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Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound*

by

Barbara Kaltenbacher[1], Irena Lasiecka[2†] and Richard Marchand[3]

[1]University of Klagenfurt, Klagenfurt, Austria

[2]University of Virginia, Charlottesville, USA
and Systems Research Institute, Polish Academy of Sciences, Warsaw, Poland

[3]U.S. Air Force Academy, Colorado Springs, Colorado, and Slippery Rock University, Pennsylvania, USA

email: barbara.kaltenbacher@aau.at, il2v@virginia.edu, richard.marchand@sru.edu

Abstract: We consider the Moore-Gibson-Thompson equation which arises, e.g., as a linearization of a model for wave propagation in viscous thermally relaxing fluids. This third order in time equation displays, even in the linear version, a variety of dynamical behaviors for their solutions that depend on the physical parameters in the equation. These range from non-existence and instability to exponential stability (in time). It will be shown that by neglecting diffusivity of the sound coefficient there arises a lack of existence of a semigroup associated with the linear dynamics. More specifically, the corresponding linear dynamics consists of three diffusions: two backward and one forward. When diffusivity of the sound is positive, the linear dynamics is described by a strongly continuous semigroup which is exponentially stable when the ratio of $\frac{\textit{sound speed}\times \textit{relaxation parameter}}{\textit{sound diffusivity}}$ is sufficiently small, and unstable in the complementary regime.

The theoretical estimates proved in the paper are confirmed by numerical validation.

Keywords: high intensity ultrasound, strongly continuous semigroup, exponential stability

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1. Introduction

Investigations on nonlinear propagation of sound in the situation of high amplitude waves have put forth extensive literature on physically well-based partial differential models, see, e.g., Crighton (1979), Coulouvrat (1992), Hamilton and Blackstock (1987), Kuznetsov (1971), Makarov and Ochmann (1997), Tjøtta (2001), or Westervelt (1963). This still highly active field of research is driven by a wide range of applications such as the medical and industrial use of high intensity ultrasound in lithotripsy, thermotherapy, ultrasound cleaning and sonochemistry.

The classical models of nonlinear acoustics are Kuznetsov's equation, the Westervelt equation, and the KZK (Kokhlov-Zabolotskaya-Kuznetsov) equation. For a mathematical well-posedness analysis of several types of initial boundary value problems for these nonlinear second order in time PDEs we refer to, e.g., Kaltenbacher and Lasiecka (2009, 2011, 2012), Kaltenbacher, Lasiecka Veljović (2011), Rozanova-Pierrat (2008). Motivated mainly by the fact that the use of classical Fourier's law leads to an infinite signal speed paradox, the use of several other constitutive relations for the heat flux within the derivation of nonlinear acoustic wave equations has been considered (see Jordan, 2009, and the references therein). Among these is the Maxwell-Cattaneo law, whose combination with the usual balance equations (conservation of mass, momentum and energy) as well as the equation of state, leads to a third order in time PDE model. A crucial prerequisite for investigating the fully nonlinear third order PDE is a thorough understanding of the well-posedness and asymptotic behavior of its linearization. Since this linearized version appears in a slightly different setting in Moore & Gibson (1960) and Thompson (1972) (see equation (11.84) on p. 556 therein), we call it here Moore-Gibson-Thompson equation, while the fully nonlinear version will be referred to as the Jordan-Moore-Gibson-Thompson equation.

The aim of this paper is to provide results for this linearized case under different relevant scenarios for the equation parameters. Numerical calculations presented in Section 6 confirm the theoretical results and quantitative estimates.

It should be noted that the analysis of the third order equations is very different from that of the second order, where a positive diffusivity coefficient provides a regularizing effect. This is no longer true in the third order equations which are of *hyperbolic type*, thus requiring a very different type of analysis than the one pertaining to second order in time equations.

1.1. PDE model

Models for thermo-viscous flow in compressible fluid relate the following quantities:

- the acoustic particle velocity $\vec{v}$ which, since assumed to be irrotational, can be expressed via a scalar acoustic velocity potential ψ in $\vec{v} = -\nabla\psi$;

- the acoustic pressure p, where $p = p_0 + p_\sim$ with $\nabla p_0 = 0$;
- the mass density $\varrho = \varrho_0 + \varrho_\sim$ with $\varrho_t = 0$;
- the temperature θ;
- the heat flux $\vec{q}$;
- the specific entropy η;

and they are based on the following relations:

- the conservation of mass, momentum and energy:

$$\varrho_t + \nabla(\varrho\vec{v}) = 0\,, \tag{1}$$

$$\varrho\Big(\vec{v}_t + \nabla(\vec{v}\cdot\vec{v})\Big) = -\nabla p + \Big(\frac{4\mu_v}{3} + \zeta_v\Big)\Delta\vec{v}\,, \tag{2}$$

$$\varrho\theta(\eta_t + \vec{v}\cdot\nabla\eta) = -\nabla\cdot\vec{q} + \mathbb{T}:\mathbb{D}, \tag{3}$$

where $\mathbb{D} = \frac{1}{2}(\nabla\vec{v} + (\nabla\vec{v})^T)$, $\mathbb{T} = -p\mathbb{I} + 2\mu_V\mathbb{D} + \lambda(\nabla\cdot\vec{v})\mathbb{I}$

- the (non-isentropic) state equation

$$p_\sim = \rho_0 c^2\left\{\frac{\rho_\sim}{\rho_0} + \frac{B}{2A}\left(\frac{\rho_\sim}{\rho_0}\right)^2 + \frac{\gamma-1}{\chi c^2}(\eta - \eta_0)\right\}, \tag{4}$$

where $\zeta_V = \lambda + \frac{2}{3}\mu_V$ is the bulk viscosity, μ_V the shear viscosity, c the speed of sound, B/A the parameter of nonlinearity, χ the coefficient of volume expansion.

