Control and Cybernetics

vol. **40** (2011) No. 4

Regularization and discretization of linear-quadratic control problems*

by

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Abstract: We analyze regularizations of a class of linear-quadratic optimal control problems with control appearing linearly. It is shown that if the optimal control is bang-bang or if a coercivity condition for the state variables is satisfied, the solutions are continuous functions of the regularization parameter. Combining error estimates for Euler discretizations of the regularized problems with those for the regularization error, we choose the regularization parameter in dependence of the meshsize to obtain optimal convergence rates for the discrete solutions. Numerical experiments confirm the theoretical findings.

Keywords: optimal control, bang-bang control, regularization, discretization.

1. Introduction

Discretizations of optimal control problems are well studied for the case when the optimal control is sufficiently smooth (see, e.g., Alt, 1997; Alt, Bräutigam, Rösch, 2007; Alt, Bräutigam, 2009; Dontchev, Hager, 1993, 2001; Dontchev, Hager, Malanowski, 2000; Dontchev, Hager, Veliov, 2000; Malanowski, Büskens, Maurer, 2005, for control problems governed by ordinary differential equations and Casas, Tröltzsch, 2010; Tröltzsch, 2010a,b, for control problems governed by partial differential equations). The results are usually based on second-order optimality conditions. Due to the lack of such conditions for bang-bang controls, there have been only few papers on discretizations of such controls (see Alt, Mackenroth, 1989, and Dhamo, Tröltzsch, 2011, and the papers cited therein). New second-order optimality conditions for bang-bang controls have been developed recently in Felgenhauer (2003), Felgenhauer, Poggiolini, Stefani (2009) and Maurer, Osmolovskii (2004), and variants of these conditions have then

*Submitted: December 2010; Accepted: July 2011

been used in Veliov (2005) and in Alt et al. (2011, 2012) to obtain error estimates for discretizations of bang-bang controls governed by ordinary differential equations and in Deckelnick, Hinze (2010) for elliptic control problems.

Discretization combined with regularization is a good alternative to direct discretization, since the problem to be solved is replaced by problems having smoother solutions. The regularization of constraints and of the cost functional of optimal control problems has been intensively studied during the last years (see, e.g., Meyer, Rösch, 2004; Meyer, Rösch, Tröltzsch, 2006; Neitzel, Tröltzsch, 2008; Tröltzsch, Yousept, 2009; Hinze, Meyer, 2010, and the papers cited therein). The dependency of solutions on regularization parameters and the combination with discretization has been investigated in Hager (1979) for multiplier methods for the solution of control problems governed by ordinary differential equations, and in Lorenz, Rösch (2010) for elliptic control problems with state constraints. Results for control problems with a sparsity functional can be found in Wachsmuth and Wachsmuth (2011). However, it seems that similar results are not known for bang-bang controls of problems governed by ordinary differential equations. The aim of the present paper is to derive such results for a class of linear-quadratic control problems.

We use the following notations: $\mathbb{R}^n$ is the n-dimensional Euclidean space with the inner product denoted by $\langle x, y\rangle$ and the norm $|x| = \langle x, x\rangle^{1/2}$. For an $m \times n$-matrix B we denote the spectral norm by $\|B\| = \sup_{|z|\le 1} |Bz|$. For $1 \le p < \infty$ we denote by $L^p(0,T;\mathbb{R}^m)$ the Banach space of measurable vector functions $u\colon [0,T] \to \mathbb{R}^m$ with

$$\|u\|_p = \left(\int_0^T |u(t)|^p \, dt\right)^{\frac{1}{p}} < \infty\,,$$

and $L^\infty(0,T;\mathbb{R}^m)$ is the Banach space of essentially bounded vector functions with the norm

$$\|u\|_\infty = \max_{1\le i\le m} \operatorname*{ess\,sup}_{t\in[0,T]} |u_i(t)|\,.$$

For $1 \le p \le \infty$ we denote by $W_p^1(0,T;\mathbb{R}^n)$ the spaces of absolutely continuous functions on $[0,T]$ with derivative in $L^p(0,T;\mathbb{R}^n)$, i.e.

$$W_p^1(0,T;\mathbb{R}^n) = \{x \in L^p(0,T;\mathbb{R}^n) \mid \dot{x} \in L^p(0,T;\mathbb{R}^n)\}$$

with

$$\|x\|_{1,p} = \left(\|x\|^p + \|\dot{x}\|_p^p\right)^{\frac{1}{p}}$$

for $1 \le p < \infty$ and

$$\|x\|_{1,\infty} = \max\{\|x\|_\infty, \|\dot{x}\|_\infty\}\,.$$

With $X = X_1 \times X_2$, $X_1 = W^1_\infty(0,T;\mathbb{R}^n)$, $X_2 = L^\infty(0,T;\mathbb{R}^m)$, we consider the following family of linear-quadratic control problems depending on the parameter $\nu \geq 0$:

$$
\begin{aligned}
(\mathrm{OQ})_\nu \qquad & \min_{(x,u)\in X_1\times X_2} f_\nu(x,u) \\
& \text{s.t.} \\
& \dot{x}(t) = A(t)x(t) + B(t)u(t) \quad \text{a.e. on } [0,T], \\
& x(0) = a\,, \\
& u(t) \in U \quad \text{a.e. on } [0,T],
\end{aligned}
$$

where f_ν is a linear-quadratic cost functional defined by

$$
\begin{aligned}
f_\nu(x,u) =& \tfrac{1}{2}x(T)^\mathsf{T} Q x(T) + q^\mathsf{T} x(T) \\
&+ \int_0^T \tfrac{1}{2} x(t)^\mathsf{T} W(t) x(t) + w(t)^\mathsf{T} x(t) + r(t)^\mathsf{T} u(t)\,dt + \tfrac{\nu}{2}\|u\|_2^2\,.
\end{aligned}
\tag{1.1}
$$

