to-state mapping. In Section 4, an optimal control problem with pointwise control constraints is introduced, conditions for the existence of at least one optimal solution are given and first-order necessary optimality conditions are obtained. Based on these results, a Pontryagin principle is derived in Section 5. Second-order necessary and sufficient optimality conditions are the topic of the final section.

In this paper we study the equation (1.1) in polygonal domains, since it is the first part of a forthcoming paper about the numerical analysis, including error estimates, of the control problem in plane polygonal domains. This paper provides the theoretical results for the numerical analysis. However, some obvious extensions to more general domains are indicated in here.

## 2. Study of the quasilinear equations

In the following, we state the main assumptions on the equation (1.1).

Assumption 2.1 *$\Omega$ is an open bounded polygonal set of $\mathbb{R}^2$ and $\Gamma$ is the boundary of $\Omega$.*

Assumption 2.2 *The functions $a : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory,*

$$\exists \alpha_a > 0 \ \text{ such that } \ a(x, y) \geq \alpha_a \ \text{ for a.e. } \ x \in \Omega \ \text{ and all } \ y \in \mathbb{R}. \tag{2.1}$$

*It holds that $a(\cdot, 0) \in L^\infty(\Omega)$, $f(\cdot, 0) \in L^p(\Omega)$, with $p \geq 2$, and for any $M > 0$ there exists a constant $C_M > 0$ and a function $\phi_M \in L^p(\Omega)$ such that, for all $|y|, |y_i| \leq M$, $i = 1, 2$,*

$$|a(x, y_2) - a(x, y_1)| \leq C_M |y_2 - y_1| \ \text{ and } \ \left|\frac{\partial f}{\partial y}(x, y)\right| \leq \phi_M(x) \ \text{ for a.e. } x \in \Omega. \tag{2.2}$$

Assumption 2.3 *$f$ is monotone non-decreasing with respect to the second variable for almost all $x \in \Omega$. There exist a positive constant $\alpha_f > 0$ and a subset $E \subset \Omega$ such that $|E| > 0$ and*

$$\frac{\partial f}{\partial y}(x, y) \geq \alpha_f \quad \forall (x, y) \in E \times \mathbb{R}.$$

*Here, $|E|$ denotes the Lebesgue measure of $E$.*

Given a Banach space $V$, we shall denote by $\langle \cdot, \cdot \rangle_{V^*,V}$ the duality product between its dual space $V^*$ and $V$. When no ambiguity arises, we will abbreviate $\langle \cdot, \cdot \rangle_{V^*,V}$ by $\langle \cdot, \cdot \rangle$. We also consider the space

$$W^{1/p',p}(\Gamma) = \left\{ y\big|_\Gamma \ \middle| \ y \in W^{1,p}(\Omega) \right\}$$

endowed with the standard norm

$$\|u\|_{W^{1/p',p}(\Gamma)} = \inf\left\{\|y\|_{W^{1,p}(\Omega)} \,\middle|\, y\big|_\Gamma = u\right\}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $y\big|_\Gamma$ is the trace of $y$ on $\Gamma$, see Lions (1969). The dual space of $W^{1/p',p}(\Gamma)$ is denoted by $W^{-1/p',p'}(\Gamma)$.

By $B_X(x,r)$ we denote the open ball in a normed space $X$ with radius $r$ centered at $x$, and by $\bar{B}_X(x,r)$ we denote its closure. By $C$ (without index) generic constants are denoted and in some formulas, the partial derivative $\partial/\partial x_j$ is sometimes abbreviated by $\partial_j$; $\sigma$ stands for the usual one-dimensional measure on $\Gamma$ induced by the associated parametrization (remember that $\Gamma$ is a Lipschitz manifold).

Throughout the paper, the solutions of PDEs are understood in the weak sense.

THEOREM 2.1 *Under the Assumptions 2.1-2.3, for any $u \in L^s(\Gamma)$, $s > 1$, problem (1.1) has a unique solution $y_u \in H^1(\Omega) \cap L^\infty(\Omega)$. Moreover there exists $\mu \in (0,1)$ independent of $u$ such that $y_u \in C^\mu(\bar{\Omega})$ and, for any bounded set $U \subset L^s(\Gamma)$,*

$$\|y_u\|_{H^1(\Omega)} + \|y_u\|_{C^\mu(\bar{\Omega})} \leq C_U \quad \forall u \in U\,, \tag{2.3}$$

*with some constant $C_U > 0$.*

*Proof. Existence of a solution.* Depending on $M > 0$, we introduce the truncated function $a_M$ by

$$a_M(x,y) = \begin{cases} a(x,y) & |y| \leq M \\ a(x,+M) & y > +M \\ a(x,-M) & y < -M\,. \end{cases}$$

In the next step we will use the following identity

$$f(x,y(x)) = f(x,0) + f_0(x,y(x))y(x), \quad \text{where } f_0(x,y) := \int_0^1 \frac{\partial f}{\partial y}(x,\theta y)\,d\theta\,.$$

In the same way, we define the truncation $f_{0M}$ of $f_0$. Next we prove the existence of a solution $y_M \in H^1(\Omega)$ of the equation with truncated coefficients

$$\begin{cases} -\operatorname{div}[a_M(x,y(x))\nabla y(x)] + f_{0M}(x,y(x))y(x) = -f(x,0) & \text{in } \Omega\,, \\ a_M(x,y(x))\nabla y(x)\cdot\vec{n}(x) = u(x) & \text{on } \Gamma\,. \end{cases} \tag{2.4}$$

To do this, we fix $u \in L^s(\Gamma)$ and $M > 0$ and define the mapping $F : L^2(\Omega) \to L^2(\Omega)$ by $F(z) = y$, where $y \in H^1(\Omega)$ is the unique solution to the linear equation

$$\begin{cases} -\operatorname{div}[a_M(x,z(x))\nabla y(x)] + f_{0M}(x,z(x))y(x) = -f(x,0) & \text{in } \Omega\,, \\ a_M(x,z(x))\nabla y(x)\cdot\vec{n}(x) = u(x) & \text{on } \Gamma\,. \end{cases} \tag{2.5}$$

Thanks to the Assumptions 2.2 and 2.3, we have $f(\cdot,0), f_{0M}(\cdot,z) \in L^p(\Omega)$ and $0 < \alpha_f \le f_{0M}(x,z)$, for each $(x,z) \in E \times \mathbb{R}$, where $E$ is introduced in Assumption 2.3. Since (2.5) is a monotone linear equation, by applying Lax-Milgram theorem we get the existence of a unique solution $y_M \in H^1(\Omega)$ of (2.5) and $F$ is well-defined. Furthermore, we have

$$\|y_M\|_{H^1(\Omega)} \le C_{a,f}\left(\|u\|_{L^s(\Gamma)} + \|f(\cdot,0)\|_{L^p(\Omega)}\right), \tag{2.6}$$

where $C_{a,f}$ depends only on $|\Omega|, \alpha_a, \alpha_f$, but neither on $a_M$ nor on $f_{0M}$. Because of the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ it is easy to apply Schauder's theorem to prove the existence of a fixed point $y_M \in H^1(\Omega)$ of $F$, which is obviously a solution of (2.4).

Now, applying Stampacchia's truncation method (see, for instance, Stampacchia, 1965, or the exposition for semilinear elliptic equations in Tröltzsch, 2010) we get

$$\|y_M\|_{L^\infty(\Omega)} \le C_\infty\big(\|u\|_{L^s(\Gamma)} + \|f(\cdot,0)\|_{L^p(\Omega)}\big), \tag{2.7}$$

where the constant $C_\infty$ depends only on $\alpha_a, \alpha_f$, but neither on $a_M(\cdot, y_M)$ nor on $f_{0M}(\cdot, y_M)$. By taking

$$M \ge C_\infty\big(\|u\|_{L^s(\Gamma)} + \|f(\cdot,0)\|_{L^p(\Omega)}\big),$$

(2.7) implies that $a_M(x,y_M(x)) = a(x,y_M(x))$ and $f_{0M}(x,y_M(x)) = f_0(x,y_M(x))$ for a.e. $x \in \Omega$, therefore $y_M \in H^1(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.1). The Hölder regularity is well known; see Murthy and Stampacchia (1972), Stampacchia (1960), or Griepentrog and Recke (2001), Gröger (1989). The inequality (2.3) follows from (2.6), (2.7) and the estimates in Stampacchia (1960).

*Uniqueness of the solution.* The proof is along the lines of the proof of Casas and Tröltzsch (2009, Theorem 2.2), where the authors follow the method by Hlaváček, Křížek and Malý (1994). The only difference in the case of Neumann boundary conditions is the use of the inequality

$$\|z\|^2_{L^2(\Omega)} \le C_E\left(\|\nabla z\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(E)}\right) \quad \forall z \in H^1(\Omega),$$

see Nečas (1967), instead of Friedrich's inequality. ∎

By assuming that $a$ is continuous on $\bar\Omega \times \mathbb{R}$, we can obtain higher regularity of the solutions of (1.1).

THEOREM 2.2 *Let us suppose that Assumptions 2.1-2.3 hold. We also assume that $a : \bar\Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. Then there exists $\bar p > 3$ such that, for any $2 < p \le \bar p$ and any $u \in L^{p/2}(\Gamma)$, (1.1) has a unique solution $y_u \in W^{1,p}(\Omega)$. Moreover, for any bounded set $U \subset L^{p/2}(\Gamma)$, there exists a constant $C_U > 0$ such that*

$$\|y_u\|_{W^{1,p}(\Omega)} \le C_U \quad \forall u \in U. \tag{2.8}$$

*In addition, if $\Omega$ is convex, then the above conclusions remain valid for some* $\bar{p} \geq \frac{6}{3-\sqrt{5}}$.

*Proof.* Here we follow the classical approach of freezing the coefficients around certain points of the domain to perform a reduction from variable coefficients to constant coefficients.

Due to Theorem 2.1, (1.1) admits a unique solution $y_u$ in $H^1(\Omega) \cap L^\infty(\Omega)$. We have to prove its $W^{1,p}(\Omega)$ regularity. Note that, thanks to our assumptions and the continuity of $y$, $\tilde{a}(\cdot) = a(\cdot, y_u(\cdot))$ is continuous in $\bar{\Omega}$.

Let $\rho > 0$, then there exists a finite number of boundary points $\{x_j\}_{j=1}^m \subset \Gamma$ such that $\Gamma \subset \bigcup_{j=1}^m B_{\mathbb{R}^2}(x_j, \rho)$. Further, let $U$ be an open set with regular boundary such that $U \subset \bar{U} \subset \Omega$ and $\Omega \subset \bigcup_{j=1}^m B_{\mathbb{R}^2}(x_j, \rho) \cup U$. We also take a partition of unity $\{\psi_j\}_{j=0}^m \subset C^\infty(\mathbb{R}^2)$ with $\sum_{j=0}^m \psi_j(x) = 1$, $0 \leq \psi_j(x) \leq 1$ $\forall x \in \bar{\Omega}$ and for $j = 0, \ldots, m$, supp $\psi_0 \subset U$ and supp $\psi_l \subset B_{\mathbb{R}^2}(x_l, \rho)$ for $l = 1, \ldots, m$. Then $y_u = \sum_{j=0}^m y_j$, where $y_j := \psi_j y_u$. We prove that $y_j \in W^{1,p}(\Omega)$ for every $j = 0, \ldots, m$.
For $j = 0$ we get

$$
\begin{aligned}
-\operatorname{div}[\tilde{a}(x)\nabla y_0] + y_0 &= -\operatorname{div}[\tilde{a}(x)\psi_0\nabla y_u] - \operatorname{div}[\tilde{a}(x) y_u \nabla\psi_0] + \psi_0 y_u \\
&= -\psi_0 \operatorname{div}[\tilde{a}(x)\nabla y_u] - \tilde{a}(x)\nabla y_u \cdot \nabla\psi_0 - \operatorname{div}[\tilde{a}(x) y_u \nabla\psi_0] + \psi_0 y_u \\
&= -\psi_0 f(x, y_u) - \tilde{a}(x)\nabla y_u \cdot \nabla\psi_0 - \operatorname{div}[\tilde{a}(x) y_u \nabla\psi_0] + \psi_0 y_u \\
&= G \quad \text{in } \Omega,
\end{aligned}
$$

with $G \in W^{-1,p}(\Omega)$ and $y_0 = 0$ on $\Gamma$. Hence the $W^{1,p}(\Omega)$ regularity of $y_0$ follows from Morrey (1996, pp. 156-157).
We fix now $j = 1, \ldots, m$ and $x_j \in \Gamma$. From the definition of the weak solution $y_u$ in $H^1(\Omega)$ of (1.1) we have for arbitrary $z \in H^1(\Omega)$

$$
\int_\Omega \tilde{a}(x)\nabla y_u \cdot \nabla(\psi_j z)\, dx = -\int_\Omega f(x, y_u)\psi_j z\, dx + \int_\Gamma u\psi_j z\, d\sigma(x),
$$

therefore

$$
\begin{aligned}
&\int_\Omega \{\tilde{a}(x)\nabla y_j \cdot \nabla z + y_j z\}\, dx = \int_\Omega \{\tilde{a}(x)\left[y_u \nabla\psi_j \cdot \nabla z + \psi_j \nabla y_u \cdot \nabla z\right] + y_u \psi_j z\}\, dx \\
&= \int_\Omega \{\tilde{a}(x)\left[y_u \nabla\psi_j \cdot \nabla z - z\nabla y_u \cdot \nabla\psi_j\right] - f(x, y_u)\psi_j z + y_u \psi_j z\} dx + \int_\Gamma u\psi_j z\, d\sigma(x) \\
&= F(z).
\end{aligned} \tag{2.9}
$$