Closing this system by Fourier's law for the heat flux

$$\vec{q} = -K\nabla\theta \tag{5}$$

with the thermal conductivity K, subtracting the time derivative of (1) from the divergence of (2) as well as inserting (4), (3), and neglecting third and higher order terms in the fluctuating quantities $\vec{v}$, $p_\sim$, $\varrho_\sim$, we obtain the Kuznetsov equation

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\frac{1}{c^2}\frac{B}{2A}(\psi_t)^2 + |\nabla\psi|^2\right)_t, \tag{6}$$

where δ is the diffusivity of the sound, see, e.g., Coulouvrat (1992), Kuznetsov (1971), for details.

Motivated by the exhibited infinite signal speed paradox, versions of this nonlinear acoustic wave equation have been recently developed that are based on a replacement of Fourier's law by the Maxwell-Cattaneo law

$$\tau\dot{q} + \vec{q} = -K\nabla\theta, \tag{7}$$

where the dot $\dot{}$ denotes the material derivative (which becomes just the time derivative if the convective term is neglected) and τ is a positive constant accounting for relaxation. An analogous procedure to the one leading to (6) then yields

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \left(\frac{1}{c^2}\frac{B}{2A}(\psi_t)^2 + |\nabla\psi|^2\right)_t, \tag{8}$$

where $b = \delta + \tau c^2$, see Jordan (2009).

In what follows we shall consider the linearized version of equation (8) written in a more general abstract form. This will lead to an abstract third order in time equation which is driven by a selfadjoint positive operator $\mathcal{A}$ defined on a Hilbert space $\mathcal{H}$

$$\tau u_{ttt} + \alpha u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = 0. \tag{9}$$

It is interesting to notice that the third order in time model has very different characteristics from the familiar second order equation ($\tau = 0, \alpha > 0$). The well-posedness of solutions fails, Fattorini (1983), even in the simplest case when $b = 0$ – see Theorem 1.1 in Subsection 1.2. Thus, the structural damping is essential for the well-posedness of third order systems. As we know, for second order (in time) equations the presence of the structural damping is irrelevant for well-posedness, it does, however, play a role in asymptotic behavior and regularity of solutions. Instead, for the third order equations structural damping ($b > 0$) is critical for the well-posedness. More specifically, it does affect both well-posedness and stability. Our main goal is to provide a complete analysis and classification of parameters leading to both well-posedness and stability of the abstract model under consideration.

More precisely, exponential stability of the trajectories depends on the critical parameter $\frac{\textit{sound speed}\times\textit{relaxation parameter}}{\textit{sound diffusivity}}$ which is required to be small enough with respect to a natural damping α in the system. More specifically, exponential stability requires $\gamma \equiv \alpha - \frac{\tau c^2}{b} > 0$. In the complementary region of the parameters the system is unstable ($\gamma < 0$) or marginally stable ($\gamma = 0$). The theoretical findings and estimates are supported by numerical calculations that reveal spectral properties of the system.

1.2. Main results

Let $\mathcal{A}$ be a self-adjoint positive operator on $\mathcal{H}$ with a dense domain $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}$. We consider the following third order in time abstract evolution equation defined on $\mathcal{H}$, where $\mathcal{H}$ is a real Hilbert space:

$$\tau u_{ttt} + \alpha u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = 0, \tag{10}$$

with the initial conditions given by

$$u(0) = u_0, \quad u_t(0) = u_1, \quad u_{tt}(0) = u_2. \tag{11}$$

This can be rewritten as a first order abstract system of the form

$$U_t(t) = AU(t),\ t > 0, \quad U(0) = U_0 = (u_0, u_1, u_2) \in H, \tag{12}$$

where $H \equiv \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{H}$ endowed with the graph norm

$$|U_0|_H^2 \equiv |\mathcal{A}^{1/2}u_0|_{\mathcal{H}}^2 + |\mathcal{A}^{1/2}u_1|_{\mathcal{H}}^2 + |u_2|_{\mathcal{H}}^2$$

which corresponds to

$$|U(t)|_H^2 = |\mathcal{A}^{1/2}u(t)|_{\mathcal{H}}^2 + |\mathcal{A}^{1/2}u_t(t)|_{\mathcal{H}}^2 + |u_{tt}(t)|_{\mathcal{H}}^2$$

and

$$U \equiv \begin{pmatrix} u \\ u_t \\ u_{tt} \end{pmatrix}; \quad A = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -c^2\tau^{-1}\mathcal{A} & -b\tau^{-1}\mathcal{A} & -\alpha\tau^{-1}I \end{pmatrix}. \tag{13}$$

Our goal is to discuss the well-posedness and asymptotic stability of the model defined in (10). Notice first that with $\tau = 0$ the model reduces to a classical wave equation with ($b > 0$) structural damping – which corresponds to an analytic semigroup. However, $\tau > 0$ makes the model of hyperbolic type and the well-posedness is no longer valid unless the parameters are appropriately selected. In what follows we shall assume that $\tau > 0$, $b \geq 0$.