Here, $u(t) \in \mathbb{R}^m$ is the control, and $x(t) \in \mathbb{R}^n$ is the state of the system at time t. Further, Q is a symmetric and positive semidefinite $n\times n$-matrix, $q \in \mathbb{R}^n$, and the functions $W\colon [0,T] \to \mathbb{R}^{n\times n}$, $w\colon [0,T] \to \mathbb{R}^n$, $r\colon [0,T] \to \mathbb{R}^m$, $A\colon [0,T] \to \mathbb{R}^{n\times n}$ $B\colon [0,T] \to R^{n\times m}$ are Lipschitz continuous. The matrices $W(t)$ are assumed to be symmetric and positive semidefinite, and the set $U \subset \mathbb{R}^m$ is defined by lower and upper bounds, i.e.,

$$
U = \{u \in \mathbb{R}^m \mid b_l \leq u \leq b_u\}
$$

with $b_l, b_u \in \mathbb{R}^m$, $b_l < b_u$, where all inequalities are to be understood componentwise. The term $\frac{\nu}{2}\|u\|_2^2$ is a regularization term.

We are interested in the behaviour of solutions (x_ν, u_ν) of $(\mathrm{OQ})_\nu$ in dependence of the regularization parameter ν. If the optimal control for $\nu = 0$ is of bang-bang type, we show that the error $\|u_\nu - u_0\|_1$ is of order ν (Theorem 4.1). Otherwise we assume that a coercivity condition for the states is satisfied and show that the error $\|x_\nu - x_0\|_2$ is of order $\sqrt{\nu}$ (Theorem 4.2). Combining these error estimates with error estimates for Euler discretizations of the regularized problems, we then choose the regularization parameter in dependence of the meshsize h in order to obtain optimal convergence rates for $\|u_{\nu,h} - u_0\|_1$, where $u_{\nu,h}$ are the discrete optimal controls.

The organization of the paper is as follows. In Section 2 we recall some basic results for Problems $(\mathrm{OQ})_\nu$. Section 3 is concerned with uniqueness of solutions of Problems $(\mathrm{OQ})_0$. Section 4 derives error estimates for the solutions of the regularized problems in dependence of the parameter ν. In Section 5 we combine these error estimates with error estimates for Euler discretizations.

2. Basic results

We denote by

$$\mathcal{U} = \{u \in X_2 \mid u(t) \in U \text{ a.e. on } [0,T]\}$$

the set of admissible controls. Furthermore,

$$\mathcal{F} = \{(x,u) \in X \mid u \in \mathcal{U},\ \dot{x}(t) = A(t)x(t) + B(t)u(t) \text{ a.e. on } [0,T],\ x(0) = a\}$$

denotes the feasible set of Problem $(\mathrm{OQ})_\nu$. Since U is nonempty, the feasible set $\mathcal{F}$ is nonempty and since U is bounded, it follows that $\dot{x}$ is bounded for any feasible pair $(x,u) \in \mathcal{F}$, and therefore $\mathcal{F} \subset X$. Moreover, there is some constant c such that

$$\|x\|_{1,\infty} \le c\,\|u\|_\infty$$

for any solution x of the system equation, which implies that $\mathcal{F}$ is bounded.

DEFINITION 2.1 *A pair* $(x_\nu, u_\nu) \in \mathcal{F}$ *is called a* minimizer for Problem $(\mathrm{OQ})_\nu$*, if* $f_\nu(x_\nu,u_\nu) \le f_\nu(x,u)$ *for all* $(x,u) \in \mathcal{F}$*, and a* strict minimizer*, if* $f_\nu(x_\nu,u_\nu) < f_\nu(x,u)$ *for all* $(x,u) \in \mathcal{F}$*,* $(x,u) \neq (x_\nu,u_\nu)$*.* ◇

Since the feasible set $\mathcal{F}$ is nonempty and bounded, and the cost functional is convex and continuous, a minimizer $(x_\nu,u_\nu) \in W_2^1(0,T;\mathbb{R}^n) \times L^2(0,T;\mathbb{R}^m)$ of $(\mathrm{OQ})_\nu$, $\nu \ge 0$, exists (see e.g. Ekeland, Temam, 1976, Chap. II, Proposition 1.2). For $\nu > 0$ the function f_ν is strictly convex, and therefore the solution (x_ν,u_ν) is a strict minimizer. In Section 3 we discuss assumptions implying uniqueness of the solution (x_0,u_0) of $(\mathrm{OQ})_0$, and in Section 4 we derive error estimates for $\|u_\nu - u_0\|_1$ and $\|x_\nu - x_0\|_\infty$.

The cost functional f_0 is Lipschitz continuous on $\mathcal{F}$, i.e., there is a constant L_f such that

$$|f_0(x,u) - f_0(z,v)| \le L_f\,(\|x-z\|_\infty + \|u-v\|_1) \quad \forall (x,u),(z,v) \in \mathcal{F}. \tag{2.1}$$

Since U is compact there exists a constant K such that for any feasible control $u \in \mathcal{U}$ and the associated solution x of the system equation we have

$$|x(t)| \le K \quad \forall t \in [0,T]\,,$$

hence, with some constant R we have

$$|\dot{x}(t)| \le R \text{ a.e. on } [0,T]. \tag{2.2}$$

This shows that the feasible trajectories are uniformly Lipschitz continuous with Lipschitz modulus R.

