**Control and Cybernetics**

vol. **40** (2011) No. 4

# Identification of matrix parameters in elliptic PDEs*

by

**Klaus Deckelnick**[1] **and Michael Hinze**[2]

[1]Institut für Analysis und Numerik,
Otto–von–Guericke–Universität Magdeburg
Universitätsplatz 2, 39106 Magdeburg, Germany
[2]Schwerpunkt Optimierung und Approximation, Universität Hamburg
Bundesstraße 55, 20146 Hamburg, Germany

**Abstract:** In the present work we treat the inverse problem of identifying the matrix-valued diffusion coefficient of an elliptic PDE from multiple interior measurements with the help of techniques from PDE constrained optimization. We prove existence of solutions using the concept of H–convergence and employ variational discretization for the discrete approximation of solutions. Using a discrete version of H–convergence we are able to establish the strong convergence of the discrete solutions. Finally we present some numerical results.

**Keywords:** parameter identification, elliptic optimal control problem, control constraints, H–convergence, variational discretization.

## 1. Introduction

In this work we consider the inverse problem of identifying the diffusion matrix $A = A(x)$ in an elliptic PDE

$$-\text{div}\ (A(x)\nabla y) = g \text{ in } \Omega,\ y = 0 \text{ on } \partial\Omega \tag{1.1}$$

from multiple interior measurements of data. Here, $\Omega \subset \mathbb{R}^n$ is a bounded open set with a Lipschitz boundary. Furthermore, we assume that $A(x) = (a_{ij}(x))_{i,j=1}^n$ satisfies $a_{ij} \in L^\infty(\Omega)$ and that there exists $a > 0$ such that $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq a|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and a.a. $x \in \Omega$. Given $g \in H^{-1}(\Omega)$, the boundary value problem (1.1) then has a unique weak solution $y \in H_0^1(\Omega)$ in the sense that

$$\int_\Omega A\nabla y \cdot \nabla v dx = \langle g, v\rangle \quad \text{for all } v \in H_0^1(\Omega), \tag{1.2}$$

*Submitted: February 2011; Accepted: June 2011

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Furthermore,

$$\|y\|_{H_0^1} \le C \|g\|_{H^{-1}}, \tag{1.3}$$

with a constant $C$, which only depends on $a$. We shall denote this solution by $y = T(A, g)$ in order to also emphasize its dependence on $A$.

In what follows we assume that measurements $(z^{(i)}, f^{(i)}) \in Z \times H^{-1}(\Omega), 1 \le i \le N$ ($Z = L^2(\Omega)$ or $Z = H_0^1(\Omega)$) are available, from which we would like to reconstruct the diffusion matrix $A$. To do so, we employ a least squares approach together with a Tikhonov regularization, i.e. we consider

$$(P) \quad \min_{A \in \mathcal{M}} J(A) := \frac{1}{2} \sum_{i=1}^{N} \|y^{(i)} - z^{(i)}\|_Z^2 + \frac{\gamma}{2} \|A\|^2 \text{ s.t. } y^{(i)} = T(A, f^{(i)}), \quad 1 \le i \le N. \tag{1.4}$$

Here, $\gamma \ge 0$ and we use the symbol $\| \cdot \|$ for the $L^2$ norm on spaces of scalar, vector or matrix–valued functions, while the admissible set $\mathcal{M}$ will be specified in Section 2. Our choice, motivated by the concept of H–convergence, guarantees the existence of a minimum of $J$. By discretizing (1.1) with the help of linear finite elements we obtain an approximation $J_h$ of $J$. Our main result, Theorem 3.2, says that each sequence of minimizers $(A_h)_{h>0}$ of $J_h$ has a subsequence that converges strongly in $L^2$ to a minimum of $J$. In order to establish this result we shall adapt a discrete version of H–convergence, introduced by Eymard and Gallouët (2003) for finite volume schemes, to our setting. The above convergence result justifies the use of $J_h$ in solving the identification problem. In practice we employ a projected steepest descent algorithm for minimizing $J_h$, see Section 4.

Let us review related work which is concerned with the identification of matrix–valued parameters in elliptic PDEs. Alt, Hoffmann and Sprekels (1984) obtained a reconstructed matrix by investigating the long time behaviour of a suitable dynamical system, see also Hoffmann, Sprekels (1984/85). Kohn and Lowe (1988) introduced a variational method that is based on a convex functional involving the variables $y$ and $A\nabla y$ and investigated its stability properties. Stability results for the reconstruction of matrices of the form $A = \nabla p \otimes \nabla y$ can be found in Hsiao, Sprekels (1988). Rannacher and Vexler (2005) proved a–priori estimates for a matrix identification problem, in which a finite number of unknown parameters is estimated from finitely many pointwise observations.

A lot of work has been devoted to the parameter estimation problem for a scalar diffusion coefficient. Identifiability results can e.g. be found in Chicone, Gerlach (1987), Richter (1981), and Vainikko, Kunisch (1993). A survey of numerical methods for parameter estimation problems can be found in Kunisch (1995). Error estimates for a least squares approach have been obtained by Falk (1983) and more recently by Wang and Zou (2010) for a functional involving a Tikhonov regularization. That paper also contains a long list of further contributions. Let us finally note that the concept of H–convergence has recently been used by Leugering and Stingl (2010) in order to treat problems in material design, in particular to identify strain tensors from displacements in linear elasticity.