Related models have been recently considered in Fernandez, Lizama, Poblete (2010, 2011). In Fernandez, Lizama, Poblete (2010), maximal regularity for the problem with zero Cauchy data was established. This result corresponds, however, to an analytic version of the model, which in our case corresponds to either $\tau = 0$ or the operator $\mathcal{A}$ assumed bounded on $\mathcal{H}$. In Fernandez, Lizama, Poblete (2011), the well-posedness of a "hyperbolic" version is considered. Here, however, the results obtained display the "loss" of regularity of transient solutions with respect to the regularity of initial conditions. Thus, the results of Fernandez, Lizama, Poblete (2011) do not lead to generation of a semigroup.

Our results, instead, show that such a loss does not occur. In fact, the model generates a strongly continuous *group* in a topological setting of both H (Theorem 1.2) and H_1 (Theorem 1.4) spaces. Moreover, we prove that the group is *exponentially stable* when $\gamma > 0, c > 0$. In the complementary region of the parameters $\gamma \geq 0, c > 0$ and also $\gamma > 0, c = 0$, it is demonstrated in Section 6 that the semigroup *is not* exponentially stable. Thus, the results obtained are complete and optimal.

Notation: $(u, v) \equiv (u, v)_{\mathcal{H}}$. $|u|^2 \equiv |u|_{\mathcal{H}}^2$, $\mathcal{A}^\theta$, $\theta \in [0, 1]$ denote fractional powers of $\mathcal{A}$; see, e.g., Pazy (1983).

THEOREM 1.1 *Let $b = 0$, $\alpha \in \mathbb{R}$ and $\mathcal{A}$ an unbounded operator (in addition to being self-adjoint and positive on $\mathcal{H}$). Then the resulting system (10) is ill-posed in the sense that it does not generate a strongly continuous semigroup on the state space H.*

THEOREM 1.2 *Let $b > 0$, $\alpha \in \mathbb{R}$. The system given in (10) generates a strongly continuous group on H.*

Our next result describes exponential decay for the energy. It turns out that the latter depends on the values of certain parameters.

Let us introduce the parameter $\gamma \equiv \alpha - \frac{c^2\tau}{b}$ and define the following energies:

$$E(t) \equiv \frac{b}{2}|\mathcal{A}^{1/2}(u_t + c^2 b^{-1} u)|^2 + \frac{\tau}{2}|u_{tt} + c^2 b^{-1} u_t|^2 + \frac{c^2}{2b}\gamma|u_t|^2, \quad (14)$$

$$E_0(t) \equiv \frac{\alpha}{2}|u_t|^2 + \frac{c^2}{2}|\mathcal{A}^{1/2}u|^2. \quad (15)$$

Theorem 1.3 *Let $b > 0$ and $\alpha > 0, c > 0$*

- *If $\gamma > 0$, there exist $\omega > 0, C > 0$ such that $\hat{E}(t) = E(t) + E_0(t)$ satisfies:*

$$\hat{E}(t) \leq Ce^{-\omega t}\hat{E}(0), \ t > 0.$$

- *If $\gamma = 0$, the energy $E(t)$ remains constant.*

Corollary 1.1 *Under the assumptions of Theorem 1.3 the semigroup e^{At} is exponentially stable on H.*

Proof. This follows from the equivalence of the norm generated by $\hat{E}(t)$ and that of H. ∎

With the motivation coming from the nonlinear model, we shall also consider solutions that exhibit more regularity. In fact, we will be able to show that with $b > 0$ the model defined in (12) generates a strongly continuous semigroup on the space

$$H_1 \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{H}$$

endowed with the norm; for $U = (u_0, u_1, u_2)$,

$$|U|^2_{H_1} \equiv |\mathcal{A}u_0|^2_{\mathcal{H}} + |\mathcal{A}^{1/2}u_1|^2_{\mathcal{H}} + |u_2|^2_{\mathcal{H}},$$

which on the trajectories corresponds to

$$|U(t)|^2_{H_1} \equiv |\mathcal{A}u(t)|^2_{\mathcal{H}} + |\mathcal{A}^{1/2}u_t(t)|^2_{\mathcal{H}} + |u_{tt}(t)|^2_{\mathcal{H}}.$$

Theorem 1.4 *Let the sound diffusivity parameter $b > 0$. The operator A given in (13) generates a strongly continuous group e^{At} on the space H_1.*

A straightforward corollary of Theorem 1.4 is the well-posedness of the nonhomogeneous problem:

Corollary 1.2 *Consider a nonhomogeneous equation driven by a forcing term $f \in L_1(0, T; \mathcal{H})$ with an arbitrary $T > 0$.*

$$\tau u_{ttt} + \alpha u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = f \quad (16)$$

with the initial data $(u_0, u_1, u_2) \in H_1$, and let $b > 0$.

Then, $(u, u_t, u_{tt}) \in C([0, T]; H_1)$.

REMARK 1.1 *It is interesting to notice the particular role played by the sound diffusivity coefficient b. While in the second order in time equations such damping leads to analytic semigroups, for the third order equation this damping is responsible for just well-posedness of the equation. The asymptotic decay depends, instead, on a damping parameter α and large values of the structural damping b (compensating for τ). This is a new feature of the abstract equations not seen in other models involving structural damping. The figures presented in Section 6 demonstrate this qualitative behavior.*

The remainder of the paper is devoted to the proofs of the main theorems (Sections 2–5), as well as numerical simulations of the spectrum to illustrate the theoretical results (Section 6).