Let $(x_\nu, u_\nu) \in \mathcal{F}$ be a minimizer of $(\mathrm{OQ})_\nu$. Then there exists a function $\lambda_\nu \in W^1_\infty(0,T;\mathbb{R}^n)$ such that the adjoint equation

$$-\dot\lambda_\nu(t) = A(t)^\mathsf{T}\lambda_\nu(t) + W(t)x_\nu(t) + w(t) \\ \text{a.e. on } [0,T],\ \lambda_\nu(T) = Qx_\nu(T) + q, \tag{2.3}$$

and the minimum principle

$$[\nu u_\nu(t)^\mathsf{T} + r(t)^\mathsf{T} + \lambda_\nu(t)^\mathsf{T}B(t)](u - u_\nu(t)) \geq 0 \quad \forall u \in U \tag{2.4}$$

hold a.e. on $[0,T]$. In case of $\nu = 0$ we denote by

$$\sigma(t) := r(t) + B(t)^\mathsf{T}\lambda_0(t) \tag{2.5}$$

the *switching function*. It is well known that (2.4) implies for $i \in \{1,\ldots,m\}$ (see, e.g., Felgenhauer, 2003; Lenhart, Workman, 2007, Chap. 17)

$$u_{0,i}(t) = \begin{cases} b_{l,i}, & \text{if } \sigma_i(t) > 0, \\ b_{u,i}, & \text{if } \sigma_i(t) < 0, \\ \text{undetermined}, & \text{if } \sigma_i(t) = 0. \end{cases} \tag{2.6}$$

Therefore, if $\nu = 0$, the optimal control u_0 is of bang-bang type or may have singular arcs.

The parameter ν is a regularization parameter. For $\nu > 0$, Problem $(\mathrm{OQ})_\nu$ admits a unique solution (x_ν, u_ν) (see, e.g., Lions, 1971, Chap. 1, Dontchev, Hager 1993, Lemma 4). Moreover, it follows from (2.4) that for $\nu > 0$ the optimal control u_ν is defined by (compare Malanowski, 1981, Sect. 1)

$$u_\nu(t) = \mathrm{Pr}_{[b_l,b_u]}\left(-\frac{1}{\nu}\left(r(t) + B(t)^\mathsf{T}\lambda_\nu(t)\right)\right), \tag{2.7}$$

where $\mathrm{Pr}_{[b_l,b_u]}$ denotes the projection onto the interval $[b_l, b_u]$, which implies that u_ν is Lipschitz continuous.

Remark *Since λ_0 satisfies the adjoint equation and A, W, w are Lipschitz continuous, λ_0 is absolutely continuous with bounded derivative and hence Lipschitz continuous, which implies that σ is also Lipschitz continuous.*

Since $(\mathrm{OQ})_\nu$ is a convex optimization problem for all $\nu \geq 0$, a pair $(x_\nu, u_\nu) \in \mathcal{F}$ satisfying the minimum principle (2.4) with some function λ_ν solving the adjoint equation (2) is a solution of $(\mathrm{OQ})_\nu$. ◇

3. Uniqueness of solutions

The analysis of parametric control problems is usually based on a second-order optimality condition (compare, e.g., Dontchev, Hager, Malanowski, 2000;

Malanowski, Büskens, Maurer, 1997). We show in the following that for Problem $(\mathrm{OQ})_0$ a similar condition holds, if the optimal control is of bang-bang type or if a coercivity condition w.r.t. the state variables holds. To this end we use recent results on second-order sufficient optimality conditions due to Felgenhauer (2003). For the bang-bang case we assume that

(A1) The set Σ of zeros of the components σ_i, $i = 1, \ldots, m$, of the switching function σ defined by (2.5) is finite and $0, T \notin \Sigma$, i.e., $\Sigma = \{s_1, \ldots, s_l\}$ with $0 < s_1 < \ldots < s_l < T$.

Let $I(s_j) := \{1 \leq i \leq m : \sigma_i(s_j) = 0\}$ be the set of active indices for the components of the switching function. In order to get stability of the bang-type structure we need an additional assumption (compare Felgenhauer, 2003):

(A2) There exist $\bar{\sigma} > 0$, $\bar{\tau} > 0$ such that
$$\sigma_i(\tau) \geq \bar{\sigma}(\tau - s_j)$$
for all $j \in \{1, \ldots, l\}$, $i \in I(s_j)$, and all $\tau \in [s_j - \bar{\tau}, s_j + \bar{\tau}]$, and
$$\sigma_i(s_j - \bar{\tau})\sigma_i(s_j + \bar{\tau}) < 0,$$
i.e., σ_i changes the sign in s_j.

Assumptions (A1) and (A2) imply uniqueness of the optimal control u_0 (see the remark following (3.6)).

The following result is extracted from the proof of Lemma 3.3 in Felgenhauer (2003). A proof can also be found in Alt et al. (2011).

LEMMA 3.1 *Let* (x_0, u_0) *be a minimizer for Problem* $(\mathrm{OQ})_0$*, and let the switching be defined by* (2.5)*. If Assumptions* (A1) *and* (A2) *are satisfied, then there are constants* $\alpha, \gamma, \bar{\delta} > 0$ *such that for any feasible pair* (x, u)

$$\int_0^T \sigma(t)^{\mathsf{T}}(u(t) - u_0(t))dt \geq \alpha \|u - u_0\|_1^2 \tag{3.1}$$

if $\|u - u_0\|_1 \leq 2\gamma\bar{\delta}$*, and*

$$\int_0^T \sigma(t)^{\mathsf{T}}(u(t) - u_0(t))dt \geq \alpha \|u - u_0\|_1 \tag{3.2}$$

if $\|u - u_0\|_1 \geq 2\gamma\bar{\delta}$. ◇

Lemma 3.1 implies a quadratic minorant for the minimal values of Problem (OQ) in a sufficiently small L^1-neighborhood, and a linear minorant outside this neighborhood.

