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Poincaré-Wirtinger inequalities in bounded variation function spaces*

by

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Abstract: The goal of this paper is to extend Poincaré-Wirtinger inequalities from Sobolev spaces to spaces of functions of bounded variation of second order.

Keywords: measure theory.

1. Introduction

A useful tool when dealing with Sobolev spaces and partial differential equations is the Poincaré-Wirtinger inequality that provides norm equivalences under appropriate assumptions. These inequalities usually provide Sobolev embeddings and compactness results (see Adams, 1978). The goal of this paper is to extend Poincaré-Wirtinger inequalities from Sobolev spaces to spaces of functions of second order bounded variation. The result is known for the space of functions of first-order bounded variation (see Attouch, Buttazzo and Michaille, 2006; Ziemer, 1980). Indeed, this space is very useful in image processing context and many variational models are developed to deal with denoising of texture extraction. Variational models in image processing can be improved using the so-called BV^2 space that we define in the next section (Bergounioux and Piffet, 2010; Bredies, Kunisch and Pock, 2010; Demengel, 1984; Piffet, 2010). Generally, these models require a priori estimates on functions while first and/or second order derivative estimates are available.

2. The spaces of functions of bounded variation

We briefly recall the definitions and the main properties of (classical) spaces of functions of bounded variation. One can refer to Ambrosio, Fusco and Pollara (2000), Attouch, Buttazzo and Michaille (2006), Evans, Gariepy (1992) for a

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complete study of the BV space and to Bergounioux and Piffet (2010), Demengel (1984), Piffet (2010) for the so-called BV^2 space.

Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$ smooth enough (Lipschitz for example). The spaces $BV(\Omega)$ and $BV^2(\Omega)$ of functions of first-order and second-order bounded variation are defined by

$$BV(\Omega) = \{u \in L^1(\Omega) \mid \Phi_1(u) < +\infty\},$$

where

$$\Phi_1(u) := \sup\left\{\int_\Omega u(x)\,\mathrm{div}\ \phi(x)\,dx \mid \phi \in \mathcal{C}_c^1(\Omega),\ \|\phi\|_\infty \leq 1\right\} \tag{2.1}$$

and

$$BV^2(\Omega) = \{u \in W^{1,1}(\Omega) \mid \Phi_2(u) < +\infty\},$$

where

- The Sobolev space $W^{1,1}(\Omega)$ is defined as

$$W^{1,1}(\Omega) = \{\ u \in L^1(\Omega) \mid \nabla u \in L^1(\Omega)\ \}$$

- The second-order total variation is:

$$\Phi_2(u) := \sup\left\{\int_\Omega u \sum_{i,j=1}^n \frac{\partial^2 \xi_{ij}}{\partial x_i \partial x_j}\ dx \mid \xi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{n\times n}),\ \|\xi\|_\infty \leq 1\right\} < \infty, \tag{2.2}$$

with $\|\xi\|_\infty = \sup_{x\in\Omega} \sqrt{\sum_{i,j=1}^n |\xi_{ij}(x)|^2}$.

The following result makes precise the connection between $BV(\Omega)$ and $BV^2(\Omega)$:

Theorem 2.1 *A function u belongs to $BV^2(\Omega)$ if and only if $u \in W^{1,1}(\Omega)$ and $\frac{\partial u}{\partial x_i} \in BV(\Omega)$ for $i \in \{1, \ldots, n\}$. In particular*

$$\Phi_2(u) \leq \sum_{i=1}^n \Phi_1\left(\frac{\partial u}{\partial x_i}\right) \leq n\ \Phi_2(u).$$

Proof. Note that we may choose $\mathcal{C}^\infty$ functions in the above definitions so that

$$\Phi_1(u) = \sup\left\{\int_\Omega u\ \mathrm{div}\xi\ dx \mid \varphi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^n),\ \|\varphi\|_\infty \leq 1\right\},\ \text{and}$$

$$\Phi_2(u) = \sup\left\{\int_\Omega u \sum_{i,j=1}^n \frac{\partial^2 \xi_{ij}}{\partial x_i \partial x_j}\ dx \mid \xi \in \mathcal{C}_c^\infty\left(\Omega, \mathbb{R}^{n\times n}\right),\ \|\xi\|_\infty \leq 1\right\}.$$

We first prove the left-hand side inequality for any $u \in BV^2(\Omega)(\subset W^{1,1}(\Omega))$: let $\xi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^{n\times n})$ and perform an integration by parts

$$\int_\Omega u(x) \sum_{i,j=1}^n \frac{\partial^2 \xi_{ij}}{\partial x_i \partial x_j}(x)\, dx = -\sum_{i,j=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial \xi_{ij}}{\partial x_j}\, dx = \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \sum_{j=1}^n \frac{\partial(-\xi_{ij})}{\partial x_j}\, dx.$$