## 2. Existence of a minimum

Let us denote by $\mathcal{S}_n$ the set of all symmetric $n \times n$ matrices endowed with the inner product $A \cdot B = \mathrm{trace}(AB)$. We consider the subset

$$K := \{A \in \mathcal{S}_n \mid a \le \lambda_i(A) \le b, i = 1, \ldots, n\}$$

where $0 < a < b < \infty$ are given constants and $\lambda_1(A), \ldots, \lambda_n(A)$ denote the eigenvalues of $A$. Since $K$ is a convex and closed subset of $\mathcal{S}_n$ we may introduce the orthogonal projection $P_K : \mathcal{S}_n \to K$ for which we can derive a formula as follows: given $A \in \mathcal{S}_n$, let $S$ be an orthogonal matrix such that $SAS^t = \mathrm{diag}(\lambda_1(A), \ldots, \lambda_n(A)) =: D$. If we let

$$\tilde{D} = \mathrm{diag}\left(P_{[a,b]}(\lambda_1(A)), \ldots, P_{[a,b]}(\lambda_n(A))\right),$$

where $P_{[a,b]}(x) := \max\{a, \min\{x, b\}\}, x \in \mathbb{R}$, then clearly $S^t\tilde{D}S \in K$ and we have for every $B \in K$

$$\begin{aligned}(A - S^t\tilde{D}S) \cdot (B - S^t\tilde{D}S) &= (D - \tilde{D}) \cdot (SBS^t - \tilde{D}) \\ &= \sum_{i=1}^{n} (\lambda_i(A) - P_{[a,b]}(\lambda_i(A)))(\tilde{b}_{ii} - P_{[a,b]}(\lambda_i(A))),\end{aligned}$$

where $\tilde{B} = SBS^t \in K$. Hence, $\tilde{b}_{ii} \in [a, b], i = 1, \ldots, n$, which immediately yields

$$(A - S^t\tilde{D}S) \cdot (B - S^t\tilde{D}S) \le 0, \quad \text{for all } B \in K$$

and therefore $P_K(A) = S^t\mathrm{diag}\left(P_{[a,b]}(\lambda_1(A)), \ldots, P_{[a,b]}(\lambda_n(A))\right) S$.

Next, let us introduce the set

$$\mathcal{M} := \{A \in L^\infty(\Omega)^{n,n} \mid A(x) \in K \text{ a.e. in } \Omega\}.$$

In proving the existence to problem $(P)$ the following compactness result is crucial, see e.g. Tartar (1997).

THEOREM 2.1 *Let $(A_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. Then there exists a subsequence $(A_{k'})_{k'\in\mathbb{N}}$ and an element $A \in \mathcal{M}$ such that for every $g \in H^{-1}(\Omega)$*

$$T(A_{k'}, g) \rightharpoonup T(A, g) \text{ in } H_0^1(\Omega) \text{ and } A_{k'}\nabla T(A_{k'}, g) \rightharpoonup A\nabla T(A, g) \text{ in } L^2(\Omega)^n. \tag{2.1}$$

*The sequence $(A_{k'})_{k'\in\mathcal{M}}$ is then said to be $H$–convergent to $A$ and one writes $A_{k'} \overset{H}{\to} A$.*

Lemma 2.1 *Suppose that $(A_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{M}$ with $A_k \overset{H}{\to} A$ and $A_k \overset{*}{\rightharpoonup} A_0$ in $L^\infty(\Omega)^{n,n}$. Then $A(x) \le A_0(x)$ a.e. in $\Omega$ and*

$$\|A\|^2 \le \|A_0\|^2 \le \liminf_{k\to\infty} \|A_k\|^2. \tag{2.2}$$

*Proof.* We refer the reader to Theorem 5, Tartar (1997), for a proof of the fact that $A(x) \le A_0(x)$ a.e. in $\Omega$. Next, from the Courant–Fischer minmax theorem we infer that $\lambda_i(A(x)) \le \lambda_i(A_0(x)), i = 1, \dots, n$ and hence taking into account that $\lambda_i(A(x)) \ge 0$

$$|A(x)|^2 = \sum_{i=1}^n \lambda_i(A(x))^2 \le \sum_{i=1}^n \lambda_i(A_0(x))^2 = |A_0(x)|^2 \quad \text{a.e. in } \Omega.$$

Integration over $\Omega$ together with the weak lower semicontinuity of the $L^2$-norm then implies (2.2). ■

Using the concept of H-convergence we can now establish the existence of a solution to the minimization problem (1) including the limit case of $\gamma = 0$. In this context we also refer to Hofmann et al. (2007), where weak topologies have been used in order to analyze variational regularization methods.