THEOREM 3.1 *Let* (x_0, u_0) *be a minimizer for Problem* $(\mathrm{OQ})_0$*. If Assumptions* (A1) *and* (A2) *are satisfied, then there are constants* $\alpha, \gamma, \bar{\delta} > 0$ *such that for any feasible pair* (x, u)

$$f_0(x, u) - f_0(x_0, u_0) \geq \alpha \|u - u_0\|_1^2 \tag{3.3}$$

if $\|u - u_0\|_1 \leq 2\gamma\bar{\delta}$, *and*

$$f_0(x,u) - f_0(x_0,u_0) \geq \alpha \|u - u_0\|_1 \tag{3.4}$$

if $\|u - u_0\|_1 \geq 2\gamma\bar{\delta}$. ◇

Proof. Let (x,u) be feasible for problem $(\mathrm{OQ})_0$, let (x_0,u_0) be optimal, and let λ_0 be the adjoint state. Defining $z = x - x_0$, $v = u - u_0$ we have

$$\begin{aligned} f_0(x,u) - f_0(x_0,u_0) &= (Qx_0(T)+q)^\top z(T) + \frac{1}{2} z(T)^\top Q z(T) \\ &\quad + \int_0^T (x_0(t)W(t) + r(t)^\top) z(t)\, dt + \frac{1}{2}\int_0^T z(t)^\top W(t) z(t)\, dt \\ &\geq (Qx_0(T)+q)^\top z(T) + \int_0^T (x_0(t)^\top W(t) + r(t)^\top) z(t)\, dt, \end{aligned}$$

since Q and $W(\cdot)$ are positive semidefinite. From $\lambda_0(T) = Qx_0(T) + q$ follows

$$f_0(x,u) - f_0(x_0,u_0) \geq \lambda_0(T)^\top z(T) + \int_0^T (x_0(t)^\top W(t) + r(t)^\top) z(t)\, dt.$$

Since $z(0) = 0$ we further obtain

$$\begin{aligned} f_0(x,u) - f_0(x_0,u_0) &\geq \int_0^T (x_0(t)^\top W(t) + r(t)^\top) z(t)\, dt + \lambda_0(T)^\top z(T) \\ &= \int_0^T (x_0(t)^\top W(t) + r(t)^\top) z(t)\, dt + \int_0^T \dot{z}(t)^\top \lambda_0(t)\, dt + \int_0^T z(t)^\top \dot{\lambda}_0(t)\, dt \\ &= \int_0^T (x_0(t)^\top W(t) + r(t)^\top) z(t)\, dt + \int_0^T [A(t)z(t) + B(t)v(t)]^\top \lambda_0(t)\, dt \\ &\quad - \int_0^T z(t)^\top \left[A(t)^\top \lambda_0(t) + W(t)x_0(t) + r(t)\right]\, dt \\ &= \int_0^T \lambda_0(t)^\top B(t) v(t) dt = \int_0^T \sigma(t)^\top v(t) dt. \end{aligned}$$

The assertion now follows from Lemma 3.1. ■

Since x_0 solves the state equation for u_0 and x solves the state equation for u, we have

$$\dot{x}(t) - \dot{x}_0(t) = A(t)(x(t) - x_0(t)) + B(t)(u(t) - u_0(t)) \quad \text{a.e. on } [0,T],$$

and $x(0) - x_0(0) = 0$. This implies

$$\|x - x_0\|_{1,1} \leq c\, \|u - u_0\|_1$$

with some constant c. Together with (3.3), (3.4) we obtain with some constant $\tilde{\alpha} > 0$

$$f_0(x,u) - f_0(x_0,u_0) \geq \tilde{\alpha}(\|u-u_0\|_1^2 + \|x-x_0\|_{1,1}^2) \tag{3.5}$$

for any feasible pair (x,u) with $\|u-u_0\|_1 \leq 2\gamma\bar{\delta}$, and

$$f_0(x,u) - f_0(x_0,u_0) \geq \tilde{\alpha}(\|u-u_0\|_1 + \|x-x_0\|_{1,1}) \tag{3.6}$$

for any feasible pair (x,u) with $\|u-u_0\|_1 \geq 2\gamma\bar{\delta}$.

Remark *(compare Felgenhauer, 2003, Theorem 2.2) These estimates also imply uniqueness of the solution of* $(\mathrm{OQ})_0$. *If* $(x,u) \in \mathcal{F}$ *is an arbitrary solution of* $(\mathrm{OQ})_0$, *then* $f_0(x,u) = f_0(x_0,u_0)$. *By (3.5), respectively (3.6) we then obtain* $(x,u) = (x_0,u_0)$. ◇

Example 3.1 (Lenhart, Workman, 2007, Example 17.2) We consider the problem

$$
\begin{aligned}
(\mathrm{OQ1})_\nu \quad & \min \int_0^2 -2x(t) + 3u(t)\,dt + \frac{\nu}{2}\|u\|^2 \\
& \text{s.t.} \\
& \dot{x}(t) = x(t) + u(t) \qquad \text{a.e. on } [0,2], \\
& x(0) = 5\,, \\
& 0 \leq u(t) \leq 2 \qquad\qquad \text{a.e. on } [0,2].
\end{aligned}
$$