$F$ is a linear continuous functional on $W^{1,p'}(\Omega)$. To verify this, consider, for instance, the terms $z\nabla y_u$ and $uz$:

$$
\|z\nabla y_u\|_{L^1(\Omega)} \leq C\|z\|_{L^{\frac{2p'}{2-p'}}(\Omega)} \|\nabla y\|_{L^{\frac{2p}{p+2}}(\Omega)} \leq C\|z\|_{W^{1,p'}(\Omega)} \|\nabla y\|_{L^2(\Omega)}
$$

the above being a consequence of the embedding $W^{1,p'}(\Omega) \subset L^{\frac{2p}{p-2}}(\Omega)$ and the fact that $\frac{2p'}{2-p'} = \frac{2p}{p-2} > 2$ for $p > 2$. Moreover, $z\big|_\Gamma \in W^{1-1/p',p'}(\Gamma) \subset L^{\frac{p}{p-2}}(\Gamma)$, hence Hölder's inequality yields

$$\begin{aligned}\|zu\|_{L^1(\Gamma)} \leq \|u\|_{L^{p/2}(\Gamma)}\|z\|_{L^{\frac{p}{p-2}}(\Gamma)} &\leq C\|u\|_{L^{p/2}(\Gamma)}\|z\|_{W^{1-1/p',p'}(\Gamma)} \\ &\leq C\|u\|_{L^{p/2}(\Gamma)}\|z\|_{W^{1,p'}(\Omega)} .\end{aligned}$$

From (2.9) we get for $z \in W^{1,p'}(\Omega)$

$$\int_\Omega \{\tilde{a}(x_j)\nabla y_j \cdot \nabla z + y_j z\}\, dx = \int_{\Omega\cap B_{\mathbb{R}^2}(x_j,\rho)} [\tilde{a}(x_j) - \tilde{a}(x)]\, \nabla y_j \cdot \nabla z\, dx + F(z)\, .$$

Consider now the mapping $\mathcal{F} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$, $\mathcal{F}(w) = y_w$, where $y_w$ is the solution of the problem

$$\int_\Omega \{\tilde{a}(x_j)\nabla y_w \cdot \nabla z + y_w z\}\, dx = \int_{\Omega\cap B_{\mathbb{R}^2}(x_j,\rho)} [\tilde{a}(x_j) - \tilde{a}(x)]\, \nabla w \cdot \nabla z\, dx + F(z)\, , \tag{2.10}$$

for every $z \in W^{1,p'}(\Omega)$. According to Dauge (1992, Corollary 3.10), $\mathcal{F}$ is well-defined, since there exists $\bar{p} > 3$ such that, for any $2 < p \leq \bar{p}$, (2.10) admits a unique solution $y_w \in W^{1,p}(\Omega)$. If $\Omega$ is assumed to be convex, then $\bar{p} \geq \frac{6}{3-\sqrt{5}}$, see Dauge (1992, Corollary 3.12). Next we prove that $\mathcal{F}$ is a contraction, so that Banach's fixed-point theorem is applicable to obtain $y_j \in W^{1,p}(\Omega)$. For $w_i \in W^{1,p}(\Omega)$, $i = 1, 2$, we have

$$\|\mathcal{F}(w_2) - \mathcal{F}(w_1)\|_{W^{1,p}(\Omega)} \leq C\|\tilde{a}(x_j) - \tilde{a}(\cdot)\|_{L^\infty(\Omega\cap B_{\mathbb{R}^2}(x_j,\rho))}\|w_2 - w_1\|_{W^{1,p}(\Omega)},$$

where $C$ depends only on $\alpha_a$ and not on $x_j$. We can choose $\rho$ sufficiently small, such that $C\|\tilde{a}(x_j) - \tilde{a}(\cdot)\|_{L^\infty(\Omega\cap B_{\mathbb{R}^2}(x_j,\rho))} < 1$, hence $\mathcal{F}$ is a contraction and the proof is complete. ∎

In the rest of the paper, $\bar{p}$ is as in the statement of the previous theorem.

REMARK 2.1 *The paper by Dauge (1992), cited in the proof of the previous theorem, deals only with problems in 3d, but the result is also valid for dimension two. Indeed, given the solution $y \in H^1(\Omega)$ of the problem*

$$\left\{\begin{array}{rcll} -\Delta y + cy &=& f & \textit{in}\, \Omega\,, \\ \nabla y \cdot \vec{n} &=& u & \textit{on}\, \Gamma\,, \end{array}\right.$$

*where $c > 0$, we introduce the prism $\tilde{\Omega} = \Omega \times (0,1)$ and consider the problem*

$$\left\{\begin{array}{rcll} -\Delta \tilde{y} + c\tilde{y} &=& \tilde{f} & \textit{in}\, \tilde{\Omega}\,, \\ \nabla \tilde{y} \cdot \vec{n} &=& \tilde{u} & \textit{on}\, \Gamma \times (0,1)\,, \\ \nabla \tilde{y} \cdot \vec{n} &=& 0 & \textit{on}\, \Omega \times \{0,1\}, \end{array}\right.$$

*where* $\tilde{f} : \tilde{\Omega} \times \mathbb{R} \to \mathbb{R}$ *and* $\tilde{u} : \Gamma \times (0,1) \longrightarrow \mathbb{R}$ *are defined by* $\tilde{f}(\tilde{x}) = f(x_1, x_2)$ *and* $\tilde{u}(\tilde{x}) = u(x_1, x_2)$*, with* $\tilde{x} = (x_1, x_2, x_3)$*. Then* $\tilde{y}_{\tilde{u}} \in W^{1,p}(\tilde{\Omega})$*, but* $\tilde{y}_{\tilde{u}}(\tilde{x}) = y_u(x_1, x_2)$ *for* $\tilde{x} \in \tilde{\Omega}$*, therefore* $y_u \in W^{1,p}(\Omega)$*.*

REMARK 2.2 *It is easy to see that Theorems 2.1 and 2.2 are still valid if we require in Assumption 2.2 that* $\phi_M$ *belongs to* $L^q(\Omega)$ *with* $q \geq 2p/(2+p) > 1$*. The reason for assuming* $\phi_M \in L^p(\Omega)$ *becomes clear in the next two theorems.*

Some additional assumptions yield higher regularity for the solutions of (1.1).

THEOREM 2.3 *Let us suppose that Assumptions 2.1-2.3 hold. Assume further that for every* $M > 0$*, there exists a constant* $C_M > 0$ *such that, for all* $x_i \in \bar{\Omega}$*,* $|y_i| \leq M$*,* $i = 1, 2$*, the following local Lipschitz property is satisfied:*

$$|a(x_2, y_2) - a(x_1, y_1)| \leq C_M \left(|x_2 - x_1| + |y_2 - y_1|\right).$$

*Then, for any* $u \in L^2(\Gamma)$*, (1.1) has a unique solution* $y_u \in H^{3/2}(\Omega)$*. Moreover, for any bounded set* $U \subset L^2(\Gamma)$*, there exists a constant* $C_U > 0$ *such that*

$$\|y_u\|_{H^{3/2}(\Omega)} \leq C_U \quad \forall u \in U. \tag{2.11}$$

*Proof.* From Theorem 2.2 we know that $y_u \in W^{1,3}(\Omega) \subset C(\bar{\Omega})$. Let us show that $y_u \in H^{3/2}(\Omega)$. Thanks to the assumptions on $a$, it follows that $a(\cdot, y_u(\cdot)) \in C(\bar{\Omega})$ and $(\partial a/\partial y)(\cdot, y_u(\cdot)) \in L^\infty(\Omega)$. Using the Lipschitz property of $a$, expanding the divergence term of the equation (1.1) and dividing by $a > 0$ we find that

$$-\Delta y_u = \underbrace{\frac{1}{a}}_{L^\infty(\Omega)} \left[ -\underbrace{f(\cdot, y_u)}_{L^p(\Omega)} + \sum_{j=1}^{2} \underbrace{\partial_j a(\cdot, y_u)}_{L^\infty(\Omega)} \underbrace{\partial_j y_u}_{L^3(\Omega)} + \underbrace{\frac{\partial a}{\partial y}}_{L^\infty(\Omega)} \underbrace{|\nabla y_u|^2}_{L^{3/2}(\Omega)} \right] \quad \text{in } \Omega, \tag{2.12}$$

$$\nabla y_u \cdot \vec{n} = \frac{u}{a} \quad \text{on } \Gamma. \tag{2.13}$$

Hence the right-hand sides of (2.12) and (2.13) are in $L^{3/2}(\Omega)$ and $L^2(\Gamma)$, respectively. Then the statement of the theorem follows from Theorem A.1, see Appendix. ■

THEOREM 2.4 *Under the assumptions of Theorem 2.3 and assuming that* $\Omega$ *is convex, there exists* $p_0 > 2$ *depending on the measure of the angles in* $\Gamma$ *such that for any* $2 \leq p \leq \min\{p_0, \bar{p}\}$ *(*$\bar{p} \geq \frac{6}{3-\sqrt{5}}$*) and any* $u \in W^{1-1/p,p}(\Gamma)$*, the solution of (1.1) belongs to* $W^{2,p}(\Omega)$*. Moreover, for any bounded set* $U \subset W^{1-1/p,p}(\Gamma)$*, there exists a constant* $C_U > 0$ *such that*

$$\|y_u\|_{W^{2,p}(\Omega)} \leq C_U, \quad \forall u \in U. \tag{2.14}$$

*Proof.* First we prove the result for $p \le \frac{\bar{p}}{2}$. Since $W^{1-1/p,p}(\Gamma) \subset L^r(\Gamma)$ for any $1 \le r < \infty$, we can apply Theorem 2.2 to get the existence of a unique solution $y_u$ in $W^{1,2p}(\Omega)$ if $p \le \frac{\bar{p}}{2}$. We have to prove the $W^{2,p}(\Omega)$ regularity.

Repeating the steps of the proof of Theorem 2.3 we get that the right-hand side of (2.12) is in $L^p(\Omega)$. Hence, it is enough to prove that $\frac{u}{a} \in W^{1-1/p,p}(\Gamma)$. Then a well-known result by Grisvard (1985, Corollary 4.4.3.8) on maximal regularity yields the existence of $p_0 > 2$ such that $y_u \in W^{2,p}(\Omega)$, if $2 \le p \le \min\{p_0, \frac{\bar{p}}{2}\}$. This $p_0$ depends on the measure of the angles in $\Gamma$.
For any $u \in W^{1-1/p,p}(\Gamma)$ there exists at least one $z \in W^{1,p}(\Omega)$ such that $z\big|_\Gamma = u$, see Nečas (1967, Theorem 5.7). Moreover $\frac{z}{a} \in W^{1,p}(\Omega)$, because $\frac{z}{a} \in L^p(\Omega)$ and for $i = 1, 2$

$$\partial_i\left[\frac{z}{a}\right] = \frac{1}{a}\partial_i z - \frac{z}{a^2}\left[\partial_i a + \frac{\partial a}{\partial y}\partial_i y_u\right]$$

belongs to $L^p(\Omega)$. Here we have used the boundedness of $a$, $\partial_i a$ and $\partial a/\partial y$ and the fact that $z \in W^{1,p}(\Omega) \subset L^{2p}(\Omega)$ for $p \ge 2$ and $\partial_i y_u \in L^{2p}(\Omega)$. From $z$, $\frac{z}{a} \in W^{1,p}(\Omega)$ and $z\big|_\Gamma = u$, it is easy to get $\frac{z}{a}\big|_\Gamma = \frac{u}{a}$, hence $\frac{u}{a} \in W^{1-1/p,p}(\Gamma)$.

Finally, we assume that $\frac{\bar{p}}{2} < p \le \min\{p_0, \bar{p}\}$. From the first part of the proof we know that $y_u \in W^{2,\frac{\bar{p}}{2}}(\Omega) \subset C^1(\bar{\Omega})$. Then, the right hand side of (2.12) is in $L^p(\Omega)$ and $\frac{u}{a} \in W^{1-1/p,p}(\Gamma)$ once again. Therefore, as above we conclude that $y_u \in W^{2,p}(\Omega)$. ∎

REMARK 2.3 *The proof of Theorem 2.1 on the existence and uniqueness of a solution of (1.1) is valid for arbitrary Lipschitz domains in $\mathbb{R}^n$. On the other hand, the statements of Theorems 2.1, 2.2 and 2.4 hold true for dimension* 2 *or* 3 *assuming that* $\Gamma$ *is of class* $C^{1,1}$. *In the case of convex and polygonal domains the* $H^{3/2}(\Omega)$ *regularity of* $y_u$, *for* $u \in L^2(\Gamma)$, *can be proved by decomposing the Neumann problem into two different problems and combining the results by Jerison and Kenig (1981, 1995). This technique is presented in Casas, Mateos and Raymond (2009) and Casas, Mateos and Tröltzsch (2005).*

## 3. Differentiability of the control-to-state mapping $u \mapsto y_u$

In order to derive the first- and second-order optimality conditions for the control problem, we need some differentiability of the functions involved in the control problem.

