Control and Cybernetics

vol. **40** (2011) No. 2

Algorithms for integral solutions of a class of diophantine equations*†

by

Agnieszka Polak

Faculty of Mathematics and Computer Science,
Nicolaus Copernicus University
ul. Chopina 12/18, 87-100 Toruń, Poland
e-mail: apolak@mat.umk.pl

Abstract: In 1970 a negative solution to the tenth Hilbert problem, concerning the determination of integral solutions of diophantine equations, has been published by Y. W. Matiyasevich (see Matiyasevich, 1970). Despite this result, we can present algorithms to compute integral solutions (roots) for a wide class of quadratic diophantine equations of the form $q(x) = d$, where $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ is a homogeneous quadratic form. We will focus on the roots of one (i.e., $d = 1$) of quadratic Euler forms of selected posets from Loupias' list (see Loupias, 1975). In particular, we will describe the roots of positive definite quadratic forms and the roots of quadratic forms that are principal (see Simson, 2010a). The algorithms and results we present here are successfully used in the representation theory of finite groups and algebras.

Keywords: integral quadratic form, unit form, diophantine equations, roots, Euler bilinear form, Euclidean diagrams, mesh quiver, algorithm, Maple.

1. Introduction

The problem of finding a general procedure solving any diophantine equation had been posed by David Hilbert in 1900 during the International Congress of Mathematicians in Paris. This problem was open for many years, until Y. W. Matiyasevich published the negative solution of the Tenth Hilbert Problem in 1970 (see Matiyasevich, 1970). Despite the result of Matiyasevich, it is possible to algorithmically describe the roots for selected classes of diophantine equations. We follow one of attempts, concerning integral quadratic forms, developed in Simson (2010b). One of problems stated in Simson (2010a) asks

*Submitted: July 2010; Accepted: May 2011.

†This is an extended and amended version of the paper, presented at the 5$^{\text{th}}$ Congress of Young IT Scientists (Międzyzdroje, 23-25 September 2010).

for a method to exhibit $\mathbb{Z}$-bilinear equivalence between poset from the lists of Loupias (1975) and Zavadskij-Shkabara (1976), and extended Dynkin diagrams. This kind of equivalence for one poset from these lists is presented in this paper. Precisely, we show that $\mathbb{L}_9$ is $\mathbb{Z}$-bilinearly equivalent to oriented extended Dynkin diagram $\widetilde{\mathbb{E}}_8$. The scheme we introduce is universal enough to be applied for the remaining posets from the mentioned lists.

We denote by $\mathbb{Z}$ the ring of integers, by $\mathbb{N}$ the set of non-negative integers. Given $n \geq 1$, we denote by $\mathbb{M}_n(\mathbb{Z})$ the $\mathbb{Z}$-algebra of all square n by n matrices with coefficients in $\mathbb{Z}$, and by $e_1, \ldots, e_n$ the standard basis of the free abelian group $\mathbb{Z}^n$. In our work we focus on unit quadratic forms $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$, with q defined by the formula $q(x) = \sum\limits_{i=1}^{n} q_{ii}x_i^2 + \sum\limits_{1 \leq i < j \leq n} q_{ij}x_i x_j$ for $q_{ij} \in \mathbb{Z}$ and $i, j \in \mathbb{N}$. A collection of algorithms describing the sets of roots of these forms, i.e., the sets $\mathcal{R}_q = \{v \in \mathbb{Z}^n;\ q(v) = 1\}$, is presented. We give the procedures allowing us to describe the roots of positive definite forms (i.e., forms satisfying $q(v) > 0$ for any $0 \neq v \in \mathbb{Z}^n$) and principal forms (see Simson, 2011). Our main result consists in showing that posets from the list of Loupias (1975) and Zavadskij-Shkabara (1976) (LZS for short) and forms associated to extended Dynkin diagrams are $\mathbb{Z}$-bilinearly equivalent (Section 4). These equivalences are presented with a few selected example posets from the mentioned list, but the procedures described in this paper can be applied to any poset from the LZS list, in accordance with theses contained in the author's prepared doctoral dissertation.

A number of criteria coupled with algorithms are recalled in Section 2, allowing for deciding if a given form q is positive definite or positive semi-definite. These criteria are utilized in Section 4 to verify if a given form is principal. In case of principal forms full description of corresponding sets of roots is possible.

Section 3 is devoted to graphs and quadratic forms associated to them. An important example of graphs is made up of Dynkin diagrams and extended Dynkin diagrams (see Simson, 2010b). A number of theorems characterizing these diagrams is given. In particular, the sets of roots of forms associated to these diagrams are characterized. Next, $\mathbb{Z}$-bilinear equivalence of two forms is defined and illustrated by an example. Then, $\mathbb{Z}$-bilinear equivalence of $\mathbb{Z}$-bilinear Euler form, associated to a poset, with $\mathbb{Z}$-bilinear Euler form of oriented Dynkin diagram is shown. The results have important applications in representation theory of groups, algebras, quivers and partially ordered sets, as well as in the study of derived categories of module categories.

In Section 4 the principal form and defect are defined, and some algorithms to automate our calculations are presented. We give a procedure to find the roots of principal forms. The $\mathbb{Z}$-bilinear equivalence between Euler forms of LZS posets and forms associated to extended Dynkin diagrams is shown, which is a result of the author's research in the field of the roots of quadratic forms, and constitutes one of the most important results contained in this article.

Diophantine equations in practical applications

The integral quadratic forms, considered by us, are nothing else than quadratic diophantine equations. The problem of determining their integral solutions such that $q(x) = d$ for d an integer, is NP-hard, as was shown in 1978 by Adleman and Manders (1976). The situation that we describe concerns the quadratic diophantine equations of the form $q(x) = 1$. If q is positive definite, then we are able to find all roots of an equation $q(x) = 1$ by applying the restrictively counting algorithm (see Simson, 2010a). In this work we focus on semi-positive definite quadratic forms which, as we known from Claim 1 (see Marczak, Polak and Simson, 2010), have infinitely many roots. In Section 4 we propose a geometric method which can be used to determine all roots of an equation $q(x) = 1$, in case when q is semi-positive definite, and Ker q is generated by a single non-zero vector.

Diophantine equations have numerous important applications in representation theory and in combinatorics (see Ringel, 1984; Simson, 1992). They are also used in public-key cryptography, as well as in the analysis of stability of nonlinear systems (see Buchmann and Vollmer, 2007). Moreover, diophantine equations can be used commercially in the economic model, described below.

Primitive commercial wandering

Assume that a producer of beer has $n \geq 2$ breweries located in different touristically attractive cities $A_1, \ldots, A_n$, situated in the mountains. We write $i < j$ if there is a one-way highway from A_i to A_j, A_j is more attractive than A_i, and the mountain of A_j is higher than that of A_i. Let us assume that $v_j \geq 0$ is the number of bottles of beer that are selling in A_j a year, and we define $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ to be the selling vector. Assume that the annual profit of selling the beer in A_j equals $q_{ii}v_i^2 + \sum_{i<j} q_{ij}v_iv_j$. It follows that the annual profit function is given by

$$q(v) = \sum_{i=1}^{n} q_{ii}v_i^2 + \sum_{1\leq i<j\leq n} q_{ij}v_iv_j.$$

It is nothing else than an integral quadratic form that we consider. In other words, the annual profit is positive (respectively non-negative) if and only if q is positive (respectively non-negative). Moreover, given a profit q being a positive integer $d \in \mathbb{N}$, we can describe the finite set of all selling vectors such that the year profit $q(v)$ equals d.

In this work we consider the case of all selling vectors $v \in \mathbb{Z}^n$ such that $q(v) = 1$, i.e., there is a profit. We demonstrate the algorithm (see Algorithm 4) that we use to describe a set of all selling vectors $v \in \mathbb{Z}^n$ such that $q(v) = 1$ in case q is positive semi-definite (so the yearly profit from selling beer is nonnegative). If q is positive definite, the number of selling vectors is finite, and may be described by the restrictively counting algorithm (see Simson, 2010a) (in exponential time). If q is positive semi-definite (i.e., non-negative), which is the case considered in this work, then the number of all selling vectors satisfying $q(v) = 1$ is infinite (see Marczak, Polak and Simson, 2010). However, we are able to describe it using the algorithms that we propose in Section 4. We determine a certain finite subset $\check{R} \subset \mathcal{R}_q$, where $\mathcal{R}_q$ is an infinite set of all selling vectors (roots), and we align the vectors from $\check{R}$ in a mesh graph, labeled with the roots. The mesh graph is infinite, but we determine a finite piece of it, labeled with selling vectors from the finite set $\check{R}$. All the remaining selling vectors for $q(v) = 1$ may be generated by means of simple operations (adding or subtracting vectors, multiplying vectors by scalars).

Our algorithms concern the case of $q(v) = d$ for $d = 1$ (or $d = 0$, which is simple). If $d \geq 2$ and q is positive definite, then all integral roots from d may be described by the restrictively counting algorithm (see Simson, 2010a). If q is positive semidefinite and $d \geq 2$, then the problem of determining all integral roots remains open.