As $\displaystyle\sum_{j=1}^n \frac{\partial(-\xi_{ij})}{\partial x_j} = -\text{div } L_i(\varphi)$ where $L_i(\xi)$ is the ith row of the matrix ξ and $L_i(\xi)$ satisfies $\|L_i(\xi)\|_\infty \le 1$, we get

$$\int_\Omega u(x) \sum_{i,j=1}^n \frac{\partial^2 \xi_{ij}}{\partial x_i \partial x_j}(x)\, dx \le \sum_{i=1}^n \Phi_1\left(\frac{\partial u}{\partial x_i}\right).$$

The inequality is obtained by taking the supremum over ξ .
Let us prove the right-hand side inequality: for every $\varphi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^n)$, we get

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)\, div\, \varphi(x)\, dx = \int_\Omega \frac{\partial u}{\partial x_i}(x) \sum_{j=1}^n \frac{\partial \varphi_j}{\partial x_j}(x)\, dx = -\int_\Omega u(x) \sum_{j=1}^n \frac{\partial^2 \varphi_j}{\partial x_j \partial x_i}(x)\, dx.$$

Let $\psi^k : \Omega \to \mathbb{R}^{n\times n}$ be defined as follows: every line of the matrix $\psi^k(x)$ but the kth is null and the line k is $[\varphi_1(x), \cdots, \varphi_n(x)]$ so that

$$\sum_{j=1}^n \frac{\partial^2 \varphi_j}{\partial x_k \partial x_j} = \sum_{i,j=1}^n \frac{\partial^2 \psi^k_{ij}}{\partial x_i \partial x_j}.$$

Therefore, $\forall k \in \{1, \ldots, N\}$,

$$\int_\Omega \frac{\partial u}{\partial x_k}(x)\ div\ \varphi(x)\ dx = -\int_\Omega u(x) \sum_{i,j=1}^n \frac{\partial \psi^k_{ij}}{\partial x_i \partial x_j}(x)\ dx \le \Phi_2(u)\ .$$

This holds for every $\varphi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^n)$, so that $\forall i \in \{1, \ldots, n\}$,

$$\Phi_1\left(\frac{\partial u}{\partial x_i}\right) \le \Phi_2(u)$$

and

$$\sum_{i=1}^n \Phi_1\left(\frac{\partial u}{\partial x_i}\right) \le n\Phi_2(u)\ . \qquad \blacksquare$$

The space $BV(\Omega)$, endowed with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1} + \Phi_1(u)$, and $BV^2(\Omega)$ endowed with the following norm

$$\|u\|_{BV^2(\Omega)} := \|u\|_{W^{1,1}(\Omega)} + \Phi_2(f) = \|u\|_{L^1} + \|\nabla u\|_{L^1} + \Phi_2(f), \tag{2.3}$$

where Φ_2 is given by (2.2), are Banach spaces.

We next recall standard properties of functions of 1st and 2nd order bounded variation. We first have embedding theorems

PROPOSITION 2.1 *(Ambrosio, Fusco and Pallara, 2000; Attouch, Buttazzo and Michaille, 2006; Demengel, 1984; Piffet, 2010) Let Ω be an open subset of $\mathbb{R}^n$ with Lipschitz boundary.*

(1) *$BV(\Omega) \subset L^2(\Omega)$ with continuous embedding, if $n = 2$.*
(2) *$BV(\Omega) \subset L^p(\Omega)$ with compact embedding, for every $p \in [1, 2)$, if $n = 2$.*
(3) *Assume $n \geq 2$. Then*

$$BV^2(\Omega) \hookrightarrow W^{1,q}(\Omega) \quad \textit{with} \quad q \leq \frac{n}{n-1},$$

with continuous embedding. Moreover the embedding is compact if $q < \frac{n}{n-1}$.

