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Basic properties of a new class of parametric intuitionistic fuzzy implications*

by

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Abstract: In this paper a new class of intuitionistic fuzzy implications is introduced. Fulfillment of some axioms and properties together with Modus Ponens and Modus Tollens inference rules is investigated. Negation induced by implication is presented.

Keywords: parametric fuzzy implication, intuitionistic fuzzy logic.

1. Introduction

When in 1965 Lotfi A. Zadeh introduced the concept of fuzzy set (FS) there were not too many enthusiasts of his idea. It seemed that the idea would be published just as a curiosity, maybe new, but of no practical use and, therefore, not worth studying. However, several enthusiasts were not offended by the fact, and built a theory that since the mid-seventies led to actual applications, bringing tangible benefits. Reflections on a generalization of fuzzy set theory led to presentation in 1983 (widely known after the publication of Atanassov, 1986) of the concept of intuitionistic fuzzy sets (IFS). The author of this idea, K.T. Atanassov, introduced some kind of independence between the degree of membership of an element to the set and its degree of non-membership to this set.

Linked to the theory of IFS is intuitionistic fuzzy logic (IFL). In this logic the truth-value of variable $x \in X$ is given by ordered pair $< a, b >$, where a, b, $a + b \in [0, 1]$. Here, X denotes the set of propositional variables. The numbers a and b are interpreted as the degrees of validity and non-validity of x. We denote the truth-value of x by $V(x)$. The variable with truth-value *true* in the classical logic we denote by $\underline{1}$ and the variable with truth-value *false* by $\underline{0}$. For these variables there also holds $V(\underline{1}) =< 1, 0 >$ and $V(\underline{0}) =< 0, 1 >$.

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We call variable x an Intuitionistic Fuzzy Tautology (shortly: IFT), if and only if for $V(x) =< a, b >$ there holds: $a \geq b$ and, similarly, an Intuitionistic Fuzzy co-Tautology (IFcT), if $a \leq b$ holds.

For every x we can define the value of negation of x in the typical form $V(\neg x) =< b, a >$.

An important operator of IFL is intuitionistic fuzzy implication. In a recent (2010) book *35 Years of Fuzzy Set Theory* the authors noticed over 138 intuitionistic fuzzy implications discovered or gathered primarily by K.T. Atanassov (2005, 2006, 2008, 2010). L. Atanassova (2009) presented an additional implication. A simple generalization of Atanassova's results was given by Dworniczak (2010).

Definition 1 *A fuzzy implication (see Baczyński and Jayaram, 2008, and Czogała and Łęski, 2001) is a mapping* $I : [0,1]^2 \to [0,1]$ *where for* p_1, p_2, p, q_1, q_2, $q \in [0,1]$ *the following hold:*

(i1FL) if $p_1 \leq p_2$ then $I(p_1, q) \geq I(p_2, q)$,

(i2FL) if $q_1 \leq q_2$ then $I(p, q_1) \leq I(p, q_2)$,

(i3FL) $I(0, q) = 1$,

(i4FL) $I(p, 1) = 1$,

(i5FL) $I(1, 0) = 0$.

Intuitionistic fuzzy sets let us process the uncertainty in a way unprecedented in mathematical modeling. In contrast to stochastic or classical fuzzy models, using the intuitionistic account, even partial knowledge of the membership or non-membership of an element to the set might be used.

In an analogous manner, in the intuitionistic logic, the degrees of validity and non-validity of a propositional variable would not sum to 1. Thus, one can talk about *true* not only to a degree $< 1, 0 >$ but about *intuitionistic fuzzy true*, defined as the IFT.

Therefore, we may consider the replacement in Definition 1 of conditions (i3FL)-(i5FL) by the weakened conditions (i3)-(i5).

When applying Definition 1 to the IFL we will first introduce (following Atanassov, 1986) some ordering relation for the intuitionistic truth-value. For $V(x) =< a, b >$ and $V(y) =< c, d >$ where $x, y \in X$, a, b, c, d, $a + b$, $c + d$ $\in [0, 1]$, we write $V(x) \preceq V(y)$ if and only if $a \leq c$ and $b \geq d$.