Theorem 2.2 *Problem $(P)$ has a solution $A \in \mathcal{M}$ for every $\gamma \ge 0$.*

*Proof.* Let $(A_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$ be a minimizing sequence for problem $(P)$ so that $J(A_k) \searrow \inf_{A\in\mathcal{M}} J(A)$ as $k \to \infty$. Combining Theorem 2.1 with the fact that $(A_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)^{n,n}$ we deduce that there exist $A \in \mathcal{M}, A_0 \in L^\infty(\Omega)^{n,n}$ such that $A_{k'} \overset{H}{\to} A$ and $A_{k'} \overset{*}{\rightharpoonup} A_0$ in $L^\infty(\Omega)^{n,n}$ for some suitable subsequence. Letting $y^{(i)}_{k'} = T(A_{k'}, f^{(i)}), y^{(i)} = T(A, f^{(i)}), 1 \le i \le N$, we therefore have $y^{(i)}_{k'} \rightharpoonup y^{(i)}$ in $H^1_0(\Omega)$. Hence,

$$\begin{aligned} J(A) &= \frac{1}{2}\sum_{i=1}^N \|y^{(i)} - z^{(i)}\|_Z^2 + \frac{\gamma}{2}\|A\|^2 \\ &\le \liminf_{k'\to\infty} \frac{1}{2}\sum_{i=1}^N \|y^{(i)}_{k'} - z^{(i)}\|_Z^2 + \frac{\gamma}{2}\liminf_{k'\to\infty}\|A_{k'}\|^2 \\ &\le \liminf_{k'\to\infty} J(A_{k'}) = \inf_{A\in\mathcal{M}} J(A), \end{aligned}$$

where we also used (2.2). ■

Let us next derive a suitable form of the necessary first order optimality conditions for a solution of $(P)$. To begin, it is not difficult to verify that $J$ is Fréchet differentiable on $\mathcal{M}$ with

$$J'(A)H = \sum_{i=1}^N (y^{(i)} - z^{(i)}, w^{(i)})_Z + \gamma(A, H)_{L^2}, \quad H \in L^\infty(\Omega)^{n,n} \tag{2.3}$$

where $y^{(i)} = T(A, f^{(i)})$ and $w^{(i)} = D_A T(A, f^{(i)})H \in H_0^1(\Omega), 1 \le i \le N$ is the partial derivative of $T$ with respect to $A$ in direction $H$ which is given as the unique solution of

$$\int_\Omega A\nabla w^{(i)} \cdot \nabla v dx = -\int_\Omega H\nabla y^{(i)} \cdot \nabla v dx \quad \text{for all } v \in H_0^1(\Omega). \tag{2.4}$$

In order to rewrite (2.3) we introduce the functions $p^{(i)} \in H_0^1(\Omega)$, $i = 1, \ldots, N$ as the unique solutions of the following adjoint problems:

$$\int_\Omega A\nabla v \cdot \nabla p^{(i)} dx = (y^{(i)} - z^{(i)}, v)_Z \quad \text{for all } v \in H_0^1(\Omega). \tag{2.5}$$

Abbreviating $(a \otimes b)_{kl} = \frac{1}{2}(a_k b_l + a_l b_k), k, l = 1, \ldots, n$ for $a, b \in \mathbb{R}^n$ we then have

$$J'(A)H = \int_\Omega \big(-\sum_{i=1}^N \nabla y^{(i)} \otimes \nabla p^{(i)} + \gamma A\big) \cdot H dx, \quad H \in L^\infty(\Omega)^{n,n}. \tag{2.6}$$

Note that the above integral exists since $\nabla y^{(i)} \otimes \nabla p^{(i)} \in L^1(\Omega)^{n,n}$. In conclusion

THEOREM 2.3 *Let $A \in \mathcal{M}$ be a solution of $(P)$. Then for every $\lambda > 0$*

$$A(x) = P_K \left( A(x) - \lambda \left( \gamma A(x) - \sum_{i=1}^N \nabla y^{(i)}(x) \otimes \nabla p^{(i)}(x) \right) \right) \quad \textit{a.e. in } \Omega.$$

*Proof.* The optimality of $A$ implies that $J'(A)(\tilde{A} - A) \ge 0$ for all $\tilde{A} \in \mathcal{M}$ which can be rewritten with the help of (2.6) as follows:

$$\int_\Omega \left( A(x) - \lambda \left( \gamma A(x) - \sum_{i=1}^N \nabla y^{(i)}(x) \otimes \nabla p^{(i)}(x) \right) - A(x) \right) \times \left( \tilde{A}(x) - A(x) \right) dx \le 0 \quad \text{for all } \tilde{A} \in \mathcal{M}.$$

A localization argument shows that $A(x)$ is the orthogonal projection of

$$A(x) - \lambda \left( \gamma A(x) - \sum_{i=1}^N \nabla y^{(i)}(x) \otimes \nabla p^{(i)}(x) \right)$$

onto $K$ a.e. in $\Omega$, which implies the result. ∎

Let us note that in the case of $\gamma > 0$ the particular choice $\lambda = \frac{1}{\gamma}$ gives

$$A(x) = P_K \left( \frac{1}{\gamma} \left( \sum_{i=1}^N \nabla y^{(i)}(x) \otimes \nabla p^{(i)}(x) \right) \right) \quad \text{a.e. in } \Omega. \tag{2.7}$$