2. Positive definite forms and positive semi-definite forms, the Sylvester criterion

By an integral quadratic form we mean a map $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ defined by the formula

$$q(x) = \sum_{i=1}^{n} q_{ii}x_i^2 + \sum_{1\leq i<j\leq n} q_{ij}x_ix_j,$$

where $q_{ij} \in \mathbb{Z}$ and $i,\ j \in \mathbb{N}$. Mostly we consider unit forms, i.e., forms satisfying $q_{11} = \ldots = q_{nn} = 1$. We recall a few necessary definitions and criteria for positive definiteness and positive semi-definiteness of given quadratic form. We present the algorithms implementing these criteria, utilized in Section 4 for describing the roots of positive definite forms and of positive semi-definite forms.

Definition 2.1 *A form $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ is called:*
(a) *positive definite, if $q(v) > 0$ for every $0 \neq v \in \mathbb{Z}^n$,*
(b) *positive semi-definite, if $q(v) \geq 0$ for every $v \in \mathbb{Z}^n$,*
(c) *weakly positive, if $q(v) > 0$ for every $0 \neq v \in \mathbb{N}^n$,*
(d) *weakly non-negative, if $q(v) \geq 0$ for every $0 \neq v \in \mathbb{N}^n$.*

Definition 2.2 *To a quadratic form $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$, defined by the formula $q(x) = q(x_1, ..., x_n) = \sum_{i=1}^{n} q_{ii}x_i^2 + \sum_{1\leq i<j\leq n} q_{ij}x_ix_j, q_{ij} \in \mathbb{Z}$, we associate the Gram matrix*

$$G_q = \begin{bmatrix} \hat{q}_{11} & \cdots & \hat{q}_{1n} \\ \vdots & \ddots & \vdots \\ \hat{q}_{n1} & \cdots & \hat{q}_{nn} \end{bmatrix} \in \mathbb{M}_n(\mathbb{Z}),$$

in which $\hat{q}_{jj} = q_{jj}$ for $j = 1, \ldots, n$, and $\hat{q}_{ij} = \hat{q}_{ji} = \frac{1}{2}q_{ij}$ for $i \neq j$.

Theorem 2.1 *A quadratic and symmetric form $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ is positive definite if and only if its Gram matrix $G_q = [\widehat{q}_{ij}]$ satisfies the following conditions of Sylvester:*

$$\widehat{q}_{11} > 0, \quad \det \begin{bmatrix} \widehat{q}_{11} & \widehat{q}_{12} \\ \widehat{q}_{21} & \widehat{q}_{22} \end{bmatrix} > 0, \ldots, \det \begin{bmatrix} \widehat{q}_{11} & \cdots & \widehat{q}_{1s} \\ \vdots & \ldots & \vdots \\ \widehat{q}_{s1} & \cdots & \widehat{q}_{ss} \end{bmatrix} > 0,$$

$$\det G_q > 0, \quad \text{for } 2 \leq s \leq n.$$

The description of this algorithm, called from now on the SylvesterCriterion, can be found in Simson (2010a). The next theorem motivates the algorithm for testing positive semi-definiteness of a form:

Theorem 2.2 *A quadratic form $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ is positive semi-definite if and only if the following inequality holds for $G_q = [\widehat{q}_{ij}]$:*

$$det \begin{bmatrix} \widehat{q}_{i_1 i_1} & \cdots & \widehat{q}_{i_1 i_r} \\ \vdots & \ddots & \vdots \\ \widehat{q}_{i_r i_1} & \cdots & \widehat{q}_{i_r i_r} \end{bmatrix} \geq 0,$$

for every $r = \{1, \ldots, n\}$ and $1 \leq i_1 < \ldots < i_r \leq n$.

We concentrate on Euler integral quadratic forms of posets. Let $I \equiv (I, \preceq)$ denote a poset, for which $|I| = n$. An incidence matrix of I is defined as

$$C_I = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \in \mathbb{M}_n(\mathbb{Z}), \text{ where } n = |I| \text{ and } c_{ij} = \begin{cases} 0, & \text{if } i \not\preceq j \\ 1, & \text{if } i \preceq j \end{cases}$$

for any $i,\ j \in I$. An Euler integral quadratic form $\overline{q}_I : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ of a given poset I is defined with the formula

$$\overline{q}_I(x) = x \cdot C_I^{-1} \cdot x^{tr},$$

where $C_I^{-1} = [\overline{c}_{ij}] \in \mathbb{M}_n(\mathbb{Z})$. For $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ a quadratic form, we call the vector $v \in \mathbb{Z}^n$, for which $q(v) = 1$, an integral root of one of this form. We denote by $\mathcal{R}_q = \{v \in \mathbb{Z}^n;\ q(v) = 1\}$ the set of all roots of q. In case of a positive semi-definite form we are also interested in its kernel, Ker q=$\{v \in \mathbb{Z}^n;\ q(v) = 0\}$. A number of known theorems characterizing the sets $\mathcal{R}_q$ for positive definite forms and for positive semi-definite forms will be stated (see Barot and de la Pena, 1999; Simson, 1992, 2004-2009).

The following algorithm (Algorithm 1) tests if a given Euler form of poset I is positive semi-definite (see Appendix for explanation of the used MAPLE procedures).

Algorithm 1 GENERALIZEDSYLVESTERCRITERION(C^{-1})

Input: C^{-1} – an inverse of incidence matrix of poset.
Output: 1 if $\overline{q}$ is positive semi-definite; 0 otherwise.

```
n ← coldim(C^{-1})
Gram ← (C^{-tr} + C^{-1})
L ←choose(n)
for i = 2 to nops(L) do // L[1]=[]
      if det (submatrix(Gram, L[i], L[i]))< 0 then
            return 0
      end if
end for
return 1
```

The algorithm presented is a consequence of the well-known generalized Sylvester criterion. We decided to present it here due to its usefulness in further algorithms.

THEOREM 2.3 *Let $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ be a quadratic form defined by the formula $q(x) = \sum_{i=1}^{n} x_i^2 + \sum_{1\le i<j\le n} q_{ij}x_ix_j$, where $q_{ij} \in \mathbb{Z}$.*

(a) *If q is positive definite, then $|\mathcal{R}_q| < \infty$.*
(b) *If q is weakly positive, then $|\mathcal{R}_q^+| < \infty$.*
(c) *If q is positive definite and $q_{ij} < 0$ for $i < j$, then $\mathcal{R}_q = \mathcal{R}_q^+ \cup \mathcal{R}_q^-$, where $\mathcal{R}_q^- = \{-v;\ v \in \mathcal{R}_q^+\}$.*
(d) *If q is positive semi-definite, $q_{ij} < 0$ for $i < j$, and $Ker\ q{=}\mathbb{Z}\cdot \mathbf{h}$, for $\mathbf{h}$ sincere, i.e., $h_i \neq 0$, for any $i \in \{1,\dots,n\}$, then $\mathcal{R}_q = \mathcal{R}_q^+ \cup \mathcal{R}_q^-$.*

DEFINITION 2.3 *Let I be a poset, $|I| = n$, and let C_I^{-1} be an inverse of incidence matrix of I such that det $C_I^{-1} = 1$, or det $C_I^{-1} = -1$.*

- *The Coxeter-Euler matrix of I is defined as $\overline{Cox_I} := -C_I^{-1}\cdot C_I^{tr} \in \mathbb{M}_n(\mathbb{Z})$,*
- *The Coxeter-Euler transformation of I is a homomorphism $\overline{\Phi}_I : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ of the $\mathbb{Z}^n$ group, defined by the formula $\overline{\Phi}_I(x){=}x\cdot \overline{Cox_I}$,*
- *The Coxeter-Euler polynomial can be calculated as $\overline{F}_I(t) = det\ (t\cdot E - \overline{Cox_I}) \in \mathbb{Z}[t]$, where $E \in \mathbb{M}_n(\mathbb{Z})$ is the identity matrix.*

DEFINITION 2.4 *Let I be a poset with $\mathbb{Z}$-invertible incidence matrix C_I. Let $\mathcal{R}_{\overline{q}_I} = \{v \in \mathbb{Z}^n;\ \overline{q}_I(v) = 1\}$ be the set of integral roots of Euler form $\overline{q}_I$. By the $\overline{\Phi}_I$-orbit of a vector $v \in \mathcal{R}_{\overline{q}_I}$ we mean the set*

$$\overline{\Phi}_I\text{-}\mathcal{O}rb(v) = \overline{\Phi}_I\text{-}\mathcal{O}(v) = \{\overline{\Phi}_I^m(v)\}_{m\in\mathbb{Z}}.$$

DEFINITION 2.5 *Let I be a poset, $C_I^{-1} \in \mathbb{M}_n(\mathbb{Z})$ a matrix with determinant equal to 1 or -1, and $\overline{\Phi}_I$ an Euler-Coxeter transformation. A $\overline{\Phi}_I$-mesh is composed of the set of vectors v, $\overline{\Phi}_I^{-1}(v)$, $v^{(1)},\dots,v^{(s)}$ for any $2 \le s \in \mathbb{N}$, satisfying:*

(i) $v + \overline{\Phi}_I^{-1}(v) = v^{(1)} + \ldots + v^{(s)}$.
(ii) *Each of vectors* v, $v^{(1)}, \ldots, v^{(s)}$ *is situated in another* $\overline{\Phi}_I$*-orbit. The following picture is a way of visualizing a* $\overline{\Phi}_I$*-mesh:*

$$\begin{tikzcd}
 & v^1 \arrow[dddr] & \\
 & v^2 \arrow[ddr] & \\
 & \cdot & \\
v \arrow[uuur] \arrow[uur] \arrow[ddr] \arrow[dddr] \arrow[rr, dashed, no head] & & \overline{\Phi}_I^{-1}(v) \\
 & \cdot & \\
 & v^{s-1} \arrow[uur] & \\
 & v^s \arrow[uuur] &
\end{tikzcd}$$

To a $\overline{\Phi}_I-$mesh one associates the $\overline{\Phi}_I-$mesh translation quiver $\Gamma(\mathcal{R}_{q_I}, \overline{\Phi}_I)$. See Simson (2010a) for a detailed description of mesh translation quivers.