In the case of IFL the conditions (i1FL)–(i5FL) for implication $\Rightarrow$ are given in the form:

(i1) if $V(x_1) \preceq V(x_2)$ then $V(x_1 \Rightarrow y) \succeq V(x_2 \Rightarrow y)$,

(i2) if $V(y_1) \preceq V(y_2)$ then $V(x \Rightarrow y_1) \preceq V(x \Rightarrow y_2)$,

(i3) $\underline{0} \Rightarrow y$ is an IFT,

(i4) $x \Rightarrow \underline{1}$ is an IFT,

(i5) $\underline{1} \Rightarrow \underline{0}$ is an IFcT.

2. Main results

Now we introduce a parametric class of fuzzy intuitionistic implications.

THEOREM 1 *Let $V(x) =< a, b >$ and $V(y) =< c, d >$. An intuitionistic logical connective with truth-value:*

$$V(x \to_\varphi y) =< \frac{b+c+\varphi}{2\varphi}, \frac{a+d+\varphi-2}{2\varphi} >,$$

where $\varphi \in \Re$, $\varphi \geq 2$, is intuitionistic fuzzy implication fulfilling Definition 1 with (i1)-(i5)[1].

Proof. Note first that $< \frac{b+c+\varphi}{2\varphi}, \frac{a+d+\varphi-2}{2\varphi} >$ fulfils IFS conditions because

1') $$0 \leq \frac{1}{2} = \frac{\varphi}{2\varphi} \leq \frac{b+c+\varphi}{2\varphi} \leq \frac{2+\varphi}{2\varphi} \leq 1,$$

2') $$0 \leq \frac{\varphi-2}{2\varphi} \leq \frac{a+d+\varphi-2}{2\varphi} \leq \frac{\varphi}{2\varphi} = \frac{1}{2} \leq 1,$$

3') $$0 \leq \frac{1}{2} \leq \frac{2\varphi-2}{2\varphi} \leq \frac{b+c+\varphi}{2\varphi} + \frac{a+d+\varphi-2}{2\varphi} \leq \frac{2\varphi}{2\varphi} = 1.$$

Further conditions:

(i1) If $< a_1, b_1 >= V(x_1) \preceq V(x_2) =< a_2, b_2 >$ then $a_1 \leq a_2$ and $b_1 \geq b_2$, so

$$\frac{b_1+c+\varphi}{2\varphi} \geq \frac{b_2+c+\varphi}{2\varphi} \quad \text{and} \quad \frac{a_1+d+\varphi-2}{2\varphi} \leq \frac{a_2+d+\varphi-2}{2\varphi}$$

and, consequently, $V(x_1 \to_\varphi y) \succeq V(x_2 \to_\varphi y)$.

(i2) If $< c_1, d_1 >= V(y_1) \preceq V(y_2) =< c_2, d_2 >$ then $c_1 \leq c_2$ and $d_1 \geq d_2$, so

$$\frac{b+c_1+\varphi}{2\varphi} \leq \frac{b+c_2+\varphi}{2\varphi} \quad \text{and} \quad \frac{a+d_1+\varphi-2}{2\varphi} \geq \frac{a+d_2+\varphi-2}{2\varphi}$$

and, consequently, $V(x_1 \to_\varphi y) \preceq V(x_2 \to_\varphi y)$.

(i3) There is, by definition, $V(\underline{0} \to_\varphi y) =< \frac{1+c+\varphi}{2\varphi}, \frac{d+\varphi-2}{2\varphi} >$.
Because $\frac{1+c+\varphi}{2\varphi} \geq \frac{d+\varphi-2}{2\varphi}$ is equivalent to inequality $c - d \geq -3$, and this holds, therefore $\underline{0} \to_\varphi y$ is an IFT.

(i4) There is $V(x \to_\varphi \underline{1}) =< \frac{b+1+\varphi}{2\varphi}, \frac{a+\varphi-2}{2\varphi} >$.

[1] Implication $\to_\varphi$ is not present in the previous bibliography, known to author.

Because $\dfrac{b+1+\varphi}{2\varphi} \geq \dfrac{a+\varphi-2}{2\varphi}$ is equivalent to inequality $b-a \geq -3$, and this holds, therefore $x \rightarrow_\varphi \underline{1}$ is an IFT.

(i5) There is $V(\underline{1} \rightarrow_\varphi \underline{0}) =< \frac{1}{2}, \frac{1}{2} >$, therefore $\underline{1} \rightarrow_\varphi \underline{0}$ is an IFcT. ∎

For given values of x and y the degrees of validity $f(\varphi) = \dfrac{b+c+\varphi}{2\varphi}$ and the degrees of non-validity $g(\varphi) = \dfrac{a+d+\varphi-2}{2\varphi}$ are the functions of the parameter φ.