ASSUMPTION 3.1 *The functions $a$ and $f$ are of class $C^2$ with respect to the second variable and, for any number $M > 0$, there exist constants $D_M, D_{M,a} > 0$*

*such that*

*(1)* $\displaystyle\sum_{j=1}^{2}\left\{\left|\frac{\partial^j a}{\partial y^j}(x,y)\right| + \left|\frac{\partial^j f}{\partial y^j}(x,y)\right|\right\} \le D_M\,,$

*(2)* $\displaystyle\left|\frac{\partial^k a}{\partial y^k}(x_1,y_1) - \frac{\partial^k a}{\partial y^k}(x_2,y_2)\right| \le D_{M,a}\left(|x_1 - x_2| + |y_1 - y_2|\right),$

*for a.a.* $x \in \Omega$ *and all* $x_i \in \bar\Omega$ *and* $|y|, |y_i| \le M$*,* $i = 1,2$*,* $k = 1,2$*.*

As a first step we study the linearized equation of (1.1) around a solution $y$

$$\left\{\begin{array}{rcll} -\operatorname{div}\left[a(x,y)\nabla z(x) + \dfrac{\partial a}{\partial y}(x,y)z\nabla y\right] + \dfrac{\partial f}{\partial y}(x,y)z & = & g & \text{in } \Omega\,, \\ \left[a(x,y)\nabla z + \dfrac{\partial a}{\partial y}(x,y)z\nabla y\right]\cdot\vec{n}(x) & = & v & \text{on } \Gamma\,. \end{array}\right. \tag{3.1}$$

We say that $z_v \in H^1(\Omega)$ is a solution of (3.1), if

$$\int_\Omega \left\{a(x,y)\nabla z_v\cdot\nabla w + \frac{\partial a}{\partial y}(x,y)z_v\nabla y\cdot\nabla w + \frac{\partial f}{\partial y}(x,y)z_v w\right\}dx$$
$$= \langle g, w\rangle_{H^1(\Omega)^*,H^1(\Omega)} + \langle v, w\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \quad \forall w \in H^1(\Omega)\,.$$

In view of the lack of the monotonicity of the linear operator, the well-posedness of (3.1) is not obvious.

THEOREM 3.1 *Suppose that the Assumptions 2.1-2.3 and 3.1-(1) hold. Given* $y \in W^{1,p}(\Omega)$*, for any* $v \in H^{-1/2}(\Gamma)$ *and* $g \in H^1(\Omega)^*$*, the linearized equation (3.1) has a unique solution* $z_v \in H^1(\Omega)$*.*

*Proof.* The proof of the uniqueness follows the same steps as the one of Casas and Tröltzsch (2009, Theorem 2.7) with obvious modifications. For the existence we modify conveniently the arguments of Casas and Tröltzsch (2009, Theorem 2.7). For every $t \in [0,1]$ let us consider the linear operator $T_t : H^1(\Omega) \longrightarrow H^1(\Omega)^*$ given by

$$\langle T_t z, w\rangle = \int_\Omega \left\{a(x,y)\nabla z\cdot\nabla w + t\frac{\partial a}{\partial y}(x,y)z\nabla y\cdot\nabla w + \frac{\partial f}{\partial y}(x,y)zw\right\}dx\,. \tag{3.2}$$

For $t = 0$, the resulting linear operator is monotone and by an obvious application of the Lax-Milgram Theorem we know that it is an isomorphism. Let us denote by $S$ the set of points $t \in [0,1]$, for which the equation $T_t$ defines an isomorphism. $S$ is not empty because $0 \in S$. Setting $t_{max}$ the supremum of $S$, then we have that $t_{max} \in S$. Indeed, we are going to prove that $T_{t_{\max}}$ is an isomorphism between $H^1(\Omega)$ and $H^1(\Omega)^*$. Obviously it is continuous and injective. It is enough to prove the surjectivity. Given $h \in H^1(\Omega)^*$, we have to find an element $z \in H^1(\Omega)$ such that $T_{t_{max}} z = h$. Take a sequence $\{t_k\}_{k=1}^\infty \subset S$ such that $t_k \to t_{max}$ when $k \to \infty$ and denote by $z_k$ the element of $H^1(\Omega)$ such that $T_{t_k} z_k = h$. Then we have

$$\|z_k\|_{H^1(\Omega)} \leq C \left( \|z_k\|_{L^{\frac{2p}{p-2}}(\Omega)} + \|h\|_{H^1(\Omega)^*} \right).$$

Arguing as in Casas and Tröltzsch (2009, Theorem 2.7), we get that $\{z_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega)$ and the weak limit $z$ of a subsequence satisfies that $T_{t_{max}} z = h$. Therefore, we conclude that $t_{\max} \in S$.

Finally, we prove that $t_{max} = 1$. If it is false, then let us consider the operators $T_{t_{max}+\varepsilon}, T_{t_{max}} \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$, for any $\varepsilon > 0$ with $t_{max} + \varepsilon \leq 1$. It is easy to check that

$$\|T_{t_{max}+\varepsilon} - T_{t_{max}}\|_{\mathcal{L}(H^1(\Omega), H^1(\Omega)^*)} \leq C\varepsilon.$$

If we take $0 < \varepsilon < 1/C$, then we have that $T_{t_{max}+\varepsilon}$ is also an isomorphism, which contradicts the fact that $t_{max}$ is the supremum of $S$. ■

REMARK 3.1 *It is easy to check that the proof of Theorem 3.1 can be modified in an obvious way to verify that for any given functions $y \in W^{1,p}(\Omega)$ and $y_i \in L^\infty(\Omega)$, $1 \leq i \leq 3$, the equation*

$$\begin{cases} -\operatorname{div}\left[a(x,y_1)\nabla z(x) + \dfrac{\partial a}{\partial y}(x,y_2) z \nabla y\right] + \dfrac{\partial f}{\partial y}(x,y_3) z \;=\; g \quad \text{in}\, \Omega\,, \\ \left[a(x,y_1)\nabla z(x) + \dfrac{\partial a}{\partial y}(x,y_2) z \nabla y\right] \cdot \vec{n}(x) \;=\; v \quad \text{on}\, \Gamma\,, \end{cases}$$

*has a unique solution $z \in H^1(\Omega)$ .*

THEOREM 3.2 *Under the Assumptions 2.1-2.3 and 3.1-(1) and supposing that $a : \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, the control-to-state mapping $G : L^{p/2}(\Gamma) \to W^{1,p}(\Omega)$, $G(u) = y_u$, is of class $C^2$ for all $p \in (2, \bar{p}]$. Moreover, for any $v, v_1, v_2 \in L^{p/2}(\Gamma)$, the functions $z_v = G'(u)v$, $z_{v_1,v_2} = G''(u)[v_1, v_2]$ are the unique solutions in $W^{1,p}(\Omega)$ of the equations*

$$\begin{cases} -\operatorname{div}\left[a(x,y_u)\nabla z + \dfrac{\partial a}{\partial y}(x,y_u) z \nabla y_u\right] + \dfrac{\partial f}{\partial y}(x,y_u) z \;=\; 0 \quad \text{in}\, \Omega\,, \\ \left[a(x,y_u)\nabla z + \dfrac{\partial a}{\partial y}(x,y_u) z \nabla y_u\right] \cdot \vec{n} \;=\; v \quad \text{on}\, \Gamma\,, \end{cases} \tag{3.3}$$

*and*

$$\begin{cases} -\operatorname{div}\left[a(x,y_u)\nabla z + \dfrac{\partial a}{\partial y}(x,y_u)z\nabla y_u\right] + \dfrac{\partial f}{\partial y}(x,y_u)z = -\dfrac{\partial^2 f}{\partial y^2}(x,y_u)z_1z_2 \\ \qquad +\operatorname{div}\left[\dfrac{\partial a}{\partial y}(x,y_u)(z_1\nabla z_2 + z_2\nabla z_1) + \dfrac{\partial^2 a}{\partial y^2}(x,y_u)z_1z_2\nabla y_u\right] \quad \text{in } \Omega\,, \\ \qquad\qquad \left[a(x,y_u)\nabla z + \dfrac{\partial a}{\partial y}(x,y_u)z\nabla y_u\right]\cdot\vec{n}(x) = \\ \qquad -\left[\dfrac{\partial a}{\partial y}(x,y_u)(z_1\nabla z_2 + z_2\nabla z_1) + \dfrac{\partial^2 a}{\partial y^2}(x,y_u)z_1z_2\nabla y_u\right]\cdot\vec{n} \quad \text{on } \Gamma\,, \end{cases} \tag{3.4}$$

*respectively, where* $z_i = G'(u)v_i$, $i = 1,2$.

*Proof.* Let us introduce the mapping $F : W^{1,p}(\Omega)\times L^{p/2}(\Gamma) \to W^{1,p'}(\Omega)^*$ by

$$\langle F(y,u),w\rangle = \int_\Omega \{a(x,y)\nabla y\cdot\nabla w + f(x,y)w\}\,dx - \int_\Gamma uw\,d\sigma(x)\,. \tag{3.5}$$

From the Assumptions 2.2 and 3.1-(1) and the embedding $W^{1-1/p',p'}(\Gamma)\subset L^{\frac{p}{p-2}}(\Gamma) = L^{(p/2)'}(\Gamma)$ it follows that $F$ is well defined, of class $C^2$, and $F(y_u,u) = 0$ for every $u\in L^{p/2}(\Gamma)$. Our goal is to prove that $(\partial F/\partial y)(y_u,u) : W^{1,p}(\Omega)\to W^{1,p'}(\Omega)^*$ defined by

$$\left\langle\frac{\partial F}{\partial y}(y_u,u)z,w\right\rangle = \int_\Omega\left\{a(x,y_u)\nabla z\cdot\nabla w + \frac{\partial a}{\partial y}(x,y_u)z\nabla y_u\cdot\nabla w + \frac{\partial f}{\partial y}(x,y_u)zw\right\}dx$$

is an isomorphism. Then we can apply the implicit function theorem to deduce the differentiability properties of $G$ stated in the theorem. The representations (3.3) and (3.4) for $G'$ and $G''$ are then obtained by simple computations as in Casas and Tröltzsch (2009). According to Theorem 3.1, for any $v\in H^{-1/2}(\Gamma)$, there exists a unique element $z\in H^1(\Omega)$ such that

$$\frac{\partial F}{\partial y}(y_u,u)z = Bv\,,$$

where the operator $B : H^{-1/2}(\Gamma)\to H^1(\Omega)^*$ is defined by $\langle Bv,w\rangle = \int_\Gamma vw\,d\sigma(x)$. It is enough to prove that $z\in W^{1,p}(\Omega)$ if $v\in L^{p/2}(\Gamma)\subset H^{-1/2}(\Gamma)$. More precisely, this means that the unique solution $z\in H^1(\Omega)$ of (3.1), associated to $v\in L^{p/2}(\Gamma)$ and $g = 0$, belongs to $W^{1,p}(\Omega)$. First of all, let us note that

$$a(\cdot,y_u)\in L^\infty(\Omega)\,,\ \frac{\partial a}{\partial y}(\cdot,y_u)\nabla y_u\in (L^p(\Omega))^2\,,\ \frac{\partial f}{\partial y}(\cdot,y_u)\in L^\infty(\Omega) \text{ and}$$
$$v\in L^{p/2}(\Gamma)\subset W^{-1/p,p}(\Gamma)\,.$$

Therefore, we can apply a result by Stampacchia (1960) (see also Murthy and Stampacchia, 1972; Gröger, 1989) to get that $z \in C(\bar{\Omega})$. For the problem

$$\begin{cases} -\operatorname{div}\left[a(x,y_u)\nabla z\right] + \dfrac{\partial f}{\partial y}(x,y_u)z_v = \operatorname{div}\left[\dfrac{\partial a}{\partial y}(x,y_u)z\nabla y_u\right] & \text{in } \Omega\,, \\ a(x,y_u)\nabla z\cdot\vec{n}(x) = v(x) - \dfrac{\partial a}{\partial y}(x,y_u)z\nabla y_u\cdot\vec{n}(x) & \text{on } \Gamma\,, \end{cases} \tag{3.6}$$

we have, along with the identity $\nabla y_u\cdot\vec{n} = \frac{u}{a}$ on $\Gamma$, that

$$v - \frac{\partial a}{\partial y}(\cdot,y_u)z\nabla y_u\cdot\vec{n} \in L^{p/2}(\Gamma) \text{ and } \operatorname{div}\left[\frac{\partial a}{\partial y}(\cdot,y_u)z\nabla y_u\right] \in W^{1,p'}(\Omega)^*\,.$$

To deduce the $W^{1,p}(\Omega)$ regularity of $z$ we can follow the same steps as in the Theorem 2.2 with obvious modifications. The existence and uniqueness of a solution of (3.4) can be deduced by the same arguments as above. ■

If we assume that $a$ is locally Lipschitz continuous and $\Omega$ is convex, then from Theorem 2.4 we know that the states $y_u$ corresponding to controls $u \in W^{1-1/p,p}(\Gamma)$, $2 \le p \le \min\{p_0,\bar{p}\}$ for some $p_0 > 2$ and $\bar{p} \ge \frac{6}{3-\sqrt{5}}$, belong to $W^{2,p}(\Omega)$. In this context, a natural question arises: Can we prove a result analogous to Theorem 3.2 with $G : W^{1-1/p,p}(\Gamma) \to W^{2,p}(\Omega)$? The answer is positive if we assume some extra regularity of $a$, namely Assumption 3.1-(2).