2. Proof of Theorem 1.1

Under the conditions of Theorem 1.1 we are led to consider

$$U \equiv \begin{pmatrix} u \\ u_t \\ u_{tt} \end{pmatrix}; \quad A = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -c^2\tau^{-1}\mathcal{A} & 0 & -\alpha\tau^{-1}I \end{pmatrix}.$$

We shall consider the eigenvalue problem for A

$$(A - \lambda I)U = 0.$$

This leads to the consideration of the operator valued characteristic polynomial

$$f(\lambda, \mathcal{A}) \equiv (\lambda^3 + \alpha\tau^{-1}\lambda^2)I + c^2\tau^{-1}\mathcal{A}.$$

Let ν_n^2, ϕ_n be eigenvalues and eigenfunctions of $\mathcal{A}$:

$$\mathcal{A}\phi_n = \nu_n^2\phi_n, \quad n = 1, 2 \ldots, \nu_n^2 \to \infty.$$

To simplify the analysis, we consider A as a perturbation of

$$A_0 \equiv \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -c^2\tau^{-1}\mathcal{A} & 0 & 0 \end{pmatrix}$$

by

$$P \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha\tau^{-1}I \end{pmatrix}.$$

Clearly, $A = A_0 + P$ with $P \in \mathcal{L}(H)$. Thus, the question of generation of the semigroup reduces, via Perturbation Theorem, Pazy (1983), to the one of generation by A_0.

On the other hand, the characteristic polynomial associated with A_0 has a simplified form

$$f_0(\lambda, \mathcal{A}) \equiv \lambda^3 I + c^2\tau^{-1}\mathcal{A}\,.$$

Inspecting the roots of the equation

$$f_0(\lambda, \nu_n^2) = \lambda^3 + c^2\tau^{-1}\nu_n^2 = 0,$$

one sees that two branches of infinitely many eigenvalues have positive real parts. This violates the necessary condition for generation of the semigroup, as long as $\mathcal{A}$ is unbounded ($\nu_n^2 \to \infty$). Thus, the necessary condition for the generation of a strongly continuous semigroup is violated, Pazy (1983), page 51. In fact, this is a special case of a general negative k-th order result, when $k \geq 3$ — see Fattorini (1983), p. 99.

In the particular case under consideration, the procedure can also be directly applied to the eigenvalue equation

$$f(\lambda, \nu_n^2) = \lambda^2(\lambda + \tau^{-1}\alpha) + c^2\tau^{-1}\nu_n^2 = 0 \tag{17}$$

for A. Abbreviating $a = \alpha\tau^{-1}$, $\mu_n = c^2\tau^{-1}\nu_n^2$ and inserting $\lambda = x + \imath y$, yields the two equations

$$\begin{aligned} x^3 - 3xy^2 + ax^2 - ay^2 + \mu_n &= 0 \\ y(3x^2 - y^2 + 2ax) &= 0 \end{aligned}$$

for the real and imaginary part, respectively, of (17), which, in addition to the real solutions $y = 0$, $x < 0$ ($x = -\mu_n^{1/3}$ when $\alpha = 0$), leads to solutions with $y^2 = 3x^2 + 2ax$ and real parts x satisfying

$$\tilde{f}(x) = 8x^3 + 8ax^2 + 2a^2x - \mu_n = 0\,.$$

For sufficiently small $0 < c = c(\mu_0) < 1$, independent of n, we have

$$\tilde{f}(c\mu_n^{1/3}) = (8c^3 - 1)\mu_n + 8ac^2\mu_n^{2/3} + 2a^2c\mu_n^{1/3} < 0$$

(note that for c small enough $\tilde{f}(c\mu^{1/3}) \leq \tilde{f}(c\mu_0^{1/3})$ for all $\mu \geq \mu_0$.) On the other hand,

$$\tilde{f}(\mu_n^{1/3}) = 7\mu_n + 8a\mu_n^{2/3} + 2a^2\mu_n^{1/3} > 0\,.$$

Hence, there exists a root $x \in [c\mu_n^{1/3}, \mu_n^{1/3}]$ for each n, which implies existence of infinitely many eigenvalues having positive real parts.

3. Proof of Theorem 1.2

Let $b > 0$ and without loss of generality we normalize $\tau = 1$. It is convenient to introduce the following variable:

$$z \equiv u_t + c^2 b^{-1} u. \tag{18}$$

Consequently, $u_t = z - c^2 b^{-1} u$, and

$$\begin{aligned} u_{ttt} &= -\alpha u_{tt} - b\mathcal{A}z \\ u_{tt} &= z_t - c^2 b^{-1} u_t = z_t - c^2 b^{-1}[z - c^2 b^{-1} u] \\ z_{tt} &= u_{ttt} + c^2 b^{-1} u_{tt} \\ &= -(\alpha - c^2 b^{-1}) u_{tt} - b\mathcal{A}z \\ &= -\gamma z_t + \gamma c^2 b^{-1}[z - c^2 b^{-1} u] - b\mathcal{A}z. \end{aligned} \tag{19}$$