3. Quadratic forms of graphs and their equivalences with Dynkin diagrams and with Euler diagrams

Let $Q = (Q_0, Q_1)$ be a finite graph, Q_0 its vertex set, and Q_1 its edge set. We briefly recall the definition of a quadratic form of a graph, introduced in Sim2. By a quadratic form of a graph Q we mean the form $q_Q : \mathbb{R}^{Q_0} \longrightarrow \mathbb{R}$ defined by the formula $q_Q(x) = \sum\limits_{i \in Q_0} x_i^2 - \sum\limits_{i-j} d_{ij}^Q x_i x_j$, for any $x = (x_j)_{j \in Q_0} \in \mathbb{R}^{Q_0}$, where $n = |Q_0|$, and $d_{ij}^Q = |Q_1(i,j)|$ counts the edges connecting i and j in Q, and the sum is taken over all unordered pairs $i-j$ of vertices $i, j \in Q_0$ connected with at least one edge. An important example of graphs is given by the Dynkin diagrams and the extended Dynkin diagrams (also known as Euclidean diagrams). A detailed study of these diagrams can be found in Simson (2010a,b). Description of the roots of Dynkin diagrams is known (see Polak and Simson, 2010).

We now recall the definition of $\mathbb{Z}-$bilinear Euler form Sim3, because we need it to exhibit the $\mathbb{Z}$-bilinear equivalence of forms. A form $\overline{b}_{q_I} : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$, defined by the formula $\overline{b}_{q_I}(x, y) = x \cdot C_I^{-1} y^{tr}$, where C_I is an incidence matrix of poset I, is called $\mathbb{Z}$-bilinear Euler form.

We focus on the Loupias-Zavadski-Shkabara posets, LZS in short, presented independently by Loupias (1975) and by Zavadski-Shkabara (1976). It is one of our main objectives in this article to show that for any subposet I of a poset from the LZS list (see Drozdowski and Simson, 1978), for which $\overline{q}_I$ is positive definite, there exists precisely one Dynkin diagram Δ, such that $\overline{F}(t)_I = \overline{F}_\Delta(t)$

and $\mathbb{Z}-$bilinear forms associated to this subposet I, and to Dynkin diagram Δ, are $\mathbb{Z}-$bilinearly equivalent. At the beginning of this section we recall the principal configuration of roots for an oriented Dynkin diagram (see Simson, 2010a). These configurations, which in essence are fragments of mesh translation quivers for oriented Dynkin diagrams and Euclidean diagrams, containing the basis vectors $e_1, \ldots, e_n$, are used in a proof of $\mathbb{Z}$-bilinear equivalence of forms. We consider Dynkin diagrams for a particular, fixed orientation. The reader may use another orientation, but one needs to keep in mind that in such case the principal configuration for Dynkin diagrams will be different. The $\mathbb{Z}$-bilinear equivalence of Euler forms for a subposet I of an LZS poset (precisely, a subposet for which $\overline{q}_I$ is positive definite), with a Dynkin diagram, when shown, delivers the full description of the set of roots $\mathcal{R}_{\overline{q}_I} = \{v \in \mathbb{Z}^n;\ \overline{q}_I(v) = 1\}$.

We are now in position to present the oriented Dynkin diagrams (Table 1) and to recall the principal configuration of roots for those diagrams (Table 2), with this particular orientation, which can be found in Simson (2010a). The mentioned principal configuration of roots will be used to prove $\mathbb{Z}$-bilinear equivalence between the forms of positive definite subposets of LZS posets and the forms associated to oriented Dynkin diagrams $\mathbb{D}_7$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.

Table 1. Canonically oriented Dynkin diagrams

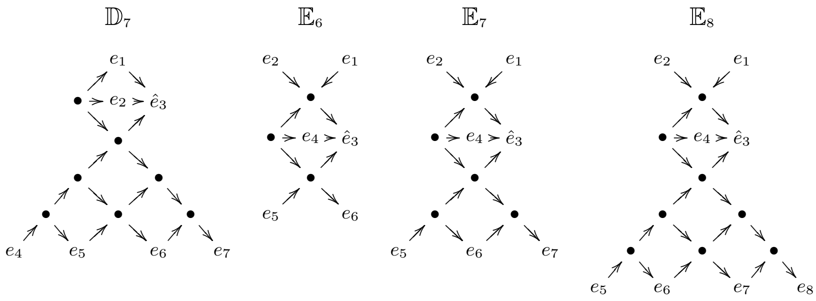
$$\mathbb{D}_7: \quad \begin{tikzcd} & 2 \arrow[d] \\ 1 \arrow[r] & 3 & 4 \arrow[l] & 5 \arrow[l] & 6 \arrow[l] & 7 \arrow[l] \end{tikzcd}$$

$$\mathbb{E}_6: \quad \begin{tikzcd} & & 4 \arrow[d] \\ 1 \arrow[r] & 2 \arrow[r] & 3 & 5 \arrow[l] & 6 \arrow[l] \end{tikzcd}$$

$$\mathbb{E}_7: \quad \begin{tikzcd} & & 4 \arrow[d] \\ 1 \arrow[r] & 2 \arrow[r] & 3 & 5 \arrow[l] & 6 \arrow[l] & 7 \arrow[l] \end{tikzcd}$$

$$\mathbb{E}_8: \quad \begin{tikzcd} & & 4 \arrow[d] \\ 1 \arrow[r] & 2 \arrow[r] & 3 & 5 \arrow[l] & 6 \arrow[l] & 7 \arrow[l] & 8 \arrow[l] \end{tikzcd}$$

Table 2. Principal configurations of roots for the Dynkin diagrams (Table 1)



We now collect a number of theorems (see Simson, 2004-2009, 2010) that we use to show $\mathbb{Z}$-bilinear equivalence of quadratic forms. Their proofs are left as easy exercises.

THEOREM 3.1 *Let* $b_A,\ b_{A'} : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$ *be forms defined by matrices* $A,\ A'$*, respectively. In addition, assume that determinants of these matrices are equal to 1 or -1. The following diagram:*

$$\begin{tikzcd} \mathbb{Z}^n \times \mathbb{Z}^n \arrow[d, "h_B \times h_B"', "\simeq"] & \mathbb{Z} \\ \mathbb{Z}^n \times \mathbb{Z}^n \arrow[ur, "b_A"'] & \end{tikzcd}, \qquad (1)$$

for $h_B(x) = x \cdot B$ *an isomorphism of the* $\mathbb{Z}^n$ *group, is commutative if and only if there exists a matrix* $B \in \mathbb{M}_n(\mathbb{Z})$ *satisfying* $\det B \in \{-1,\ 1\}$ *and* $A' = B \cdot A \cdot B^{tr}$.

THEOREM 3.2 *Let* $\Phi_A,\ \Phi_{A'} : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ *be Coxeter-Euler transformations, defined respectively for matrices* $A,\ A' \in \mathbb{M}_n(\mathbb{Z})$*, whose determinants are equal to 1 or -1. If the diagram* (1) *is commutative, then the following diagram is commutative as well:*

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, shift left, "\Phi_{A'}"] & \mathbb{Z}^n \arrow[l, shift left, "\Phi_{A'}^{-1}"] \\ \mathbb{Z}^n \arrow[u, "h_B", "\simeq"'] \arrow[r, shift left, "\Phi_A"] & \mathbb{Z}^n \arrow[u, "h_B", "\simeq"'] \arrow[l, shift left, "\Phi_A^{-1}"] \end{tikzcd}.$$

COROLLARY 3.1 *Let* $A,\ A' \in \mathbb{M}_n(\mathbb{Z})$ *be matrices defined as previously. If the diagram* (1) *is commutative, then*

(a) $Cox_{A'} = B^{-1} \cdot Cox_A \cdot B$,

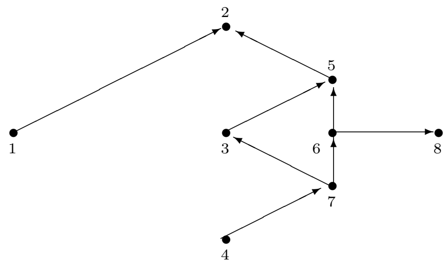
(b) $F_{\Phi_{A'}}(t) = F_{\Phi_A}(t)$,

(c) $\mathbf{c}_A = \mathbf{c}_{A'}$.