THEOREM 3.3 *Under the Assumptions 2.1-2.3 and 3.1 and assuming that $\Omega$ is convex, there exists $p_0 > 2$ depending on the measure of the angles in $\Gamma$ such that the control-to-state mapping $G : W^{1-1/p,p}(\Gamma) \to W^{2,p}(\Omega)$, $G(u) = y_u$, is of class $C^2$ for $2 \le p \le \min\{p_0,\bar{p}\}$, with some $\bar{p} \ge \frac{6}{3-\sqrt{5}}$. Moreover, for any $v, v_1, v_2 \in W^{1-1/p,p}(\Gamma)$, the functions $z_v = G'(u)v$, $z_{v_1,v_2} = G''(u)[v_1,v_2]$ are the unique solutions in $W^{2,p}(\Omega)$ of the equations (3.3) and (3.4), respectively.*

*Proof.* Consider the Banach space

$$V(\Omega) = \{y \in W^{2,p}(\Omega) \,\big|\, \nabla y\cdot\vec{n} \in W^{1-1/p,p}(\Gamma)\}\,,$$

endowed with the graph norm. Then, all the elements $y_u = G(u)$ belong to this space provided that $u \in W^{1-1/p,p}(\Gamma)$. Let us define the mapping

$$F : V(\Omega) \times W^{1-1/p,p}(\Gamma) \longrightarrow L^p(\Omega) \times W^{1-1/p,p}(\Gamma)$$

by $F(y,u) = (-\operatorname{div}[a(x,y)\nabla y] + f(x,y), a(x,y)\nabla y\cdot\vec{n} - u)$. Next, we verify that $F$ is well defined. By expanding the divergence term, we find

$$\operatorname{div}\left[a(x,y)\nabla y\right] = [\nabla_x a](x,y)\cdot\nabla y + \frac{\partial a}{\partial y}(x,y)\,|\nabla y|^2 + a(x,y)\Delta y\,.$$

For $y \in W^{2,p}(\Omega)$, $p \geq 2$, the right hand side of the previous equality is in $L^p(\Omega)$, therefore it remains to show that $a(\cdot,y)\nabla y \cdot \vec{n} \in W^{1-1/p,p}(\Gamma)$ in order to deduce that $F$ is well defined. To this end, we use the following fact:

$$\text{If } b \in C^{0,\mu}(\Gamma),\ v \in W^{1-1/p,p}(\Gamma) \text{ with } \mu > 1 - 1/p, \text{ then } bv \in W^{1-1/p,p}(\Gamma), \tag{3.7}$$

see Grisvard (1985, Theorem 1.4.1.1). Now, taking into account the Lipschitz property of $a$ with respect to $x$ and $y$ and the embedding $H^2(\Omega) \subset C^{0,\mu}(\bar{\Omega})$ for every $\mu \in (0,1)$, see Nečas (1967, §2 Theorem 3.8), we have $a(\cdot, y(\cdot)) \in C^{0,\mu}(\Gamma)$ for all $\mu \in (1 - \frac{1}{p}, 1)$. This, along with (3.7), yields $a(\cdot,y)\nabla y \cdot \vec{n} \in W^{1-1/p,p}(\Gamma)$.

On the other hand, it is obvious that $F$ is a $C^2$ mapping. We are going to apply the implicit function theorem. To this end we need to prove that the linear operator $\partial_y F(y,u) : V(\Omega) \longrightarrow L^p(\Omega) \times W^{1-1/p,p}(\Gamma)$ is an isomorphism. First we note that

$$\frac{\partial F}{\partial y}(y,u)z =$$
$$\left(-\mathrm{div}\left[a(x,y)\nabla z + \frac{\partial a}{\partial y}(x,y)z\nabla y\right] + \frac{\partial f}{\partial y}(x,y)z,\ \left[a(x,y)\nabla z + \frac{\partial a}{\partial y}(x,y)z\nabla y\right]\cdot\vec{n}\right),$$

for every $z \in W^{2,p}(\Omega)$. We have to prove the existence of a unique solution $z \in V(\Omega)$ of (3.1) for any $g \in L^p(\Omega)$ and any $v \in W^{1-1/p,p}(\Gamma)$. First, the existence and uniqueness of a solution in $H^1(\Omega) \cap L^\infty(\Omega)$ follows from Theorem 3.1. Moreover, due to the inclusion $W^{1-1/p,p}(\Gamma) \subset L^r(\Gamma)$ for every $r < \infty$, the $W^{1,\bar{p}}(\Omega)$ regularity follows from Theorem 3.2. Finally we prove that $z \in W^{2,p}(\Omega)$. For this purpose, we follow the steps of the proof of Theorem 2.4 and consider first the case when $p \leq \frac{\bar{p}}{2}$. From (3.1) we get

$$\begin{aligned}
-\Delta z &= \frac{1}{a}\left[g + \mathrm{div}\left[\frac{\partial a}{\partial y}(\cdot,y)z\nabla y\right] + \nabla_x a(\cdot,y)\cdot\nabla z + \frac{\partial a}{\partial y}(\cdot,y)\nabla z\cdot\nabla y - \frac{\partial f}{\partial y}(\cdot,y)z\right] \\
&= \frac{1}{a}\left[g + z\nabla_x\frac{\partial a}{\partial y}(\cdot,y)\cdot\nabla y + \frac{\partial^2 a}{\partial y^2}(\cdot,y)z\,|\nabla y|^2 + 2\frac{\partial a}{\partial y}(\cdot,y)\nabla z\cdot\nabla y\right. \\
&\qquad \left.+\frac{\partial a}{\partial y}(\cdot,y)z\Delta y + \nabla_x a(\cdot,y)\cdot\nabla z - \frac{\partial f}{\partial y}(\cdot,y)z\right] \qquad \text{in } \Omega, \qquad (3.8)\\
\nabla z\cdot\vec{n} &= \frac{1}{a}\left[v - \frac{\partial a}{\partial y}(\cdot,y)z\nabla y\cdot\vec{n}\right] \qquad \text{on } \Gamma. \qquad (3.9)
\end{aligned}$$

The right-hand sides of the above equations belong to $L^p(\Omega)$ and $W^{1-1/p,p}(\Gamma)$, respectively. Finally, as in the proof of Theorem 2.4, we apply again the regularity result by Grisvard (1985, Corollary 4.4.3.8) that yields the existence of a value $p_0 > 2$ depending on the measure of the angles in $\Gamma$ such that $z \in W^{2,p}(\Omega)$ for $2 \leq p \leq \min\{p_0, \frac{\bar{p}}{2}\}$. When $p > \frac{\bar{p}}{2}$ we can proceed as at the end of the proof of Theorem 2.4 to obtain the $W^{2,p}(\Omega)$ regularity.

## 4. The control problem

In the following, we assume that $2 < p \le \bar{p}$ with $\bar{p}$ taken from Theorem 2.2. We associate with the state equation (1.1) the following optimal control problem

$$
(\mathcal{P}) \quad \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x))\, dx + \int_\Gamma l(x, y_u(x), u(x))\, d\sigma(x)\,, \\ u \in L^\infty(\Gamma)\,, \\ u_a(x) \le u(x) \le u_b(x) \text{ for a.e. } x \in \Gamma\,, \end{cases}
$$

where $L : \Omega \times \mathbb{R} \to \mathbb{R}$ and $l : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and $u_a, u_b \in L^\infty(\Gamma)$, with $u_a \le u_b$ a.e. on $\Gamma$.

Assumption 4.1 *The functions $L : \Omega \times \mathbb{R} \to \mathbb{R}$ and $l : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are of class $C^2$ with respect to the second variable and to the last two variables, respectively. For any $M > 0$, there exist constants $C_{L,M}, C_{l,M} > 0$ and functions $\psi_{\Omega,M} \in L^p(\Omega)$, $\psi_{1,\Gamma,M} \in L^p(\Gamma)$ and $\psi_{2,\Gamma,M} \in L^2(\Gamma)$, such that*

$$
\left|\frac{\partial L}{\partial y}(x,y)\right| \le \psi_{\Omega,M}(x)\,, \quad \left|\frac{\partial l}{\partial y}(s,y,u)\right| \le \psi_{1,\Gamma,M}(s)\,, \quad \left|\frac{\partial l}{\partial u}(s,y,u)\right| \le \psi_{2,\Gamma,M}(s)\,,
$$

$$
\left\|\frac{\partial^2 L}{\partial y^2}(x,y)\right\| \le C_{L,M}\,, \quad \left\|D^2_{(y,u)} l(s,y,u)\right\| \le C_{l,M}\,,
$$

$$
\left|\frac{\partial^2 L}{\partial y^2}(x,y_2) - \frac{\partial^2 L}{\partial y^2}(x,y_1)\right| \le C_{L,M}\,|y_2 - y_1|\,,
$$

$$
\left\|D^2_{(y,u)} l(s,y_2,u_2) - D^2_{(y,u)} l(s,y_1,u_1)\right\| \le C_{l,M}\,(|y_2 - y_1| + |u_2 - u_1|)\,,
$$

*for a.a. $x \in \Omega, s \in \Gamma$ and $|y|, |y_i|, |u|, |u_i| \le M$, $i = 1, 2$, where $D^2_{(y,u)} l$ denotes the second derivative of $l$ w.r. to $(y, u)$, i.e. the associated Hessian matrix.*

The next theorem concerns the existence of a solution for problem $(\mathcal{P})$. The proof of this theorem is standard.

Theorem 4.1 *Suppose that the Assumptions 2.1-2.3 hold and assume that $a : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and $l$ is convex w.r. to $u$. Assume further that, for any $M > 0$, there exist functions $\psi_{l,M} \in L^1(\Gamma)$, $\psi_{L,M} \in L^1(\Omega)$ such that*

$$
|L(x,y)| \le \psi_{L,M}(x) \quad \text{and} \quad |l(s,y,u)| \le \psi_{l,M}(s)\,,
$$

*for a.e. $x \in \Omega$, $s \in \Gamma$ and $|y|, |u| \le M$. Then $(\mathcal{P})$ has at least one optimal solution $\bar{u}$.*

Theorem 4.2 *Assume that $a : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and the Assumptions 2.1-2.3, 3.1-(1) and 4.1 hold. Then the function $J : L^\infty(\Gamma) \to \mathbb{R}$ is of class $C^2$. Moreover, for every $u, v, v_1, v_2 \in L^\infty(\Gamma)$, we have*

$$
J'(u)v = \int_\Gamma \left(\frac{\partial l}{\partial u}(x, y_u, u) + \varphi_u\right) v\, d\sigma(x) \tag{4.1}
$$

*and*

$$J''(u)v_1v_2 = \int_\Gamma \left\{ \frac{\partial^2 l}{\partial y^2}(x, y_u, u) z_{v_1} z_{v_2} + \frac{\partial^2 l}{\partial y \partial u}(x, y_u, u)(z_{v_1} v_2 + z_{v_2} v_1) \right.$$
$$\left. + \frac{\partial^2 l}{\partial u^2}(x, y_u, u) v_1 v_2 \right\} d\sigma(x) + \int_\Omega \left[ \frac{\partial^2 L}{\partial y^2}(x, y_u) - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u) \right] z_{v_1} z_{v_2}\, dx$$
$$- \int_\Omega \nabla\varphi_u \cdot \left[ \frac{\partial^2 a}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} \nabla y_u + \frac{\partial a}{\partial y}(x, y_u) \left( z_{v_1} \nabla z_{v_2} + z_{v_2} \nabla z_{v_1} \right) \right] dx\,, \qquad (4.2)$$

*where $\varphi_u \in W^{1,p}(\Omega)$ is the unique solution of the adjoint equation*

$$\begin{cases} -\mathrm{div}[a(x, y_u)\nabla\varphi] + \dfrac{\partial a}{\partial y}(x, y_u)\nabla y_u \cdot \nabla\varphi + \dfrac{\partial f}{\partial y}(x, y_u)\varphi = \dfrac{\partial L}{\partial y}(x, y_u) & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad a(x, y_u)\nabla\varphi \cdot \vec{n}(x) = \dfrac{\partial l}{\partial y}(x, y_u, u) & \text{on } \Gamma, \end{cases} \qquad (4.3)$$

*and $z_{v_i} = G'(u)v_i$, is the solution of (3.3) for $v = v_i$, $i = 1, 2$.*

*Proof.* The first and second order derivatives of $J$ can be obtained by an elementary calculus. We only show the existence and uniqueness of a solution of the adjoint state equation (4.3).
To show the existence of $\varphi_u \in H^1(\Omega)$ we consider the operator

$$T \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)\,,$$
$$\langle Tz, \varphi \rangle = \int_\Omega \left\{ a(x, y_u)\nabla z \cdot \nabla\varphi + \frac{\partial a}{\partial y}(x, y_u) z \nabla y_u \cdot \nabla\varphi + \frac{\partial f}{\partial y}(x, y_u) z\varphi \right\} dx\,.$$

According to Theorem 3.1, $T$ is an isomorphism and consequently its adjoint operator $T^* \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ is also an isomorphism. But this is equivalent to the well-posedness of the adjoint equation (4.3) in $H^1(\Omega)$. To obtain the $W^{1,p}(\Omega)$ regularity of $\varphi_u$ we can proceed as in Theorem 2.2, provided that the term

$$w = \frac{\partial a}{\partial y}(x, y_u)\nabla y_u \cdot \nabla\varphi_u$$

belongs to $W^{1,p'}(\Omega)^*$. By using that $\nabla y_u \in L^p(\Omega)$, $\nabla\varphi_u \in L^2(\Omega)$ and invoking the Hölder inequality, we get that $\nabla y_u \cdot \nabla\varphi_u \in L^{\frac{2p}{p+2}}(\Omega)$, hence $w$ belongs also to $L^{\frac{2p}{p+2}}(\Omega)$. Finally, the inclusion $L^{\frac{2p}{p+2}}\Omega) \subset W^{1,p'}(\Omega)^*$ and the same argumentation as in the proof of the $W^{1,p}(\Omega)$ regularity of the solution of (3.3) yields $\varphi_u \in W^{1,p}(\Omega)$.

Remark 4.1 *Let us remark that the previous theorem is still valid under the weaker assumptions $\psi_{\Omega,M} \in L^q(\Omega)$, $q \geq 2p/(p+2)$, and $\psi_{1,\Gamma,M} \in L^{p/2}(\Gamma)$; see Assumption 4.1. The stronger assumptions made in 4.1 are only necessary to prove a regularity result for the optimal solutions of $(\mathcal{P})$; see Theorem 4.4.*

REMARK 4.2 *From the expression (4.2) for $J''(u)$, it is obvious that $J''(u)$ can be extended to a continuous bilinear form $J''(u) : L^2(\Gamma) \times L^2(\Gamma) \to \mathbb{R}$.*

The first order optimality conditions stated in the next theorem follow from the variational inequality $J'(\bar{u})(u - \bar{u}) \geq 0$ for every $u_a \leq u \leq u_b$, along with the expression of the derivative of $J$ given by (4.1) and (4.3).