## 3. Finite element discretization

Let $\mathcal{T}_h$ be a regular triangulation of $\Omega$ with maximum mesh size

$$h := \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$$

and suppose that $\bar{\Omega}$ is the union of the elements of $\mathcal{T}_h$; boundary elements are allowed to have one curved face. We define the space of linear finite elements,

$$X_h := \{v_h \in H_0^1(\Omega) \,|\, v_h \text{ is a linear polynomial on each } T \in \mathcal{T}_h\}.$$

It is well known that there exists an interpolation operator $\Pi_h : H_0^1(\Omega) \to X_h$ such that

$$\Pi_h w \to w \text{ in } H^1(\Omega) \text{ as } h \to 0 \quad \text{for every } w \in H_0^1(\Omega). \tag{3.1}$$

For given $A \in \mathcal{M}$ and $g \in H^{-1}(\Omega)$, the problem

$$\int_\Omega A\nabla y_h \cdot \nabla v_h dx = \langle g, v_h \rangle \quad \text{for all } v_h \in X_h$$

has a unique solution $y_h = T_h(A, g) \in X_h$. Furthermore, a standard argument yields the error bound

$$\|y - y_h\|_{H_0^1} \leq \frac{b}{a} \inf_{v_h \in X_h} \|y - v_h\|_{H_0^1}, \quad \text{where } y = T(A, g). \tag{3.2}$$

In order to set up an approximation of $(P)$ we use variational discretization as in Hinze (2005) and for $\gamma \geq 0$ consider

$$(P_h) \ \min_{A \in \mathcal{M}} J_h(A) := \frac{1}{2}\sum_{i=1}^N \|y_h^{(i)} - z^{(i)}\|_Z^2 + \frac{\gamma}{2}\|A\|^2 \text{ s.t.} y_h^{(i)} = T_h(A, f^{(i)}), \quad 1 \leq i \leq N. \tag{3.3}$$

Similar arguments as in Section 2 show that $J_h$ is Fréchet differentiable and that for $A \in \mathcal{M}$

$$J_h'(A)H = \int_\Omega \left( -\sum_{i=1}^N \nabla y_h^{(i)} \otimes \nabla p_h^{(i)} + \gamma A \right) \cdot H dx, \quad H \in L^\infty(\Omega)^{n,n}. \tag{3.4}$$

Since $\dim X_h < \infty$, it is straightforward to see that $(P_h)$ has a solution $A_h \in \mathcal{M}$. Furthermore, Theorem 2.3 holds accordingly. In particular, we have the analogue of (2.7), so that every solution $A_h$ of $(P_h)$ satisfies

$$A_h(x) = P_K \left( \frac{1}{\gamma} \sum_{i=1}^N \nabla y_h^{(i)}(x) \otimes \nabla p_h^{(i)}(x) \right) \text{ a.e. in } \Omega, \tag{3.5}$$

provided that $\gamma > 0$. Here, $y_h^{(i)} = T_h(A_h, f^{(i)})$ and $p_h^{(i)} \in X_h$ are the solutions of the adjoint problems

$$\int_\Omega A_h \nabla v_h \cdot \nabla p_h^{(i)} dx = (y_h^{(i)} - z^{(i)}, v_h)_Z \quad \text{for all } v_h \in X_h, 1 \leq i \leq N. \tag{3.6}$$

Remark 3.1 *Let us note that in view of (3.5) $A_h$ is piecewise constant so that a discretization of the set $\mathcal{M}$ is not required. Variational discretization automatically yields solutions to* (3.3) *which allow for a finite-dimensional representation.*

In order to investigate the convergence of the approximate solutions we shall employ a discrete version of Theorem 2.1.

Theorem 3.1 *Let $(A_h)_{h>0}$ be a sequence in $\mathcal{M}$. Then there exists a subsequence $(A_{h'})_{h'>0}$ and $A \in \mathcal{M}$ such that for every $g \in H^{-1}(\Omega)$*

$$T_{h'}(A_{h'}, g) \rightharpoonup T(A, g) \text{ in } H_0^1(\Omega) \text{ and } A_{h'} \nabla T_{h'}(A_{h'}, g) \rightharpoonup A \nabla T(A, g) \text{ in } L^2(\Omega)^n. \tag{3.7}$$

*We then say that the sequence $(A_{h'})_{h' \in \mathcal{M}}$ Hd–converges to $A$ and write $A_{h'} \stackrel{Hd}{\to} A$.*

*Proof.* The line of argument follows the corresponding proof in the continuous case (see Tartar, 1997) and a similar result for a finite volume scheme, see Eymard, Gallouët (2003). We therefore only sketch the main steps.
*Step 1:* One first shows that there exists a subsequence, for ease of notation again denoted by $(A_h)_{h>0}$, and continuous linear operators $S : H^{-1}(\Omega) \to H_0^1(\Omega)$, $R : H^{-1}(\Omega) \to L^2(\Omega)^n$ such that for every $g \in H^{-1}(\Omega)$

$$T_h(A_h, g) \rightharpoonup S(g) \text{ in } H_0^1(\Omega), \quad A_h \nabla T_h(A_h, g) \rightharpoonup R(g) \text{ in } L^2(\Omega)^n \quad \text{as } h \to 0. \tag{3.8}$$