By differentiating the function f we obtain $f'(\varphi) = \dfrac{-(b+c)}{2\varphi^2} \leq 0$, where equality holds only for $b = c = 0$.

Similarly, in the case of the function $g(\varphi)$ we have $g'(\varphi) = \dfrac{2-(a+d)}{2\varphi^2} \geq 0$, where equality holds only for $a = d = 1$.

This means that $f(\varphi)$ is a non-increasing function of the parameter, while $g(\varphi)$ a non-decreasing one.

The sum

$$h(\varphi) = \frac{b+c+\varphi}{2\varphi} + \frac{a+d+\varphi-2}{2\varphi}$$

has a derivative

$$h'(\varphi) = \frac{2-(a+b+c+d)}{2\varphi^2} \geq 0.$$

The derivative is non-negative, reaching the value of 0 only for $a+b = c+d = 1$, which means: when x and y are classical fuzzy values (they are IFS, too). The function h is therefore a non-decreasing function of parameter φ (and for IFS which are not FS – strictly increasing).

For any variable x, with $V(x) =< a, b >$, following Atanassov (1986), we can define the degree of indeterminacy[2] of the truth-value of the variable x:

$$\pi(x) = 1 - a - b.$$

This degree is interpreted as some kind of measure of lack of knowledge whether the value of variable x is "true" or not. This degree could also be called a measure of uncertainty of the adjudicating whether the variable x belongs or does not belong to the true variables set.

For classical fuzzy truth-values there is $b = 1 - a$ and always $\pi(x) = 0$, meaning lack of uncertainty of judgment.

In the case of $\rightarrow_\varphi$ implication the degree of indeterminacy of the value of implication is a decreasing (and for an FS – always equal 0) function of parameter φ.

[2] In the literature this degree is called also hesitation margin, hesitancy degree, intuitionistic fuzzy index (J. Kacprzyk, E. Szmidt), degree of uncertainty (A. Ban, J. Kacprzyk, K. Atanassov, E. Szmidt, L. Todorova).

Parameter φ could be therefore interpreted as some kind of „factor" or „index" of uncertainty of the truth-value of intuitionistic fuzzy implication, wherein together with the increase of this parameter φ the degree of indeterminacy decreases (for FS – it is equal 0).

Moreover, property

$$\pi(\varphi) = \frac{2-(a+b+c+d)}{2\varphi} \overset{\varphi\to\infty}{\longrightarrow} 0$$

holds.

In the recent literature[3], besides (i1)–(i5), also the following axioms are postulated:

(i6) $V(\underline{1} \Rightarrow y) = V(y)$,

(i7) $V(x \Rightarrow x) = V(\underline{1})$,

(i8) $V(x \Rightarrow (y \Rightarrow z)) = V(y \Rightarrow (x \Rightarrow z))$

(i9) $V(x \Rightarrow y) = V(\underline{1}) \Leftrightarrow V(x) \preceq V(y)$,

(i10) $V(x \Rightarrow y) = V(N(y) \Rightarrow N(x))$, where N is some negation,

(i11) $\Rightarrow$ is a continuous function,

where x, y, z are variables with the truth-values $V(x) =< a, b >$, $V(y) = < c, d >$, $V(z) =< e, f >$ and a, b, c, d, e, f, $a+b$, $c+d$, $e+f \in [0,1]$.

THEOREM 2 *Implication* $\to_\varphi$

a) does not satisfy (i6)*,* (i7)*,* (i8)*,*

b) does not satisfy (i9)*, but if* $V(x \to_\varphi y) = V(\underline{1})$ *then* $V(x) \preceq V(y)$*,*

c) satisfies (i11) *and* (i10) *with* $N = \neg$*.*

Proof.

a) (i6) $V(\underline{1} \to_\varphi y) =< \dfrac{c+\varphi}{2\varphi}, \dfrac{1+d+\varphi-2}{2\varphi} > \neq < c, d >$.

(i7) $V(x \to_\varphi x) =< \dfrac{b+a+\varphi}{2\varphi}, \dfrac{a+b+\varphi-2}{2\varphi} > \neq < 1, 0 >$.