THEOREM 4.3 *Let the assumptions of Theorem 4.2 be satisfied. Then, if $\bar{u}$ is a local minimum of $(\mathcal{P})$, there exists $\bar{\varphi} \in W^{1,p}(\Omega)$ such that*

$$\begin{cases} -\operatorname{div}[a(x,\bar{y})\nabla\bar{\varphi}(x)] + \dfrac{\partial a}{\partial y}(x,\bar{y})\nabla\bar{y}\cdot\nabla\bar{\varphi} + \dfrac{\partial f}{\partial y}(x,\bar{y})\bar{\varphi} = \dfrac{\partial L}{\partial y}(x,\bar{y}) & \text{in } \Omega\,, \\ a(x,\bar{y})\nabla\bar{\varphi}\cdot\vec{n}(x) = \dfrac{\partial l}{\partial y}(x,\bar{y},\bar{u}) & \text{on } \Gamma\,, \end{cases} \tag{4.4}$$

$$\int_\Gamma \left(\frac{\partial l}{\partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}(x)\right)(u(x) - \bar{u}(x))\, d\sigma(x) \geq 0 \quad \text{for all} \quad u_a \leq u \leq u_b\,, \tag{4.5}$$

*where $\bar{y}$ is the state associated to $\bar{u}$.*

If we define the Riesz representation of $J'$,

$$\bar{d}(x) = \frac{\partial l}{\partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}(x)\,, \tag{4.6}$$

then we deduce from (4.5) that

$$\bar{d}(x) = \begin{cases} = 0 & \text{if } u_a(x) < \bar{u}(x) < u_b(x) \\ \leq 0 & \text{if } \bar{u}(x) = u_b(x) \\ \geq 0 & \text{if } \bar{u}(x) = u_a(x)\,. \end{cases} \tag{4.7}$$

Finally, we give a result concerning the regularity of the optimal solutions of $(\mathcal{P})$. The statement and the proof of this result follow the ideas of Casas and Tröltzsch (2009, Theorem 3.5), the main difference concerns the $W^{1-1/p,p}(\Gamma)$ boundary regularity of the optimal control and its proof.

THEOREM 4.4 *Let the assumptions of Theorem 4.2 be fulfilled and suppose that*

$$\frac{\partial l}{\partial u} : \Gamma \times (\mathbb{R}\times\mathbb{R}) \to \mathbb{R} \quad \text{be continuous}, \tag{4.8}$$

$$\exists \Lambda_l > 0 \ \text{ such that } \ \frac{\partial^2 l}{\partial u^2}(x,y,u) \geq \Lambda_l \quad \text{for a.a. } x \in \Gamma \ \text{ and } \ \forall (y,u) \in \mathbb{R}^2\,. \tag{4.9}$$

*Then, for every $x \in \Gamma$, the equation*

$$\frac{\partial l}{\partial u}(x,\bar{y}(x),t) + \bar{\varphi}(x) = 0 \tag{4.10}$$

*has a unique solution* $\bar{t} = \bar{s}(x)$. *The function* $\bar{s} : \Gamma \to \mathbb{R}$ *is continuous and is related to* $\bar{u}$ *by the formula*

$$\bar{u}(x) = Proj_{[u_a(x),u_b(x)]}\bar{s}(x) = \max\{\min\{u_b(x), \bar{s}(x)\}, u_a(x)\}\,. \tag{4.11}$$

*If* $u_a, u_b$ *are continuous on* $\Gamma$, *then* $\bar{u}$ *is continuous, too. If* $u_a, u_b \in C^{0,1}(\Gamma)$ *and for every* $M > 0$ *there exists a constant* $C_{l,M} > 0$ *such that*

$$\left|\frac{\partial l}{\partial u}(x_2, y, u) - \frac{\partial l}{\partial u}(x_1, y, u)\right| \le C_{l,M}\,|x_2 - x_1| \quad \forall x_i \in \Gamma \text{ and } \forall |y|, |u| \le M\,, \tag{4.12}$$

*then* $\bar{s}, \bar{u} \in W^{1-1/p,p}(\Gamma)$ *for* $2 < p \le \min\{p_0, \bar{p}\}$. *In addition, if* $\Omega$ *is convex and* $a : \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ *is locally Lipschitz, then* $\bar{s}, \bar{u} \in C^{0,1}(\Gamma)$ *and* $\bar{y}, \bar{\varphi} \in W^{2,p}(\Omega)$.

*Proof.* Let us recall that $\bar{y}, \bar{\varphi} \in W^{1,p}(\Omega) \subset C(\bar{\Omega})$, for some $2 < p \le \bar{p}$. We fix $x \in \Gamma$ and consider the real function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \bar{\varphi}(x) + \frac{\partial l}{\partial u}(x, \bar{y}(x), t)\,.$$

Then $g$ is $C^1$ with $g'(t) \ge \Lambda_l > 0$. Therefore, $g$ is strictly increasing and

$$\lim_{t\to-\infty} g(t) = -\infty \quad \text{and} \quad \lim_{t\to+\infty} g(t) = +\infty\,.$$

Hence, there exists a unique element $\bar{t} \in \mathbb{R}$ satisfying $g(\bar{t}) = 0$, i.e. $\bar{s}$ is well defined.

Taking $\bar{d}$ as defined in (4.6) and using (4.7) along with the strict monotonicity of $(\partial l/\partial u)$ with respect to the third variable, we obtain

$$\begin{cases} \text{if } \bar{d}(x) = 0 & \text{then } \bar{u}(x) = \bar{s}(x) \\ \text{if } \bar{d}(x) < 0 & \text{then } u_b(x) = \bar{u}(x) < \bar{s}(x) \\ \text{if } \bar{d}(x) > 0 & \text{then } u_a(x) = \bar{u}(x) > \bar{s}(x)\,, \end{cases}$$

which implies (4.11).

To show the boundedness of $\bar{s}$ on $\Gamma$ we use the mean value theorem along with (4.8), (4.9) and (4.10):

$$\Lambda_l\,|\bar{s}(x)| \le \left|\frac{\partial l}{\partial u}(x, \bar{y}(x), \bar{s}(x)) - \frac{\partial l}{\partial u}(x, \bar{y}(x), 0)\right| = \left|\bar{\varphi}(x) + \frac{\partial l}{\partial u}(x, \bar{y}(x), 0)\right|,$$

therefore

$$|\bar{s}(x)| \le \frac{1}{\Lambda_l}\max_{x\in\Gamma}\left|\bar{\varphi}(x) + \frac{\partial l}{\partial u}(x, \bar{y}(x), 0)\right| < \infty\,.$$

The continuity of $\bar{s}$ at every point $x \in \Gamma$ follows easily from the continuity of $\bar{y}$ and $(\partial l/\partial u)$ by using the following inequality

$$\begin{aligned}|\bar{s}(x) - \bar{s}(x')| &\leq \frac{1}{\Lambda_l}\left|\frac{\partial l}{\partial u}(x', \bar{y}(x'), \bar{s}(x)) - \frac{\partial l}{\partial u}(x', \bar{y}(x'), \bar{s}(x'))\right| \\ &\leq \frac{1}{\Lambda_l}\left[|\bar{\varphi}(x') - \bar{\varphi}(x)| + \left|\frac{\partial l}{\partial u}(x', \bar{y}(x'), \bar{s}(x)) - \frac{\partial l}{\partial u}(x, \bar{y}(x), \bar{s}(x))\right|\right]. \quad (4.13)\end{aligned}$$

If $u_a, u_b \in C(\Gamma)$, then the identity (4.11) and the continuity of $\bar{s}$ imply that $\bar{u} \in C(\Gamma)$.

The traces of $\bar{y}$ and $\bar{\varphi}$ belong to $W^{1-1/p,p}(\Gamma)$. Assuming that (4.12) holds and $u_a, u_b \in C^{0,1}(\Gamma)$, taking the norm

$$\|z\|_{W^{1-1/p,p}(\Gamma)} = \left(\|z\|^p_{L^p(\Gamma)} + \int_\Gamma\int_\Gamma \frac{|z(x) - z(x')|^p}{|x - x'|^p}\, d\sigma(x) d\sigma(x')\right)^{1/p},$$

the $W^{1-1/p,p}(\Gamma)$ regularity of $\bar{u}$ and $\bar{s}$ follows from (4.13) and (4.11). Now, if $\Omega$ is assumed convex and the local Lipschitz property of $a$ holds, then $\bar{y} \in W^{2,p}(\Omega)$. The same is also true for $\bar{\varphi}$, provided that (4.12) is satisfied. To this aim, we prove that $(\partial l/\partial y)(\cdot, \bar{y}(\cdot), \bar{u}(\cdot)) \in W^{1-1/p,p}(\Gamma)$. Together with $(\partial L/\partial y)(\cdot, \bar{y}(\cdot)) \in L^p(\Omega)$ we can follow Theorem 2.4 to deduce $\bar{\varphi} \in W^{2,p}(\Omega)$. Using (4.12) and Assumption 4.1 we get $(\partial l/\partial y)(\cdot, \bar{y}(\cdot), \bar{u}(\cdot)) \in L^p(\Gamma)$ and

$$\begin{aligned}&\left|\frac{\partial l}{\partial y}(x, \bar{y}(x), \bar{u}(x)) - \frac{\partial l}{\partial y}(x', \bar{y}(x'), \bar{u}(x'))\right| \\ &\qquad\qquad \leq C\left(|x - x'| + |\bar{y}(x) - \bar{y}(x')| + |\bar{u}(x) - \bar{u}(x')|\right),\end{aligned}$$

for a.a. $x, x' \in \Gamma$. It follows that

$$\begin{aligned}&\frac{\left|\frac{\partial l}{\partial y}(x, \bar{y}(x), \bar{u}(x)) - \frac{\partial l}{\partial y}(x', \bar{y}(x'), \bar{u}(x'))\right|^p}{|x - x'|^p} \\ &\qquad\qquad \leq C_p\left[1 + \frac{|\bar{y}(x) - \bar{y}(x')|^p}{|x - x'|^p} + \frac{|\bar{u}(x) - \bar{u}(x')|^p}{|x - x'|^p}\right] =: z(x, x')\end{aligned}$$

and $\int_\Gamma\int_\Gamma z(x, x') d\sigma(x) d\sigma(x') < \infty$, therefore $(\partial l/\partial y)(\cdot, \bar{y}(\cdot), \bar{u}(\cdot)) \in W^{1-1/p,p}(\Gamma)$. Using the embedding $W^{2,p}(\Omega) \subset C^{0,1}(\bar{\Omega})$, see Nečas (1967, Chapter 2, Theorem 3.8), and (4.13) we get the Lipschitz regularity of $\bar{u}$ and $\bar{s}$. ∎

Remark 4.3 *The previous theorem is valid for domains $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, non-convex and non-polygonal, assuming the $C^{1,1}$ regularity of $\Gamma$.*

## 5. Pontryagin's principle

In this section we derive the Pontryagin's principle satisfied by a local solution of $(\mathcal{P})$, which is needed for the second-order analysis we study in Section 6. For this purpose, we impose the following assumption:

ASSUMPTION 5.1 *$L : \Omega \times \mathbb{R} \to \mathbb{R}$ and $l : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are of class $C^1$ with respect to the second variable, $2 < p \leq \bar{p}$ and for any $M > 0$, there exist functions $\psi_{\Omega,M} \in L^q(\Omega)$, $q \geq 2p/(p+2)$, and $\psi_{\Gamma,M} \in L^{p/2}(\Gamma)$, such that*

$$\left|\frac{\partial L}{\partial y}(x,y)\right| \leq \psi_{\Omega,M}(x)\,, \quad \left|\frac{\partial l}{\partial y}(s,y,u)\right| \leq \psi_{\Gamma,M}(s)$$

*hold for a.a. $x \in \Omega$ and all $s \in \Gamma$, $|y|, |u| \leq M$.*

Let us introduce the Hamiltonian $H$ associated to the control problem $(\mathcal{P})$,

$$H(x,y,u,\varphi) = l(x,y,u) + \varphi u\,.$$

The Pontryagin's principle is formulated as follows.

THEOREM 5.1 *Let $\bar{u}$ be a local solution of $(\mathcal{P})$ and suppose that the Assumptions 2.1-2.3, 3.1-(1) and 5.1 hold. We also assume that $a : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous. Then there exists $\bar{\varphi} \in W^{1,p}(\Omega)$ satisfying the adjoint equation (4.3) and the minimum condition*

$$H(x,\bar{y}(x),\bar{u}(x),\bar{\varphi}(x)) = \min_{s \in [u_{a_{\bar{\varepsilon}}}(x), u_{b_{\bar{\varepsilon}}}(x)]} H(x,\bar{y}(x),s,\bar{\varphi}(x)) \text{ for a.a. } x \in \Gamma\,, \quad (5.1)$$

*where*

$$u_{a_{\bar{\varepsilon}}}(x) = \max\{u_a(x), \bar{u}(x) - \bar{\varepsilon}\} \quad \text{and} \quad u_{b_{\bar{\varepsilon}}}(x) = \min\{u_b(x), \bar{u}(x) + \bar{\varepsilon}\}\,,$$

*$\bar{\varepsilon} > 0$ is the radius of the $L^\infty(\Gamma)$-ball where $J$ achieves the (local) minimum value at $\bar{u}$ among all feasible controls.*

If $l$ is convex with respect to the 3rd variable, (5.1) follows immediately from the variational inequality (4.5).

To prove this theorem, first the sensitivity of the state with respect to certain pointwise perturbations of the control is studied. The following propositions are crucial to accomplish these perturbations.