The next theorem is the result of the author's work on the roots of quadratic forms and constitutes one of the most important results contained in this paper.

Theorem 3.3 *Let $\mathbb{L}_9$ be the poset from the LZS list, presented in Fig. 9 in Drozdowski and Simson (1978, page 16), and I its subposet, visible in the following figure:*

I :



(a) $F_I(t) = F_{\mathbb{E}_8}(t)$,
(b) *There is a matrix $B = B_I \in \mathbb{M}_8(\mathbb{Z})$, for which det $B = 1$ or det $B = -1$ and the following diagram is commutative:*

$$\begin{tikzcd} \mathbb{Z}^8 \times \mathbb{Z}^8 \arrow[r, "b_{\mathbb{E}_8}"] & \mathbb{Z} \\ \mathbb{Z}^{|I|} \times \mathbb{Z}^{|I|} \arrow[u, "h_{B_I} \times h_{B_I}", "\cong"'] \arrow[ur, "b_I"'] & \end{tikzcd} , \qquad (2)$$

i.e., b_I is $\mathbb{Z}$-equivalent to $b_{\mathbb{E}_8}$.

One can easily show that $\overline{q}_I$ is positive definite. It suffices to apply the Sylvester criterion, described in Section 2. In order to show that the Dynkin diagram $\mathbb{E}_8$ satisfies the following conditions, we act according to the following scheme:

A scheme of dealing with the $\mathbb{E}_8$ Dynkin diagram

Stage 0° Take $n = |I|$.
Stage 1° Calculate the Coxeter polynomial $F_I(t) \in \mathbb{Z}[t]$.
Stage 2° Verify if the equality $F_I(t) = F_{\mathbb{E}_8}(t)$ holds; if so, go to 3°; otherwise stop.
Stage 3° Fix an orientation for the Dynkin diagram $\mathbb{E}_8$; in our examples we take the orientation shown in Table 1. In addition, list the principal configuration of roots (see Table 2) for chosen orientation.
Stage 4° Apply the mesh algorithm (described in Simson, 2010a) to build a fragment of $\overline{\Phi}_I$ - mesh translation quiver of roots $\Gamma(\mathcal{R}_{q_I}, \overline{\Phi}_I)$.
Stage 5° Look at the obtained fragment of a mesh translation quiver and find "‘hypothetical vectors"’ $e'_1 = h_B(e_1), \ldots, e'_n = h_B(e_n)$ of the principal configuration of roots, through a "‘hypothetical isomorphism"’ $h_B : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ in diagram (2). Write a matrix $B = [e'_1, \ldots, e'_n]^{tr} \in \mathbb{M}_n(\mathbb{Z})$.

Stage 6° Verify, if det $B \in \{1, -1\}$, and if the equality $C_{\mathbb{E}_8}^{-1} = B \cdot C_I^{-1} \cdot B^{tr}$ is satisfied, i.e., if diagram (2) is commutative (see Theorem 3.1).

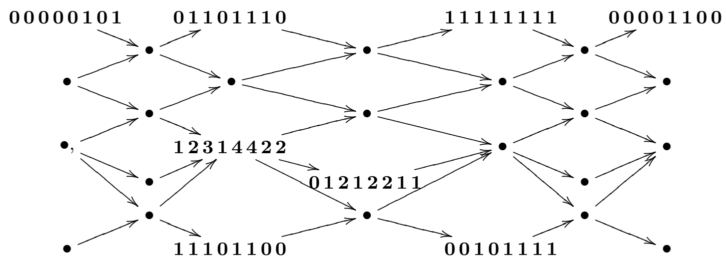
Proof. (a) and (b) We notice that Euler matrix C_I^{-1} for poset I has the following form:

$$C_I^{-1} = \begin{bmatrix} 1 & \hat{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \hat{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \hat{1} & 0 \\ 0 & \hat{1} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{1} & 1 & 0 & \hat{1} \\ 0 & 0 & \hat{1} & 0 & 1 & \hat{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In our example, $n = 8$ (stage 0°). We easily compute the Coxeter polynomial for I: $\overline{F}_I(t)$=det $(tE - \overline{Cox}_I)$=$t^8 + t^7 - t^5 - t^4 - t^3 + t + 1$. Coxeter polynomials for Dynkin diagrams are presented in Simson (2010b), including $\overline{F}_{\mathbb{E}_8}$. Thus, the equality

$$\overline{F}_I(t) = \overline{F}_{\mathbb{E}_8}(t) = t^8 + t^7 - t^5 - t^4 - t^3 + t + 1$$

holds (stage 2°). We fix an orientation for Dynkin diagram $\mathbb{E}_8$, as demonstrated in Table 1. A principal configuration of roots for a Dynkin diagram, oriented in this way, can be found in Table 2 (stage 3°). The following picture visualizes a fragment of a mesh translation quiver $\Gamma(\mathcal{R}_{q_I}, \overline{\Phi}_I)$ for poset I (stage 4°).



Hence,

$$h(e_1) = 11101100,\ h(e_2) = 00101111,\ h(\hat{e}_3) = \hat{1}\hat{2}\hat{3}\hat{1}\hat{4}\hat{4}\hat{2}\hat{2},\ h(e_4) = 01212211,$$
$$h(e_5) = 00001100,\ h(e_6) = 11111111,\ h(e_7) = 01101110,\ h(e_8) = 00000101.$$

Therefore, the quested matrix B, corresponding to above fragment of a $\overline{\Phi}_I$-mesh translation quiver, has the following form (see stage 5°),

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ \hat{1} & \hat{2} & \hat{3} & \hat{1} & \hat{4} & \hat{4} & \hat{2} & \hat{2} \\ 0 & 1 & 2 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

where $\hat{a} = -a$ for $0 < a \in \mathbb{N}$. We calculate A' as $A' = B \cdot C_I^{-1} \cdot B^{tr}$.

$$A' =$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ \hat{1} & \hat{2} & \hat{3} & \hat{1} & \hat{4} & \hat{4} & \hat{2} & \hat{2} \\ 0 & 1 & 2 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & \hat{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \hat{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \hat{1} & 0 \\ 0 & \hat{1} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{1} & 1 & 0 & \hat{1} \\ 0 & 0 & \hat{1} & 0 & 1 & \hat{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & \hat{1} & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & \hat{2} & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & \hat{3} & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & \hat{1} & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & \hat{4} & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & \hat{4} & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & \hat{2} & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & \hat{2} & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

By performing multiplication, we get

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{1} & 1 & \hat{1} & \hat{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \hat{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \hat{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \hat{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which is equal to $C_{\mathbb{E}_8}^{-1}$ for the $\mathbb{E}_8$ diagram (stage 6°), with the following orientation:

$$\begin{tikzcd}
 & & 4 & & & & \\
1 & 2 \arrow[l] & 3 \arrow[l] \arrow[u] \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8
\end{tikzcd}$$

4. Principal forms and their roots

In this section we are concerned about principal forms, defined in Simson (2010a). We are able to give a full algorithmic description of a set of roots for this kind of form. It is our main objective in this article to show that for any poset I from the LZS list (see Drozdowski and Simson, 1978), for which $\overline{q}_I$ is principal, there exists precisely one extended Dynkin diagram $\widetilde{\Delta}$, such that $\overline{F}(t)_I = \overline{F}_{\widetilde{\Delta}}(t)$ and $\mathbb{Z}-$bilinear forms associated to this poset I, and to extended Dynkin diagram $\widetilde{\Delta}$, are $\mathbb{Z}$-bilinearly equivalent. A proof of this fact is an unpublished result of the author. We focus on a few selected posets from the LZS list, but presented procedures and algorithms can be applied to any poset from this list (for which $\overline{q}_I$ is principal). This result is a part of a doctoral dissertation, written by the author. We prove that the Euler form associated to $\mathbb{L}_9$ poset (Fig. 9 in Drozdowski and Simson, 1978, page 16) is $\mathbb{Z}$-bilinearly equivalent to Euler form of oriented extended Dynkin diagram. These proofs of $\mathbb{Z}-$bilinear equivalence of Euler forms of selected principal posets from the LZS list, with Euler forms of oriented Euclidean diagrams, are author's own contribution to the research on quadratic diophantine equations.

Similarly to the previous section, we recall a principal configuration of roots for oriented extended Dynkin diagrams (also called Euclidean diagrams), see Simson (2010a). These configurations are used in a proof of Z-bilinear equivalence of forms.

We begin this section with a presentation of oriented extended Diagrams, together with a principal configuration of roots for this orientation. Then, we recall the definition of principal forms, for which we exhibit the mentioned $\mathbb{Z}-$bilinear equivalence. We also define a defect of Coxeter-Euler transformation (see Simson, 2010a, 2011), which is helpful for description of a full set of roots of a principal form.