(i8) $V(x \to_\varphi (y \to_\varphi z)) =$

$$=< \frac{2\varphi b + d + e + \varphi(2\varphi+1)}{4\varphi^2}, \frac{2\varphi a + c + f + (\varphi-2)(2\varphi+1)}{4\varphi^2} > \neq$$

$$\neq < \frac{2\varphi d + b + e + \varphi(2\varphi+1)}{4\varphi^2}, \frac{2\varphi c + a + f + (\varphi-2)(2\varphi+1)}{4\varphi^2} > =$$

$$= V(y \to_\varphi (x \to_\varphi z)).$$

The equality holds only for $b = d$ and $a = c$, i.e., if $V(x) = V(y)$.

[3]Various authors give these axioms following Klir and Yuan (1995), pp. 308, 310. See also Atanassov (2005, 2006, 2008, 2010), Atanassova (2009), and Baczyński and Jayaram (2008).

b) (i9) If $V(x \to_\varphi y) = V(\underline{1})$, i.e. $\dfrac{b+c+\varphi}{2\varphi} = 1$ and $\dfrac{a+d+\varphi-2}{2\varphi} = 0$, then $b+c=\varphi$ and $a+d=2-\varphi$, and this holds only for $\varphi = 2$, $a=d=0$, $b=c=1$.

So, there is $V(x) =< 0,1 >\preceq< 1,0 >= V(y)$.

In the other direction, if we assume that $V(x) \preceq V(y)$ i.e. $a \leq c$ and $b \geq d$, then not necessarily $V(x \to_\varphi y) = V(\underline{1})$. Counterexample: $a=d=0.1$, $b=c=0.2$, $\varphi \in \Re$, $\varphi \geq 2$.

c) (i10) Because $V(N(x)) =< b,a >$, and $V(N(y)) =< d,c >$, then

$$V(N(y) \to_\varphi N(x)) =< \frac{b+c+\varphi}{2\varphi}, \frac{a+d+\varphi-2}{2\varphi} >= V(x \to_\varphi y).$$

(i11) Arithmetic operations are continuous due to both arguments. ∎

It is also easy to check that the implication $\to_\varphi$ does not satisfy the classical (two-valued) logic axioms.

Namely $V(\underline{0} \to_\varphi \underline{0}) = V(\underline{1} \to_\varphi \underline{1}) =< \dfrac{1+\varphi}{2\varphi}, \dfrac{\varphi-1}{2\varphi} >\neq V(\underline{1}), V(\underline{1} \to_\varphi \underline{0}) = < \dfrac{\varphi}{2\varphi}, \dfrac{\varphi}{2\varphi} >\neq V(\underline{0})$ and $V(\underline{0} \to_\varphi \underline{1}) =< \dfrac{\varphi+2}{2\varphi}, \dfrac{\varphi-2}{2\varphi} >\neq V(\underline{1})$ (except for $\varphi = 2$). But we notice that $\underline{0} \to_\varphi \underline{0}$, $\underline{1} \to_\varphi \underline{1}$ and $\underline{0} \to_\varphi \underline{1}$ are IFTs, however $\underline{1} \to_\varphi \underline{0}$ is an IFcT. All these values are classical fuzzy truth-values.

As we can see, the implication $\to_\varphi$ is not a generalization of the classical implication.

Let us introduce now some IFL-case of axioms (i6)-(i10) in the form:

(i6IFL)	1^0)	$\underline{1} \Rightarrow y$ is an IFT iff y is an IFT,
	2^0)	$\underline{1} \Rightarrow y$ is an IFcT iff y is an IFcT,
(i7IFL)		$x \Rightarrow x$ is an IFT,
(i8IFL)	1^0)	$x \Rightarrow (y \Rightarrow z)$ is an IFT iff $y \Rightarrow (x \Rightarrow z)$ is an IFT,
	2^0)	$x \Rightarrow (y \Rightarrow z)$ is an IFcT iff $y \Rightarrow (x \Rightarrow z)$ is an IFcT,
(i9IFL)		$x \Rightarrow y$ is an IFT iff $V(x) \preceq V(y)$,
(i10IFL)	1^0)	$x \Rightarrow y$ is an IFT iff $N(y) \Rightarrow N(x)$ is an IFT,
	2^0)	$x \Rightarrow y$ is an IFcT iff $N(y) \Rightarrow N(x)$ is an IFcT.