PROPOSITION 5.1 *Let $\rho \in (0,1)$, then a sequence of $\sigma$-measurable sets $\{E_k\}_{k=1}^\infty$, with $E_k \subset \Gamma$ and $\sigma(E_k) = \rho\sigma(\Gamma)$, exists such that $(1/\rho)\chi_{E_k} \rightharpoonup 1$ weakly$^*$ in $L^\infty(\Gamma)$ when $k \to \infty$.*

For the proof of Proposition 5.1, the reader is referred to Casas (1996).

PROPOSITION 5.2 *Under the assumptions of Theorem 5.1, for any $u \in L^\infty(\Gamma)$ there exists a number $0 < \hat\rho < 1$ and $\sigma$-measurable sets $E_\rho$, with $\sigma(E_\rho) = \rho\sigma(\Gamma)$ for all $0 < \rho < \hat\rho$, that have the following properties: If we define*

$$u_\rho(x) = \begin{cases} \bar u(x) & \text{if } x \in \Gamma \setminus E_\rho \\ u(x) & \text{if } x \in E_\rho\,, \end{cases}$$

*then*

$$y_\rho = \bar y + \rho z + r_\rho\,, \quad \lim_{\rho \searrow 0} \frac{1}{\rho}\|r_\rho\|_{W^{1,p}(\Omega)} = 0\,, \tag{5.2}$$

$$J(u_\rho) = J(\bar u) + \rho z^0 + r_\rho^0\,, \quad \lim_{\rho \searrow 0} \frac{1}{\rho}|r_\rho^0| = 0 \tag{5.3}$$

*hold true, where $\bar y$ and $y_\rho$ are the states associated to $\bar u$ and $u_\rho$ respectively, $z$ is the unique element of $W^{1,p}(\Omega)$ satisfying the linearized equation*

$$\left\{\begin{array}{rcll} -\operatorname{div}\left[a(x,\bar y)\nabla z + \dfrac{\partial a}{\partial y}(x,\bar y) z \nabla \bar y\right] + \dfrac{\partial f}{\partial y}(x,\bar y) z & = & 0\,, & \text{in } \Omega\,, \\ \left[a(x,\bar y)\nabla z + \dfrac{\partial a}{\partial y}(x,\bar y) z \nabla \bar y\right]\cdot \vec n(x) & = & u - \bar u\,, & \text{on } \Gamma \end{array}\right. \tag{5.4}$$

*and*

$$z^0 = \int_\Omega \frac{\partial L}{\partial y}(x,\bar y) z\,dx + \int_\Gamma \left\{ \frac{\partial l}{\partial y}(x,\bar y,\bar u) z + l(x,\bar y,u) - l(x,\bar y,\bar u)\right\} d\sigma(x)\,.$$

*Proof.* Since the proof is similar to that of Casas and Tröltzsch (2009, Proposition 4.3) we only comment upon the main differences. We define $g \in L^1(\Gamma)$ by

$$g(x) = l(x,\bar y(x),u(x)) - l(x,\bar y(x),\bar u(x))\,.$$

Given $\rho \in (0,1)$, we take a sequence $\{E_k\}_{k=1}^\infty$ as in Proposition 5.1. Since $L^\infty(\Gamma)$ is compactly embedded in $W^{-1/p,p}(\Gamma)$, there exists $k_\rho$ such that

$$\left|\int_\Gamma \left[1 - \frac{1}{\rho}\chi_{E_k}(x)\right] g(x)\,d\sigma(x)\right| + \left\|(1 - \frac{1}{\rho}\chi_{E_k})(u - \bar u)\right\|_{W^{-1/p,p}(\Gamma)} < \rho \ \ \forall k \ge k_\rho. \tag{5.5}$$

The inequality (5.5) is the analog of Casas and Tröltzsch (2009, Eq. (4.7)). The same argumentation as in the proof of Casas and Tröltzsch (2009, Proposition 4.3) yields (5.2). To prove (5.3) we first introduce

$$L_\rho(x) = \int_0^1 \frac{\partial L}{\partial y}(x, \bar y(x) + \theta(y_\rho(x) - \bar y(x)))\,d\theta\,,$$

$$l_\rho(x) = \int_0^1 \frac{\partial l}{\partial y}(x, \bar y(x) + \theta(y_\rho(x) - \bar y(x)), u_\rho(x))\,d\theta\,.$$

Now, recalling the definition of $g$, and using (5.5), we have

$$
\begin{aligned}
&\frac{J(u_\rho)-J(\bar u)}{\rho} = \\
&= \int_\Omega \frac{L(x,y_\rho(x)) - L(x,\bar y(x))}{\rho}dx + \int_\Gamma \frac{l(x,y_\rho(x),u_\rho(x)) - l(x,\bar y(x),\bar u(x))}{\rho}d\sigma(x) \\
&= \int_\Omega \frac{L(x,y_\rho(x)) - L(x,\bar y(x))}{\rho}dx + \int_\Gamma \frac{l(x,y_\rho(x),u_\rho(x)) - l(x,\bar y(x),u_\rho(x))}{\rho}d\sigma(x) \\
&\qquad\qquad + \int_\Gamma \frac{l(x,\bar y(x),u_\rho(x)) - l(x,\bar y(x),\bar u(x))}{\rho}d\sigma(x) \\
&= \int_\Omega L_\rho(x) z_\rho(x)\,dx + \int_\Gamma \left\{ l_\rho(x) z_\rho(x) + \frac{1}{\rho}\chi_{E_\rho}(x) g(x) \right\} d\sigma(x) \to \\
&\to \int_\Omega \frac{\partial L}{\partial y}(x,\bar y(x)) z(x)\,dx + \int_\Gamma \left\{ \frac{\partial l}{\partial y}(x,\bar y(x),\bar u(x)) z(x) + g(x) \right\} d\sigma(x) = z^0 ,
\end{aligned}
$$

which implies (5.3). ■

*Proof of Theorem 5.1.* Since $\bar u$ is a local solution of problem $(\mathcal{P})$, there exists $\bar\varepsilon > 0$ such that $J$ achieves the minimum at $\bar u$ among all feasible controls of $\bar B_{L^\infty(\Gamma)}(\bar u, \bar\varepsilon)$. Let us take $u \in \bar B_{L^\infty(\Gamma)}(\bar u,\bar\varepsilon)$ with $u_a \le u \le u_b$ a.e. on $\Gamma$. Following Proposition 5.2, we consider sets $\{E_\rho\}_{\rho>0}$ such that (5.2) and (5.3) hold. Then $u_\rho \in \bar B_{L^\infty(\Gamma)}(\bar u,\bar\varepsilon)$ and (5.3) lead to

$$
0 \le \lim_{\rho\searrow 0} \frac{J(u_\rho) - J(\bar u)}{\rho} = z^0 .
$$

By using the variational formulation of (5.4) and the adjoint state given by (4.4), we get from the previous inequality after a integration by parts

$$
\begin{aligned}
0 &\le \int_\Gamma \{\bar\varphi(x)(u(x) - \bar u(x)) + l(x,\bar y, u) - l(x,\bar y,\bar u)\}\, d\sigma(x) \\
&= \int_\Gamma \{H(x,\bar y(x),u(x),\bar\varphi(x)) - H(x,\bar y(x),\bar u(x),\bar\varphi(x))\}\, d\sigma(x) . \qquad (5.6)
\end{aligned}
$$

Since $u$ is an arbitrary feasible control in the ball $\bar B_{L^\infty(\Gamma)}(\bar u,\bar\varepsilon)$, taking into account the definitions of $u_{a_{\bar\varepsilon}}$ and $u_{b_{\bar\varepsilon}}$ given in the statement of Theorem 5.1, we deduce from (5.6)

$$
\int_\Gamma H(x,\bar y(x),\bar u(x),\bar\varphi(x))\, d\sigma(x) = \min_{u_{a_{\bar\varepsilon}} \le u \le u_{b_{\bar\varepsilon}}} \int_\Gamma H(x,\bar y(x),u(x),\bar\varphi(x))\, d\sigma(x) . \qquad (5.7)
$$

It remains to prove that (5.7) implies (5.1). To do this we can follow the lines of the proof of Casas (1996, Theorem 1). ■

## 6. Second-order optimality conditions

In this section we prove at first the necessary and next the sufficient second-order optimality conditions. We suppose that the Assumptions 2.1-2.3, 3.1-(1) and 4.1 hold, $2 < p \leq \bar{p}$ with $\bar{p}$ taken from Theorem 2.2, and the function $a : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous. Observe that Assumption 4.1 implies 5.1, therefore Theorem 5.1 holds.

If $\bar{u}$ is a feasible control for problem $(\mathcal{P})$ and there exists $\bar{\varphi} \in W^{1,p}(\Omega)$ satisfying (4.4) and (4.5), then we introduce the cone of critical directions

$$C_{\bar{u}} = \left\{ h \in L^\infty(\Gamma) \,|\, h(x) \begin{cases} \geq 0 & \text{if } \bar{u}(x) = u_a(x) \\ \leq 0 & \text{if } \bar{u}(x) = u_b(x) \\ = 0 & \text{if } \bar{d}(x) \neq 0 \end{cases} \quad \text{for } x \in \Gamma \right\}, \tag{6.1}$$

where $\bar{d}$ is defined by (4.6). In the previous section we introduced the Hamiltonian $H$ associated to our problem $(\mathcal{P})$. It follows easily that

$$\frac{\partial H}{\partial u}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)) = \bar{d}(x)\,.$$

In the sequel, we will use the notation

$$\bar{H}_u(x) = \frac{\partial H}{\partial u}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x)) \quad \text{and} \quad \bar{H}_{uu}(x) = \frac{\partial^2 H}{\partial u^2}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x))\,.$$

In the following theorem we state the necessary second-order optimality conditions.

Theorem 6.1 *Let $\bar{u}$ be a local optimal solution of $(\mathcal{P})$. Then the following inequalities hold*

$$\begin{cases} J''(\bar{u})h^2 \geq 0, & \forall h \in C_{\bar{u}} \\ \bar{H}_{uu}(x) \geq 0, & \textit{for a.a. } x \textit{ with } \bar{H}_u(x) = 0\,. \end{cases} \tag{6.2}$$

*Proof.* The first inequality of (6.2) can be proved by the arguments of Casas and Tröltzsch (2009, Theorem 5.1). The second inequality follows easily from (5.1). Indeed, it is an easy and well known conclusion of (5.1) that

$$\bar{H}_u(x) \begin{cases} \geq 0 & \text{if } \bar{u}(x) = u_a(x) \\ \leq 0 & \text{if } \bar{u}(x) = u_b(x) \\ = 0 & \text{if } u_a(x) < \bar{u}(x) < u_b(x) \end{cases} \qquad \text{for a.a. } x \in \Gamma$$

and

$$\bar{H}_{uu}(x) \geq 0 \;\text{ if }\; \bar{H}_u(x) = 0 \;\text{ for a.a. } x \in \Gamma\,. \qquad \blacksquare$$

Let us define the Lagrange function associated to $(\mathcal{P})$,

$$\mathcal{L} : L^\infty(\Gamma) \times W^{1,p}(\Omega) \times W^{1,p}(\Omega) \to \mathbb{R}$$

given by

$$\begin{aligned}\mathcal{L}(u, y, \varphi) &= \mathcal{J}(y, u) - \int_\Omega \{a(x, y)\nabla y \cdot \nabla\varphi + \varphi f(x, y)\}\, dx + \int_\Gamma \varphi u\, d\sigma(x)\\ &= \int_\Gamma H(x, y(x), u(x), \varphi(x))\, d\sigma(x) + \int_\Omega \{L(x, y) - [a(x, y)\nabla y \cdot \nabla\varphi + \varphi f(x, y)]\}\, dx,\end{aligned}$$

where

$$\mathcal{J}(y, u) = \int_\Omega L(x, y)\, dx + \int_\Gamma l(x, y, u)\, d\sigma(x)\,.$$

By using the Lagrange function, we can express the second-order optimality conditions in the form (6.3) below, which is more convenient in optimization theory. This form is obtained as follows: Defining $\bar H_y$, $\bar H_{yy}$ and $\bar H_{yu}$ similarly to $\bar H_u$ and $\bar H_{uu}$ we can write the first and second order derivatives of $\mathcal{L}$ w.r. to $(y, u)$ as follows

$$\begin{aligned}D_{(y,u)}\mathcal{L}(\bar u, \bar y, \bar\varphi)(z, h) &= \int_\Gamma \left\{\bar H_y(x)z(x) + \bar H_u(x)h(x)\right\} d\sigma(x)\\ &+ \int_\Omega \left\{\frac{\partial L}{\partial y}(x, \bar y)z - \bar\varphi\frac{\partial f}{\partial y}(x, \bar y)z - \nabla\bar\varphi \cdot \left[a(x, \bar y)\nabla z + \frac{\partial a}{\partial y}(x, \bar y)z\nabla\bar y\right]\right\} dx\,.\end{aligned}$$

If we take $z$ as the solution of (3.1) associated to $v = h$ and $g = 0$, along with the adjoint state (4.4), we get

$$D_{(y,u)}\mathcal{L}(\bar u, \bar y, \bar\varphi)(z, h) = \int_\Gamma \bar H_u(x)h(x)d\sigma(x)\,.$$

Moreover, we find

$$\begin{aligned}D^2_{(y,u)}\mathcal{L}(\bar u, \bar y, \bar\varphi)(z, h)^2 &= \int_\Gamma \left\{\bar H_{yy}(x)z^2(x) + 2\bar H_{yu}(x)zh + \bar H_{uu}(x)h^2(x)\right\} d\sigma(x)\\ &+ \int_\Omega \left\{\frac{\partial^2 L}{\partial y^2}(x, \bar y)z^2 - \bar\varphi\frac{\partial f^2}{\partial y^2}(x, \bar y)z^2 - \nabla\bar\varphi \cdot \left[\frac{\partial^2 a}{\partial y^2}(x, \bar y)z^2\nabla\bar y + 2\frac{\partial a}{\partial y}(x, \bar y)z\nabla z\right]\right\} dx\,.\end{aligned}$$

Once again, taking $z$ as above, we deduce from (4.2)

$$J''(\bar u)h^2 = D^2_{(y,u)}\mathcal{L}(\bar u, \bar y, \bar\varphi)(z, h)^2\,. \tag{6.3}$$

The next theorem provides the second order sufficient optimality conditions of $(\mathcal{P})$.