*Step 2:* We show that $S$ is invertible. For $g \in H^{-1}(\Omega)$ denote by $w \in H_0^1(\Omega), w_h \in X_h$ the solutions of

$$\int_\Omega \nabla w \cdot \nabla v dx = \langle g, v \rangle, \quad v \in H_0^1(\Omega), \qquad \int_\Omega \nabla w_h \cdot \nabla v_h dx = \langle g, v_h \rangle, \quad v_h \in X_h.$$

Clearly, $\|w\|_{H_0^1} = \|g\|_{H^{-1}}$ and $w_h \to w$ in $H_0^1(\Omega)$ in view of (3.1). Setting $y_h = T_h(A_h, g)$ we have in addition that

$$\int_\Omega A_h \nabla y_h \cdot \nabla v_h dx = \langle g, v_h \rangle = \int_\Omega \nabla w_h \cdot \nabla v_h dx, \quad v_h \in X_h,$$

from which we infer that $\|w_h\|_{H_0^1} \leq b \|y_h\|_{H_0^1}$ recalling the definition of $\mathcal{M}$. Combining this bound with (3.8) and using again the properties of $\mathcal{M}$ we deduce that

$$\|g\|^2_{H^{-1}} = \|w\|^2_{H^1_0} = \lim_{h\to 0} \|w_h\|^2_{H^1_0} \le b^2 \liminf_{h\to 0} \|y_h\|^2_{H^1_0}$$
$$\le \frac{b^2}{a} \liminf_{h\to 0} \int_\Omega A_h \nabla y_h \cdot \nabla y_h dx = \frac{b^2}{a} \liminf_{h\to 0} \langle g, y_h\rangle = \frac{b^2}{a}\langle g, S(g)\rangle, \qquad (3.9)$$

which implies that $S$ is invertible.
*Step 3:* Let $C : H^1_0(\Omega) \to L^2(\Omega)^n$ be defined by $Cv := RS^{-1}v$. For a given $g \in H^{-1}(\Omega)$ the function $y_h = T_h(A_h, g)$ satisfies by definition

$$\int_\Omega A_h \nabla y_h \cdot \nabla v_h dx = \langle g, v_h\rangle \quad \text{for all } v_h \in X_h. \qquad (3.10)$$

Sending $h \to 0$ and taking into account (3.8) and (3.1) we infer

$$\int_\Omega Cy \cdot \nabla v dx = \langle g, v\rangle \quad \text{for all } v \in H^1_0(\Omega), \quad \text{where } y = S(g). \qquad (3.11)$$

Next, let $g, \tilde g \in H^{-1}(\Omega)$ be arbitrary and define $y = S(g), \tilde y = S(\tilde g)$ as well as $y_h = T_h(A_h, g), \tilde y_h = T_h(A_h, \tilde g)$. Recalling (3.10) we have for every $\varphi \in C^\infty_0(\Omega)$

$$\int_\Omega \varphi A_h \nabla y_h \cdot \nabla \tilde y_h dx = \int_\Omega A_h \nabla y_h \cdot \nabla r_h dx + \langle g, \varphi \tilde y_h\rangle - \langle g, r_h\rangle$$
$$- \int_\Omega A_h \nabla y_h \cdot \nabla \varphi\, \tilde y_h dx,$$

where we have abbreviated $r_h = \varphi \tilde y_h - I_h(\varphi \tilde y_h)$ and $I_h$ denotes the Lagrange interpolation operator. A standard interpolation estimate implies

$$\|\varphi \tilde y_h - I_h(\varphi \tilde y_h)\|_{H^1(T)} \le Ch\|D^2(\varphi \tilde y_h)\|_{L^2(T)} \le Ch\|\varphi\|_{W^{2,\infty}(T)}\|\tilde y_h\|_{H^1(T)},\ T \in \mathcal{T}_h,$$

so that $r_h \to 0$ in $H^1_0(\Omega)$ as $h \to 0$ since $\|\tilde y_h\|_{H^1} \le C$. Observing in addition that $A_h \nabla y_h \rightharpoonup Cy$ in $L^2(\Omega)^n$, $\varphi \tilde y_h \rightharpoonup \varphi \tilde y$ in $H^1_0(\Omega)$ and $\tilde y_h \to \tilde y$ in $L^2(\Omega)$ we obtain

$$\lim_{h\to 0} \int_\Omega \varphi A_h \nabla y_h \cdot \nabla \tilde y_h dx = \langle g, \varphi \tilde y\rangle - \int_\Omega Cy \cdot \nabla \varphi\, \tilde y dx,$$

which, combined with (3.11), yields

$$\lim_{h\to 0} \int_\Omega \varphi A_h \nabla y_h \cdot \nabla \tilde y_h dx = \int_\Omega \varphi Cy \cdot \nabla \tilde y dx. \qquad (3.12)$$

Similarly as in Eymard, Gallouët (2003), Proof of Theorem 2, one now deduces from (3.11) and (3.12) that there exists $A \in \mathcal{M}$ such that

$$(Cy)(x) = A(x)\nabla y(x) \quad \text{a.e. in } \Omega. \qquad (3.13)$$

This completes the proof of the theorem. ∎

COROLLARY 3.1 *Suppose that* $(A_h)_{h>0}$ *is a sequence in* $\mathcal{M}$ *with* $A_h \stackrel{Hd}{\to} A$ *and* $A_h \stackrel{*}{\rightharpoonup} A_0$ *in* $L^\infty(\Omega)^{n,n}$. *Then* $A \le A_0$ *a.e. in* $\Omega$ *and* $\|A\|^2 \le \|A_0\|^2 \le \liminf_{h\to 0} \|A_h\|^2$.