With this notation we introduce the vector

$$Y \equiv \begin{pmatrix} \mathcal{A}^{1/2} u \\ \mathcal{A}^{1/2} z \\ z_t \end{pmatrix}$$

and we consider $Z \equiv \mathcal{H} \times \mathcal{H} \times \mathcal{H}$. The original model can now be rewritten in operator form as:

$$Y_t(t) = BY(t), \quad Y(0) = Y_0 \in Z$$

where the matrix operator B with a natural domain takes the form:

$$B \equiv \begin{pmatrix} -c^2 b^{-1} I & I & 0 \\ 0 & 0 & \mathcal{A}^{1/2} \\ -\gamma c^4 b^{-2} \mathcal{A}^{-1/2} & -b\mathcal{A}^{1/2} + \gamma c^2 b^{-1} \mathcal{A}^{-1/2} & -\gamma I \end{pmatrix}, \tag{20}$$

$$\mathcal{D}(B) = \{Y = (y_1, y_2, y_3) \in Z; y_1 \in \mathcal{H}, y_2 \in \mathcal{D}(\mathcal{A}^{1/2}), y_3 \in \mathcal{D}(\mathcal{A}^{1/2})\}.$$

The operator matrix B can be represented as a bounded perturbation of B_0, i.e.: $B = B_0 + K$ where

$$B_0 \equiv \begin{pmatrix} -c^2 b^{-1} I & I & 0 \\ 0 & 0 & \mathcal{A}^{1/2} \\ 0 & -b\mathcal{A}^{1/2} & -\gamma I \end{pmatrix},$$

and the bounded perturbation denoted by K is given by

$$K \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\gamma c^4 b^{-2} \mathcal{A}^{-1/2} & \gamma c^2 b^{-1} \mathcal{A}^{-1/2} & 0 \end{pmatrix}. \tag{21}$$

Moreover, $B_0 = B_1 + P_B$, where the perturbation P_B is bounded on Z and given by

$$P_B \equiv \begin{pmatrix} -c^2b^{-1}I & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\gamma I \end{pmatrix} \tag{22}$$

and

$$B_1 \equiv \mathcal{A}^{1/2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -bI & 0 \end{pmatrix} \tag{23}$$

and $\mathcal{D}(B_1) = \mathcal{D}(B) \equiv \mathcal{H} \times \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2})$. We shall topologize Z with a weighted inner product

$$(Y, \hat{Y})_Z \equiv (y_1, \hat{y}_1)_{\mathcal{H}} + b(y_2, \hat{y}_2)_{\mathcal{H}} + (y_3, \hat{y}_3)_{\mathcal{H}}. \tag{24}$$

Thus we have the following decomposition

LEMMA 3.1 *Let $b > 0, \tau = 1$. Then*

$$B = B_1 + P_B + K,$$

where B, B_1, P_B, K are given by (20), (23), (22), (21), respectively, with the following properties

- *$B_1 : \mathcal{D}(B_1) \subset Z \to Z$ is skew-adjoint, hence maximally dissipative on Z.*
- *P_B and K are bounded on Z.*
- *If, in addition $\mathcal{A}$ has compact resolvent on $\mathcal{H}$ then K is compact on Z.*

Proof. The first statement follows from the obvious fact that the operator

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -bI & 0 \end{pmatrix} \tag{25}$$

is skew-adjoint with respect to the weighted inner product (24), hence B_1 is skew-adjoint, therefore maximally dissipative generator of a group.
The boundedness of P_B and K follows from the fact that each entry of the corresponding matrix operators represent bounded operators in $\mathcal{L}(\mathcal{H})$.
The third statement follows from the fact that $\mathcal{A}^{-1/2}$ is compact on $\mathcal{H}$. ∎

The well-posedness on Z is an immediate consequence of the fact that B is a bounded perturbation of a maximally dissipative operator B_1. Standard semigroup theory implies that B itself generates a strongly continuous semigroup on Z. On the other hand, it is evident that the topology on Z is equivalent (since $b > 0$) to the topology on H. This completes the proof of Theorem 1.2.

REMARK 3.1 *The stability of e^{Bt} depends on the lower order perturbations $P_B + K$. We will show exponential decay by deriving energy estimates in the next section.*

4. Proof of Theorem 1.3

The proof of Theorem 1.3 follows through several lemmas.

REMARK 4.1 *Recalling that* $\gamma \equiv \alpha - \frac{c^2\tau}{b}$, *we have that* $E(t) \geq 0$ *for* $\gamma \geq 0$. *Thus* $\hat{E}(t)$ *is equivalent in norm to the one induced by*

$$|u_{tt}(t)|^2 + |\mathcal{A}^{1/2}u_t(t)|^2 + |\mathcal{A}^{1/2}u(t)|^2.$$

In fact, the evolution $(u(t), u_t(t), u_{tt}(t)) \equiv S_t(u_0, u_1, u_2)$ *defines a continuous flow on*

$$H \equiv D(\mathcal{A}^{1/2}) \times D(\mathcal{A}^{1/2}) \times \mathcal{H}.$$

Step 1: The energy dissipation.

LEMMA 4.1 *The following identity holds*

$$\frac{d}{dt}E(t) + \gamma|u_{tt}|^2 = 0. \tag{26}$$

Thus, when $\gamma \geq 0$, the problem is dissipative with a strict dissipation when $\gamma > 0$. Instead, when $\gamma = 0$, the problem is conservative. This is to say:

- $E(t) + \gamma\int_0^t |u_{tt}|^2 ds = E(0), \gamma > 0$
- $E(t) = E(0), \gamma = 0.$

Proof. Step 1: Multiply (10) by u_{tt} and integrate by parts. This gives

$$\frac{d}{dt}\Big[\tau|u_{tt}|^2 + b|\mathcal{A}^{1/2}u_t|^2 + 2c^2(\mathcal{A}u, u_t)\Big] + 2\alpha|u_{tt}|^2 - 2c^2|\mathcal{A}^{1/2}u_t|^2 = 0. \tag{27}$$