THEOREM 3 *Implication* $\to_\varphi$
a) does not satisfy (i6IFL), *neither* 1^0) *nor* 2^0), *however*
– *if* y *is an IFT then* $\underline{1} \to_\varphi y$ *is an IFT,*
– *if* $\underline{1} \to_\varphi y$ *is an IFcT then* y *is an IFcT,*
b) satisfies (i7IFL),
c) satisfies (i8IFL),
d) does not satisfy (i9IFL), *however*
– *if* $V(x) \preceq V(y)$ *then* $x \to_\varphi y$ *is an IFT.*
e) satisfies (i10IFL) *with* $N = \neg$.

Proof.

a) If y is an IFT, i.e. $c \geq d$, so then $\dfrac{c+\varphi}{2\varphi} \geq \dfrac{d+\varphi}{2\varphi} \geq \dfrac{d+\varphi-1}{2\varphi}$, therefore $\underline{1} \to_\varphi y$ is an IFT,

and if $\underline{1} \to_\varphi y$ is an IFcT, i.e. $\dfrac{c+\varphi}{2\varphi} \leq \dfrac{d+\varphi-1}{2\varphi}$ then $c \leq d-1$, therefore $c \leq d$, hence y is an IFcT,

b) there is $V(x \to_\varphi x) =< \dfrac{b+a+\varphi}{2\varphi}, \dfrac{a+b+\varphi-2}{2\varphi} >$, and $\dfrac{b+a+\varphi}{2\varphi} \geq \dfrac{a+b+\varphi-2}{2\varphi}$ holds, therefore $x \to_\varphi x$ is an IFT,

c) let $x \to_\varphi (y \to_\varphi z)$ be an IFT.

Therefore, $\dfrac{2\varphi b+d+e+\varphi(2\varphi+1)}{4\varphi^2} \geq \dfrac{2\varphi a+c+f+(\varphi-2)(2\varphi+1)}{4\varphi^2}$ what is equivalent to $c+f-d-e \leq 2\varphi(b-a+2)+2$ and this inequality holds for every x, y, z.

Likewise, $\dfrac{2\varphi d+b+e+\varphi(2\varphi+1)}{4\varphi^2} \geq \dfrac{2\varphi c+a+f+(\varphi-2)(2\varphi+1)}{4\varphi^2}$ holds.

So $x \to_\varphi (y \to_\varphi z)$ is an IFT iff $y \to_\varphi (x \to_\varphi z)$ is an IFT, and also $x \to_\varphi (y \to_\varphi z)$ is an IFcT iff $y \to_\varphi (x \to_\varphi z)$ is an IFcT,

d) condition: $x \to_\varphi y$ is an IFT does not entail $V(x) \preceq V(y)$.

For example: if $a = c = 0.4$, $b = 0.5$, $d = 0.6$, $\varphi \in \Re$, $\varphi \geq 2$, then $\dfrac{b+c+\varphi}{2\varphi} \geq \dfrac{a+d+\varphi-2}{2\varphi}$ but not $V(x) \preceq V(y)$ because $a \leq c$ and not $b \geq d$.

In turn, if $V(x) \preceq V(y)$ i.e. $c \geq a$ and $b \geq d$, is also $b+c \geq a+d$, therefore $\dfrac{b+c+\varphi}{2\varphi} \geq \dfrac{a+d+\varphi}{2\varphi} \geq \dfrac{a+d+\varphi-2}{2\varphi}$, which means that $x \to_\varphi y$ is an IFT,

e) it is a simple consequence of (i10). ■

There exist two basic rules of inference. They are Modus Ponens and Modus Tollens rules. These are the tautologies, given in the two-valued logic in the form: $(p \wedge (p \Rightarrow q)) \Rightarrow q$ and $((p \Rightarrow q) \wedge \neg q) \Rightarrow \neg p$, respectively. The Modus Ponens in the IFL-case takes the following form: if x is an IFT and $(x \Rightarrow y)$ is an IFT then y is an IFT. Similarly, Modus Tollens in the IFL-case takes the form: if $(x \Rightarrow y)$ is an IFT and y is an IFcT then x is an IFcT.