*Proof.* We use the same notation as in the proof of Theorem 3.1. By *Step 2* above there exists for every $y \in H_0^1(\Omega)$ an element $g \in H^{-1}(\Omega)$ such that $y = S(g)$. Defining $y_h = T_h(A_h, g)$ there holds $y_h \rightharpoonup y$ in $H_0^1(\Omega)$, $A_h\nabla y_h \rightharpoonup A\nabla y$ in $L^2(\Omega)^n$. Furthermore, we have for any $\varphi \in C_0^\infty(\Omega), \varphi \ge 0$ that

$$\begin{aligned} 0 &\le \int_\Omega \varphi A_h \nabla(y_h - y)\cdot\nabla(y_h - y)dx \\ &= \int_\Omega \varphi A_h\nabla y_h\cdot\nabla y_h dx - 2\int_\Omega \varphi A_h\nabla y_h\cdot\nabla y dx + \int_\Omega \varphi A_h \nabla y\cdot\nabla y dx. \end{aligned}$$

Recalling (3.12), (3.13) and the fact that $A_h \stackrel{*}{\rightharpoonup} A_0$ in $L^\infty(\Omega)^{n,n}$ we obtain upon sending $h \to 0$

$$0 \le -\int_\Omega \varphi A\nabla y\cdot\nabla y dx + \int_\Omega \varphi A_0 \nabla y\cdot\nabla y dx,$$

from which we infer that $A\nabla y\cdot\nabla y \le A_0\nabla y\cdot\nabla y$ a.e. in $\Omega$. Since $y \in H_0^1(\Omega)$ is arbitrary, we deduce that $A \le A_0$ a.e. in $\Omega$. The remaining estimate is obtained in the same way as in the proof of Lemma 2.1. ∎

We are now in position to prove a convergence result for a sequence $(A_h)_{h>0}$ of solutions of $(P_h)$ in the case of $\gamma > 0$.

THEOREM 3.2 *Let* $\gamma > 0$ *and let* $A_h \in \mathcal{M}$ *be a solution of* $(P_h)$. *Then there exists a subsequence* $(A_{h'})_{h'>0}$ *and* $A \in \mathcal{M}$ *such that* $A_{h'} \to A$ *in* $L^2(\Omega)^{n,n}$, $T_{h'}(A_{h'}, f^{(i)}) \to T(A, f^{(i)})$ *in* $Z$, $1 \le i \le N$ *and* $A$ *is a solution of* $(P)$.

*Proof.* In view of Theorem 3.1 and Corollary 3.1 there exists a subsequence, again denoted by $(A_h)_{h>0}$, and $A \in \mathcal{M}$ such that $A_h \stackrel{Hd}{\to} A$ and $A_h \stackrel{*}{\rightharpoonup} A_0$ in $L^\infty(\Omega)^{n,n}$ with $A \le A_0$ a.e. in $\Omega$. Let $y_h^{(i)} = T_h(A_h, f^{(i)})$, $y^{(i)} = T(A, f^{(i)})$, $1 \le i \le N$. Then $y_h^{(i)} \rightharpoonup y^{(i)}$ in $H_0^1(\Omega)$, $A_h\nabla y_h^{(i)} \rightharpoonup A\nabla y^{(i)}$ in $L^2(\Omega)^n$, so that we may deduce similarly as in the proof of Theorem 2.2 that $J(A) \le \liminf_{h\to 0} J_h(A_h)$. Next, Theorem 2.2 implies that $(P)$ has a solution $\bar{A} \in \mathcal{M}$. Then we have

$$J(\bar{A}) \le J(A) \le \liminf_{h\to 0} J_h(A_h) \le \limsup_{h\to 0} J_h(A_h) \le \limsup_{h\to 0} J_h(\bar{A}) = J(\bar{A}),$$