THEOREM 6.2 *Let $\bar{u}$ be a feasible control for problem $(\mathcal{P})$ and $\bar{\varphi} \in W^{1,p}(\Omega)$ satisfying (4.4) and (4.5). We also assume that there exist $\mu > 0$ and $\tau > 0$ such that*

$$\begin{cases} J''(\bar{u})h^2 > 0 & \forall h \in C_{\bar{u}} \setminus \{0\} \\ \bar{H}_{uu}(x) \geq \mu & \text{if } \left|\bar{H}_u(x)\right| \leq \tau \text{ for a.a. } x \in \Gamma\,. \end{cases} \tag{6.4}$$

*Then there exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Gamma)} \leq J(u)$$

*for every feasible control $u$ for $(\mathcal{P})$, with $\|u - \bar{u}\|_{L^\infty(\Gamma)} \leq \varepsilon$.*

*Proof.* The proof follows the same steps as that of Casas and Tröltzsch (2009, Theorem 5.2). Let us indicate some minor changes. We will argue by contradiction. Let us assume that there exists a sequence $\{u_k\}_{k=1}^\infty \subset L^\infty(\Gamma)$ of feasible controls for $(\mathcal{P})$ with

$$\|u_k - \bar{u}\|_{L^\infty(\Gamma)} < \frac{1}{k} \quad \text{and} \quad J(\bar{u}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Gamma)} > J(u_k)\,. \tag{6.5}$$

Let us define

$$y_k = G(u_k) = y_{u_k},\ \bar{y} = G(\bar{u}) = y_{\bar{u}},\ \rho_k = \|u_k - \bar{u}\|_{L^2(\Gamma)} \text{ and } v_k = \frac{1}{\rho_k}(u_k - \bar{u})\,,$$

then

$$\lim_{k\to\infty} \|y_k - \bar{y}\|_{W^{1,p}(\Omega)} = 0,\ \lim_{k\to\infty} \rho_k = 0 \text{ and } \|v_k\|_{L^2(\Gamma)} = 1 \quad \forall k \in \mathbb{N}\,. \tag{6.6}$$

By taking a subsequence, if necessary, we can assume that $v_k \rightharpoonup v$ weakly in $L^2(\Gamma)$. We will prove that $v \in C_{\bar{u}}$. For this, we will need the following result

$$\lim_{k\to\infty} \frac{1}{\rho_k}(y_k - \bar{y}) = z \quad \text{in } H^1(\Omega)\,,$$

where $z \in H^1(\Omega)$ is the solution of (3.1) corresponding to the state $\bar{y}$ and $g = 0$, which we prove now. By setting $z_k = (y_k - \bar{y})/\rho_k$, subtracting the state equations satisfied by $(y_k, u_k)$ and $(\bar{y}, \bar{u})$, dividing by $\rho_k$ and applying the mean value theorem we get

$$\begin{cases} -\operatorname{div}\left[a(x, y_k)\nabla z_k + \dfrac{\partial a}{\partial y}(x, \bar{y} + \theta_k(y_k - \bar{y}))z_k \nabla\bar{y}\right] & \\ \qquad\qquad + \dfrac{\partial f}{\partial y}(x, \bar{y} + \nu_k(y_k - \bar{y}))z_k = 0 & \text{in } \Omega\,, \\ \left[a(x, y_k)\nabla z_k + \dfrac{\partial a}{\partial y}(x, \bar{y} + \theta_k(y_k - \bar{y}))z_k \nabla\bar{y}\right]\cdot\vec{n}(x) = v_k & \text{on } \Gamma\,. \end{cases} \tag{6.7}$$

Notice that $\theta_k$, $\nu_k$ are functions depending on the space variable and their measurability can be shown by applying Ekeland and Temam (1976, Theorem 1.2, p. 236, and Proposition 1.1, p. 234) to the positive functions

$$g_1 : \bar{\Omega} \times [0,1] \to \mathbb{R}\,,$$
$$g_1(x,t) = \left| a(x, y_k(x)) - a(x, \bar{y}(x)) - \frac{\partial a}{\partial y}(x, \bar{y}(x) + t(y_k(x) - \bar{y}(x))) \right|,$$

and

$$g_2 : \Omega \times [0,1] \to \mathbb{R}\,,$$
$$g_2(x,t) = \left| f(x, y_k(x)) - f(x, \bar{y}(x)) - \frac{\partial f}{\partial y}(x, \bar{y}(x) + t(y_k(x) - \bar{y}(x))) \right|,$$

respectively. Using similar arguments as in the proof of Casas and Tröltzsch (2009, Theorem 5.2), we can pass to the limit in (6.7) and deduce

$$\begin{cases} -\operatorname{div}\left[a(x,\bar{y})\nabla z + \dfrac{\partial a}{\partial y}(x,\bar{y}) z \nabla \bar{y}\right] + \dfrac{\partial f}{\partial y}(x,\bar{y}) z = 0 & \text{in } \Omega\,, \\ \left[a(x,\bar{y})\nabla z + \dfrac{\partial a}{\partial y}(x,\bar{y}) z \nabla \bar{y}\right] \cdot \vec{n}(x) = v & \text{on } \Gamma\,. \end{cases} \tag{6.8}$$

It remains to show the strong convergence $z_k \to z$ in $H^1(\Omega)$. This can be done by using (6.7), (6.8), and the uniform convergence $y_k \to \bar{y}$. It can be shown easily that

$$\int_\Omega a(x,\bar{y})\,|\nabla z_k|^2\,dx \to \int_\Omega a(x,\bar{y})\,|\nabla z|^2\,dx\,.$$

Finally, this convergence, along with the weak convergence of $\{z_k\}_{k=1}^\infty$ in $H^1(\Omega)$ implies the strong convergence $z_k \to z$ in $H^1(\Omega)$. With the help of this convergence we can prove that $v \in C_{\bar{u}}$; see Casas and Tröltzsch (2009, Theorem 5.2).

Our next goal is to prove that $v$ does not satisfy the first condition of (6.4), which leads immediately to the identity $v = 0$ and then to the final contradiction. By using the definition of $\mathcal{L}$, (6.5) and the fact that $(y_k, u_k)$ and $(\bar{y}, \bar{u})$ satisfy the state equation, we get

$$\begin{aligned} \mathcal{L}(u_k, y_k, \bar{\varphi}) = \mathcal{J}(y_k, u_k) &< \mathcal{J}(\bar{y}, \bar{u}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Gamma)} \\ &= \mathcal{L}(\bar{u}, \bar{y}, \bar{\varphi}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Gamma)}\,. \end{aligned} \tag{6.9}$$

By performing a Taylor expansion up to the second order, we obtain

$$\begin{aligned} \mathcal{L}(u_k, y_k, \bar{\varphi}) &= \mathcal{L}(\bar{u} + \rho_k v_k, \bar{y} + \rho_k z_k, \bar{\varphi}) = \mathcal{L}(\bar{u}, \bar{y}, \bar{\varphi}) + \rho_k D_{(u,y)}\mathcal{L}(\bar{u}, \bar{y}, \bar{\varphi})(z_k, v_k) \\ &+ \frac{\rho_k^2}{2} D^2_{(u,y)}\mathcal{L}(\bar{u} + \theta_k \rho_k v_k, \bar{y} + \nu_k \rho_k z_k, \bar{\varphi})(z_k, v_k)^2. \end{aligned}$$

Taking into account (6.9) and (6.5), the last equality leads to

$$\rho_k D_{(u,y)}\mathcal{L}(\bar{u},\bar{y},\bar{\varphi})(z_k,v_k)+\frac{\rho_k^2}{2}D^2_{(u,y)}\mathcal{L}(w_k,\xi_k,\bar{\varphi})(z_k,v_k)^2<\frac{1}{k}\|u_k-\bar{u}\|^2_{L^2(\Gamma)}=\frac{\rho_k^2}{k},$$

where $\xi_k := \bar{y}+\nu_k\rho_k z_k$ and $w_k := \bar{u}+\theta_k\rho_k v_k$. Obviously, $\xi_k \to \bar{y}$ in $W^{1,p}(\Omega)$ and $w_k \to \bar{u}$ in $L^\infty(\Gamma)$. Taking into account the expressions obtained for the derivatives of $\mathcal{L}$ we get, after dividing the previous inequality by $\rho_k^2$,

$$\begin{aligned}\frac{1}{\rho_k}\int_\Gamma \bar{H}_u(x)v_k(x)\,d\sigma(x) &+ \frac{1}{2}\Bigg[\int_\Gamma \big\{H^k_{yy}(x)z_k^2(x)+2H^k_{yu}(x)(z_k,v_k)\\ &+H^k_{uu}(x)v_k^2(x)\big\}\,d\sigma(x)+\int_\Omega\Bigg\{\frac{\partial^2 L}{\partial y^2}(x,\xi_k)z_k^2-\bar{\varphi}\frac{\partial f^2}{\partial y^2}(x,\xi_k)z_k^2\\ &-\nabla\bar{\varphi}\cdot\left[\frac{\partial^2 a}{\partial y^2}(x,\xi_k)z_k^2\nabla\xi_k+2\frac{\partial a}{\partial y}(x,\xi_k)z_k\nabla z_k\right]\Bigg\}\,dx\Bigg]<\frac{1}{k}\,, \end{aligned}\tag{6.10}$$

where

$$H^k_{yy}(x)=H_{yy}(x,\xi_k(x),w_k(x),\bar{\varphi}(x))\,,$$

with analogous definitions for $H^k_{yu}(x)$ and $H^k_{uu}(x)$. It is easy to check that

$$\begin{cases}(H^k_{yy}(x),H^k_{yu}(x),H^k_{uu}(x))\to(\bar{H}_{yy}(x),\bar{H}_{yu}(x),\bar{H}_{uu}(x))\\ |H^k_{yy}(x)|+|H^k_{yu}(x)|+|H^k_{uu}(x)|\le C\end{cases}\quad\text{for a.a. } x\in\Gamma\,,$$

for some constant $C<\infty$. The following convergence properties can also be verified easily

$$\begin{cases}\dfrac{\partial^j a}{\partial y^j}(x,\xi_k)z_k\nabla\bar{\varphi}\to\dfrac{\partial^j a}{\partial y^j}(x,\bar{y})z\nabla\bar{\varphi},\ j=1,2\,,\\ \nabla z_k\to\nabla z\quad\text{and}\quad z_k\nabla\xi_k\to z\nabla\bar{y}\,, & \text{in } L^2(\Omega)^2\quad\text{and}\\ \bar{\varphi}\dfrac{\partial^2 f}{\partial y^2}(x,\xi_k)z_k\to\bar{\varphi}\dfrac{\partial^2 f}{\partial y^2}(x,\bar{y})z\,, & \text{in } L^2(\Omega)\,.\end{cases}$$

Using the above properties we can pass to the limit in (6.10) as follows

$$\begin{aligned}&\limsup_{k\to\infty}\left\{\frac{1}{\rho_k}\int_\Gamma\bar{H}_u(x)v_k(x)\,d\sigma(x)+\frac{1}{2}\int_\Gamma H^k_{uu}(x)v_k^2(x)\,d\sigma(x)\right\}\\ &+\frac{1}{2}\Bigg[\int_\Gamma\big\{\bar{H}_{yy}(x)z^2(x)+2\bar{H}_{yu}(x)zv\big\}\,d\sigma(x)+\int_\Omega\Bigg\{\frac{\partial^2 L}{\partial y^2}(x,\bar{y})z^2\\ &-\bar{\varphi}\frac{\partial f^2}{\partial y^2}(x,\bar{y})z^2-\nabla\bar{\varphi}\cdot\left[\frac{\partial^2 a}{\partial y^2}(x,\bar{y})z^2\nabla\bar{y}+2\frac{\partial a}{\partial y}(x,\bar{y})z\nabla z\right]\Bigg\}\,dx\Bigg]\le 0.\end{aligned}\tag{6.11}$$

Now, we prove that the above upper limit is bounded from below by

$$\frac{1}{2}\int_\Gamma \bar{H}_{uu}(x)v^2(x)\,d\sigma(x).$$

This can be done as in Casas and Tröltzsch (2009, Theorem 5.2) by a convexity argument for which the second condition of (6.4) plays an essential role. The difficulty in achieving this goal is due to the fact that we only have a weak convergence $v_k \rightharpoonup v$. Once the mentioned lower estimate is proved, from (6.11) and (6.3) we deduce that $J''(\bar{u})v^2 = D^2_{(y,u)}\mathcal{L}(\bar{u},\bar{y},\bar{\varphi})(z,v)^2 \leq 0$. Finally, according to (6.4) this is possible only if $v = 0$. To get this estimate we use the following inequality, proved in Casas and Tröltzsch (2009, Theorem 5.2) for distributed controls, but there is no difference in the arguments,

$$\begin{aligned}
&\limsup_{k\to\infty}\left\{\frac{1}{\rho_k}\int_\Gamma \bar{H}_u(x)v_k(x)\,d\sigma(x)+\frac{1}{2}\int_\Gamma H^k_{uu}(x)v_k^2(x)\,d\sigma(x)\right\}\\
&\geq \limsup_{k\to\infty}\left\{\int_{\{|\bar{H}_u|>\tau\}}\left[\|\bar{H}_{uu}\|_{L^\infty(\Gamma)}+\frac{1}{2}\bar{H}_{uu}\right]v_k^2\,d\sigma(x)+\frac{1}{2}\int_{\{|\bar{H}_u|\leq\tau\}}\bar{H}_{uu}v_k^2\,d\sigma(x)\right\}\\
&\quad\geq \frac{1}{2}\int_\Gamma \bar{H}_{uu}v^2\,d\sigma(x)\,.
\end{aligned}\tag{6.12}$$