THEOREM 4 *Implication* $\to_\varphi$
a) *does not satisfy Modus Ponens in the IFL-case,*
b) *does not satisfy Modus Tollens in the IFL-case.*

Proof. Proof by counterexample. Let $a = d = 0.5$, $b = c = 0.4$, $\varphi \in \Re$, $\varphi \geq 2$;

a) we have that x is an IFT and $x \to_\varphi y$ is an IFT, because $\dfrac{b+c+\varphi}{2\varphi} \geq \dfrac{a+d+\varphi-2}{2\varphi}$, while y is not an IFT.

b) we have that $x \to_\varphi y$ is an IFT and y is an IFcT, while x is not an IFcT. ■

One of the fundamental tautologies of classical logic is the relationship between implication and negation. This relationship says that the truth-value of negation of the variable x is equal to the value of the logical implications of the antecedent x and the consequent *false*.

Symbolically, this tautology is written in the form of $N(x) \Leftrightarrow (x \Rightarrow 0)$. Using this relationship we can, for every intuitionistic fuzzy implication, designate a corresponding negation, called a generated (induced) negation.

THEOREM 5 *Negation N_φ generated by $\to_\varphi$ is expressed by formula:*

$$V(N_\varphi(x)) =< \frac{b+\varphi}{2\varphi}, \frac{a+\varphi-1}{2\varphi} > .$$

Proof. Proof by definition of $\to_\varphi$. ■

REMARKS

R1) If x is an IFcT, then $N_\varphi(x)$ is an IFT, and, if $N_\varphi(x)$ is an IFcT, then x is an IFT.

R2) Negation N_φ is not involutive.

R3) Negation N_φ does not satisfy the classical axioms $N_\varphi(\underline{0}) = \underline{1}$ and $N_\varphi(\underline{1}) = \underline{0}$. Moreover, $N_\varphi(\underline{0})$ is never equal to $\underline{1}$ and $N_\varphi(\underline{1})$ is never equal to $\underline{0}$. But $N_\varphi(\underline{0})$ is an IFT and $N_\varphi(\underline{1})$ is an IFcT. The values $N_\varphi(\underline{0})$ and $N_\varphi(\underline{1})$ are classical fuzzy truth-values.

R4) Axiom (i10) is satisfied for negation N_φ only for $a + d = b + c = 1$, which means that $V(x) = V(y)$. Generally, (i10) does not hold because

$$V(x \to_\varphi y) =< \frac{b+c+\varphi}{2\varphi}, \frac{a+d+\varphi-2}{2\varphi} > \neq < \frac{b+c+2\varphi(\varphi+1)-1}{4\varphi^2}, \frac{a+d+2\varphi(\varphi-1)-1}{4\varphi^2} >= V(N_\varphi(y) \to_\varphi N_\varphi(x)).$$

R5) Axiom (i10IFL) does not hold, but we note that $N_\varphi(y) \to_\varphi N_\varphi(x)$ is always an IFT, therefore properties

— if $x \to_\varphi y$ is an IFT then $N_\varphi(y) \to_\varphi N_\varphi(x)$ is an IFT,
— if $x \to_\varphi y$ is an IFcT then $N_\varphi(y) \to_\varphi N_\varphi(x)$ is an IFT,
— if $N_\varphi(y) \to_\varphi N_\varphi(x)$ is an IFcT then $x \to_\varphi y$ is an IFT,
— if $N_\varphi(y) \to_\varphi N_\varphi(x)$ is an IFcT then $x \to_\varphi y$ is an IFcT

are formally valid.

Now we denote $N^1_\varphi(x) = N_\varphi(x)$ and $N^{m+1}_\varphi(x) = N_\varphi(N^m_\varphi(x))$ for any $m \in \mathbb{N}_+$.

THEOREM 6 *For a natural number $n \geq 1$ the negation N_φ satisfies the relationships*

a) $$V(N_\varphi^{2n-1}(x)) =< \frac{b[4\varphi^2-1]+\varphi[(2\varphi)^{2n}-1]+2\varphi(\varphi-1)[(2\varphi)^{2n-2}-1]}{(2\varphi)^{2n-1}[4\varphi^2-1]}, \frac{a[4\varphi^2-1]+2\varphi^2[(2\varphi)^{2n-2}-1]+(\varphi-1)[(2\varphi)^{2n}-1]}{(2\varphi)^{2n-1}[4\varphi^2-1]} > .$$

b) $$V(N_\varphi^{2n}(x)) =< \frac{a[4\varphi^2-1]+2\varphi^2[(2\varphi)^{2n}-1]+(\varphi-1)[(2\varphi)^{2n}-1]}{(2\varphi)^{2n}[4\varphi^2-1]}, \frac{b[4\varphi^2-1]+\varphi[(2\varphi)^{2n}-1]+2\varphi(\varphi-1)[(2\varphi)^{2n}-1]}{(2\varphi)^{2n}[4\varphi^2-1]} > .$$