where the last equality follows from (3.2) and (3.1). We deduce that

$$\lim_{h\to 0} J_h(A_h) = J(A) = J(\bar{A}), \tag{3.14}$$

in particular, $A$ is a minimum of $J$. Furthermore, we have

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{N}\|y_h^{(i)}-y^{(i)}\|_Z^2+\frac{\gamma}{2}\|A_h-A\|^2 &= \frac{1}{2}\sum_{i=1}^{N}\|(y_h^{(i)}-z^{(i)})-(y^{(i)}-z^{(i)})\|_Z^2 \\
&\quad+\frac{\gamma}{2}\|A_h-A\|^2 \\
&= \frac{1}{2}\sum_{i=1}^{N}\|y_h^{(i)}-z^{(i)}\|_Z^2-\sum_{i=1}^{N}(y_h^{(i)}-z^{(i)},y^{(i)}-z^{(i)})_Z+\frac{1}{2}\sum_{i=1}^{N}\|y^{(i)}-z^{(i)}\|_Z^2 \\
&\quad+\frac{\gamma}{2}\|A_h\|^2-\gamma(A_h,A)+\frac{\gamma}{2}\|A\|^2 \\
&= J_h(A_h)+J(A)-\sum_{i=1}^{N}(y_h^{(i)}-z^{(i)},y^{(i)}-z^{(i)})_Z-\gamma(A_h,A) \\
&\to 2J(A)-\sum_{i=1}^{N}\|y^{(i)}-z^{(i)}\|_Z^2-\gamma(A_0,A) \\
&\le 2J(A)-\sum_{i=1}^{N}\|y^{(i)}-z^{(i)}\|_Z^2-\gamma\|A\|^2=0,
\end{aligned}$$

where we have used (3.14) and the fact that $A \le A_0$ a.e. in $\Omega$. The theorem is proven. ∎

## 4. Numerical examples

In this section we consider (3.3) with $Z = L^2(\Omega)$ and $\Omega := (-1,1)^2 \subset \mathbb{R}^2$. We take a finite element approximation with piecewise linear, continuous functions defined on a triangulation containing 512 triangles, constructed with the POIMESH environment of MATLAB. Let $N = 1$ with data $(z, f)$ given by $z = I_h y$ where

$$y(x_1,x_2)=(1-x_1^2)(1-x_2^2) \text{ and } f(x_1,x_2)=(1-x_2^2)(6x_1^2+2)+2(1-x_1^2).$$

Note that $y$ is the solution of (1.1) when

$$A(x_1,x_2)=\begin{bmatrix} 1+x_1^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

In the definition of $K$ we have chosen $a = 0.5$ and $b = 10$. The discrete problem (3.3) is solved using the projected steepest descent method with Armijo step size rule, see, e.g., Kelley (1999). In view of Remark 3.1 it is sufficient to iterate within the class of matrices in $\mathcal{M}$ that are piecewise constant. Given such an $A$, the new iterate is computed according to

$$A^+ = A(\tau) \text{ with } \tau=\max_{l\in\mathbb{N}}\{\beta^l; J_h(A(\beta^l))-J_h(A)\le-\frac{\sigma}{\beta^l}\|A(\beta^l)-A\|^2\}$$

where $\beta \in (0,1)$ and

$$A(\tau)_{|T} := P_K\Big(A_{|T} + \tau\big(\nabla y_{h|T} \otimes \nabla p_{h|T} - \gamma A_{|T}\big)\Big), \quad T \in \mathcal{T}_h.$$

Here, $y_h = T_h(A,f)$ and $p_h$ is the solution of the adjoint problem (3.6). In our calculations we chose $\gamma = 0.001$, $\sigma = 10^{-4}$, $\beta = 0.5$ and as initial matrix

$$A^0 := \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

The iteration was stopped if $\|A^+ - A(1)\| \le \tau_a + \tau_r \|A^0 - A^0(1)\|$ or the maximum number of 5000 iterations was reached. For $\tau_a = 10^{-3}$ and $\tau_r = 10^{-2}$ we have $\|A^0 - A^0(1)\| = 7.94 \times 10^{-2}$, $J_h(A^0) = 2.18 \times 10^{-1}$ and the algorithm terminated after 400 iterations with $\tilde{A}$ and $\tilde{y}_h = T_h(\tilde{A}, f)$ such that

$$\|\tilde{y}_h - z\| = 1.02 \times 10^{-2}, \quad \|A - \tilde{A}\| = 2.05 \text{ and } J_h(\tilde{A}) = 2.77 \times 10^{-2}.$$

Note that we cannot expect the difference $A - \tilde{A}$ to become small since the diffusion matrix will not be determined uniquely by just one set of data. Upon performing 5000 iterations we obtained $\tilde{A}$ and $\tilde{y}_h$ such that

$$\|\tilde{y}_h - z\| = 8.22 \times 10^{-3}, \quad \|A - \tilde{A}\| = 1.53 \text{ and } J_h(\tilde{A}) = 2.32 \times 10^{-2}.$$

Fig. 1 from left to right shows $\tilde{y}_h, z$ and $\tilde{y}_h - z$ after 400 iterations.

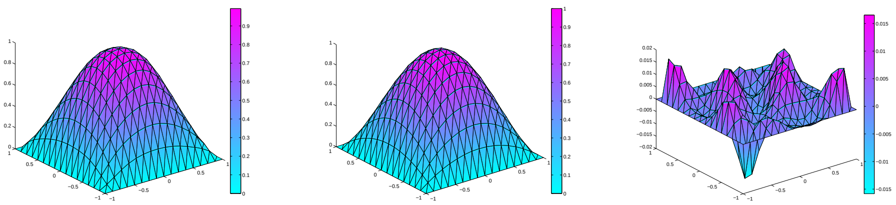

Figure 1. Numerical solution, desired state, and error $\tilde{y}_h - z$ for $\gamma = 1. \times 10^{-3}$ after the stopping criterion of the projected steepest descent method is met

By combining the projected gradient method with a homotopy in the parameter $\gamma$ we were also able to treat the case of $\gamma = 0$. We started with $\gamma = 1$ and reduced $\gamma$ by a factor of 0.8 after every ten iterations. Using the same notation as above we obtained after 5000 iterations

$$\|\tilde{y}_h - z\| = 9.61 \times 10^{-4}, \quad \|A - \tilde{A}\| = 1.40$$

and the corresponding results are displayed in Fig. 2. One observes that the difference between $\tilde{y}_h$ and $z$ is comparatively large in regions where $\nabla y$ is small, which is in agreement with classical results on the identifiability of scalar diffusion coefficients, see, e.g., Richter (1981).