Step 2: Multiply (10) by u_t and integrate by parts to obtain

$$\frac{d}{dt}\Big[c^2|\mathcal{A}^{1/2}u|^2 + \alpha|u_t|^2 + 2\tau(u_{tt}, u_t)\Big] + 2b|\mathcal{A}^{1/2}u_t|^2 - 2\tau|u_{tt}|^2 = 0. \tag{28}$$

Step 3. Combine the equalities and perform some algebraic manipulations to complete squares,

$$\frac{d}{dt}\Big[\tau|u_{tt}|^2 + b|\mathcal{A}^{1/2}u_t|^2 + 2c^2(\mathcal{A}u, u_t) + \frac{c^2}{b}(c^2|\mathcal{A}^{1/2}u|^2 + \alpha|u_t|^2 + 2\tau(u_{tt}, u_t))\Big] + 2(\alpha - \frac{\tau c^2}{b})|u_{tt}|^2 = 0. \tag{29}$$

Using notation (14), we obtain the claimed equality. ∎

Step 2: Equipartition of the energy. Recall that multiplying the original equation by u_t and integrating by parts leads to:

$$b|\mathcal{A}^{1/2}u_t|^2 = \tau|u_{tt}|^2 - \frac{d}{dt}\Big[\frac{1}{2}\alpha|u_t|^2 + \frac{1}{2}c^2|\mathcal{A}^{1/2}u|^2 + \tau(u_{tt}, u_t)\Big]\,, \tag{30}$$

see (28).

Step 3: Boundedness of the energy. Our first goal is to establish the boundedness of the total energy $E(t) + E_0(t)$ where

$$E_0(t) \equiv \frac{1}{2}\alpha|u_t(t)|^2 + \frac{1}{2}c^2|A^{1/2}u(t)|^2.$$

From Lemma 4.1 and (30) we conclude that

$$\frac{d}{dt}E(t) + \frac{\gamma}{2\tau}\frac{d}{dt}E_0(t) + \frac{1}{2}\gamma|u_{tt}|^2 + \frac{1}{2}\gamma b\tau^{-1}|\mathcal{A}^{1/2}u_t|^2 = -\frac{\gamma}{2}\frac{d}{dt}(u_{tt}, u_t). \tag{31}$$

In particular, with C denoting a generic constant (independent of t), the following inequality holds for all $s < t$

Lemma 4.2 *Let $\gamma \geq 0$. Then the total energy is bounded for all times.*

$$\begin{aligned} E(t) + \frac{\gamma}{2\tau}E_0(t) + \frac{1}{2}\gamma\int_s^t [|u_{tt}|^2 + b\tau^{-1}|\mathcal{A}^{1/2}u_t|^2]dz \\ \leq E(s) + \frac{\gamma}{2\tau}E_0(s) + C_{\gamma,\tau,b,\alpha}[E(t) + E(s)] \leq C_{\gamma,b,\tau,\alpha}[E(s) + E_0(s)]. \end{aligned} \tag{32}$$

This means that the total energy is bounded in time by the initial *total* energy. Our goal is to show that the energy is exponentially decaying provided $\gamma > 0$. In view of Lemma 4.2 it only remains to provide an estimate for $\int_0^T |\mathcal{A}^{1/2}u|^2dt$.

Step 4: Reconstruction of $\int_0^T |\mathcal{A}^{1/2}u|^2dt$. In order to derive exponential decays one needs to reconstruct the time integral of $|\mathcal{A}^{1/2}u|^2$. For this we apply the multiplier u, which leads to

$$\frac{b}{2}\frac{d}{dt}|\mathcal{A}^{1/2}u|^2 + c^2|\mathcal{A}^{1/2}u|^2 = \alpha|u_t|^2 + \frac{d}{dt}[\frac{\tau}{2}|u_t|^2 - \tau(u_{tt}, u) - \alpha(u_t, u)]. \tag{33}$$

Integrating in time (33) gives

$$\begin{aligned} &\frac{b}{2}|\mathcal{A}^{1/2}u(t)|^2 + c^2\int_s^t |\mathcal{A}^{1/2}u|^2 \\ &= \frac{b}{2}|\mathcal{A}^{1/2}u(s)|^2 + \alpha\int_s^t |u_t|^2 + \Big[\frac{\tau}{2}|u_t|^2 - \tau(u_{tt}, u) - \alpha(u_t, u)\Big]_s^t \\ &\leq \alpha\int_s^t |\mathcal{A}^{1/2}u_t|^2 + CE(t) + CE_0(t) + CE(s) + CE_0(s) \\ &\leq C[E(t) + E_0(t) + E(s) + E_0(s)] \leq C[E(0) + E_0(0)] = \hat{E}(0)\,, \end{aligned} \tag{34}$$

where the last inequality holds by Lemma 4.2. Combining (34) and Lemma 4.2 we obtain

$$\frac{1}{2}\gamma\int_s^t [|u_{tt}|^2 + b\tau^{-1}|\mathcal{A}^{1/2}u_t|^2]dz + c^2\int_s^t |\mathcal{A}^{1/2}u|^2 \leq C[E(0) + E_0(0)],$$

which implies

$$\int_0^T [E(t) + E_0(t)] \leq C_{\gamma,b,\alpha,\tau,c}[E(0) + E_0] < \infty,$$

proving exponential decays on the strength of Theorem 4.1, p.116, Pazy (1983), which generalizes (from $p = 2$ to $1 \leq p < \infty$) the result of R. Datko (1970).