The following table (Table 3) contains extended Dynkin diagrams $\widetilde{\mathbb{D}}_7$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, with fixed orientation. Principal configurations of roots with a negative defect (see Simson, 2010a, 2011) for extended Dynkin diagrams oriented this way are presented below (Table 4).

Table 3. Oriented extended Dynkin diagrams

$\widetilde{\mathbb{D}}_7$:

$$\begin{tikzcd} & 2 \arrow[d] & & & 8 \arrow[d] & \\ 1 \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] \arrow[r] & 7 \end{tikzcd}$$

$\widetilde{\mathbb{E}}_6$:

$$\begin{tikzcd} & & 4 \arrow[d] & & \\ & & 5 \arrow[d] & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 & 6 \arrow[l] & 7 \arrow[l] \end{tikzcd}$$

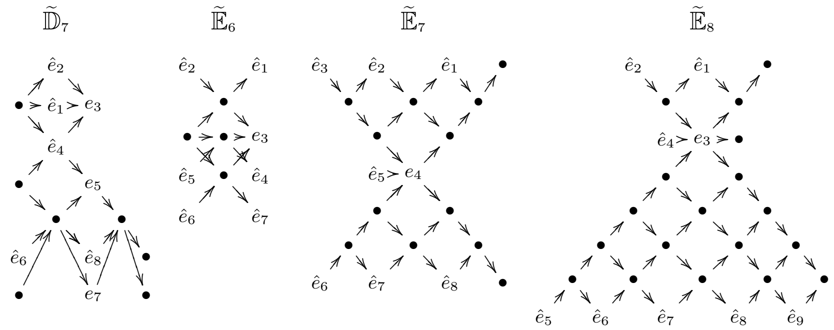
$\widetilde{\mathbb{E}}_7$:

$$\begin{tikzcd} & & & 5 \arrow[d] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 & 6 \arrow[l] & 7 \arrow[l] & 8 \arrow[l] \end{tikzcd}$$

$\widetilde{\mathbb{E}}_8$:

$$\begin{tikzcd} & & 4 \arrow[d] & & & & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 & 5 \arrow[l] & 6 \arrow[l] & 7 \arrow[l] & 8 \arrow[l] & 9 \arrow[l] \end{tikzcd}$$

Table 4. Principal configurations of roots for extended Dynkin diagrams



Definition 4.1 *Let I be a poset, $\overline{q}_I : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ an Euler form, defined by the formula $\overline{q}_I(x) = x \cdot C_I^{-1} x^{tr}$, and $G_{\overline{q}_I}$ the Gram matrix of this form.*

(a) *A form $\overline{q}_I : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ is called principal, if*

(a1) *$\overline{q}_I$ is positive semi-definite,*

(a2) *its kernel has the form $Ker\ \overline{q}_I = \mathbb{Z} \cdot \mathbf{h}_{q_I}$, where $0 \neq \mathbf{h}_{q_I} \in \mathbb{Z}^n$.*

(b) *A bilinear Euler form* $\overline{b}_{q_I} : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$ *is called principal, if* $\overline{q}_I : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ *results in* $\overline{q}_I(x) = \overline{b}_{q_I}(x, x)$ *being principal.*

We now present an algorithm verifying whether an Euler form C_I^{-1} of poset I, encoded with a matrix C_I^{-1}, is principal. This algorithm makes use of the Sylvester criterion, described in Theorem 2.1, and a function Kernel, computing the kernel. A function computing the kernel of Euler form (i.e., a set Ker $\overline{q}_I = \{v \in \mathbb{Z}^n;\ \overline{q}_I(v) = 0\}$), utilizes Lagrange algorithm (see Simson, 2010a, 2011), which reduces given form to a cannonical form (a sum of squares). With a cannonical form, computation of kernel reduces to solving a system of linear equations.

Algorithm 2 IsPrincipal(C^{-1})

Input: C^{-1} – an inverse of incidence matrix of poset.
Output: 1 - if $\overline{q}$ is principal; 0 - in the other case.

```
if SylvesterCriterion(C^{-1})=1 then
    return 0
end if
if GeneralizedSylvesterCriterion(C^{-1})=1 then
    h=Kernel(C^{-1})
    if nops(h)>1 or nops(h)=∅ then
        return 0
    end if
    return 1
else
    return 0
end if
```

The following theorem is useful for proving $\mathbb{Z}$-bilinear equivalences, see Simson (2010b) for the proof.

Theorem 4.1 *Let* I *be a poset, and* $C_I^{-1} \in \mathbb{M}_n(\mathbb{Z})$ *a nonsingular incidence matrix of* I*, satisfying det* $C_I^{-1} = 1$ *or det* $C_I^{-1} = -1$*. Let* $\overline{\Phi} : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ *be a Coxeter-Euler transformation defined by this matrix. If a bilinear form* $\overline{b}_I : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$*, defined with the formula* $\overline{b}_I(x, y) = x \cdot C_I^{-1} \cdot y^{tr}$*, is principal, and the kernel is of the form* $Ker\overline{q}_I = \mathbb{Z} \cdot \mathbf{h}$*, then there exists a natural number* $c \geq 1$ *and a group homomorphism* $\overline{\partial}_I : \mathbb{Z}^n \longrightarrow \mathbb{Z}$*, yielding* $\overline{\Phi}^c(x) = x + \overline{\partial}_I(x) \cdot \mathbf{h}$ *for any* $x \in \mathbb{Z}^n$*.*

Definition 4.2 *Minimal* c *from Theorem 4.1 will be called a reduced Coxeter number of* C_I^{-1}*, and denoted by* $\mathbf{\check{c}_I}$*.* $\overline{\partial}_I$ *will be called a defect of the Coxeter transformation* $\overline{\Phi}$ *or a defect of* C_I^{-1} *.*

Algorithm 3 determines a defect of the Coxeter transformation for an Euler form $\overline{q}_I$ encoded by C_I^{-1}. A reduced Coxeter number $\check{\mathbf{c}}_{\mathbf{I}}$ is also computed by this procedure.

Algorithm 3 CalculateDefect(C^{-1})

Input: C^{-1} – an inverse of incidence matrix of poset.
Output: A defect if it exists; 0 in the other case.

```
h ←Kernel(C^{-1})
if nops(h)>1 or nops(h)=∅ then
    return 0
end if
Cox1 ← −C^{-1} · C^{tr}; X ← [x[1], . . . , x[n]] // n is size of matrix C^{-1}
if IsPrincipal(C^{-1})=0 then
    return 0
end if
c ← 1; Cox ← Cox1
while true do
    T ← X · Cox − X; defect ← gcd(T[1, 1], . . . , T[1, n])
    if defect = 1 then
        c ← c + 1; Cox ← Cox · Cox1
    else
        for j = 1 to nops(defect) do
            h1[1, j] ← T[1,j] / defect
        end for
        if h1 = h or h1 = −h then
            return defect
        else
            c ← c + 1; Cox ← Cox · Cox1
        end if
    end if
end while
return 0
```

An algorithm computing, in particular, the defect of the Coxeter transformation and reduced Coxeter number, has been presented in Simson (2010a). Our procedure is simpler as it contains only the computations required in this paper. Nonetheless, it is based on the same ideas as the algorithm from Simson (2011). No better algorithms are known that would compute these invariants.

The next two theorems characterize the sets of roots for principal forms (see Simson, 2011). They have been proved in Simson (2004-2010).

THEOREM 4.2 *If Q is one of canonically oriented Euclidean diagrams $\widetilde{\mathbb{D}}_6$, $\widetilde{\mathbb{D}}_7$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, presented in Table 3, then*

(a) $\overline{q}_Q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ *is principal and* $Ker\, \overline{q}_Q = \mathbb{Z} \cdot \mathbf{h_Q}$, *where*

$$\mathbf{h_Q} = \begin{cases} (1\ 1 \ldots 1\ 1), & \text{when } Q = \widetilde{\mathbb{A}}_n, \\ (1\ 1\ 2 \ldots 2\ 1\ 1), & \text{when } Q = \widetilde{\mathbb{D}}_n, \quad n \geq 4, \\ (3\ 2\ 1\ 2\ 1\ 2\ 1), & \text{when } Q = \widetilde{\mathbb{E}}_6, \\ (4\ 3\ 2\ 1\ 2\ 3\ 2\ 1), & \text{when } Q = \widetilde{\mathbb{E}}_7, \\ (6\ 4\ 2\ 3\ 5\ 4\ 3\ 2\ 1), & \text{when } Q = \widetilde{\mathbb{E}}_8, \end{cases}$$