Proof. For $n = 1$ we have

$$V(N_\varphi^{2n-1}(x)) =< \frac{b[4\varphi^2-1]+\varphi[(2\varphi)^{2n}-1]}{(2\varphi)^{2n-1}[4\varphi^2-1]}, \frac{a[4\varphi^2-1]+(\varphi-1)[(2\varphi)^{2n}-1]}{(2\varphi)^{2n-1}[4\varphi^2-1]} >= =< \frac{b+\varphi}{2\varphi}, \frac{a+\varphi-1}{2\varphi} >= V(N_\varphi(x)),$$

and

$$V(N_\varphi^{2n}(x)) =< \frac{a+2\varphi^2+\varphi-1}{4\varphi^2}, \frac{b+2\varphi^2-\varphi}{4\varphi^2} >= V(N_\varphi(N_\varphi(x))).$$

We assume that for $n > 1$ there is

$$V(N_\varphi^{2n-1}(x)) =< \frac{b[4\varphi^2-1]+\varphi[(2\varphi)^{2n}-1]+2\varphi(\varphi-1)[(2\varphi)^{2n-2}-1]}{(2\varphi)^{2n-1}[4\varphi^2-1]}, \frac{a[4\varphi^2-1]+2\varphi^2[(2\varphi)^{2n-2}-1]+(\varphi-1)[(2\varphi)^{2n}-1]}{(2\varphi)^{2n-1}[4\varphi^2-1]} > .$$

Therefore,

$$V(N_\varphi^{2n}(x)) = V(N_\varphi(N_\varphi^{2n-1}(x))) =$$
$$=< \frac{1}{2\varphi}\left(\frac{a[4\varphi^2-1]+2\varphi^2[(2\varphi)^{2n-2}-1]+(\varphi-1)[(2\varphi)^{2n}-1]}{(2\varphi)^{2n-1}[4\varphi^2-1]}+\varphi\right), \frac{1}{2\varphi}\left(\frac{b[4\varphi^2-1]+\varphi[(2\varphi)^{2n}-1]+2\varphi(\varphi-1)[(2\varphi)^{2n-2}-1]}{(2\varphi)^{2n-1}[4\varphi^2-1]}+\varphi-1\right) >=$$
$$=< \frac{a[4\varphi^2-1]+2\varphi^2[(2\varphi)^{2n}-1]+(\varphi-1)[(2\varphi)^{2n}-1]}{(2\varphi)^{2n}[4\varphi^2-1]}, \frac{b[4\varphi^2-1]+\varphi[(2\varphi)^{2n}-1]+2\varphi(\varphi-1)[(2\varphi)^{2n}-1]}{(2\varphi)^{2n}[4\varphi^2-1]} > .$$

The proof for $V(N_\varphi^{2k-1}(x))$ is analogous.

Thus, due to mathematical induction Theorem 6 is valid for every $n \in \mathbb{N}_+$. ■

Corollary 1 $\lim_{m\to\infty} V(N_\varphi^m(x)) =< \frac{\varphi+1}{2\varphi+1}, \frac{\varphi}{2\varphi+1} >$.

Remarks

R1) $\lim_{m\to\infty} V(N_\varphi^m(x))$ is an IFT and a classical fuzzy set.

R2) $\lim_{m\to\infty} V(N_\varphi^m(x)) \preceq < \frac{3}{5}, \frac{2}{5} >$.

Corollary 2 $\lim_{\varphi\to\infty}(\lim_{m\to\infty} V(N_\varphi^m(x))) =< \frac{1}{2}, \frac{1}{2} >$.

3. The intuitionistic fuzzy implication in some basic problem of MCDM

Suppose that in the problem of multicriteria decision making (MCDM) each variant x_i, $i = 1, ...n$, from a finite set of variants (alternatives), is assessed according to k criteria of evaluation. Suppose that we can give values a_{ij}, $b_{ij} \in [0,1]$, with $a_{ij} + b_{ij} \in [0,1]$, interpreted as the degrees of validity and non-validity of judgment "the variant x_i satisfies the criterion K_j". This means that we can get an ordered pair of assessments that is equal to the intuitionistic fuzzy value $< a_{ij}, b_{ij} >$.