Finally, using that $\|v_k\|_{L^2(\Gamma)} = 1$, along with (6.11), (6.12), the second condition of (6.4) and the fact that $v = 0$, we deduce

$$\begin{aligned}
0&\geq \limsup_{k\to\infty}\left\{\int_{\{|\bar{H}_u|>\tau\}}\left[\|\bar{H}_{uu}\|_{L^\infty(\Gamma)}+\frac{1}{2}\bar{H}_{uu}\right]v_k^2\,d\sigma(x)+\frac{1}{2}\int_{\{|\bar{H}_u|\leq\tau\}}\bar{H}_{uu}v_k^2\,d\sigma(x)\right\}\\
&\geq \limsup_{k\to\infty}\left\{\frac{\|\bar{H}_{uu}\|_{L^\infty(\Gamma)}}{2}\int_{\{|\bar{H}_u|>\tau\}}v_k^2\,d\sigma(x)+\frac{\mu}{2}\int_{\{|\bar{H}_u|\leq\tau\}}v_k^2\,d\sigma(x)\right\}\\
&\geq \frac{\min\{\|\bar{H}_{uu}\|_{L^\infty(\Gamma)},\mu\}}{2}\limsup_{k\to\infty}\left\{\int_\Gamma v_k^2\,d\sigma(x)\right\} = \frac{\min\{\|\bar{H}_{uu}\|_{L^\infty(\Gamma)},\mu\}}{2} > 0\,,
\end{aligned}$$

yielding the contradiction we were looking for. ■

REMARK 6.1 *Let us note that* $\bar{H}_{uu}(x) = (\partial^2 l/\partial u^2)(x,\bar{y}(x),\bar{u}(x))$. *Therefore, if the second derivative of* $l$ *w.r. to* $u$ *is strictly positive, then the second condition of (6.4) is satisfied. A standard example is given by the function*

$$l(x,y(x),u(x)) = l_0(x,y(x))+\frac{\lambda}{2}u^2\,,\quad \text{with } \lambda>0\,.$$

*In this case the control* $\bar{u}$ *given in the statement of Theorem 6.2 is locally optimal even in the sense of* $L^2(\Gamma)$. *To prove this, we follow the same lines of the* *previous proof with the following differences. Instead of (6.5) we assume the existence of a sequence* $\{u_k\}_{k=1}^{\infty} \subset U_{ad}$ *with*

$$\|u_k - \bar{u}\|_{L^2(\Gamma)} < \frac{1}{k} \quad \textit{and} \quad J(\bar{u}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Gamma)} > J(u_k)\,.$$

*Then, by using the identity* $\bar{H}_{uu}(x) \equiv \lambda$ *and* $\|v_k\|_{L^2(\Gamma)} = 1$*, we can considerably shorten the previous proof in the following way*

$$\begin{aligned} 0 &\geq \limsup_{k\to\infty} \left\{ \frac{1}{\rho_k} \int_\Gamma \bar{H}_u(x) v_k(x)\, d\sigma(x) + \frac{1}{2} \int_\Gamma H^k_{uu}(x) v_k^2(x)\, d\sigma(x) \right\} \\ &= \limsup_{k\to\infty} \left\{ \frac{1}{\rho_k} \int_\Gamma |\bar{H}_u(x)||v_k(x)|\, d\sigma(x) + \frac{1}{2} \int_\Gamma \bar{H}_{uu}(x) v_k^2(x)\, d\sigma(x) \right\} \\ &\geq \limsup_{k\to\infty} \left\{ \frac{\lambda}{2} \int_\Gamma v_k^2(x)\, d\sigma(x) \right\} = \frac{\lambda}{2}\,. \end{aligned}$$

*This yields the desired contradiction.*

We give another sufficient optimality condition equivalent to (6.4), which is very useful for numerical purposes. The proof of this equivalence is carried out in Casas and Mateos (2002, Theorem 4.4).

THEOREM 6.3 *Let* $\bar{u}$ *be a feasible control for problem* $(\mathcal{P})$*. We assume that there exists* $\bar{\varphi} \in W^{1,p}(\Omega)$ *satisfying (4.4) and (4.5). Then (6.4) holds if and only if there exist* $\delta, \rho > 0$ *such that*

$$J''(\bar{u})h^2 \geq \delta \|h\|^2_{L^2(\Gamma)} \quad \forall h \in C^\rho_{\bar{u}}\,, \tag{6.13}$$

*where*

$$C^\rho_{\bar{u}} = \left\{ h \in L^2(\Gamma) \,|\, h(x) \begin{cases} \geq 0 & \textit{if } \ \bar{u}(x) = u_a(x) \\ \leq 0 & \textit{if } \ \bar{u}(x) = u_b(x) \\ = 0 & \textit{if } \ |\bar{d}(x)| > \rho \end{cases} \quad \textit{for a.a. } x \in \Gamma \right\}.$$

REMARK 6.2 *Comparing the first inequality of (6.4) with the analogous one of (6.2), we notice that the gap is minimal between the necessary and sufficient conditions. On the other hand, the second inequality of (6.4) is stronger than the corresponding one of (6.2). In general we cannot take* $\tau = 0$ *in (6.4). The reader is referred to Dunn (1998) for a simple example proving the impossibility of taking* $\tau = 0$*.*

## A. Regularity results for solutions of elliptic PDEs in non-convex polygons

In this appendix we study the following Neumann problem

$$\left\{ \begin{array}{rcll} -\Delta y &=& f & \text{in } \Omega\,, \\ \nabla y \cdot \vec{n} &=& g & \text{on } \Gamma\,, \end{array} \right. \tag{A.1}$$

where $\Omega \subset \mathbb{R}^2$ is a (non necessarily convex) polygonal open and bounded set, $\Gamma = \partial\Omega$, $f \in L^p(\Omega)$ with $p > 4/3$ and $g \in L^2(\Gamma)$. We assume the compatibility condition

$$\int_\Omega f(x)\,dx + \int_\Gamma g(x)\,d\sigma(x) = 0\,. \tag{A.2}$$

It is well known that (A.1) has a solution in $H^1(\Omega)$ that is unique up to an additive constant.

Theorem A.1 *Let $y \in H^1(\Omega)$ be a solution of (A.1), then $y \in H^{3/2}(\Omega)$ and there exists a constant $C > 0$ independent of $f$ and $g$ such that*

$$\|y\|_{H^{3/2}(\Omega)} \le C\left(\|f\|_{L^p(\Omega)} + \|g\|_{L^2(\Gamma)} + \left|\int_\Omega y(x)\,dx\right|\right). \tag{A.3}$$

The last term in the inequality (A.3) is a consequence of the uniqueness of $y$ up to an additive constant. To prove the previous theorem we will use the following

Lemma A.1 *There exists $s > 3/2$ independent of $f$ such that the problem*

$$\left\{\begin{array}{rcll} -\Delta y_1 & = & f & \text{in}\,\Omega\,, \\ y_1 & = & 0 & \text{on}\,\Gamma\,, \end{array}\right. \tag{A.4}$$

*has a unique solution in $H^s(\Omega)$. Moreover*

$$\|y_1\|_{H^s(\Omega)} \le C_s\|f\|_{L^p(\Omega)}\,.$$

*Proof.* According to Dauge (1988, Theorem 23.3), see also Dauge (1989, Theorem 3), there exists $s_0 \in (3/2, 5/2)$ depending on the angles of $\Omega$ and the minimum positive eigenvalue of the Laplace operator in $\Omega$, such that $y_1 \in H^s(\Omega)$ if $f \in H^{s-2}(\Omega)$ and $3/2 < s < s_0$. Moreover, the following estimate holds

$$\|y_1\|_{H^s(\Omega)} \le C_s\|f\|_{H^{s-2}(\Omega)}\,.$$

The proof is concluded if we prove that $L^p(\Omega) \subset H^{s-2}(\Omega)$ for $3/2 < s < s_0$. For $0 < 2 - s < 1/2$, we have that $H^{s-2}(\Omega) = H^{2-s}(\Omega)^*$. Taking into account that $H^{1/2}(\Omega) \subset L^4(\Omega)$, we can get $\varepsilon > 0$ small enough and $s$ close to $3/2$ such that $p > (4-\varepsilon)/(3-\varepsilon)$ and $H^{2-s}(\Omega) \subset L^{4-\varepsilon}(\Omega)$. Then, $L^p(\Omega) \subset L^{(4-\varepsilon)/(3-\varepsilon)}(\Omega) = L^{4-\varepsilon}(\Omega)^* \subset H^{2-s}(\Omega)^* = H^{s-2}(\Omega)$. ∎

*Proof of Theorem A.1.* Let $y_1 \in H^s(\Omega)$ be the solution of (A.4), then $\nabla y_1 \cdot \vec{n} \in H^{s-3/2}(\Gamma) \subset L^2(\Gamma)$. Using (A.2) and integrating the equation (A.1) we obtain

$$-\int_\Gamma g(x)\,d\sigma(x) = \int_\Omega f(x)\,dx = -\int_\Omega \Delta y_1(x)\,dx = -\int_\Gamma \nabla y_1(x)\cdot\vec{n}(x)\,d\sigma(x)\,,$$

therefore the Neumann problem

$$\left\{\begin{array}{rcll} -\Delta y_2 & = & 0 & \text{in } \Omega\,, \\ \nabla y_2\cdot\vec{n} & = & g-\nabla y_1\cdot\vec{n} & \text{on } \Gamma\,, \end{array}\right. \tag{A.5}$$

has a solution in $H^1(\Omega)$. Moreover, this solution is in $H^{3/2}(\Omega)$ (see Jerison and Kenig, 1981, and Kenig, 1994) and from Lemma A.1 it follows that

$$\begin{aligned} \|y_2\|_{H^{3/2}(\Omega)} &\le C\left(\|g-\nabla y_1\cdot\vec{n}\|_{L^2(\Gamma)}+\left|\int_\Omega y_2(x)\,dx\right|\right) \\ &\le C\left(\|g\|_{L^2(\Gamma)}+\|y_1\|_{H^s(\Omega)}+\left|\int_\Omega y_2(x)\,dx\right|\right) \\ &\le C\left(\|g\|_{L^2(\Gamma)}+\|f\|_{L^p(\Omega)}+\left|\int_\Omega y_2(x)\,dx\right|\right). \end{aligned}$$

Finally, it is clear that $y_1+y_2\in H^{3/2}(\Omega)$ and $y=y_1+y_2+\mathit{constant}$, which concludes the proof. ∎

THEOREM A.2 *If $y\in H^{3/2}(\Omega)$ and $\Delta y\in L^p(\Omega)$, then $y\big|_\Gamma\in H^1(\Gamma)$ and $\nabla y\cdot\vec{n}\in L^2(\Gamma)$ and the following estimate holds*

$$\|y\big|_\Gamma\|_{H^1(\Omega)}+\|\nabla y\cdot\vec{n}\|_{L^2(\Gamma)}\le C\left(\|y\|_{H^{3/2}(\Omega)}+\|\Delta y\|_{L^p(\Omega)}\right). \tag{A.6}$$

*Proof.* As we stated in the proof of Theorem A.1 $y=y_1+y_2+\mathit{constant}$, with $y_1\in H^s(\Omega)\cap H^1_0(\Omega)$ and $y_2\in H^{3/2}(\Omega)$ harmonic. Following Jerison and Kenig, 1995, Theorem 5.6 and Corollary 5.7, we have that $y_2\big|_\Gamma\in H^1(\Gamma)$, $\nabla y_2\vec{n}\in L^2(\Gamma)$ and

$$\|y_2\big|_\Gamma\|_{H^1(\Gamma)}+\|\nabla y_2\cdot\vec{n}\|_{L^2(\Gamma)}\le C\|y_2\|_{H^{3/2}(\Omega)}\,. \tag{A.7}$$

On the other hand, $y_1\big|_\Gamma=0$, therefore $y\big|_\Gamma=y_2\big|_\Gamma+\mathit{constant}\in H^1(\Gamma)$. Since $y_1\in H^s(\Omega)$, with $s>3/2$, it holds that $\nabla y_1\cdot\vec{n}\in L^2(\Gamma)$ and

$$\|\nabla y_1\cdot\vec{n}\|_{L^2(\Gamma)}\le C\|y_1\|_{H^s(\Omega)}\le C\|f\|_{L^p(\Omega)}\,. \tag{A.8}$$

Thus we have that $\nabla y\cdot\vec{n}=\nabla y_1\cdot\vec{n}+\nabla y_2\cdot\vec{n}\in L^2(\Gamma)$ and (A.6) follows from (A.7) and (A.8). ∎

COROLLARY A.1 *Let $a\in L^p(\Omega)$ satisfy $0\le a$ and $a\not\equiv 0$ at least on a subset of $\Omega$ with positive measure. Then the problem*

$$\left\{\begin{array}{rcll} -\Delta y+a(x)y & = & f & \text{in } \Omega\,, \\ \nabla y\cdot\vec{n} & = & g & \text{on } \Gamma\,, \end{array}\right.$$

*has a unique solution $y\in H^{3/2}(\Omega)$ such that*

$$\|y\|_{H^{3/2}(\Omega)}\le C\left(\|f\|_{L^p(\Omega)}+\|g\|_{L^2(\Gamma)}\right),$$

*where $C>0$ depends on $\|a\|_{L^p(\Omega)}$ but it is independent of $f$ and $g$.*

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