We get lower semi-continuity results as well:

THEOREM 2.2 (1) *The mapping $u \mapsto \Phi_1(u)$ is lower semi-continuous from $BV(\Omega)$ to $\mathbb{R}^+$ for the $L^1(\Omega)$ topology.*

(2) *The mapping $u \mapsto \Phi_2(u)$ is lower semi-continuous from $BV^2(\Omega)$ endowed with the strong topology of $W^{1,1}(\Omega)$ to $\mathbb{R}$. More precisely, if $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of $BV^2(\Omega)$ that strongly converges to u in $W^{1,1}(\Omega)$ then*

$$\Phi_2(u) \leq \liminf_{k\to\infty} \Phi_2(u_k).$$

We end this section with a "density " result in $BV(\Omega)$:

THEOREM 2.3 *(Attouch, Buttazzo and Michaille, 2006, Theorem 10.1.2., p. 375) The space $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $BV(\Omega)$ in the following sense: for every $u \in BV(\Omega)$ there exist a sequence $(u_n)_{n\geq 0} \in \mathcal{C}^\infty(\overline{\Omega})$ such that*

$$\lim_{n\to+\infty} \|u_n - u\|_{L^1} = 0 \qquad \textit{and} \lim_{n\to+\infty} \Phi_1(u_n) = \Phi_1(u)\ .$$

This convergence is called the ***strict*** *convergence as in Ambrosio, Fusco and Pallara (2000).*

Let us define the space $BV_0(\Omega)$ as the space of functions of bounded variation that vanish on the boundary $\partial\Omega$ of Ω. More precisely, as Ω is bounded and $\partial\Omega$ is Lipschitz, functions of $BV(\Omega)$ have a trace of class L^1 on $\partial\Omega$ (Ambrosio, Fusco and Pallara, 2000; Attouch, Buttazzo and Michaille, 2006; Ziemer, 1980), and the trace mapping $T : BV(\Omega) \to L^1(\partial\Omega)$ is linear, continuous from $BV(\Omega)$ equiped with the strict convergence to $L^1(\partial\Omega)$ endowed with the strong topology (Attouch, Buttazzo and Michaille, 2006, Theorem 10.2.2, p. 386). The space $BV_0(\Omega)$ is then defined as the kernel of T. It is a Banach space, endowed with the induced norm.

REMARK 1 *We may also define $\widetilde{BV_0}$ as the closure of $\mathcal{D}(\Omega)$ (the space of $\mathcal{C}^\infty$ fuctions with compact support in Ω) for the strict convergence of $BV(\Omega)$.*

It is easy to see that $\widetilde{BV_0(\Omega)} \subset BV_0(\Omega)$. The converse inclusion is more technical to prove: the proof of Attouch, Buttazzo and Michaille (2006), p. 189 has to be adapted. Though we do not prove it in the present paper we conjecture it is true.

3. Poincaré-Wirtinger inequalities

3.1. Poincaré-Wirtinger inequality in $BV(\Omega)$

We first recall the classical Poincaré-Wirtinger inequality for the Sobolev-space $W^{1,1}(\Omega)$(see Attouch, Buttazzo and Michaille, 2006, p. 161–180 for example, or Attouch, Buttazzo and Michaille (2006).

THEOREM 3.1 *Let Ω be an open subset of $\mathbb{R}^n$, which is bounded in one direction. Then, there exists a constant C_Ω such that*

$$\forall v \in W_0^{1,1}(\Omega) \qquad \|v\|_{L^1(\Omega)} \leq C_\Omega \, \|\nabla v\|_{L^1(\Omega)} \ .$$

Moreover, if Ω is an open bounded set of class $\mathcal{C}^1$, then there exists a constant C_Ω such that

$$\forall u \in W^{1,1}(\Omega) \qquad \|u - m(u)\|_{L^1(\Omega)} \leq C_\Omega \, \|\nabla u\|_{L^1(\Omega)} \ ,$$

where where $m(u) := \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u(x)dx$ is the mean-value of u.

A consequence of the previous results is a Poincaré-Wirtinger inequality in the BV- space

THEOREM 3.2 *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz open bounded set. Then there exists a constant $C > 0$ such that*

$$\forall u \in BV(\Omega) \qquad \|u - m(u)\|_{L^1(\Omega)} \leq C\Phi_1(u) \ .$$

Proof. The result is mentioned in Attouch, Buttazzo and Michaille (2006), p. 399 (proof of Lemma 10.3.2), but we give the proof for convenience. Let $u \in BV(\Omega)$ and $(u_n)_{n\geq 0} \in \mathcal{C}^\infty(\overline{\Omega})$ be a sequence such that

$$\lim_{n\to+\infty} \|u_n - u\|_{L^1} = 0 \qquad \text{and} \lim_{n\to+\infty} \Phi_1(u_n) = \Phi_1(u) \ .$$

It is clear that $m(u_n) \to m(u)$. In addition, $u_n \in W^{1,1}(\Omega)$ since Ω is bounded. We use Theorem 3.1 to infer

$$\forall n \qquad \|u_n - m(u_n)\|_{L^1(\Omega)} \leq C\|\nabla u_n\|_{L^1} = \Phi_1(u_n) \ .$$