Assume that the criteria are not equally important and each of them is assigned to one of the linguistic assessments, included in Table 1. These assessments must be made by a supervisor, or, in the case of many experts, must be some aggregation of their opinions.

Psychological research suggested that it is often convenient to evaluate the criteria, using terms derived from natural language. In the natural language, therefore, the degrees of criteria validity can be correspondingly expressed. In 1956, G.A. Miller argued that the number of levels of evaluation of a term can not be too great. An average person distinguishes among no more than 7 ± 2 levels of intensity features. In the case of more levels there is some kind of excess of the number of options, which causes the merging of opinions and the lack of differentiation (Miller's Law, Miller's magical number 7).

Let the linguistic degrees of validity of the criteria and their corresponding intuitionistic fuzzy values (IFV) be given as in Table 1.

Based on the application of the implication to the assessments $< a_{ij}, b_{ij} >$, using IFV$_j$ from Table 1, we will evaluate the revised degrees of validity. More precisely, we give the degrees of validity and non-validity of the expression "if the criterion is valid then it is satisfied".

The type of intuitionistic implication affects of course the value obtained after its application.

Suppose that three alternatives were evaluated according to four criteria by intuitionistic assessment, and these criteria were considered to be either strongly important or rather important or insignificant or not known type 2, respectively (Table 2). Let the implication be $\to_\varphi$ for $\varphi = 2$. Calculated corrected degrees of validity are contained in Table 3.

Table 1. Linguistic assessments and their intuitionistic counterparts

Linguistic assessments of the criterion K_j	IFV$_j$
strongly important	$< 1.0, 0.0 >$
important	$< 0.8, 0.0 >$
rather important	$< 0.6, 0.0 >$
insignificant	$< 0.0, 0.5 >$
almost totally unimportant	$< 0.0, 0.9 >$
I do not know, Type 1; *I have no opinion,* *I can not regard this criterion as valid or invalid*	$< 0.0, 0.0 >$
I do not know, Type 2; *some prerequisites suggest that the criterion is* *important and some prerequisites, on the contrary*	$< 0.5, 0.5 >$

Table 2. The values of degrees of the criteria met by the variants

Criteria / Alternatives	K_1 *strongly important*	K_2 *rather important*	K_3 *insignificant*	K_4 *I do not know, Type 2*
x_1	$< 1.0, 0.0 >$	$< 0.9, 0.0 >$	$< 0.7, 0.0 >$	$< 0.7, 0.2 >$
x_2	$< 1.0, 0.0 >$	$< 0.3, 0.7 >$	$< 0.4, 0.3 >$	$< 1.0, 0.0 >$
x_3	$< 0.6, 0.3 >$	$< 0.5, 0.3 >$	$< 0.9, 0.0 >$	$< 0.9, 0.1 >$

Table 3. The assessments corrected by the degrees of validity of the criteria

Criteria / Alternatives	K_1 *strongly important*	K_2 *rather important*	K_3 *insignificant*	K_4 *I do not know, Type 2*
x_1	$< 0.750, 0.250 >$	$< 0.725, 0.150 >$	$< 0.800, 0.000 >$	$< 0.800, 0.175 >$
x_2	$< 0.750, 0.250 >$	$< 0.575, 0.325 >$	$< 0.725, 0.075 >$	$< 0.875, 0.125 >$
x_3	$< 0.650, 0.325 >$	$< 0.625, 0.225 >$	$< 0.850, 0.000 >$	$< 0.850, 0.150 >$

The results given in Table 3 are the basis for calculating the aggregate assessment and designation of weak preference relations in the set of alternatives. Considerations on this particular issue go beyond the established framework of this article.

Use of IFS and the implications for data processing takes advantage of even partial information about the degrees of fulfilling (and not-fulfilling) criteria by the different variants.

4. Conclusions

In the paper a new class of fuzzy intuitionistic implications with their basic properties is presented. The influence of changes of the value parameter on the value of the validity degree, non-validity degree and the degree of indeterminacy for this implication is shown. This implication may be the subject of further research, both in terms of its properties or comparisons with other intuitionistic fuzzy implications, and possible applications. In the broad field of economic applications, for example, this class of implications may relate to reasoning with incomplete or uncertain information, or multiple criteria decision making, especially with varying degrees of importance of the criteria.

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