(b) $\Gamma(\mathcal{R}_{\overline{q}_{\mathbb{L}}} \cup Ker\, \overline{q}_{\mathbb{L}}, \overline{\Phi}_Q)$ *has a shape of a* $\overline{\Phi}_Q$*-mesh translation quiver and the following equality holds:*

$$\Gamma(\mathcal{R}_{\overline{q}_{\mathbb{L}}} \cup Ker\, \overline{q}_{\mathbb{L}}, \overline{\Phi}_Q) = \Gamma(\overline{\partial}_{\mathbb{L}}^{-} \mathcal{R}_{\overline{q}_{\mathbb{L}}}, \overline{\Phi}_Q) \cup \Gamma(\overline{\partial}_{\mathbb{L}}^{+} \mathcal{R}_{\overline{q}_{\mathbb{L}}}, \overline{\Phi}_Q) \cup \Gamma(\overline{\partial}_{\mathbb{L}}^{0} \cup Ker\, \overline{q}_{\mathbb{L}}, \overline{\Phi}_Q),$$

where $\Gamma(\overline{\partial}_{\mathbb{L}}^{-} \mathcal{R}_{\overline{q}_{\mathbb{L}}}, \overline{\Phi}_Q) = -\Gamma(\overline{\partial}_{\mathbb{L}}^{+} \mathcal{R}_{\overline{q}_{\mathbb{L}}}, \overline{\Phi}_Q)$, *and* $\Gamma(\overline{\partial}_{\mathbb{L}}^{0} \cup Ker\, \overline{q}_{\mathbb{L}}, \overline{\Phi}_Q)$ *is a sand-glass tube of rank*

$$m_Q = \begin{cases} (2, 2, n-2), & \text{when } Q = \widetilde{\mathbb{D}}_n, \\ (2, 3, 3), & \text{when } Q = \widetilde{\mathbb{E}}_6, \\ (2, 3, 4), & \text{when } Q = \widetilde{\mathbb{E}}_7, \\ (2, 3, 5), & \text{when } Q = \widetilde{\mathbb{E}}_8 \end{cases}.$$

An exhaustive study on sand-glass tubes can be found in Simson (2010a, 2011).

DEFINITION 4.3 *Let I be a poset, and $\overline{q}_I : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ a unit Euler form $x \cdot C_I^{-1} \cdot x^{tr}$. Assume that $\overline{q}_I$ is principal, its kernel has the form $Ker\, \overline{q}_I = \mathbb{Z} \cdot \mathbf{h}$, and j-th coordinate h_j of $\mathbf{h}$ is positive. By a root reducer of $\mathcal{R}_q$ modulo h_j we mean the set $\mathcal{R}_q^{(h_j)} = \{v \in \mathcal{R}_q;\ 0 \leq v_j \leq h_j - 1\}$.*

THEOREM 4.3 *Let I be a poset, $\overline{q}_I : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ a quadratic Euler form $\overline{q}_I(x) = x \cdot C_I^{-1} \cdot x^{tr}$, and det $C_I^{-1} \in \{1,\ -1\}$. In addition, assume that $\overline{q}_I$ is principal, its kernel has the form $Ker \overline{q}_I = \mathbb{Z} \cdot \mathbf{h}$ and j-th coordinate h_j of $\mathbf{h}$ is positive. Then, the following conditions are satisfied:*

(a) $\mathcal{R}_{\overline{q}} = \bigcup\limits_{m \in \mathbb{Z}} t_{\mathbf{h}}^m \mathcal{R}_{\overline{q}}^{(h_j)}$, *where $t_{\mathbf{h}}^m : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ is a translation defined by the formula $t_{\mathbf{h}}^m(v) = v + m \cdot \mathbf{h}$,*

(b) *the set $\mathcal{R}_{\overline{q}}^{(h_j)}$ is finite,*

(c) $\overline{\partial}_I^0 \mathcal{R}_{\overline{q}} = \bigcup\limits_{m \in \mathbb{Z}} t_{\mathbf{h}}^m \overline{\partial}_I^0 \mathcal{R}_{\overline{q}}^{(h_j)}$, *where $t_{\mathbf{h}}^m : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ is a translation defined by the formula $t_{\mathbf{h}}^m(v) = v + m \cdot \mathbf{h}$.*

We are ready to present the algorithm determining the roots of an Euler form, which is principal, and for which a defect of Coxeter-Euler transformation equals zero.

Theorem 4.3 asserts that this set is finite. Algorithm 4 makes use of the restrictively counting algorithm, described in Simson (2010a, 2011), which computes a list of roots for a positive definite form.

Algorithm 4 CalculateRootsWithZeroDefect(C^{-1})

Input: C^{-1} – an inverse of incidence matrix of poset.
Output: list of roots in $\partial^0\mathcal{R}_q$; 0 - if $\overline{q}$ is not principal or list of roots is empty.

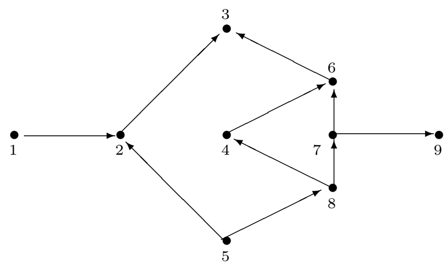
```
if GeneralizedSylvesterCriterion(C^{-1})=0 then
      return 0
end if
h ←Kernel (C^{-1})
if nops(h)>1 or nops(h)=∅ then
      return 0
end if
h ← h[1]
if first j such that h[j] ≠ 0 is h[j] < 0 then
      h ← −h
end if
h_j ← j where j is first index such that h[j] ≠ 0
create matrix OE with matrix C by inserting 0 in C[i, h_j] for i ≠ j
OE ← OE^{-1}
L ← RestrictivelyCountingAlgorithm(C^{-1})
if L = ∅ then
      return 0
end if
G ← []
for i = 1 to nops(L) do
      // algorithm 3 gives the defect, and here we evaluate its value for L[i]
      if defect(L[i])=0 then
            G ← G ∪ L[i]
      end if
end for
return G
```

No algorithms are known to compute the roots in a zero defect case that would be superior to this one. The notion of defect is quite new and has been introduced in Simson (2010a).

The next theorem is the result of author's own research on the roots of quadratic forms and constitutes one of the most important results of this paper. We show a $\mathbb{Z}$-bilinear equivalence between an Euler form of a poset I from the LZS list, and an Euler form, associated to one of extended Dynkin diagrams $\widetilde{\mathbb{D}}_6$, $\widetilde{\mathbb{D}}_7$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$.

Theorem 4.4 *Let $\mathbb{L}$ be the Loupias's poset $\mathbb{L}_9$, drawn in the following image:*

$\mathbb{L}_9$:



(a) $\overline{b}_{q_{\mathbb{L}}} : \mathbb{Z}^{|\mathbb{L}|} \times \mathbb{Z}^{|\mathbb{L}|} \longrightarrow \mathbb{Z}$ *is a principal form, the defect is of the form* $\overline{\partial}_{\mathbb{L}} = x_2 - x_3 + x_4 - x_5 + x_7$*, and the reduced Coxeter number* $\check{\mathbf{c}}_{\mathbb{L}} = 30$.

(b) *There exists precisely one extended (oriented) Dynkin diagram,* $\widetilde{\Delta}_{\mathbb{L}} = \widetilde{\mathbb{E}}_8$*, and there exists a matrix* $B = B_{\mathbb{L}} \in \mathbb{M}(\mathbb{Z})$*, with determinant equal to 1 or -1, such that*

$$F_{\mathbb{L}}(t) = F_{\widetilde{\Delta}_{\mathbb{L}}}(t),$$

and the following diagram is commutative:

$$\begin{tikzcd} \mathbb{Z}^{|\widetilde{\Delta}_{\mathbb{L}}|} \times \mathbb{Z}^{|\widetilde{\Delta}_{\mathbb{L}}|} \arrow[r, "b_{\widetilde{\Delta}_{\mathbb{L}}}"] & \mathbb{Z} \\ \mathbb{Z}^{|\mathbb{L}|} \times \mathbb{Z}^{|\mathbb{L}|} \arrow[u, "h_{B_{\mathbb{L}}} \times h_{B_{\mathbb{L}}}", "\cong"'] \arrow[ur, "b_{\mathbb{L}}"'] & \end{tikzcd} , \tag{3}$$

i.e., $b_{\mathbb{L}}$ *and* $b_{\Delta_{\mathbb{L}}}$ *are* $\mathbb{Z}$*-bilinearly equivalent.*

(c) $\overline{\Phi}_{\mathbb{L}}$*-orbits of the set* $\mathcal{R}_{q_{\mathbb{L}}} \cup$ *Ker* $q_{\mathbb{L}}$ *constitute a* $\Phi_{\mathbb{L}}$*-mesh translation quiver of the form*

$$\Gamma(\mathcal{R}_{q_{\mathbb{L}}} \cup Ker\, q_{\mathbb{L}}, \overline{\Phi}_{\mathbb{L}}) = \Gamma(\overline{\partial}_{\mathbb{L}}^{-} \mathcal{R}_{q_{\mathbb{L}}}, \overline{\Phi}_{\mathbb{L}}) \cup \Gamma(\overline{\partial}_{\mathbb{L}}^{+} \mathcal{R}_{q_{\mathbb{L}}}, \overline{\Phi}_{\mathbb{L}}) \cup \Gamma(\overline{\partial}_{\mathbb{L}}^{0} \cup Ker\, q_{\mathbb{L}}, \overline{\Phi}_{\mathbb{L}}),$$

where $\Gamma(\overline{\partial}_{\mathbb{L}}^{+} \mathcal{R}_{q_{\mathbb{L}}}, \overline{\Phi}_{\mathbb{L}}) = \Gamma(\overline{\partial}_{\mathbb{L}}^{-} \mathcal{R}_{q_{\mathbb{L}}}, \overline{\Phi}_{\mathbb{L}})$ *are connected infinite subquivers consisting of* $|\mathbb{L}|$ *orbits and* $\Gamma(\overline{\partial}_{\mathbb{L}}^{0} \cup Ker\ q_{\mathbb{L}}, \overline{\Phi}_{\mathbb{L}})$ *is a sand-glass tube of rank* $m_{\mathbb{L}}$*=(2,3,5) for* $\mathbb{L} = \mathbb{L}_9$.