5. Proof of Theorem 1.4

The proof relies on a suitable decomposition of the state that is compatible with the topology generated by H_1.

To this end we introduce the following variables

$$z \equiv u_t + c^2 b^{-1} u$$

as in the previous section (see (18)) and

$$v \equiv \mathcal{A}u + b^{-1} z_t.$$

Consequently, $u_t = z - c^2 b^{-1} u$ and, instead of (19), we get

$$\begin{aligned}
u_{tt} &= z_t - c^2 b^{-1} u_t = z_t - c^2 b^{-1}[z - c^2 b^{-1} u] \\
u_{ttt} &= z_{tt} - c^2 b^{-1}[z_t - c^2 b^{-1} u_t] = z_{tt} - c^2 b^{-1}[z_t - c^2 b^{-1}[z - c^2 b^{-1} u]] \\
v_t &= (\mathcal{A}u + b^{-1} z_t)_t = b^{-1}[-u_{ttt} - \alpha u_{tt} - c^2 \mathcal{A}u] + b^{-1} z_{tt} \\
&= -b^{-1}\alpha u_{tt} - b^{-1} c^2 \mathcal{A}u + c^2 b^{-2}[z_t - c^2 b^{-1} z + c^4 b^{-2} u] \\
&= -c^2 b^{-1} v + [2c^2 b^{-2} - \alpha b^{-1}] z_t - b^{-3} c^4 [z - c^2 b^{-1} u] + \alpha c^2 b^{-2}[z - c^2 b^{-1} u] \\
&= -c^2 b^{-1} v + [2c^2 b^{-2} - \alpha b^{-1}] z_t + \gamma c^2 b^{-2}[z - c^2 b^{-1} u]. \qquad (35)
\end{aligned}$$

With this notation we introduce the vector

$$Y \equiv \begin{pmatrix} v \\ \mathcal{A}^{1/2} z \\ z_t \end{pmatrix}$$

and we consider $Z \equiv \mathcal{H} \times \mathcal{H} \times \mathcal{H}$. The original model can now be rewritten in operator form as:

$$Y_t(t) = DY(t), \quad Y(0) = Y_0 \in Z$$

where the matrix operator D with a natural domain takes the form $D = D_0 + K_1$ with

$$D_0 \equiv \begin{pmatrix} -c^2 b^{-1} I & 0 & (2c^2 b^{-2} - \alpha b^{-1}) I \\ 0 & 0 & \mathcal{A}^{1/2} \\ 0 & -b\mathcal{A}^{1/2} & -\gamma I \end{pmatrix}$$

$$\mathcal{D}(D) = \mathcal{D}(D_0) = \{Y = (y_1, y_2, y_3) \in Z; y_1 \in \mathcal{H}, y_2 \in \mathcal{D}(\mathcal{A}^{1/2}), y_3 \in \mathcal{D}(\mathcal{A}^{1/2})\}$$

and K_1 is a suitable bounded (respectively compact if $\mathcal{A}^{-1}$ is compact) perturbation comprised of lower order terms resulting from the decomposition (5) and is given by

$$K_1 \equiv \begin{pmatrix} -\gamma c^4 b^{-3} \mathcal{A}^{-1} & \gamma c^2 b^{-2} \mathcal{A}^{-1/2} & \gamma c^4 b^{-4} \mathcal{A}^{-1} \\ 0 & 0 & 0 \\ -\gamma c^4 b^{-2} \mathcal{A}^{-1} & \gamma c^2 b^{-1} \mathcal{A}^{-1/2} & \gamma c^4 b^{-3} \mathcal{A}^{-1} \end{pmatrix}. \tag{36}$$

The same argument as in the proof of Theorem 1.2 applies to inference that D_0 generates a continuous group on Z. Indeed, D_0 is a bounded perturbation of the same operator B_1 as introduced in the proof of Theorem 1.2. Since K_1 is also bounded on Z (in fact compact when $\mathcal{A}^{-1}$ is compact), the entire D is a bounded perturbation of a skew adjoint operator on Z. This proves that D is the generator of a group on Z. On the other hand, direct inspection reveals that with the given change of variables defined by Y, we have that $Y \in Z$ is equivalent to $U \in H_1$.

6. Spectral computations

The figures presented here illustrate the behavior of the spectrum of the operator A as depending on the parameters c^2 (sound speed), $\alpha \geq 0$ (viscous damping), $b \geq 0$ (sound diffusivity), a parameter $n \to \infty$ representing the natural modes of an elliptic operator, i.e., $\lambda_n = n^2$ where λ_n are the eigenvalues of $\mathcal{A}$. Taking $\tau = 1$, the *Mathematica* software package (by *Wolfram*) was used to derive a symbolic form of the solutions to the characteristic polynomial associated with the operator A in terms of the parameters. This approach made it possible to create a parametric plot showing the variation in the eigenvalues as one parameter varied while the other three parameters remained fixed at prescribed values as indicated in the captions of the respective figures. As we recall, the parameter critical for stability is $\gamma = \alpha - \frac{\tau c^2}{b}$, so we consider four different ranges for it.