For $\nu = 0$ the unique solution of the adjoint equation is given by

$$\lambda_0(t) = 2 - 2e^{2-t} \quad \forall t \in [0,2]. \tag{3.7}$$

Therefore, the switching function is defined by

$$\sigma(t) = r(t) + \lambda_0(t) = 3 + \lambda_0(t) = 5 - 2e^{2-t}. \tag{3.8}$$

The unique zero of this function in $[0,2]$ is $s_1 = 2 - \ln(5/2)$, which implies that the optimal control for $\nu = 0$ is of bang-bang type and given by

$$u_0(t) = \begin{cases} 2 & \text{for } 0 \leq t < s_1, \\ 0 & \text{for } s_1 < t \leq 2, \end{cases}$$

with associated state function

$$x_0(t) = \begin{cases} 7e^t - 2 & \text{for } 0 \leq t < s_1\,, \\ 7e^t - 5e^{t-2} & \text{for } s_1 < t \leq 2\,. \end{cases}$$

Since $\sigma'(s_1) = 5$, Assumptions (A1) and (A2) are satisfied, and the solution is uniquely determined. ◇

If Assumptions (A1) and (A2) are not satisfied, an optimal solution u_0 of $(\mathrm{OQ})_0$ may have singular arcs. This case is considered next. Using the fact that for any $(x, u) \in \mathcal{F}$ we have

$$f_0'(x_0, u_0)((x, u) - (x_0, u_0)) \geq 0$$

by the optimality of (x_0, u_0), we obtain

$$\begin{aligned} f_0(x, u) - f_0(x_0, u_0) \geq & \frac{1}{2}(x(T) - x_0(T))^\mathsf{T} Q(x(T) - x_0(T)) \\ & + \frac{1}{2}\int_0^T (x(t) - x_0(t))^\mathsf{T} W(t)(x(t) - x_0(t))dt. \end{aligned} \tag{3.9}$$

In order to assure uniqueness of the solution we assume that

(A3) The matrices $W(t)$, $t \in [0, T]$, are uniformly positive definite, i.e., there is some $\alpha > 0$ such that for all $t \in [0, T]$

$$x^T W(t)x \geq \alpha\, |x|^2 \quad \forall x \in \mathbb{R}^n.$$

Now let $(z_0, v_0) \in \mathcal{F}$ be any solution of $(\mathrm{OQ})_0$ with associated multiplier μ_0. Then $f_0(z_0, v_0) = f_0(x_0, u_0)$. If Assumption (A3) is satisfied, it follows from (3.9) and the positive semi-definiteness of Q that

$$0 \geq \int_0^T (z_0(t) - x_0(t))^\mathsf{T} W(t)(z_0(t) - x_0(t))dt \geq \alpha\, \|z_0 - x_0\|_2^2,$$

and therefore $z_0 \equiv x_0$. By the adjoint equations this further implies $\mu_0 \equiv \lambda_0$, and hence uniqueness of the switching function. Especially, the sets

$$\Sigma_{0,i} = \{t \in [0, T] \mid \sigma_i(t) = 0\}\,, \quad i = 1, \ldots, m\,,$$

are independent of the special solution, and hence by (2.6)

$$v_{0,i}(t) = u_{0,i}(t) \quad \text{a.e. on} \ \ [0, T] \setminus \Sigma_{0,i}.$$

Uniqueness of the control $u_{0,i}$ on an interval $[t_1, t_2] \subset \Sigma_{0,i}$, $t_1 < t_2$, can only be guaranteed under additional assumptions. By the system equation we have

$$B(t)^\mathsf{T} B(t)u_0(t) = B(t)^\mathsf{T}\left[\dot{x}_0(t) - A(t)x_0(t)\right]. \tag{3.10}$$

This uniquely determines u on $[t_1, t_2]$ for instance, if $B(t)^\mathsf{T} B(t)$ is invertible on $[t_1, t_2]$. In case of scalar controls ($m = 1$) this is satisfied, if $B(t) \neq 0_n$ for all $t \in [t_1, t_2]$.

Example 3.2 (Lenhart, Workman, 2007, Example 17.3) We consider the problem

$$\begin{aligned} (\mathrm{OQ2})_\nu \quad & \min \frac{1}{2}\int_0^2 (x(t) - x_d(t))^2\, dt + \frac{\nu}{2}\, \|u\|^2 \\ & \text{s.t.} \\ & \dot{x}(t) = u(t) \qquad \text{a.e. on } [0, 2], \\ & x(0) = 5\,, \\ & 0 \leq u(t) \leq 2 \qquad \text{a.e. on } [0, 2], \end{aligned}$$

with $x_d(t) = t$. For $\nu = 0$, the functions u_0, x_0, λ_0, defined by

$$u_0(t) = 0\,,\ x_0(t) = 1\,,\ \lambda_0(t) = \frac{1}{2}(t-1)^2$$

for $t \in [0,1]$ and

$$u_0(t) = 1\,,\ x_0(t) = t\,,\ \lambda_0(t) = 0$$

for $t \in [1,2]$ satisfy the optimality conditions. Therefore, (x_0, u_0) is a solution of $(\mathrm{OQ2})_0$. On $[0,1]$ the optimal control is of bang-bang type, and on $[0,2]$ the solution is given by (3.10). Hence the solution is uniquely determined. $\diamond$

4. Error estimates for solutions of regularized problems

In order to derive error estimates for $\|u_\nu - u_0\|_1$ and $\|x_\nu - x_0\|_\infty$ we use standard techniques from parametric optimization.