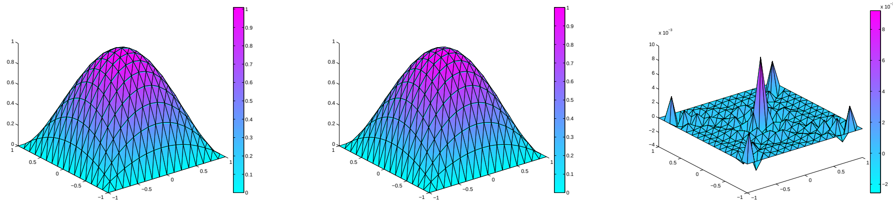

Figure 2. Numerical solution, desired state, error $\tilde{y}_h - z$ for $\gamma = 0$ after 5000 iterations of the steepest descent method

**Acknowledgements**

The presentation of the numerical results is partly based on a MATLAB code developed by Ronny Hoffmann in his diploma thesis (Hoffmann, 2005) written under the supervision of the second author. The authors gratefully acknowledge the support of the DFG Priority Program 1253 entitled *Optimization With Partial Differential Equations*.

## References

Alt, H.W., Hoffmann, K.H. and Sprekels, J. (1984) A numerical procedure to solve certain identification problems. *Intern. Ser. Numer. Math.* **68**, 11–43.

Chicone, C. and Gerlach, J. (1987) A note on the identifiability of distributed parameters in elliptic equations. *SIAM J. Math. Anal.* **18**, 1378–1384.

Eymard, R. and Gallouët, T. (2003) H–convergence and numerical schemes for elliptic problems. *SIAM J. Numer. Anal.* **41**, 539–562.

Falk, R.S. (1983) Error estimates for the numerical identification of a variable coefficient. *Math. Comput.* **40**, 537–546.

Hinze, M. (2005) A variational discretization concept in control constrained optimization: the linear-quadratic case. *Comput. Optim. Appl.* **30**, 45–61.

Hoffmann, K.H. and Sprekels, J. (1984/85) On the identification of coefficients of elliptic problems by asymptotic regularization. *Numer. Funct. Anal. Optim.* **7**, 157–177.

Hoffmann, R. (2005) *Entwicklung numerischer Methoden zur Schätzung matrixwertiger verteilter Parameter bei elliptischen Differentialgleichungen*, Diploma thesis, TU Dresden.

Hofmann, B., Kaltenbacher, B., Pöschl, C. and Scherzer, O. (2007) A convergence rates result for Tikhonov regularization in Banach spaces with non–smooth operators. *Inverse Problems* **23**, 987–1010.

HSIAO, G.C. and SPREKELS, J. (1988) A stability result for distributed parameter identification in bilinear systems. *Math. Meth. Appl. Sciences* **10**, 447–456.

KELLEY, C.T. (1999) *Iterative Methods for Optimization.* SIAM.

KOHN, R.V. and LOWE, B.D. (1988) A variational method for parameter identification. *RAIRO Modél. Math. Anal. Numér.* **22**, 119–158.

KUNISCH, K. (1995) Numerical methods for parameter estimation problems. In: *Inverse Problems in Diffusion Processes (Lake St. Wolfgang, 1994).* SIAM, Philadelphia, PA, 199–216.

LEUGERING, G. and STINGL, M. (2010) PDE-constrained optimization for advanced materials. In: G. Leugering et al., eds, *Constrained Optimization and Optimal Control for Partial Differential Equations*. Birkhäuser.

RANNACHER, R. and VEXLER, B. (2005) A priori estimates for the finite element discretization of elliptic parameter identification problems with pointwise measurements. *SIAM J. Cont. Optim.* **44**, 1844–1863.

RICHTER, G.R. (1981) An inverse problem for the steady state diffusion equation. *SIAM J. Appl. Math.* **41**, 210–221.

TARTAR, L. (1997) Estimation of homogenized coefficients. In: A. Cherkaev, R. Kohn, eds., *Topics in the Mathematical Modelling of Composite Materials*. Birkhäuser, 9-20.

VAINIKKO, G. and KUNISCH, K. (1993) Identifiability of the transmissivity coefficient in an elliptic boundary value problem. *Z. Anal. Anwendungen* **12**, 327–341.

WANG, L. and ZOU, J. (2010) Error estimates of finite element methods for parameter identification problems in elliptic and parabolic systems. *Discrete Contin. Dyn. Syst. Ser. B* **14**, 1641–1670.