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Singular fractional discrete-time linear systems*

by

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Abstract: A new class of singular fractional linear discrete-time systems is introduced. Using the Weierstrass regular pencil decomposition the solution to the state equation of singular fractional linear systems is derived. The considerations are illustrated by numerical examples.

Keywords: decomposition, singular, fractional, linear system, solution.

1. Introduction

Singular (descriptor) linear systems have been addressed in many papers and books (Dodig and Stosic, 2009; Dai, 1989; Fahmy and O'Reill, 1989; Kaczorek, 1992, 2004, 2007a, 2011; Kucera and Zagalak, 1988; Van Dooren, 1979. The eigenvalue and invariant assignment by state and output feedbacks have been investigated in Dodig and Stosic (2009), Dai (1989), Fahmy and O'Reill (1989) and Kaczorek (1992, 2004), while the realization problem for singular positive continuous-time systems with delays in Kaczorek (2007b). The computation of Kronecker's canonical form of a singular pencil has been analyzed in Van Dooren (1979).

Fractional positive continuous-time linear systems have been addressed in Kaczorek (2008) and Podlubny (1999), and positive linear systems with different fractional orders in Kaczorek (2010b). An analysis of fractional linear electrical circuits has been presented in Kaczorek (2010a), and some selected problems in theory of fractional linear systems were treated in the monograph Kaczorek (2011a). A new class of singular fractional linear continuous-time systems and singular electrical circuits has been addressed in Kaczorek (2011b).

In this paper a new class of singular fractional linear discrete-time systems will be introduced and their state equations solution will be derived.

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The paper is organized as follows. In Section 2 the fractional singular discrete-time linear systems are introduced and Weierstrass regular pencil decomposition is recalled. The solution of the state equation of singular fractional linear discrete-time system is derived using the Weierstrass pencil decomposition in Section 3. Illustrating numerical examples are given in Section 4 and concluding remarks in Section 5.

To the best of the author's knowledge the singular fractional linear discrete-time systems have not been considered yet.

Following notation will be throughout in the paper: the set of $n \times m$ real matrices will be denoted $\Re^{n\times m}$ and $\Re^n := \Re^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n \times n$ identity matrix by I_n.

2. Preliminaries

Consider the singular fractional discrete-time linear system described by the state equation

$$E\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ = \{0, 1, ...\} \tag{2.1}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$ are the state and input vectors, $A \in \Re^{n\times n}$, $E \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, and the fractional difference of the order α is defined by

$$\Delta^\alpha x_i = \sum_{k=0}^{i} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} x_{i-k}, \quad 0 < \alpha < 1 \tag{2.2}$$

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \begin{cases} 1 & \text{for} \quad k = 0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} & \text{for} \quad k = 1, 2, ... \end{cases} \tag{2.3}$$

It is assumed that

$$\det E = 0 \tag{2.4a}$$

and

$$\det[Ez - A] \neq 0 \tag{2.4b}$$

for some $z \in C$ (the field of complex numbers).

LEMMA 1 *(Gantmacher, 1960, p.92) If (2.4) holds then there exist nonsingular matrices $P, Q \in \Re^{n\times n}$ such that*

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{2.5}$$

where $N \in \Re^{n_2\times n_2}$ is a nilpotent matrix with the index μ (i.e. $N^\mu = 0$ and $N^{\mu-1} \neq 0$), $A_1 \in \Re^{n_1\times n_1}$, n_1 is equal to degree of the polynomial

$$\det[Es - A] = a_{n_1} z^{n_1} + ... + a_1 z + a_0 \tag{2.6}$$

and $n_1 + n_2 = n$.

A method for computation of matrices P and Q was given in Van Dooren (1979).

Using Lemma 2.1 we shall derive the solution x_i to the equation (2.1) for given initial conditions x_0 and input vector u_i, $i \in Z_+$.