Passing to the limit gives the result. ■

COROLLARY 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz open bounded set. Then there exists a constant $C > 0$ such that*

$$\forall u \in BV(\Omega) \text{ such that } \int_\Omega u(x)dx = 0 \qquad \|u\|_{L^1(\Omega)} \leq C\Phi_1(u) \ .$$

Remark 2 *We have the same result on the set* Ω *for functions in* $\widetilde{BV_0}$. *The proof is the same and we use the definition of* $\widetilde{BV_0}$ *to approximate any function of* $\widetilde{BV_0}$ *by a sequence of* $\mathcal{D}(\Omega)$ *functions: let* Ω *be a Lipschitz open bounded subset of* $\mathbb{R}^n$, *then there exists a constant* $C_\Omega > 0$ *such that*

$$\forall u \in \widetilde{BV_0} \qquad \|u\|_{L^1(\Omega)} \leq C_\Omega \Phi_1(u) \ .$$

Another result can be found in Ziemer (1980) for functions in $BV_0(\Omega)$. More precisely

Theorem 3.3 *Let* Ω *be a connected and Lipschitz open bounded subset of* $\mathbb{R}^n$. *There exists a constant* $C_\Omega > 0$ *such that*

$$\forall u \in \ BV_0(\Omega) \qquad \|u\|_{L^1(\Omega)} \leq C_\Omega \Phi_1(u) \ .$$

Proof. We use the fact that $\partial\Omega$ is a borelian set so that Corollary 5.12.8 of Ziemer, 1980, can be used. Indeed the capacity of $\partial\Omega$ is different from 0 since the $n-1$ dimensional Hausdorff measure of A is nonnegative (Ziemer, 1980, Lemma 5.12.3). ∎

3.2. Poincaré-Wirtinger inequality in $BV^2(\Omega)$

We may extend the previous inequalities to functions in $BV^2(\Omega)$:

Corollary 3.2 *Let* $\Omega \subset \mathbb{R}^n$ *be a connected Lipschitz open, bounded set. Then there exists a constant* $C > 0$ *such that*

$$\forall u \in BV^2(\Omega) \qquad \|\nabla u - M(\nabla u)\|_{L^1(\Omega),\mathcal{N}} \leq C_{\mathcal{N}} \Phi_2(u),$$

where $M(V) := \{m(V_1), \cdots, m(V_n))$ *is the (vectorial) mean-value of* V *and*

$$\|V\|_{L^1(\Omega),\mathcal{N}} = \mathcal{N}((\|V_1\|_{L^1(\Omega)}, \cdots, \|V_n\|_{L^1(\Omega)})),$$

where $\mathcal{N}$ *denotes any norm in* $\mathbb{R}^n$ *(for example the* ℓ^1*-norm).*

Proof. As $u \in BV^2(\Omega)$ we use Proposition 2.1 to infer that for every $i \in \{1, \cdots, n\}$ $\dfrac{\partial u}{\partial x_i} \in BV(\Omega)$. With Theorem 3.2 we get the existence of C such that

$$\forall i \in \{1, \cdots, n\} \qquad \|\frac{\partial u}{\partial x_i} - m(\frac{\partial u}{\partial x_i})\|_{L^1(\Omega)} \leq C\Phi_1(\frac{\partial u}{\partial x_i}) \leq nC\,\Phi_2(u). \quad (3.1)$$

This gives the result. ∎

Let us consider the particular case where $u \in BV^2_m(\Omega)$, defined as follows

$$BV^2_m(\Omega) := \left\{ u \in BV^2(\Omega) \mid \int_\Omega \frac{\partial u}{\partial x_i}\, dx = 0 \ i = 1, \cdots, n \right\} \ .$$

Corollary 3.3 *Let $\Omega \subset \mathbb{R}^n$ be a connected Lipschitz open, bounded set. Then, there exists a constant $C_\Omega > 0$ only depending on Ω such that*

$$\forall u \in BV^2_m(\Omega), \quad \forall i = 1, \cdots, n \qquad \|\frac{\partial u}{\partial x_i}\|_{L^1(\Omega)} \leq C_\Omega \Phi_2(u) \ . \tag{3.2}$$

At last a direct corollary of Theorem 3.3 is the following:

Corollary 3.4 *Let $\Omega \subset \mathbb{R}^n$ be a connected Lipschitz open, bounded set. Then, there exists a constant $C_\Omega > 0$ only depending on Ω such that*

$$\forall u \in BV^2(\Omega) \text{ such that } \frac{\partial u}{\partial x_i} = 0 \text{ on } \partial\Omega, \quad \forall i = 1, \cdots, n$$

$$\|\frac{\partial u}{\partial x_i}\|_{L^1(\Omega)} \leq C_\Omega \Phi_2(u) \ . \tag{3.3}$$

Proof. We use Theorem 3.3 with $\frac{\partial u}{\partial x_i}$ to infer

$$\|\frac{\partial u}{\partial x_i}\|_{L^1(\Omega)} \leq C\Phi_1(\frac{\partial u}{\partial x_i}) \ ,$$

and we conclude with Theorem 2.1. ∎

3.3. Example

Assume $n = 2$ and $\Omega =]a_1, b_1[\times]a_2, b_2[$. Then $BV^2(\Omega) \subset W^{1,2}(\Omega)$, the trace of any function in $BV^2(\Omega)$ belongs to $L^2(\Omega)$ and the Green formula gives

$$\int_\Omega \frac{\partial u}{\partial x_1} dx_1\, dx_2 = \int_{a_2}^{b_2} (u(b_1, x_2) - u(a_1, x_2))\, dx_2 \ ,$$

and

$$\int_\Omega \frac{\partial u}{\partial x_2} dx_2\, dx_1 = \int_{a_1}^{b_1} (u(x_1, b_2) - u(x_1, a_2)\, dx_1 \ .$$

Therefore we get: for all $u \in BV^2(\Omega)$ such that $u = 0$ on $\partial\Omega$

$$\Phi_1(u) = \|\nabla u\|_{L^1(\Omega)} \leq C_\Omega \Phi_2(u) \ . \tag{3.4}$$

4. An application in image processing

In Bergounioux and Piffet (2010) we have investigated a second order variational model for image processing:

$$\min_{v \in BV^2(\Omega)} F(v) := \frac{1}{2}\|u_d - v\|^2_{L^2(\Omega)} + \lambda\Phi_2(v) + \delta\|v\|_{W^{1,1}(\Omega)},$$

where Ω is a square open set of $\mathbb{R}^2$, $\lambda, \delta > 0$ and $u_d \in L^2(\Omega)$. We chose $\delta > 0$ because we were not able to prove existence of solution without this assumption. However, we may now avoid the use of the penalization term $\delta \|v\|_{W^{1,1}(\Omega)}$ if we look for solutions in

$$BV_0^2(\Omega) := \{u \in BV^2(\Omega) \mid u_{|\partial\Omega} = 0 \} ,$$

solving

$$(\mathcal{P}) \qquad \inf\{ \ F(v) \mid v \in BV_0^2(\Omega) \ \} .$$

More precisely

THEOREM 4.1 *Assume $\lambda > 0$ and $\delta = 0$. Then problem $(\mathcal{P})$ has at least a solution.*

Proof. Let $v_n \in BV_0^2(\Omega)$ be a minimizing sequence, i.e.

$$\lim_{n \to +\infty} F(v_n) = \inf\{ \ F(v) \mid v \in BV_0^2(\Omega) \ \} < +\infty.$$

The sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $BV^2(\Omega)$.

Indeed $\Phi_2(v_n)$ is bounded and $\Phi_1(v_n)$ as well with relation (3.4). As v_n is L^2-bounded, it is also bounded in $W^{1,1}(\Omega)$. This yields that v_n is bounded in $BV^2(\Omega)$. With Proposition 2.1 we get the strong convergence (up to a subsequence) of $(v_n)_{n\in\mathbb{N}}$ in $W^{1,1}(\Omega)$ to v^*, the weak convergence in $W^{1,2}(\Omega)$ and the strong convergence in $L^2(\Omega)$. With Theorem 2.2 we get

$$\Phi_2(v^*) \le \liminf_{n\to+\infty} \Phi_2(v_n),$$

so that

$$F(v^*) \le \liminf_{n\to+\infty} F(v_n) = \min_{v \in BV_0^2(\Omega)} F(v).$$

It remains to prove that $v^* \in BV_0^2(\Omega)$. The compactness of the trace operator γ_0 from $W^{1,2}(\Omega)$to $L^2(\partial\Omega)$ (see Biergert, 2009 for example) gives the result, since $\gamma_0(v_n) = 0$ for every n and v_n weakly converges to v^* in $W^{1,2}(\Omega)$. ∎

A full study of this model with $\delta > 0$ (including numerical tests) has been performed in Bergounioux and Piffet (2010).

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