THEOREM 4.1 *Let Assumptions* (A1) *and* (A2) *be satisfied. Then there exist constants* c_1, c_2 *independent of* ν *such the estimates*

$$\|u_\nu - u_0\|_1 \le c_1\,\nu\,, \quad \|x_\nu - x_0\|_{1,1} \le c_2\,\nu \tag{4.1}$$

hold. $\diamond$

Proof. For $\nu > 0$ we have by Lemma 3.1

$$\int_0^T \sigma(t)^\mathsf{T}(u_\nu(t) - u_0(t))dt \ge \alpha\,\|u_\nu - u_0\|_1^2, \tag{4.2}$$

if $\|u_\nu - u_0\|_1 \le 2\gamma\bar{\delta}$, and

$$\int_0^T \sigma(t)^\mathsf{T}(u_\nu(t) - u_0(t))dt \ge \alpha\,\|u_\nu - u_0\|_1, \tag{4.3}$$

if $\|u_\nu - u_0\|_1 \ge 2\gamma\bar{\delta}$, where $\alpha \ge 0$ and $\alpha > 0$, if Assumptions (A1) and (A2) are satisfied. From the minimum principle (2.4) we obtain

$$\int_0^T [\nu u_\nu(t)^\mathsf{T} + r(t)^\mathsf{T} + \lambda_\nu(t)^\mathsf{T} B(t)](u_0(t) - u_\nu(t)) \ge 0\,. \tag{4.4}$$

Adding (4.4) and (4.2), respectively (4.3), we obtain

$$\alpha\,\|u_\nu - u_0\|_1^2 \le \int_0^T [-\nu u_\nu(t)^\mathsf{T} + (\lambda_0(t)^\mathsf{T} - \lambda_\nu(t)^\mathsf{T})B(t)](u_\nu(t) - u_0(t))dt, \tag{4.5}$$

if $\|u_\nu - u_0\|_1 \le 2\gamma\bar{\delta}$, and

$$\alpha\,\|u_\nu - u_0\|_1 \le \int_0^T [-\nu u_\nu(t)^\mathsf{T} + (\lambda_0(t)^\mathsf{T} - \lambda_\nu(t)^\mathsf{T})B(t)](u_\nu(t) - u_0(t))dt, \tag{4.6}$$

if $\|u_\nu - u_0\|_1 \geq 2\gamma\bar{\delta}$. Since x_ν, x_0 satisfy the system equation, and λ_ν, λ_0 satisfy the adjoint equation we obtain

$$\begin{aligned}\int_0^T &[(\lambda_0(t)^\mathsf{T} - \lambda_\nu(t)^\mathsf{T})B(t)](u_\nu(t) - u_0(t))dt \\ &= (x_0(T) - x_\nu(T))^\mathsf{T} Q(x_\nu(T) - x_0(T)) \\ &+ \int_0^T (x_0(t) - x_\nu(t))^\mathsf{T} W(t)(x_\nu(t) - x_0(t))\, dt.\end{aligned}$$

Together with (4.5), (4.6) this implies

$$\begin{aligned}&\alpha\, \|u_\nu - u_0\|_1^2 + (x_\nu(T) - x_0(T))^\mathsf{T} Q(x_\nu(T) - x_0(T)) \\ &\quad + \int_0^T (x_\nu(t) - x_0(t))^\mathsf{T} W(t)(x_\nu(t) - x_0(t))\, dt \\ &\quad \leq -\nu \int_0^T u_\nu(t)^\mathsf{T}(u_\nu(t) - u_0(t))dt,\end{aligned} \tag{4.7}$$

if $\|u_\nu - u_0\|_1 \leq 2\gamma\bar{\delta}$, and

$$\begin{aligned}&\alpha\, \|u_\nu - u_0\|_1 + (x_\nu(T) - x_0(T))^\mathsf{T} Q(x_\nu(T) - x_0(T)) \\ &\quad + \int_0^T (x_\nu(t) - x_0(t))^\mathsf{T} W(t)(x_\nu(t) - x_0(t))\, dt \\ &\quad \leq -\nu \int_0^T u_\nu(t)^\mathsf{T}(u_\nu(t) - u_0(t))dt,\end{aligned} \tag{4.8}$$

if $\|u_\nu - u_0\|_1 \geq 2\gamma\bar{\delta}$.

If Assumptions (A1) and (A2) are satisfied, we have $\alpha > 0$. Since the matrices Q, $W(t)$, $t \in [0, T]$, are assumed to be positive semidefinite, we obtain by (4.7)

$$\alpha\, \|u_\nu - u_0\|_1^2 \leq \nu\, \|u_\nu\|_\infty \|u_\nu - u_0\|_1,$$

if $\|u_\nu - u_0\|_1 \leq 2\gamma\bar{\delta}$, and by (4.7)

$$\alpha\, \|u_\nu - u_0\|_1 \leq \nu T \|u_\nu\|_\infty (\|u_\nu\|_\infty + \|u_0\|_\infty),$$

if $\|u_\nu - u_0\|_1 \geq 2\gamma\bar{\delta}$. In both cases we obtain the first estimate of (4.1) with some constant c_1 independent of ν. This also implies the second estimate of (4.1) since $z = x_\nu - x_0$ satisfies the linear differential equation

$$\dot{z}(t) = A(t)z(t) + B(t)(u_\nu(t) - u_0(t))$$

with initial condition $z(0) = 0$. ∎

Remark *Using the proof technique of Theorem 5.5 in Alt et al. (2011) one can show in addition that there exists a constant κ independent of ν such that for sufficiently small ν the optimal controls u_ν coincide with u_0 except on a set of measure $\leq \kappa\nu$.* $\diamond$