- *Figure 1*: Corresponds to the values $\alpha = b = 1$, $n = 5$. Hence, $\gamma_c = 1 - c^2$. As the values of c vary from 0 to 1, the real eigenvalues corresponding to the mode $n = 5$ vary from 0 to -1. The real parts of the complex conjugate eigenvalues vary, instead, from $-\frac{1}{2}$ to 0. The values of $c = 0$ and $c = 1$ (hence $\gamma_c = 0$) produce eigenvalues with real parts equal zero

– marginal stability regions. Indeed, when $\gamma_c = 0$, the wave part of the operator B (or B_0) is unstable (marginally stable). Instead, when $c = 0$, then the diffusional part of the flow represented by the first entry in the matrix B (or B_0) is unstable.

- *Figure 2*: Corresponds to the values $\alpha = c = 0.5$, $n = 5$. Hence $\gamma_b = \frac{1}{2} - \frac{1}{4b}$, with b varying, $b \in (0, 2)$. Strong instability (real parts of eigenvalues positive) occurs when $b < \frac{1}{2}$ or $\gamma_b < 0$.
- *Figure 3*: Corresponds to the values $b = c = \frac{1}{2}$, $n = 5$. Thus, $\gamma_\alpha = \alpha - \frac{1}{2}$, with α varying, $\alpha \in (0, 1)$. Strong instability corresponding to the wave operator occurs when $\alpha < \frac{1}{2}$ or $\gamma_\alpha < 0$.
- *Figure 4*: Corresponds to the values $\alpha = c = 1$, $b = 2$. Then, $\gamma_n = \frac{1}{2}$ with values n ranging, $n \in (0.1, 8)$. The numbers n represent natural modes of the elliptic operator $\mathcal{A}$. Fig. 4 clearly displays the behavior of a damped equation with the spectrum located on two branches of hyperbolas. The real parts of the corresponding eigenvalues converge to $-\frac{1}{2}\gamma = -\frac{1}{4}$, while the diffusive part has the accumulation point (with respect to running n) at $-\frac{c^2}{b} = -0.5$. This picture reveals characteristics of a full spectrum of the operator B as combing a hyperbolic part with a diffusive part. Clearly, the overall stability depends on the values encoded by the matrix K, which, however, is responsible for the finite dimensional behavior only.

As seen below, the spectrum of the generator A reveals a hyperbolic behavior with stability exhibited when $\gamma > 0$ and $c > 0$. This confirms the findings of Theorem 1.3.

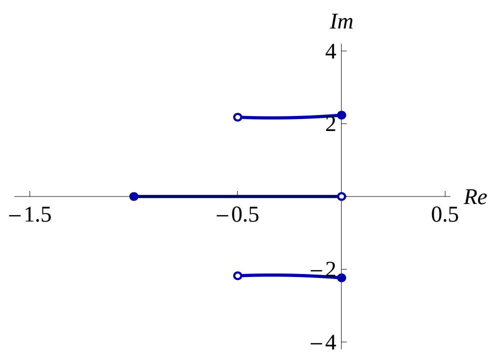

Figure 1. Variation in the spectrum as **c varies** from 0 (circles) to 1 (dots) with $\alpha = b = 1, n = 5$; $\boldsymbol{\gamma_c = 1 - c^2}$.

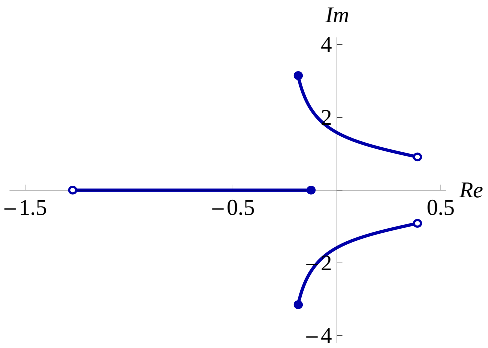

Figure 2. Variation in the spectrum as $\boldsymbol{b}$ **varies** from 0 (circles) to 2 (dots) with $\alpha = c = 0.5, n = 5$; $\boldsymbol{\gamma_b = \frac{1}{2} - \frac{1}{4b}}$.

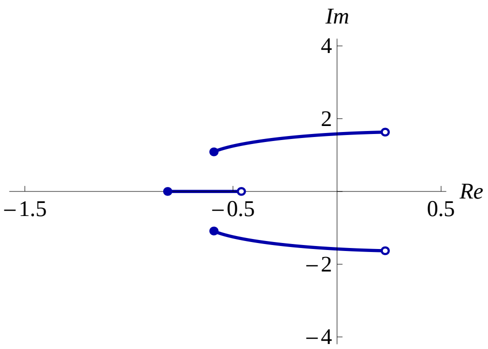

Figure 3. Variation in the spectrum as $\boldsymbol{\alpha}$ **varies** from 0 (circles) to 1 (dots) with $b = c = 0.5, n = 5$; $\boldsymbol{\gamma_\alpha = \alpha - \frac{1}{2}}$.

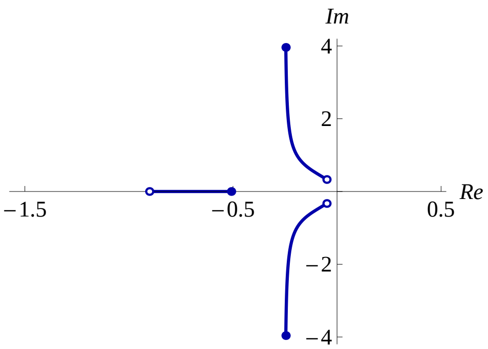

Figure 4. Variation in the spectrum as $\boldsymbol{n}$ **varies** from 0.1 (circles) to 8 (dots) with $\alpha = c = 1, b = 2$; $\boldsymbol{\gamma_n = \frac{1}{2}}$.

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