3. Solution of the singular fractional linear systems

Premultiplying the equation (2.1) by the matrix $P \in \Re^{n\times n}$ and introducing the new state vector

$$\bar{x}_i = \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix} = Q^{-1}x_i, \quad \bar{x}_i^{(1)} \in \Re^{n_1}, \quad \bar{x}_i^{(2)} \in \Re^{n_2}, \quad i \in Z_+ \tag{3.1}$$

we obtain

$$PEQQ^{-1}\Delta^\alpha x_{i+1} = PEQ\Delta^\alpha Q^{-1}x_{i+1} = PAQQ^{-1}x_i + PBu_i \tag{3.2}$$

and after using (2.5) and (3.1)

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \Delta^\alpha \begin{bmatrix} \bar{x}_{i+1}^{(1)} \\ \bar{x}_{i+1}^{(2)} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i, i \in Z_+ \tag{3.3}$$

where

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB, \quad B_1 \in \Re^{n_1\times m}, \quad B_2 \in \Re^{n_2\times m} \tag{3.4}$$

Taking into account (2.2), from (3.3) we obtain

$$\begin{aligned} \bar{x}_{i+1}^{(1)} &= -\sum_{k=1}^{i+1}(-1)^k \binom{\alpha}{k} \bar{x}_{i-k+1}^{(1)} + A_1\bar{x}_i^{(1)} + B_1u_i \\ &= A_{1\alpha}\bar{x}_i^{(1)} + \sum_{k=2}^{i+1}(-1)^{k-1}\binom{\alpha}{k}\bar{x}_{i-k+1}^{(1)} + B_1u_i \end{aligned} \tag{3.5}$$

and

$$N\left[\bar{x}_{i+1}^{(2)} + \sum_{k=1}^{i+1}(-1)^k \binom{\alpha}{k} \bar{x}_{i-k+1}^{(2)}\right] = \bar{x}_i^{(2)} + B_2u_i \tag{3.6}$$

where $A_{1\alpha} = A_1 + I_{n_1}\alpha$.

The solution $\bar{x}_i^1$ to equation (3.5) is well known, Kaczorek (2008, 2011a), and it is given by the following theorem.

THEOREM 3.1 *The solution $\bar{x}_i^1$ of the equation* (3.5) *is given by the formula*

$$\bar{x}_i^{(1)} = \Phi_i \bar{x}_0^{(1)} + \sum_{k=0}^{i-1} \Phi_{i-k-1} B_1 u_k, \quad i \in Z_+ \tag{3.7}$$

where the matrices Φ_i are determined by the equation

$$\Phi_{i+1} = \Phi_i A_{1\alpha} + \sum_{k=2}^{i+1} (-1)^{k-1} \binom{\alpha}{k} \Phi_{i-k+1}, \quad \Phi_0 = I_{n_1}. \tag{3.8}$$

To find the solution $\bar{x}_i^2$ of the equation (3.6) for $N \neq 0$ it is assumed that

$$N = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \in \Re^{n_2}. \tag{3.9}$$

For (3.9) the equation (3.6) can be written in the form

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \left(\sum_{j=0}^{i+1} (-1)^j \binom{\alpha}{j} \begin{bmatrix} \bar{x}_{i-j+1}^{(21)} \\ \bar{x}_{i-j+1}^{(22)} \\ \vdots \\ \bar{x}_{i-j+1}^{(2,n_2)} \end{bmatrix} \right) = \begin{bmatrix} \bar{x}_i^{(21)} \\ \bar{x}_i^{(22)} \\ \vdots \\ \bar{x}_i^{(2,n_2)} \end{bmatrix} + \begin{bmatrix} B_{21} \\ B_{22} \\ \vdots \\ B_{2,n_2} \end{bmatrix} u_i, \ i \in Z_+ \tag{3.10}$$