In order to prove Theorem 4.4, we act according to the following scheme:

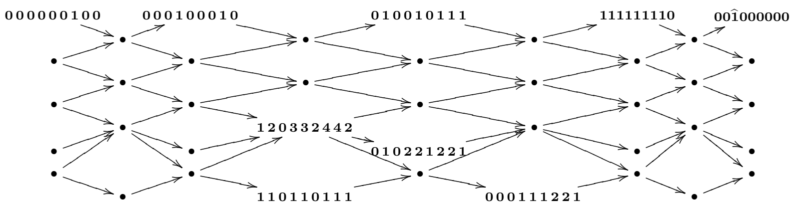
A scheme of dealing with extended Dynkin diagrams

An Euler form $\overline{q}_{\mathbb{L}}$ is given. We aim at finding an extended (oriented) Dynkin diagram $\Delta_{\mathbb{L}}$, such that a diagram from the above theorem is commutative.

Stage 0° Take $n = |\mathbb{L}|$.
Stage 1° Calculate the Coxeter polynomial $F_{\mathbb{L}}(t)$.
Stage 2° Let $\Delta_{\mathbb{L}}$ be an extended Dynkin diagram, for which Coxeter polynomials $F_{\mathbb{L}}(t)$ and $F_{\Delta_{\mathbb{L}}}(t)$ are identical.
Stage 3° Fix an orientation for $\Delta_{\mathbb{L}}$; in our examples we take the orientation shown in Table 1. In addition, list the principal configuration of roots (see Table 2) for chosen orientation.
Stage 4° Apply the mesh algorithm to describe a fragment of $\overline{\Phi}_{\mathbb{L}}$-mesh translation quiver $\Gamma(\mathcal{R}_{q_{\mathbb{L}}}, \overline{\Phi}_{\mathbb{L}})$ of roots.
Stage 5° Look at the obtained fragment of a mesh translation quiver and find 'hypothetical vectors' $e'_1 = h_B(e_1), \ldots, e'_n = h_B(e_n)$, corresponding to vectors $e_1, \ldots, e_n$ of the principal configuration of roots, through a 'hypothetical isomorphism' $h_B : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ in diagram (3). Write a matrix $B = [e'_1, \ldots, e'_n]^{tr} \in \mathbb{M}_n(\mathbb{Z})$.
Stage 6° Verify, if $\det B \in \{1, -1\}$, and if the equality $C^{-1}_{\widetilde{\mathbb{E}}_8} = B \cdot C^{-1}_{\mathbb{L}} \cdot B^{tr}$ is satisfied, i.e., if diagram from the above theorem is commutative.

Proof. (a) Use the IsPrincipal procedure to verify, if $\overline{q}_{\mathbb{L}} = \overline{q}_{\mathbb{L}_9}$ is a principal form. Hence, a defect and reduced Coxeter number exist, and we calculate them with the help of CalculateDefect. We obtain $\overline{\partial}_{\mathbb{L}} = x_2 - x_3 + x_4 - x_5 + x_7$ and $\check{\mathbf{c}}_{\mathbb{L}} = 30$.

(b) Let $\mathbb{L} = \mathbb{L}_9$. Notice that $\mathbb{L}_9$ and $\widetilde{\mathbb{E}}_8$ have the same Coxeter polynomial. Below we show a fragment of $\overline{\Phi}_{\mathbb{L}}$-mesh translation quiver $\Gamma(\partial^+_{\mathbb{L}}\mathcal{R}_{q_{\mathbb{L}}}, \overline{\Phi}_{\mathbb{L}})$, upon which we write down a matrix B.



Therefore,

$h(\hat{e}_1) = \hat{1}\hat{1}0\hat{1}\hat{1}0\hat{1}\hat{1}\hat{1}$, $h(\hat{e}_2) = 000\hat{1}\hat{1}\hat{1}\hat{2}\hat{2}\hat{1}$, $h(e_3) = 120332442$, $h(\hat{e}_4) = 0\hat{1}0\hat{2}\hat{2}\hat{1}\hat{2}\hat{2}\hat{1}$, $h(\hat{e}_5) = 001000000$, $h(\hat{e}_6) = \hat{1}\hat{1}\hat{1}\hat{1}\hat{1}\hat{1}\hat{1}\hat{1}0$, $h(\hat{e}_7) = 0\hat{1}00\hat{1}0\hat{1}\hat{1}\hat{1}$, $h(\hat{e}_8) = 000\hat{1}000\hat{1}0$, $h(\hat{e}_9) = 000000\hat{1}00$.

The B matrix, obtained in stage 5°, has the following form:

$$B = \begin{bmatrix} \hat{1} & \hat{1} & 0 & \hat{1} & \hat{1} & 0 & \hat{1} & \hat{1} & \hat{1} \\ 0 & 0 & 0 & \hat{1} & \hat{1} & \hat{1} & \hat{2} & \hat{2} & \hat{1} \\ 1 & 2 & 0 & 3 & 3 & 2 & 4 & 4 & 2 \\ 0 & \hat{1} & 0 & \hat{2} & \hat{2} & \hat{1} & \hat{2} & \hat{2} & \hat{1} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{1} & \hat{1} & \hat{1} & \hat{1} & \hat{1} & \hat{1} & \hat{1} & \hat{1} & 0 \\ 0 & \hat{1} & 0 & 0 & \hat{1} & 0 & \hat{1} & \hat{1} & \hat{1} \\ 0 & 0 & 0 & \hat{1} & 0 & 0 & 0 & \hat{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{1} & 0 & 0 \end{bmatrix},$$

and its determinant equals -1 and $C_{\mathbb{L}}^{-1} = \begin{bmatrix} 1 & \hat{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \hat{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \hat{1} & 0 & 0 & 0 \\ 0 & \hat{1} & 1 & 0 & 1 & 0 & 0 & \hat{1} & 0 \\ 0 & 0 & \hat{1} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{1} & 1 & 0 & \hat{1} \\ 0 & 0 & 0 & \hat{1} & 0 & 1 & \hat{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

The $B \cdot C_{\mathbb{L}}^{-1} \cdot B^{tr} = A'$, where

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{1} & 1 & \hat{1} & \hat{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \hat{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \hat{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \hat{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \hat{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and is equal to $C_{\widetilde{\mathbb{E}}_8}^{-1}$ matrix for $\widetilde{\mathbb{E}}_8$ diagram, oriented in the following way:

$$\begin{tikzcd} & & 4 & & & & & \\ 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[u] \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 \arrow[r] & 9 \end{tikzcd}$$

(c) The roots of the mesh translation quiver $\Gamma(\overline{\partial}_{\mathbb{L}}^{+}\mathcal{R}_{q_{\mathbb{L}}}, \overline{\Phi}_{\mathbb{L}})$ for $\mathbb{L} = \mathbb{L}_{(9)}$ have been described in (b) of the above theorem. We notice that

$$\Gamma(\overline{\partial}_{\mathbb{L}}^{-}\mathcal{R}_{q_{\mathbb{L}}}, \overline{\Phi}_{\mathbb{L}}) = -\Gamma(\overline{\partial}_{\mathbb{L}}^{+}\mathcal{R}_{q_{\mathbb{L}}}, \overline{\Phi}_{\mathbb{L}}).$$

Hence, it suffices to describe the roots for the mesh translation quiver $\Gamma(\overline{\partial}_{\mathbb{L}}^{0} \cup \operatorname{Ker} q_{\mathbb{L}}, \overline{\Phi}_{\mathbb{L}})$. We know from (a) and (b) that an Euler form is $\mathbb{Z}$-bilinearly equivalent to the form of an extended Dynkin diagram $\widetilde{\widetilde{\mathbb{E}}}_8$, and that the defect for $\mathbb{L} = \mathbb{L}_9$ equals $\overline{\partial}_{\mathbb{L}}(x) = x_2 - x_3 + x_4 - x_5 + x_7$. The kernel Ker $q_{\mathbb{L}}$ is of the form Ker $q_{\mathbb{L}} = \mathbb{Z} \cdot \mathbf{h}$, for $\mathbf{h} = 0010\hat{1}10\hat{1}0$. Theorem 4.3 implies that in order to determine all roots with zero defect $\overline{\partial}^0_{\mathbb{L}}\mathcal{R}_{q_{\mathbb{L}}}$, it suffices to find a defect and the roots with zero defect for a root reducer $\overline{\partial}^0_{\mathbb{L}}\mathcal{R}^{(h_j)}_{q_{\mathbb{L}}}$. For the $\mathbb{L}_9$ we take $j = 3$. The roots for a root reducer $\mathcal{R}^{(h_3=1)}_{q_{\mathbb{L}}}$ with zero defect can be calculated by CalculateRootsWithZeroDefect. We align these roots in root orbits so as to obtain the following fourteen orbits (Table 5):