Example 4.1 We consider Problem $(\mathrm{OQ1})_\nu$ of Example 3.1. As for $\nu = 0$ the unique solution of the adjoint equation for $\nu > 0$ is given by (3.7), i.e., in this special case λ_ν is independent of ν. Therefore, with the switching function σ defined by (3.8) it follows from (2.7) that the optimal control u_ν is the projection onto the interval $[0, 2]$ of the function

$$-\frac{1}{\nu}\sigma(t) = -\frac{1}{\nu}\left(5 - 2e^{2-t}\right),$$

which is given by

$$u_\nu(t) = \begin{cases} 2 & \text{for } 0 \leq t < s_0, \\ -\frac{1}{\nu}\left(5 - 2e^{2-t}\right) & \text{for } s_0 \leq t < s_1, \\ 0 & \text{for } s_1 < t \leq 2, \end{cases}$$

where

$$s_0 = 2 - \ln(\frac{5}{2} + \nu), \quad s_1 = 2 - \ln(\frac{5}{2}).$$

Since $s_1 - s_0 \leq \frac{2}{5}\nu$ and u_ν coincides with u_0 on $[0, s_0] \cup [s_1, 2]$, this confirms the result of Theorem 4.1. $\diamond$

If Assumptions (A1) and (A2) are not satisfied, we assume that (A3) holds. Then it follows from (4.7) or (4.8) that with some constant $\alpha > 0$

$$\alpha \|x_\nu - x_0\|_2^2 \leq \nu T \|u_\nu\|_\infty (\|u_\nu\|_\infty + \|u_0\|_\infty),$$

which implies

$$\|x_\nu - x_0\|_2 \leq c\sqrt{\nu}$$

with some constant c independent of ν. Thus, we have shown the following result.

Theorem 4.2 *Let Assumption* (A3) *be satisfied. Then for sufficiently small ν the error estimate*

$$\|x_\nu - x_0\|_2 \leq c\sqrt{\nu} \tag{4.9}$$

holds with a constant c independent of ν. $\diamond$

5. Euler discretization

Given a natural number N, let $h = T/N$ be the mesh size. We approximate the space X_2 of controls by functions in the subspace $X_{2,N} \subset X_2$ of piecewise constant functions represented by their values $u(t_j) = u_j$ at the gridpoints $t_j = jh$, $j = 0, 1, \ldots, N-1$. Further, we approximate state and adjoint state variables by functions in the subspace $X_{1,N} \subset X_1$ of continuous, piecewise linear functions represented by their values $x(t_j) = x_j$, $\lambda(t_j) = \lambda_j$ at the gridpoints t_j, $j = 0, 1, \ldots, N$. Then the Euler discretization of (OQ) is given by (see, e.g., Dontchev, Hager, 1993)

$$
\begin{aligned}
(\mathrm{OQ})_{\nu,h} \quad & \min_{(x,u)\in X_{1,N}\times X_{2,N}} \quad f_{\nu,h}(x,u) \\
& \text{s.t.} \\
& x_{j+1} = x_j + h\left[A(t_j)x_j + B(t_j)u_j\right], \quad j = 0, 1, \ldots, N-1\,, \\
& x_0 = a\,, \\
& u_j \in U, \quad j = 0, 1, \ldots, N-1\,,
\end{aligned}
$$

where f_h is the linear-quadratic cost functional defined by

$$
\begin{aligned}
f_{\nu,h}(x,u) = {} & \frac{1}{2} x_N^{\mathsf T} Q x_N + q^{\mathsf T} x_N \\
& + h \sum_{j=0}^{N-1} \left[\frac{1}{2} x_j^{\mathsf T} W(t_j) x_j + w(t_j)^{\mathsf T} x_j + r(t_j)^{\mathsf T} u_j \right] + \frac{\nu}{2} h \sum_{j=0}^{N-1} |u_j|^2.
\end{aligned}
$$

We now combine known results for Euler discretizations of the regularized problems with the error estimate (4.1) for bang-bang controls. The proofs for error estimates for Euler approximations (see Dontchev, Hager, 1993; Seydenschwanz, 2010) shows that for the solution $(x_{\nu,h}, u_{\nu,h})$ of $(\mathrm{OQ})_{\nu,h}$ and the associated multiplier $\lambda_{\nu,h}$ we have the estimate

$$
\max\{\|u_{\nu,h} - u_\nu\|_\infty, \|x_{\nu,h} - x_\nu\|_\infty, \|\lambda_{\nu,h} - \lambda_\nu\|_\infty\} \le c_1 \frac{h}{\nu} \tag{5.1}
$$

with a constant c_1 independent of the mesh size h. If Assumptions (A1) and (A2) are satisfied it then follows from (4.1) that

$$
\begin{aligned}
\|u_{\nu,h} - u_0\|_1 &\le \|u_{\nu,h} - u_\nu\|_1 + \|u_\nu - u_0\|_1 \\
&\le c_2 \|u_{\nu,h} - u_\nu\|_\infty + \|u_\nu - u_0\|_1 \\
&\le c_3 \frac{h}{\nu} + c_4\, \nu
\end{aligned}
$$

with constants c_2, c_3, c_4 independent of h and ν. Therefore, the optimal convergence rate is obtained if we choose $\nu = \sqrt{h}$ which implies for $u_h := u_{\nu,h}$

$$
\|u_h - u_0\|_1 \le c\sqrt{h} \tag{5.2}
$$

with a constant c independent of the mesh size h. The following numerical experiments confirm this estimate.