From (3.10) we have

$$\begin{aligned}
\bar{x}_i^{(21)} &= -B_{21} u_i \\
\bar{x}_i^{(22)} &= \sum_{j=0}^{i+1} (-1)^j \binom{\alpha}{j} \bar{x}_{i-j+1}^{(21)} - B_{22} u_i = -\sum_{j=0}^{i+1} (-1)^j \binom{\alpha}{j} B_{21} u_{i-j+1} - B_{22} u_i \\
\bar{x}_i^{(23)} &= \sum_{j=0}^{i+1} (-1)^j \binom{\alpha}{j} \bar{x}_{i-j+1}^{(22)} - B_{23} u_i \\
&= -\sum_{j=0}^{i+1} (-1)^j \binom{\alpha}{j} \sum_{k=0}^{i-j+2} (-1)^k \binom{\alpha}{k} B_{21} u_{i-j-k+2} \\
&\quad - \sum_{j=0}^{i+1} (-1)^j \binom{\alpha}{j} B_{22} u_{i-j+1} - B_{23} u_i
\end{aligned}$$

$$\vdots$$

$$\bar{x}_i^{(2,n_2)} = \sum_{j=0}^{i+1} (-1)^j \binom{\alpha}{j} \bar{x}_{i-j+1}^{(2,n_2-1)} - B_{2,n_2} u_i \tag{3.11}$$

If $N = 0$ then from (3.6) we have

$$\bar{x}_i^{(2)} = -B_2 u_i, \quad i \in Z_+ \tag{3.12}$$

This approach can be easily extended for

$$N = blockdiag\,[\, N_1 \ N_2 \ \dots \ N_h \,] \tag{3.13}$$

where $N_k \in \Re^{n_k}$ has the form (3.9) and $\sum_{k=1}^{h} n_k = n_2$.

If the matrix N has the form

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \Re^{n_2} \tag{3.9'}$$

the considerations are similar (dual).

Note that the matrices (3.9) and (3.9') are related by $N = S\bar{N}S$ where

$$S = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Knowing $\bar{x}_i^1$ and $\bar{x}_i^2$ we can find the desired solution of the equation (2.1) from (3.1)

$$x_i = Q \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix}, i \in Z_+. \tag{3.14}$$

4. Examples

Example 4.1 Find the solution x_i of the singular fractional linear system (2.1) with the matrices

$$E = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 4 & 2 \\ 1 & 4 & 1 \end{bmatrix}, \ A = \begin{bmatrix} 0.8 & 1.7 & 2.8 \\ 0.4 & 0.8 & 1.4 \\ 2.2 & 4.6 & 2.2 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{4.1}$$

for $\alpha = 0.5$, $u_i = u$, $i \in Z_+$ and $x_0 = [\,1 \ 2 \ -1\,]^T$ (T denotes the transpose).

It is easy to check that the matrices (4.1) satisfy the assumptions (2.4). In this case the matrices P and Q have the forms

$$P = \frac{1}{11}\begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & 1 \\ 4 & 3 & -2 \end{bmatrix}, \; Q = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2}$$

and

$$\begin{aligned} &\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} = PEQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} = PAQ = \begin{bmatrix} 0.1 & 1 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ &PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -4 \\ -3 \\ 6 \end{bmatrix}, \quad A_{1\alpha} = A_1 + I_{n_1}\alpha = \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix}, \\ &(n_1 = 2, n_2 = 1) \end{aligned} \tag{4.3}$$

Equations (3.5) and (3.6) have the forms

$$\bar{x}_{i+1}^{(1)} = \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix} \bar{x}_i^{(1)} + \sum_{k=2}^{i+1} (-1)^{k-1} \binom{0.5}{k} \bar{x}_{i-k+1}^{(1)} - \frac{1}{11}\begin{bmatrix} 4 \\ 3 \end{bmatrix} u_i, \quad i \in Z_+ \tag{4.4}$$

and

$$\bar{x}_i^{(2)} = -B_2 u_i = -\frac{6}{11} u_i, i \in Z_+ \tag{4.5}$$

Solution $\bar{x}_i^1$ of equation (4.4) has the form

$$\bar{x}_i^{(1)} = \Phi_i \bar{x}_0^{(1)} + \sum_{k=0}^{i-1} \Phi_{i-k-1} B_1 u_k, i \in Z_+ \tag{4.6}$$

where

$$\begin{aligned} &\Phi_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Phi_1 = A_{1\alpha} = \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix}, \\ &\Phi_2 = A_{1\alpha}^2 - I_{n_1}\frac{\alpha(\alpha-1)}{2!} = \begin{bmatrix} 0.485 & 1.300 \\ 0 & 0.615 \end{bmatrix}, \ldots \end{aligned} \tag{4.7}$$

and

$$\bar{x}_0 = Q^{-1} x_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \quad \bar{x}_0^{(1)} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \bar{x}_0^{(2)} = [-1]. \tag{4.8}$$