Table 5. Root orbits for $\mathbb{L}_9$

I	II	III	IV	V	VI	VII
001101000	$00\hat{1}\hat{1}0\hat{1}000$	000000001	001001101	$00\hat{1}00\hat{1}\hat{1}00$	$00\hat{1}00\hat{1}\hat{1}0\hat{1}$	$0010\hat{1}0000$
$000\hat{1}\hat{1}00\hat{1}0$	000110010	001001100	$0010\hat{1}10\hat{1}\hat{1}$	000010111	$00\hat{1}01\hat{1}011$	011001000
		$0000\hat{1}0\hat{1}\hat{1}\hat{1}$	$0000\hat{1}0\hat{1}\hat{1}0$	$00000000\hat{1}$	000010110	$1110010\hat{1}0$
						$0\hat{1}00\hat{1}10\hat{1}0$
						$\hat{1}\hat{1}00\hat{1}00\hat{1}0$

$VIII$	IX	X	XI	XII	$XIII$	XIV
011000000	$0010\hat{1}00\hat{1}0$	111000000	000000010	010010010	110010010	$11001\hat{1}010$
111001000	$0010\hat{1}1000$	000001000	010010000	110010000	$00\hat{1}010000$	$00\hat{1}010010$
$0000010\hat{1}0$	$0110010\hat{1}0$	$0000000\hat{1}0$	100000000	$0\hat{1}\hat{1}000000$	$0\hat{1}\hat{1}00\hat{1}000$	$00\hat{1}01\hat{1}000$
$0\hat{1}00\hat{1}00\hat{1}0$	$1010\hat{1}10\hat{1}0$	$0\hat{1}00\hat{1}0000$	$\hat{1}\hat{1}\hat{1}000000$	$\hat{1}\hat{1}\hat{1}00\hat{1}000$	$\hat{1}\hat{1}\hat{1}00\hat{1}010$	$0\hat{1}\hat{1}00\hat{1}010$
$\hat{1}\hat{1}00\hat{1}0000$	$\hat{1}\hat{1}00\hat{1}10\hat{1}0$	$\hat{1}00000000$	$00000\hat{1}000$	$00000\hat{1}010$	$01001\hat{1}010$	$\hat{1}0\hat{1}01\hat{1}010$

After aligning the roots in $\overline{\Phi}_{\mathbb{L}}-$orbits, we obtain two tubes of rank 2, four tubes of rank 3, and eight tubes of rank 5. The following image contains the "crown" of one of rank 2 tubes:

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[dr] & \\
001101000 \arrow[ur] \arrow[dr] & & 000\hat{1}\hat{1}00\hat{1}0 \arrow[ur] \arrow[dr] & & 001101000 \arrow[ur] \arrow[dr] & & 00\hat{1}\hat{1}00\hat{1}0 \arrow[ur] \arrow[dr] & & 01101000. \\
 & \mathbf{h} \arrow[ur] & & \mathbf{h} \arrow[ur] & & \mathbf{h} \arrow[ur] & & \mathbf{h} \arrow[ur] &
\end{tikzcd}$$

It remains to verify that vectors in tubes, constructed with mesh algorithm (see Simson, 2010a), are in $\overline{\partial}^0_{\mathbb{L}}\mathcal{R}_{q_{\mathbb{L}}}$. It suffices to notice that for a rank 2 tube, and $v^{(1)}$=01101000 $\in \overline{\partial}^0_{\mathbb{L}}\mathcal{R}_{q_{\mathbb{L}}}$, the following formula holds:

$$v^{(m)} = \begin{cases} -v^{(1)} + m \cdot \mathbf{h}, & \text{for} \quad m \quad \text{even}, \\ v^{(1)} + (m-1) \cdot \mathbf{h}, & \text{for} \quad m \quad \text{odd}. \end{cases}$$

Similar formulas can be derived for higher rank tubes. ∎

5. Concluding remarks

A number of algorithms useful for computing the roots of integral quadratic forms have been described in this paper. Beside the generalized Sylvester algorithm, which results from the well known Sylvester criterion, the presented algorithms are novel. Described algorithms are based on methods covered in detail in Simson (2010a).

One of the problems stated in Simson (2011) asked for a method to exhibit $\mathbb{Z}$-bilinear equivalence between posets from the lists of Loupias (1975) and Zavadskij-Shkabara (1976), and extended Dynkin diagrams. A scheme of dealing with these cases has been presented in this article. It has been shown that for the $\mathbb{L}_9$ poset from Loupias (1975) and Zavadskij-Shkabara (1976) lists there exists precisely one Dynkin diagram (in this case, $\Delta = \widetilde{\mathbb{E}}_8$), such that $cox_I = cox_\Delta$ and $\overline{b}_I$ is Z-bilinearly equivalent to $\overline{b}_\Delta$. Nevertheless, presented schemes are universal and can be successfully applied to show Z-bilinear equivalence for the remaining posets from the above list.

Appendix

The algorithms presented in this paper utilize the capabilities built into the MAPLE symbolic computations package. We now remind the purpose of functions used in our listings:

- choose(n) — gets a list of subsets of n-element set,
- coldim(C) — returns the number of columns of supplied matrix,
- gcd($a_1, a_2, \ldots, a_n$) — computes the greatest common divisor of numbers $a_1, \ldots, a_n$,
- nops(h) — takes a list and returns its size,
- submatrix(C,[1,2,3],[2,3,4]) — gets a submatrix of C, containing rows, whose indices are given as the second parameter ([1,2,3]), and whose columns are given as the third parameter ([2,3,4]).

References

Adleman, L. and Manders, K. (1976) Diophantine complexity, *Proc. 17th IEEE Symposium on Foundations of Computer Science*, Proc. IEEE, New York, 81-88.

Barot, M. and de la Pena, J.A. (1999) The Dynkin type of a non-negative unit form, *Expo. Math.*, **17**, 339-348

Buchmann, J. and Vollmer, U. (2007) *Binary Quadratic Forms: An Algorithmic Approach (Algorithms and Computation in Mathematics)*. Springer, Berlin-Heidelberg.

Drozdowski, G. and Simson, D. (1978) Remarks on posets of finite representation type, *http : //www − users.mat.umk.pl/ simson/Drozdowski-Simson1978.pdf*. Wydział Matematyki i Informatyki UMK, Toruń.

LOUPIAS, M. (1975) Indecomposable representations of finite partially ordered sets. *Lecture Notes in Math.*, **488**, Springer-Verlag, Berlin-Heidelberg-New York, 201-209.

MARCZAK, M., POLAK, A. and SIMSON, D. (2010) P-critical integral quadratic forms and positive unit forms: An algorithmic approach. *Linear Algebra and its Applications*, **433**, 1873-1888.

MATIYASEVICH, Y. (1970) Enumerable sets are Diophantine. *Doklady Akademii Nauk SSSR*, 191, English translation in *Soviet Mathematics Doklady*, **11** (2), 279-282.

POLAK, A. and SIMSON, D. (2010) Peak P-critical Tits forms of one-peak posets, preprint.

RINGEL, C.M. (1984) Tame Algebras and Integral Quadratic Forms, *Lecture Notes in Math.*, **1099**, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo.

SIMSON, D. (1992) *Linear Representations of Partially Ordered Sets and Vectors Space Categories. Algebra, Logic and Applications*, **4**. Gordon & Breach Science Publishers, Switzerland.

SIMSON, D. (1992) *Diagramy Coxetera-Dynkina i problemy macierzowe (Coxeter-Dynkin diagrams and matrix problems; in Polish).* Instytut Matematyki UMK, Toruń.

SIMSON, D. (2004-2009) *Pierwiastki funkcjonałów kwadratowych, diagramy Dynkina i zbiory częściowo uporządkowane (Roots of square functionals, Dynkin diagrams and partially ordered sets; in Polish).* Wyklad monograficzny, Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Toruń.

SIMSON, D. (2010a) Mesh geometries of root orbits of integraf quadratic forms. *J. Pure Appl. Algebra*, **215**, 13-34 .

SIMSON, D. (2010b) Integral bilinear forms, Coxeter transformations and Coxeter polynomials of finite posets. *Linear Algebra and Appl.*, **433**, 699-717.

SIMSON, D. (2011) Mesh algorithms for solving principal diophantine equations, sand-glass tubes and tori of roots. *Fundamenta Informaticae*, **109**, 425-462.

ZAVADSKIJ, A.G. and SHKABARA, A.S. (1976) *Commutative quivers and matrix algebras of finite type.* Preprint IM-76-3, Institute of Mathematics AN USSR, Kiev (in Russian).