Example 5.1 We consider the control problem (OQ1) of Example 3.1. Fig. 1 shows a control computed by Euler discretization and Table 1 shows some error estimates for $\|u_0 - u_h\|_1$ which confirm the theoretical findings. $\diamond$

Table 1. Error for regularized solutions with $\nu = \sqrt{h}$

N	10	25	50	75	100	150	200
h	0.2	0.08	0.04	0.02667	0.02	0.01333	0.01
$\sqrt{h}$	0.4472	0.2828	0.2	0.1633	0.1414	0.1155	0.1
$\|u_h - u_0\|_1$	0.9827	0.4491	0.303	0.1978	0.1811	0.135	0.1101
$\frac{\|u_h - u_0\|_1}{\sqrt{h}}$	2.197	1.588	1.515	1.211	1.281	1.169	1.101

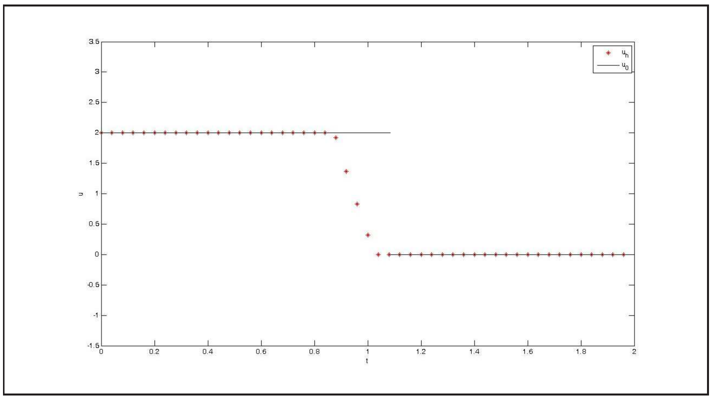

Figure 1. Optimal control for regularization with $\nu = \sqrt{h}$, $N = 50$

Example 5.2 We consider the following control problem

$$(\text{OQ1a})_\nu \ \min \int_0^1 2x_1(t) + 6x_2(t) - u_1(t) - u_2(t)\,dt + \frac{\nu}{2}\|u\|^2$$

s.t.

$$\dot{x}(t) = \begin{pmatrix} x_1(t) + u_1(t) \\ x_2(t) + u_2(t) \end{pmatrix} \qquad \text{a.e. on } [0,1],$$

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$-1 \le u_1(t) \le 1,\ -2 \le u_2(t) \le 2 \qquad \text{a.e. on } [0,1].$$

For $\nu = 0$ the optimal control is

$$u_{0,1} = \begin{cases} -1 & t < 1 - \ln(\frac{3}{2}) \\ 1 & t > 1 - \ln(\frac{3}{2}) \end{cases}, \quad u_{0,2} = \begin{cases} -2 & t < 1 - \ln(\frac{7}{6}) \\ 2 & t > 1 - \ln(\frac{7}{6}) \end{cases}.$$

Table 2 shows some error estimates for $\|u_0 - u_h\|_1$ which again confirm the theoretical findings. ◇

Table 2. Error for regularized solutions with $\nu = \sqrt{h}$

N	10	25	50	75	100	150	200
h	0.1	0.04	0.02	0.01333	0.01	0.006667	0.005
$\sqrt{h}$	0.3162	0.2	0.1414	0.1155	0.1	0.08165	0.07071
$\|u_h - u_0\|_1$	0.7535	0.4313	0.2781	0.2201	0.1872	0.1506	0.131
$\frac{\|u_h - u_0\|_1}{\sqrt{h}}$	2.383	2.157	1.966	1.906	1.872	1.844	1.853

If Assumptions (A1) and (A2) are not satisfied, we assume that (A3) holds. Then it follows from (4.9) and (5.1) that

$$\|x_{\nu,h} - x_0\|_2 \le c_3 \frac{h}{\nu} + c_4 \sqrt{\nu}$$

with constants c_3, c_4 independent of h and ν. Therefore, the optimal convergence rate is obtained if we choose $\nu = h^{\frac{2}{3}}$ which implies for $x_h := x_{\nu,h}$

$$\|x_h - x_0\|_2 \le c\, h^{\frac{1}{3}} \tag{5.3}$$

with a constant c independent of the mesh size h.

Example 5.3 We consider the control problem (OQ2) of Example 3.2. The results of Table 3 show that in this case the theoretical error estimates are not optimal. The results of Table 4 show that the optimal choice for the regularization parameter here seems to be $\nu = \sqrt{h}$. ◇

Table 3. Discretization error for regularization with $\nu = 0.1 h^{\frac{2}{3}}$

N	10	25	50	75	100	150	200
h	0.2	0.08	0.04	0.02667	0.02	0.01333	0.01
$\|x_h - x_0\|_2$	0.5275	0.2294	0.1364	0.1035	0.08614	0.06726	0.05682
$\frac{\|x_h - x_0\|_2}{h^{\frac{1}{3}}}$	0.902	0.5323	0.3989	0.3463	0.3174	0.2837	0.2637

Table 4. Discretization error for regularization with $\nu = 0.1\sqrt{h}$

N	10	25	50	75	100	150	200
h	0.2	0.08	0.04	0.02667	0.02	0.01333	0.01
$\|x_h - x_0\|_2$	0.5677	0.2779	0.1836	0.1484	0.1292	0.1073	0.09461
$\frac{\|x_h - x_0\|_2}{\sqrt{h}}$	1.269	0.9827	0.9178	0.9086	0.9133	0.9289	0.9461

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