The desired solution of the singular fractional system with (4.1) is given by

$$x_i = Q\bar{x}_i = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix} \tag{4.9}$$

where $\bar{x}_i^1$ and $\bar{x}_i^2$ are determined by (3.7) and (4.5), respectively.

EXAMPLE 4.2 Find the solution x_i of the singular fractional linear system (2.1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \ A = \begin{bmatrix} 0.2 & 2 & -2 \\ 2 & 1 & 0 \\ -1.8 & 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix} \tag{4.10}$$

for $\alpha = 0.8$, arbitrary u_i, $i \in Z_+$ and $x_0 = [\,1\ 1\ 1\,]^T$.

It is easy to check that the matrices (4.10) satisfy the assumptions (2.4). In this case the matrices P and Q have the forms

$$P = \begin{bmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}, \ Q = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \tag{4.11}$$

and

$$\begin{gathered} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} = PEQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} = PAQ = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}, \ A_{1\alpha} = A_1 + I_{n_1}\alpha = [1], \ (n_1 = 1, n_2 = 2) \end{gathered} \tag{4.12}$$

In this case equations (3.5) and (3.6) have the forms

$$\bar{x}_{i+1}^{(1)} = \bar{x}_i^{(1)} + \sum_{k=2}^{i+1} (-1)^{k-1} \binom{0.8}{k} \bar{x}_{i-k+1}^{(1)} + [\,1\ 0\,]u_i, \quad i \in Z_+ \tag{4.13}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \left(\sum_{j=0}^{i+1} (-1)^j \binom{0.8}{j} \begin{bmatrix} \bar{x}_{i-j+1}^{(21)} \\ \bar{x}_{i-j+1}^{(22)} \end{bmatrix} \right) = \begin{bmatrix} \bar{x}_i^{(21)} \\ \bar{x}_i^{(22)} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} u_i, \quad i \in Z_+ \tag{4.14}$$

and

$$\bar{x}_0 = Q^{-1}x_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \bar{x}_0^{(1)} = [1], \quad \bar{x}_0^{(2)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}. \tag{4.15}$$

The solution $\bar{x}_i^1$ of the equation (4.13) with $\bar{x}_0^1 = 1$ can be easily found using (3.7) and (3.8).

From (4.14) we have

$$
\begin{aligned}
\bar{x}_i^{(21)} &= [\,0\ -1\,]u_i, \quad i \in Z_+ \\
\bar{x}_i^{(22)} &= \sum_{j=0}^{i+1} (-1)^j \binom{0.8}{j} [\,0\ -1\,]u_{i-j+1} + [\,1\ -1\,]u_i, \quad i \in Z_+
\end{aligned}
\tag{4.16}
$$

The desired solution of the singular fractional system with (4.10) is given by

$$
x_i = Q\bar{x}_i = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(21)} \\ \bar{x}_i^{(22)} \end{bmatrix}
\tag{4.17}
$$

where $\bar{x}_i^1$, $\bar{x}_i^{21}$ and $\bar{x}_i^{22}$ are determined by (4.13) and (4.2), respectively.

5. Concluding remarks

The singular fractional linear discrete-time systems have been introduced. Using the Weierstrass regular pencil decomposition the solution to the state equation of singular fractional linear discrete-time system has been derived. The method of finding the solution to the singular fractional systems has been illustrated by two examples. These considerations can be extended to singular fractional linear discrete-time systems with singular pencils. Open problems are constituted by the extensions of these considerations to positive singular fractional linear systems and to singular positive linear systems with different fractional orders, Kaczorek (2010b, 2011b).

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